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# ON SMALL BASES FOR WHICH 1 HAS COUNTABLY MANY EXPANSIONS

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*Abstract.* Let $q \in (1, 2)$. A $q$-expansion of a number $x$ in $[0, 1/(q-1)]$ is a sequence $(\delta_i)_{i=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ satisfying

$$x = \sum_{i=1}^{\infty} \frac{\delta_i}{q^i}.$$

Let $\mathcal{B}_{\aleph_0}$ denote the set of $q$ for which there exists $x$ with a countable number of $q$-expansions, and let $\mathcal{B}_{1,\aleph_0}$ denote the set of $q$ for which 1 has a countable number of $q$-expansions. In Erdős *et al* [On the uniqueness of the expansions $1 = \sum_{i=1}^{\infty} q^{-n_i}$. *Acta Math. Hungar.* **58** (1991), 333–342] it was shown that $\min \mathcal{B}_{\aleph_0} = \min \mathcal{B}_{1,\aleph_0} = (1+\sqrt{5})/2$, and in S. Baker [On small bases which admit countably many expansions. *J. Number Theory* **147** (2015), 515–532] it was shown that $\mathcal{B}_{\aleph_0} \cap ((1+\sqrt{5})/2, q_1] = \{q_1\}$, where $q_1$ ($\approx 1.64541$) is the positive root of $x^6 - x^4 - x^3 - 2x^2 - x - 1 = 0$. In this paper we show that the second smallest point of $\mathcal{B}_{1,\aleph_0}$ is $q_3$ ($\approx 1.68042$), the positive root of $x^5 - x^4 - x^3 - x + 1 = 0$. En route to proving this result, we show that $\mathcal{B}_{\aleph_0} \cap (q_1, q_3] = \{q_2, q_3\}$, where $q_2$ ($\approx 1.65462$) is the positive root of $x^6 - 2x^4 - x^3 - 1 = 0$.

§1. *Introduction.* Let $q \in (1, 2)$ and $I_q := [0, 1/(q-1)]$. For each $x \in I_q$, there exists a sequence $(\delta_i)_{i=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ such that

$$x = \sum_{i=1}^{\infty} \frac{\delta_i}{q^i}.$$

The sequence $(\delta_i)_{i=1}^{\infty}$ is called a $q$-expansion for $x$. Without confusion, we simplify $(\delta_i)_{i=1}^{\infty}$ as $(\delta_i)$. It is straightforward to show that a real number $x$ has a $q$-expansion if and only if $x \in I_q$.

We now introduce some notation. The so-called *coding map* is defined to be $\Pi : \{0, 1\}^{\mathbb{N}} \to I_q$, where

$$\Pi((\delta_i)) = \sum_{i=1}^{\infty} \frac{\delta_i}{q^i}. \tag{1}$$

Throughout, we let $(\varepsilon_1 \dots \varepsilon_n)^k$ denote the $k$-fold concatenation of $(\varepsilon_1 \dots \varepsilon_n) \in \{0, 1\}^n$, and similarly let $(\varepsilon_1 \dots \varepsilon_n)^{\infty}$ denote the infinite concatenations of

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$(\varepsilon_1 \dots \varepsilon_n)$. Given $x \in I_q$, let $\Sigma_q(x)$ denote the set of all $q$-expansions of $x$, that is,

$$\Sigma_q(x) = \{(\delta_i) \in \{0,1\}^{\mathbb{N}} : \Pi((\delta_i)) = x\}.$$

The cardinality of the set $\Sigma_q(x)$ plays an important role in the investigation of representations of real numbers in non-integer bases. It was shown in [12] that if $q \in (1, (1+\sqrt{5})/2)$, then, for each $x \in (0, 1/(q-1))$, there are $2^{\aleph_0}$ different $q$-expansions. Sidorov showed in [16, 17] that if $q \in (1,2)$, then Lebesgue almost every $x \in (0, 1/(q-1))$ has $2^{\aleph_0}$ different $q$-expansions. Points belonging to $I_q$ with a unique $q$-expansion were investigated in [7, 13] for $q \in ((1+\sqrt{5})/2, 2)$. Some results concerning points having a fixed number of $q$-expansions were established in [2, 3, 8, 9, 18–20].

Let $m \in \mathbb{N} \cup \{\aleph_0\}$ and define

$$\mathcal{B}_m := \left\{ q \in \left[ \frac{1+\sqrt{5}}{2}, 2 \right) : \exists x \in I_q \text{ satisfying } \#\Sigma_q(x) = m \right\}.$$

Here and hereafter $\#A$ denotes the cardinality of a set $A$. The following results are known to hold:

- $\min \mathcal{B}_{\aleph_0} = (1+\sqrt{5})/2$ [9];
- $\min \mathcal{B}_2 = \check{q} \approx 1.71064$ (the positive root of $x^4 - 2x^2 - x - 1 = 0$) [18];
- $\min \mathcal{B}_k = q_f \approx 1.75488$, $k \geqslant 3$ (the positive root of $x^3 - 2x^2 - 1 = 0$) [3];
- $\mathcal{B}_2 \cap (\check{q}, q_f] = \{q_f\}$ [3];
- the smallest element of $\mathcal{B}_{\aleph_0}$ strictly greater than $(1+\sqrt{5})/2$ is $q_1$ ($\approx 1.64541$) (the positive root of $x^6 - x^4 - x^3 - 2x^2 - x - 1 = 0$) [2].

Understanding the $q$-expansions of 1 is a classical problem; see [13–15] and the references therein. The motivation of this paper is to provide a clearer understanding of what values $\#\Sigma_q(x)$ can take. Let

$$\mathcal{B}_{1,m} := \left\{ q \in \left[ \frac{1+\sqrt{5}}{2}, 2 \right) : \#\Sigma_q(1) = m \right\}.$$

It was shown in [14] that $\min \mathcal{B}_{1,1} \approx 1.78723$ (the Komornik–Loreti constant). In [9], it was proved that $\min \mathcal{B}_{1,\aleph_0} = (1+\sqrt{5})/2$. For any $n \in (\mathbb{N}\backslash\{1\}) \cup \{\aleph_0\}$, Erdős and Joó [10, 11] constructed a continuum of real numbers $q \in [q_0, 2)$ ($q_0 > 1.99803$) for which the number 1 has precisely $n$ $q$-expansions.

Motivated by the results listed above, a natural question arises: what is the second smallest point of $\mathcal{B}_{1,\aleph_0}$? In this paper we will answer this question.

Throughout the paper, we let $q_1, q_2, q_3$ be as follows: $q_1 \approx 1.64541$, $q_2 \approx 1.65462$ and $q_3 \approx 1.68042$, which are the positive roots of $x^6 - x^4 - x^3 - 2x^2 - x - 1 = 0$, $x^6 - 2x^4 - x^3 - 1 = 0$ and $x^5 - x^4 - x^3 - x + 1 = 0$, respectively. Our main result is the following.

THEOREM 1.1. *The smallest element of $\mathcal{B}_{1,\aleph_0}$ strictly greater than $(1+\sqrt{5})/2$ is $q_3$.*

En route to proving this result, we show the following.

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THEOREM 1.2. $\mathcal{B}_{\aleph_0} \cap ((1+\sqrt{5})/2, q_3] = \{q_1, q_2, q_3\}$.

We remark that $\mathcal{B}_{\aleph_0} \cap ((1+\sqrt{5})/2, q_1] = \{q_1\}$ is known [2]. The following corollary is an immediate consequence of Theorem 1.1, $\min \mathcal{B}_2 \approx 1.71064$ [18], $\min \mathcal{B}_k \approx 1.75488$ for $k \geqslant 3$ [3], $\mathcal{B}_2 \cap (\check{q}, q_f] = \{q_f\}$ [3] and $\min \mathcal{B}_{1,1} \approx 1.78723$ [14].

COROLLARY 1.3. *If* $q \in (1, q_3) \backslash \{(1+\sqrt{5})/2\}$, *then* 1 *has* $2^{\aleph_0}$ *different* $q$*-expansions.*

The paper is arranged as follows. Some definitions and results from [2] will be recalled in §2. Some results from that paper will be extended to our setting. In §3, we prove Theorem 1.2. The final section is devoted to the proof of Theorem 1.1.

§2. *Preliminaries.* In this section we shall recall some definitions and results from [2]. An interpretation of $q$-expansions from the perspective of dynamical systems was given in [2]; see also [4–6]. Let $T_{q,0}(x) = qx$ if $x \in [0, 1/(q^2-q)]$, and let $T_{q,1}(x) = qx - 1$ if $x \in [1/q, 1/(q-1)]$. We see that if $x \in L_q := [0, 1/q)$ or $x \in R_q := (1/(q^2-q), 1/(q-1)]$, then only one $T_{q,i}$ can be applied. However, when $x \in S_q := [1/q, 1/(q^2-q)]$, which is usually referred to as the switch region, we have a choice between $T_{q,0}$ and $T_{q,1}$. An element of $\bigcup_{n=0}^{\infty}\{T_{q,0}, T_{q,1}\}^n$ is denoted by $\mathbf{a}$; here $\{T_{q,0}, T_{q,1}\}^0$ denotes the identity map. Moreover, if $\mathbf{a} = a_1, \ldots, a_n$, we shall use $\mathbf{a}(x)$ to denote $a_n(\cdots(a_1(x))\cdots)$. Given $x \in I_q$, we call a finite sequence of transformations $\mathbf{a} = a_1 \ldots a_n$ *minimal* for $x$ if $\mathbf{a}(x) \in S_q$ and $\mathbf{a}|i(x) \notin S_q$ for all $i < n$. Here $\mathbf{a}|i = a_1 \ldots a_i$. We call $\mathbf{a}(x)$ a *branching point* of $x$ if $\mathbf{a}(x) \in S_q$.

Define

$$\Omega_q(x) := \{(a_i)_{i=1}^{\infty} \in \{T_{q,0}, T_{q,1}\}^{\mathbb{N}} : a_n(\cdots(a_1(x))\cdots) \in I_q \ \forall n \in \mathbb{N}\}.$$

The set $\Omega_q(x)$ is significant because $\#\Sigma_q(x) = \#\Omega_q(x)$, where our bijection is given by mapping $(\delta_i)$ to $(T_{q,\delta_i})$; see [1].

*Construction of the branching tree.* The branching tree was constructed in [2] to study $\mathcal{B}_{\aleph_0}$. We now provide details of its construction. Suppose that $x \in I_q$ and $\Omega_q(x)$ (or $\Sigma_q(x)$) is infinite. There exists a unique minimal $\mathbf{a} \in \bigcup_{n=0}^{\infty}\{T_{q,0}, T_{q,1}\}^n$ such that $\mathbf{a}(x) \in S_q$. Then there are two possibilities.

*Case 1: there exists a unique* $i \in \{0, 1\}$ *such that* $\Omega_q(T_{q,i}(\mathbf{a}(x)))$ *is finite and* $\Omega_q(T_{q,1-i}(\mathbf{a}(x)))$ *is infinite.* In this case, we draw a horizontal line of finite length that then bifurcates with an upper and a lower branch. The lower branch corresponds to $T_{q,i}(\mathbf{a}(x))$ and stops bifurcating; the upper branch corresponds to $T_{q,1-i}(\mathbf{a}(x))$ and goes on bifurcating.

*Case 2: both* $\Omega_q(T_{q,0}(\mathbf{a}(x)))$ *and* $\Omega_q(T_{q,1}(\mathbf{a}(x)))$ *are infinite.* In this case, we draw a horizontal line of finite length that then bifurcates with an upper and a lower branch. The lower branch corresponds to $T_{q,0}(\mathbf{a}(x))$; the upper branch corresponds to $T_{q,1}(\mathbf{a}(x))$. Both of them go on bifurcating.

If $\Omega_q(T_{q,i}(\mathbf{a}(x)))$ is infinite, as in Case 1 or Case 2, then there exists a unique minimal $\mathbf{a}' \in \bigcup_{n=0}^{\infty}\{T_{q,0}, T_{q,1}\}^n$ such that point $\mathbf{a}'(T_{q,i}(\mathbf{a}(x)))$ goes to

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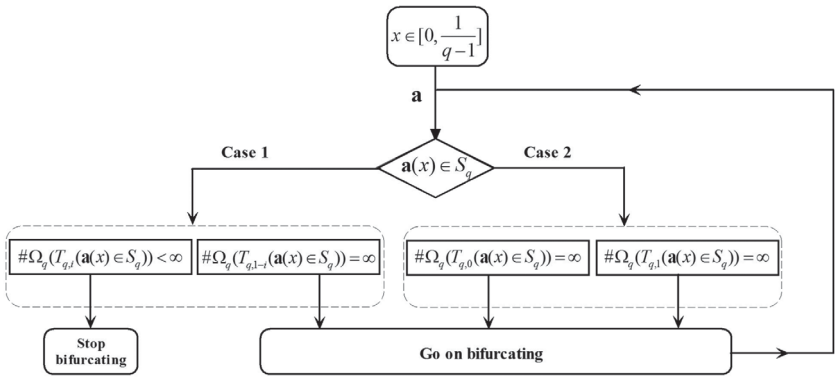



Figure 1: The flow chart of the bifurcating procedure corresponding to $x \in I_q$.

Case 1 or Case 2 again. This procedure continues indefinitely. The infinite tree we construct by repeating this process is known as the infinite branching tree corresponding to $x$. Figure 1 illustrates the bifurcating procedure.

A point $x$ for which $\Sigma_q(x)$ is infinite is said to be a *$q$ null infinite point* if, for each branching point of $x$, $\mathbf{a}(x)$ never goes to Case 2. It is easy to check that if $x$ is a *$q$ null infinite point*, then $\#\Sigma_q(x) = \aleph_0$. The $q$ null infinite points have a critical role in the proofs of the main results of [2].

LEMMA A [2, Proposition 2.7]. *$q \in \mathcal{B}_{\aleph_0}$ if and only if $I_q$ contains a $q$ null infinite point.*

Given $\min \mathcal{B}_2 = \check{q} \approx 1.71064$, $\min \mathcal{B}_k = q_f \approx 1.75488$ for $k \geqslant 3$ and $\mathcal{B}_2 \cap (\check{q}, q_f] = \{q_f\}$, we have that if $q \in ((1+\sqrt{5})/2, q_f)\backslash\{\check{q}\}$ and $x$ is a $q$ null infinite point, then, for each branching point of $x$, $\mathbf{a}(x)$, we have $\#\Omega_q(T_{q,i}(\mathbf{a}(x))) = \aleph_0$ and $\#\Omega_q(T_{q,1-i}(\mathbf{a}(x))) = 1$. This implies the following inclusion:

$$\left\{\mathbf{a}(x) : \mathbf{a}(x) \in S_q, \mathbf{a} \in \bigcup_{n=0}^{\infty}\{T_{q,0}, T_{q,1}\}^n\right\} \subseteq \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap S_q. \tag{2}$$

Here $U_q$ denotes the set of $x \in I_q$ having a unique $q$-expansion.

Unfortunately, it is difficult to deal with the set $S_q$. However, by some deductions we can restrict ourselves to a smaller set $J_q := [(q+q^2)/(q^4-1), (1+q^3)/(q^4-1)] \subseteq S_q$. We note that $J_q = S_q$ if and only if $q = q_f$.

LEMMA B [2, Lemma 3.1]. *Let $q \in [(1+\sqrt{5})/2, q_f)$. Suppose that $x \in I_q$ satisfies $\#\Sigma_q(x) > 1$; then there exists a finite sequence of transformations $\mathbf{a}$ such that $\mathbf{a}(x) \in J_q$. Here $q_f$ ($\approx 1.75488$) is the positive root of $x^3 - 2x^2 - 1 = 0$.*

Proposition 2.2 is devoted to characterizing the set $\bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q$ when $q \in [q_1, q_3]$. To prove this proposition, we need Lemmas C and 2.1.

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LEMMA C [13, Theorem 2]. *Let $q \in ((1+\sqrt{5})/2, q_f]$. Then*

$$U_q = \left\{\Pi(0^k(10)^\infty), \Pi(1^k(10)^\infty), 0, \frac{1}{q-1}\right\},$$

*where $k \geqslant 0$ and $\Pi$ is given as in (1).*

Set

$$\mathbf{y}_j = \Pi(01^j(10)^\infty) \quad \text{and} \quad \mathbf{z}_j = \Pi(10^j(01)^\infty)$$

for $j \geqslant 1$. Here and hereafter we let $(S_q \backslash J_q)_L = [1/q, (q+q^2)/(q^4-1))$ and $(S_q \backslash J_q)_R = ((1+q^3)/(q^4-1), 1/(q^2-q)]$.

LEMMA 2.1. *Let $q \in [q_1, q_3]$. Then the following hold.*

(i) $\mathbf{y}_j \in J_q$ *if and only if* $\mathbf{z}_j \in J_q$, *and* $\mathbf{y}_j \in (S_q \backslash J_q)_R$ *if and only if* $\mathbf{z}_j \in (S_q \backslash J_q)_L$.

(ii) $T_{q,0}^m(T_{q,1}(\mathbf{y}_j)) \in J_q$ *if and only if* $T_{q,1}^m(T_{q,0}(\mathbf{z}_j)) \in J_q$, *and* $T_{q,0}^m(T_{q,1}(\mathbf{y}_j)) \in (S_q \backslash J_q)_R$ *if and only if* $T_{q,1}^m(T_{q,0}(\mathbf{z}_j)) \in (S_q \backslash J_q)_L$.

(iii) $T_{q,0}^m(T_{q,1}(\mathbf{y}_j)) = \mathbf{y}_k$ *if and only if* $T_{q,1}^m(T_{q,0}(\mathbf{z}_j)) = \mathbf{z}_k$, *and* $T_{q,0}^m(T_{q,1}(\mathbf{y}_j)) = \mathbf{z}_k$ *if and only if* $T_{q,1}^m(T_{q,0}(\mathbf{z}_j)) = \mathbf{y}_k$. *Here $j \geqslant 1$ and $k \geqslant 1$.*

*Proof.* Direct computation shows that the following equations hold.

$$\mathbf{y}_j - \frac{1+q^3}{q^4-1} = -\mathbf{z}_j + \frac{q+q^2}{q^4-1}, \qquad \mathbf{y}_j - \frac{q+q^2}{q^4-1} = \frac{1+q^3}{q^4-1} - \mathbf{z}_j,$$
$$\frac{1}{q^2-q} - \mathbf{y}_j = \mathbf{z}_j - \frac{1}{q}, \tag{3}$$

$$T_{q,0}^m(T_{q,1}(\mathbf{y}_j)) - \frac{1+q^3}{q^4-1} = -T_{q,1}^m(T_{q,0}(\mathbf{z}_j)) + \frac{q+q^2}{q^4-1}, \tag{4}$$

$$T_{q,0}^m(T_{q,1}(\mathbf{y}_j)) - \frac{q+q^2}{q^4-1} = \frac{1+q^3}{q^4-1} - T_{q,1}^m(T_{q,0}(\mathbf{z}_j)),$$
$$\frac{1}{q^2-q} - T_{q,0}^m(T_{q,1}(\mathbf{y}_j)) = T_{q,1}^m(T_{q,0}(\mathbf{z}_j)) - \frac{1}{q}. \tag{5}$$

(i) is implied by (3), and (ii) is implied by (4) and (5).

Simplifying $T_{q,0}^m(T_{q,1}(\mathbf{y}_j)) = \mathbf{y}_k$ and $T_{q,1}^m(T_{q,0}(\mathbf{z}_j)) = \mathbf{z}_k$, we see that they are both equivalent to

$$(1 + 2q^{-1} - q - q^{-1-j} - q^{-2-m} - q^{-1-m} + q^{-2-k-m})(q^2-1)^{-1} = 0.$$

Similarly, simplifying $T_{q,0}^m(T_{q,1}(\mathbf{y}_j)) = \mathbf{z}_k$ and $T_{q,1}^m(T_{q,0}(\mathbf{z}_j)) = \mathbf{y}_k$, we see that they are both equivalent to

$$(-1 - 2q^{-1} + q + q^{-1-j} - q^{-2-m} + q^{-2-k-m} + q^{-m})(q^2-1)^{-1} = 0.$$

Thus, we obtain (iii). □

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PROPOSITION 2.2. *Let $q \in [q_1, q_3]$. Then we have*

$$\begin{aligned}
\bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q &= \{\mathbf{y}_j, \mathbf{z}_j, 1 \leqslant j \leqslant 3\} \\
&= \{\Pi(01^j(10)^\infty), \Pi(10^j(01)^\infty), 1 \leqslant j \leqslant 3\}, \\
\bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap (S_q \backslash J_q)_R &= \{\mathbf{y}_j, j \geqslant 4,\ 1/(q^2-q)\} \\
&= \{\Pi(01^j(10)^\infty), j \geqslant 4,\ 1/(q^2-q)\}, \\
\bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap (S_q \backslash J_q)_L &= \{\mathbf{z}_j, j \geqslant 4,\ 1/q\} \\
&= \{\Pi(10^j(01)^\infty), j \geqslant 4,\ 1/q\}.
\end{aligned}$$

*Proof.* It follows from Lemma C that

$$\bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) = \Big\{\Pi(0^j(10)^\infty), \Pi(01^{j-1}(10)^\infty), \Pi(10^{j-1}(10)^\infty), \Pi(1^j(10)^\infty), 0, \frac{1}{q^2-q}, \frac{1}{q}, \frac{1}{q-1}\Big\}.$$

Here $j \geqslant 1$. Then, by some straightforward computation, we have

$$\bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap S_q = \Big\{\mathbf{y}_j,\ \mathbf{z}_j,\ j \geqslant 1, \frac{1}{q^2-q},\ \frac{1}{q}\Big\}. \tag{6}$$

Next, we prove that

$$\mathbf{y}_j - \frac{1+q^3}{q^4-1} \begin{cases} \leqslant 0 & \text{when } j < 4, \\ > 0 & \text{when } j \geqslant 4, \end{cases} \tag{7}$$

when $q \in [q_1, q_3]$. It suffices to show that

$$q^{-1} + q + q^2 - q^3 - q^{-1-j} - q^{1-j} \begin{cases} \leqslant 0 & \text{when } j < 4, \\ > 0 & \text{when } j \geqslant 4. \end{cases} \tag{8}$$

By noting that

$$\mathbf{y}_j - \frac{1+q^3}{q^4-1} = (q^{-1} + q + q^2 - q^3 - q^{-1-j} - q^{1-j})(q^4-1)^{-1},$$

we see that (8) is equivalent to

$$\ln \frac{(q^2+1)}{-q^4+q^3+q^2+1} (\ln q)^{-1} \begin{cases} \geqslant j & \text{when } j < 4, \\ < j & \text{when } j \geqslant 4. \end{cases} \tag{9}$$

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The inequalities in (9) are true when $q \in [q_1, q_3]$, since $\ln((q^2+1)/(-q^4+q^3+q^2+1))(\ln q)^{-1}$ is strictly increasing on the interval $[q_1, q_4]$, and $\ln((q_4^2+1)/(-q_4^4+q_4^3+q_4^2+1))(\ln q_4)^{-1} = 4$ and $\ln((q_1^2+1)/(-q_1^4+q_1^3+q_1^2+1))(\ln q_1)^{-1} = 3$. Here $q_4$ ($\approx 1.69784$) is the positive root of $x^7 - x^5 - x^4 - 2x^3 - 2x^2 - x - 1 = 0$. A direct computation shows that $\mathbf{y}_j \in [(q+q^2)/(q^4-1), 1/(q^2-q)]$ for all $j \geqslant 1$. This statement combined with (7) shows that $\mathbf{y}_j \in J_q$ if $1 \leqslant j \leqslant 3$ and $\mathbf{y}_j \in (S_q \backslash J_q)_R$ if $j \geqslant 4$. Using Lemma 2.1(i), we also have $\mathbf{z}_j \in J_q$ if $1 \leqslant j \leqslant 3$ and $\mathbf{z}_j \in (S_q \backslash J_q)_L$ if $j \geqslant 4$. Our proof now follows from (6). □

§3. *Proof of Theorem 1.2.* In this section we shall give an algorithm to find all elements of the set $\mathcal{B}_{\aleph_0} \cap [q_1, q_3]$. Recall that $\min \mathcal{B}_2 = \check{q} \approx 1.71064$.

THEOREM 3.1. *Let $q \in [q_1, q_f)\backslash\{\check{q}\}$ and suppose that $\bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q$ is a finite set. Then $q \in \mathcal{B}_{\aleph_0}$ if and only if there exists $w \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q$ satisfying the following two properties.*

(i) *There exists a finite sequence of transformations $\mathbf{b} \in \bigcup_{n=0}^{\infty}\{T_{q,0}, T_{q,1}\}^n$ such that*

$$\mathbf{b}(w) = w.$$

(ii) *Let $\mathbf{b}$ be as above. Define*

$$B(w) := \{\mathbf{b}|i(w) : \mathbf{b}|i(w) \in S_q, 1 \leqslant i \leqslant |\mathbf{b}|\}.$$

*Then $B(w) \subset \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap S_q$. Here $|\mathbf{b}|$ denotes the length of $\mathbf{b}$.*

*Proof.* We begin with the rightwards implication. Suppose that $q \in \mathcal{B}_{\aleph_0}$. By Lemma A, there exists $x \in (0, 1/(q-1))$ such that $x$ is a $q$ null infinite point. Furthermore, we may assume that $x \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q$ by Lemma B and (2). Repeatedly applying Lemma B and (2), there exist $\mathbf{a}^1, \ldots, \mathbf{a}^m \in \bigcup_{n=0}^{\infty}\{T_{q,0}, T_{q1}\}^n$ satisfying

$$\mathbf{a}^i(\mathbf{a}^{i-1}(\ldots(\mathbf{a}^1(x))\ldots)) \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q$$

for each $1 \leqslant i \leqslant m$. Note that the set $\bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q$ is finite. Therefore, by the pigeonhole principle, for $m$ sufficiently large, there must exist $i, j \in \mathbb{N}$ with $1 \leqslant i < j \leqslant m$ such that

$$\mathbf{a}^j(\mathbf{a}^{j-1}(\ldots(\mathbf{a}^1(x))\ldots)) = \mathbf{a}^i(\mathbf{a}^{i-1}(\ldots(\mathbf{a}^1(x))\ldots)). \tag{10}$$

Set $\mathbf{b} = \mathbf{a}^{i+1}\cdots\mathbf{a}^j$, $\hat{\mathbf{a}} = \mathbf{a}^1\cdots\mathbf{a}^i$ and suppose that $\hat{\mathbf{a}}(x) = w$; here $w \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q$. Then (10) implies that

$$\mathbf{b}(w) = w.$$

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Since $x$ is a $q$ null infinite point, so are all its branching points. Thus, we have the second property.

To complete our 'if and only if' part, it suffices to remark that (i) and (ii) imply that $w$ is a $q$ null infinite point. So, $q \in \mathcal{B}_{\aleph_0}$. □

Now we search for all points belonging to $\mathcal{B}_{\aleph_0} \cap [q_1, q_3]$ by applying Theorem 3.1. Suppose that $x$ is a $q$ null infinite point and $x \in J_q$. First, we point out that

$$\bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q = \{\mathbf{y}_j = \Pi(01^j(10)^\infty), \mathbf{z}_j = \Pi(10^j(01)^\infty), 1 \leqslant j \leqslant 3\}$$

when $q \in [q_1, q_3]$ by Proposition 2.2. By Theorem 3.1, we only need to consider the behavior of elements of $\bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q$ under maps belonging to $\bigcup_{n=0}^{\infty}\{T_{q,0}, T_{q,1}\}^n$. By symmetry shown in Lemma 2.1, we only need to consider the points $\mathbf{y}_j$ for $1 \leqslant j \leqslant 3$. We establish the following lemma.

LEMMA 3.2.

(i) *If $q \in [q_1, q_3]$ and $\mathbf{y}_1$ is a $q$ null infinite point, then*

$$T_{q,1}(\mathbf{y}_1) \in L_q, \qquad T_{q,0}^j(T_{q,1}(\mathbf{y}_1)) \in L_q, \quad j = 1, 2,$$
$$T_{q,0}^3(T_{q,1}(\mathbf{y}_1)) \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q. \tag{11}$$

(ii) *If $q \in [q_1, q')$, then $\mathbf{y}_2$ is not a $q$ null infinite point. If $q \in [q', q_3]$ and $\mathbf{y}_2$ is a $q$ null infinite point, then*

$$T_{q,1}(\mathbf{y}_2) \in L_q, \qquad T_{q,0}(T_{q,1}(\mathbf{y}_2)) \in L_q,$$
$$T_{q,0}^2(T_{q,1}(\mathbf{y}_2)) \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q. \tag{12}$$

*Here $q'$ ($\approx 1.66184$) is the positive root of $x^5 - x^3 - x^2 - 2x - 2 = 0$.*

(iii) *If $q = q_1$ and $\mathbf{y}_3$ is a $q$ null infinite point, then*

$$T_{q,1}(\mathbf{y}_3) \in L_q, \qquad T_{q,0}(T_{q,1}(\mathbf{y}_3)) \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q. \tag{13}$$

*If $q \in (q_1, q_3]$ and $\mathbf{y}_3$ is a $q$ null infinite point, then $q$ must be $q_3$ or $q''$ ($\approx 1.67365$), the positive root of $x^5 - 2x^4 + x^3 - x^2 + x - 1 = 0$.*

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*Proof.* Direct computation yields (i). We prove (ii) now. A simple computation yields

$$T_{q,1}(\mathbf{y}_2) \in L_q, \qquad T_{q,0}(T_{q,1}(\mathbf{y}_2)) \in L_q,$$
$$T_{q,0}^2(T_{q,1}(\mathbf{y}_2)) \in \begin{cases} \left[\dfrac{q+q^2}{q^4-1}, \dfrac{1+q^3}{q^4-1}\right] & \text{if } q \in [q', q_3], \\ \left(\dfrac{1+q^3}{q^4-1}, \dfrac{1}{q^2-q}\right] & \text{if } q \in [q_1, q'). \end{cases}$$

Here $q'$ ($\approx 1.66184$) is the positive root of $x^5 - x^3 - x^2 - 2x - 2 = 0$. Thus, we obtain (12). We now assume that $\mathbf{y}_2$ is $q$ null infinite for some $q \in [q_1, q')$ and derive a contradiction. If $\mathbf{y}_2$ is a $q$ null infinite point for some $q \in [q_1, q')$, then $T_{q,0}^2(T_{q,1}(\mathbf{y}_2)) \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap (S_q \backslash J_q)_R$ and it can be shown that

$$T_{q,0}(T_{q,1}(T_{q,0}^2(T_{q,1}(\mathbf{y}_2)))) \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q.$$

Therefore, by Proposition 2.2, there exist $s \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap (S_q \backslash J_q)_R = \{\mathbf{y}_k, k \geqslant 4,\ 1/(q^2-q)\}$ and $u \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q$ such that

$$T_{q,0}^2(T_{q,1}(\mathbf{y}_2)) = s \quad \text{and} \quad T_{q,0}(T_{q,1}(T_{q,0}^2(T_{q,1}(\mathbf{y}_2)))) = u. \tag{14}$$

We will show that this is not possible, that is, $\mathbf{y}_2$ is not a $q$ null infinite point when $q \in [q_1, q')$. In fact,

$$T_{q,0}^2(T_{q,1}(\mathbf{y}_2)) = s$$

means that

$$q^5 - q^4 - 2q^3 + 2q + 1 = q^{-k}.$$

That is,

$$k = \frac{-\ln(q^5 - q^4 - 2q^3 + 2q + 1)}{\ln q}.$$

The function $-\ln(q^5 - q^4 - 2q^3 + 2q + 1)(\ln q)^{-1}$ is strictly decreasing on the interval $[q_1, q')$. Table 1 therefore implies that the only possible value of $k$ that may occur within the interval $[q_1, q')$ is $k = 4$. It is not possible that $s = 1/(q^2 - q)$, since $\lim_{k\to\infty} \mathbf{y}_k = 1/(q^2 - q)$ and $-\ln(q^5 - q^4 - 2q^3 + 2q + 1)(\ln q)^{-1}$ is monotonic. Table 1 includes the values of $q$ for which $T_{q,0}(T_{q,1}(T_{q,0}^2(T_{q,1}(\mathbf{y}_2)))) \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q$. Inspecting Table 1 shows that there are no values of $q$ for which both equations in (14) hold. Therefore, we may conclude (ii).

It remains to prove (iii). By direct computation, we have

$$T_{q,1}(\mathbf{y}_3) \in L_q, \quad T_{q,0}(T_{q,1}(\mathbf{y}_3)) \in \begin{cases} \left[\dfrac{q+q^2}{q^4-1}, \dfrac{1+q^3}{q^4-1}\right] & \text{if } q = q_1, \\ \left[\dfrac{1}{q}, \dfrac{q+q^2}{q^4-1}\right) & \text{if } q \in (q_1, q_3] \end{cases}$$

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Table 1: The values of $q$ for $\mathbf{y}_2$.

| | $k$ | $q$ | Polynomials |
|---|---|---|---|
| $T_{q,0}^2(T_{q,1}(\mathbf{y}_2)) = \mathbf{y}_k$ | 4 | 1.65027 | $x^4 - x^3 - x^2 - x + 1$ |
| | 5 | 1.63923 | $x^6 - 2x^4 - 2x^3 + x + 1$ |
| $T_{q,0}(T_{q,1}(T_{q,0}^2(T_{q,1}(\mathbf{y}_2)))) = \mathbf{y}_k$ | 1 | 1.65637 | $x^5 - 2x^3 - x^2 - x - 1$ |
| | 2 | 1.64308 | $x^5 - x^4 - x^3 - 1$ |
| | 3 | 1.63420 | $x^7 - 2x^5 - x^4 - x^2 - x - 1$ |
| $T_{q,0}(T_{q,1}(T_{q,0}^2(T_{q,1}(\mathbf{y}_2)))) = \mathbf{z}_k$ | 1 | 1.64114 | $x^2 - x - 1$ |
| | 2 | 1.65363 | $x^5 - 2x^3 - x^2 + 1$ |
| | 3 | 1.66065 | $x^5 - x^4 - x^3 + 1$ |

and we obtain (13). Furthermore, if $\mathbf{y}_3$ is a $q$ null infinite point for some $q \in (q_1, q_3]$, then it is straightforward to show that

$$T_{q,1}(T_{q,0}(T_{q,0}(T_{q,1}(\mathbf{y}_3)))) \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q.$$

Moreover, there must exist $s \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap (S_q \backslash J_q)_L = \{\mathbf{z}_k, k \geqslant 4,\ 1/q\}$ and $u \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \bigcap J_q$ such that

$$T_{q,0}(T_{q,1}(\mathbf{y}_3)) = s \quad \text{and} \quad T_{q,1}(T_{q,0}(T_{q,0}(T_{q,1}(\mathbf{y}_3)))) = u. \tag{15}$$

The equation $T_{q,0}(T_{q,1}(\mathbf{y}_3)) = s$ means that

$$k = \frac{-\ln(1 - q^{-1} + q^2 + q^3 - q^4)}{\ln q}.$$

The function $-\ln(1 - q^{-1} + q^2 + q^3 - q^4)(\ln q)^{-1}$ is strictly increasing on the interval $(q_1, q_3)$ and

$$\lim_{q \to q_3} -\ln(1 - q^{-1} + q^2 + q^3 - q^4)(\ln q)^{-1} = +\infty. \tag{16}$$

Table 2 records the first few solutions of $-\ln(1-q^{-1}+q^2+q^3-q^4)(\ln q)^{-1} = k$. It follows from (16) that the case where $s = 1/q$ is only possible when $q = q_3$. In Table 2, we also list the $q$ for which $T_{q,1}(T_{q,0}(T_{q,0}(T_{q,1}(\mathbf{y}_3)))) = u$ holds. By inspecting Table 2 and using the fact that $-\ln(1 - q^{-1} + q^2 + q^3 - q^4)(\ln q)^{-1}$ is increasing with $q$, we see that the only values of $q$ for which both equations in (15) hold simultaneously are $q \approx 1.67365$ and $q = q_3$. □

Table 3 lists the values of $q$ for which (11)–(13) and (15) hold true independently. In fact, it follows from Lemma 3.2 and the symmetric property of $\mathbf{y}_k$ and $\mathbf{z}_k$ shown in Lemma 2.1 that the equations in (11)–(13) and (15) give all the possible values of $q$ such that $q \in \mathcal{B}_{\aleph_0} \cap [q_1, q_3]$.

We are now ready to prove Theorem 1.2.

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Table 2: The values of $q$ for $\mathbf{y}_3$.

| | $k$ | $q$ | Polynomials |
|---|---|---|---|
| $T_{q,0}(T_{q,1}(\mathbf{y}_3)) = \mathbf{z}_k$ | 4 | 1.66041 | $x^6 - x^5 - x^3 - x^2 - 1$ |
| | 5 | 1.66883 | $x^8 - x^6 - x^5 - 2x^4 - x^3 - x^2 - x - 1$ |
| | 6 | 1.67365 | $x^5 - 2x^4 + x^3 - x^2 + x - 1$ |
| | 7 | 1.67644 | $x^{10} - x^8 - x^7 - 2x^6 - x^5 - x^4 - x^3 - x^2 - x - 1$ |
| $T_{q,0}(T_{q,1}(\mathbf{y}_3)) = q^{-1}$ | | 1.68042 | $x^5 - x^4 - x^3 - x + 1$ |
| $T_{q,1}(T_{q,0}(T_{q,0}(T_{q,1}(\mathbf{y}_3)))) = \mathbf{y}_k$ | 1 | 1.68042 | $x^5 - x^4 - x^3 - x + 1$ |
| | 2 | 1.65963 | $x^7 - 2x^5 - x^4 - x^3 + x + 1$ |
| | 3 | 1.64541 | $x^6 - x^4 - x^3 - 2x^2 - x - 1$ |
| $T_{q,1}(T_{q,0}(T_{q,0}(T_{q,1}(\mathbf{y}_3)))) = \mathbf{z}_k$ | 1 | 1.65462 | $x^6 - 2x^4 - x^3 - 1$ |
| | 2 | 1.67365 | $x^5 - 2x^4 + x^3 - x^2 + x - 1$ |
| | 3 | 1.68400 | $x^8 - 2x^6 - x^5 - x^2 - x - 1$ |

*Proof of Theorem 1.2.* By applying Theorem 3.1 and Lemma 3.2, we can find all points belonging to $\mathcal{B}_{\aleph_0} \cap [q_1, q_3]$ in Table 3. One can see from Table 3 that

$$T_{q,0}(T_{q,1}(\mathbf{y}_3)) = \mathbf{z}_3, \qquad T_{q,1}(T_{q,0}(\mathbf{z}_3)) = \mathbf{y}_3,$$
$$T_{q,1}(T_{q,0}(T_{q,0}(T_{q,1}(\mathbf{y}_3)))) = \mathbf{y}_3 \tag{17}$$

when $q = q_1$ ($\approx 1.64541$), which is the positive root of $x^6 - x^4 - x^3 - 2x^2 - x - 1 = 0$,

$$T_{q,0}^3(T_{q,1}(\mathbf{y}_1)) = \mathbf{z}_1, \qquad T_{q,1}^3(T_{q,0}(\mathbf{z}_1)) = \mathbf{y}_1, \qquad T_{q,0}^3(T_{q,1}^4(T_{q,0}(\mathbf{z}_1))) = \mathbf{z}_1 \tag{18}$$

when $q = q_2$ ($\approx 1.65462$), which is the positive root of $x^6 - 2x^4 - x^3 - 1 = 0$, and

$$T_{q,0}^3(T_{q,1}(\mathbf{y}_1)) = \mathbf{y}_1$$

when $q = q_3$ ($\approx 1.68042$), which is the positive root of $x^5 - x^4 - x^3 - x + 1 = 0$. So, the conditions in Theorem 3.1 are satisfied when $q = q_1, q_2, q_3$, respectively. That is, $q_j \in \mathcal{B}_{\aleph_0}$ for each $1 \leqslant j \leqslant 3$.

Finally, it is easy to check that $q \notin \mathcal{B}_{\aleph_0}$ if $q$ takes the values listed in Table 3, except when $q = q_j$, $1 \leqslant j \leqslant 3$. For example, we have

$$T_{q,0}^3(T_{q,1}(\mathbf{y}_1)) = \mathbf{y}_2$$

when $q$ ($\approx 1.65963$) is the positive root of $x^5 - x^4 - x^3 - x + 1 = 0$. However, there exists no point in $\bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \bigcap J_q$ such that condition (i) of Theorem 3.1 holds for this value of $q$. The other values of $q$ are dealt with similarly. □

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Table 3: The values of $q$ for $\mathbf{y}_k$ and $\mathbf{z}_k$, $1 \leqslant k \leqslant 3$.

| | $k$ | $q$ | Polynomials |
|---|---|---|---|
| $T^3_{q,0}(T_{q,1}(\mathbf{y}_1)) = \mathbf{y}_k$ (or $T^3_{q,1}(T_{q,0}(\mathbf{z}_1)) = \mathbf{z}_k$) | 1 | 1.68042 | $x^5 - x^4 - x^3 - x + 1$ |
| | 2 | 1.65963 | $x^7 - 2x^5 - x^4 - x^3 + x + 1$ |
| | 3 | 1.64541 | $x^6 - x^4 - x^3 - 2x^2 - x - 1$ |
| $T^3_{q,0}(T_{q,1}(\mathbf{y}_1)) = \mathbf{z}_k$ (or $T^3_{q,1}(T_{q,0}(\mathbf{z}_1)) = \mathbf{y}_k$) | 1 | 1.65462 | $x^6 - 2x^4 - x^3 - 1$ |
| | 2 | 1.67365 | $x^5 - 2x^4 + x^3 - x^2 + x - 1$ |
| | 3 | 1.68400 | $x^8 - 2x^6 - x^5 - x^2 - x - 1$ |
| $T^2_{q,0}(T_{q,1}(\mathbf{y}_2)) = \mathbf{y}_k$ (or $T^2_{q,1}(T_{q,0}(\mathbf{z}_2)) = \mathbf{z}_k$) | 1 | 1.72208 | $x^4 - x^3 - x^2 - x + 1$ |
| | 2 | 1.68929 | $x^6 - 2x^4 - 2x^3 + x + 1$ |
| | 3 | 1.6663 | $x^6 - x^5 - x^4 - x^3 + x^2 + 1$ |
| $T^2_{q,0}(T_{q,1}(\mathbf{y}_2)) = \mathbf{z}_k$ (or $T^2_{q,1}(T_{q,0}(\mathbf{z}_2)) = \mathbf{y}_k$) | 1 | 1.67602 | $x^5 - 2x^3 - x^2 - 1$ |
| | 2 | 1.7049 | $x^5 - x^4 - x^3 - 1$ |
| | 3 | 1.72004 | $x^7 - 2x^5 - x^4 - x^2 - x - 1$ |
| $T_{q,0}(T_{q,1}(\mathbf{y}_3)) = \mathbf{z}_k$ (or $T_{q,1}(T_{q,0}(\mathbf{z}_3)) = \mathbf{y}_k$) | 3 | 1.64541 | $x^6 - x^4 - x^3 - 2x^2 - x - 1$ |
| $T_{q,0}(T_{q,1}(\mathbf{y}_3)) = \mathbf{z}_6$, $T_{q,1}(T_{q,0}(\mathbf{z}_6)) = \mathbf{z}_2$ (or $T_{q,1}(T_{q,0}(\mathbf{z}_3)) = \mathbf{y}_6$, $T_{q,0}(T_{q,1}(\mathbf{y}_6)) = \mathbf{y}_2$) | | 1.67365 | $x^5 - 2x^4 + x^3 - x^2 + x - 1$ |
| $T_{q,0}(T_{q,1}(\mathbf{y}_3)) = 1/q$, $T_{q,1}(T_{q,0}(1/q)) = \mathbf{y}_1$ (or $T_{q,1}(T_{q,0}(\mathbf{z}_3)) = 1/(q^2 - q)$, $T_{q,0}(T_{q,1}(1/(q^2 - q))) = \mathbf{z}_1$) | | 1.68042 | $x^5 - x^4 - x^3 - x + 1$ |

COROLLARY 3.3. *The points* $\mathbf{y}_3$ *and* $\mathbf{z}_3$ *have a countable number of* $q_1$*-expansions; the points* $\mathbf{y}_1$ *and* $\mathbf{z}_1$ *have a countable number of* $q_2$*-expansions.*

*Proof.* We remark that the first result was shown in [**2**, Proposition 3.5]. For self-containedness, we give a proof here. It can be seen from (13) and (17) that

$$\Sigma_{q_1}(\mathbf{y}_3) = \{(1001)^k 01^3 (10)^\infty, (1001)^\infty, (1001)^k 1010^3 (01)^\infty\}$$

and

$$\Sigma_{q_1}(\mathbf{z}_3) = \{(0110)^k 10^3 (01)^\infty, (0110)^\infty, (0110)^k 0101^3 (10)^\infty\}$$

for some $k \geqslant 0$, which lead to the first result.

Similarly, it can be seen from (11) and (18) that

$$\Sigma_{q_2}(\mathbf{y}_1) = \{(10^4 1^3)^k 01(10)^\infty, (10^4 1^3)^\infty, (10^4 1^3)^k 10^3 10(01)^\infty\}$$

and

$$\Sigma_{q_2}(\mathbf{z}_1) = \{(01^4 0^3)^k 10(01)^\infty, (01^4 0^3)^\infty, (01^4 0^3)^k 01^3 01(10)^\infty\}$$

for some $k \geqslant 0$, which lead to $\mathbf{y}_1$ and $\mathbf{z}_1$ having a countable number of $q_2$-expansions. □

§4. *Proof of Theorem 1.1.* To prove Theorem 1.1, it suffices to prove that 1 has $\aleph_0$ $q$-expansions when $q = q_3$ and 1 has a continuum of $q$-expansions when $q = q_1$ and $q = q_2$. We only prove the case of $q = q_1$; the case of $q = q_2$ can be verified in a similar way.

THEOREM 4.1. *If* $q_3$ ($\approx 1.68042$) *is the positive root of* $x^5 - x^4 - x^3 - x + 1 = 0$*, then* 1 *is a* $q_3$ *null infinite point and therefore has* $\aleph_0$ $q_3$*-expansions.*

*Proof.* It is straightforward to show that for all $k \geqslant 0$,

$$(T_{q_3,0}^3 \circ T_{q_3,1})^k (T_{q_3,1}(1)) \in S_{q_3}.$$

Moreover, these are the only $\mathbf{a} \in \bigcup_{n=0}^\infty \{T_{q_3,0}, T_{q_3,1}\}^n$ such that $\mathbf{a}(1) \in S_{q_3}$. This is sufficient to imply that 1 is a $q_3$ null infinite point. Namely, for each $k \geqslant 0$, we have

$$T_{q_3,0}((T_{q_3,0}^3 \circ T_{q_3,1})^k (T_{q_3,1}(1))) = \Pi(1(10)^\infty).$$

Therefore,

$$\Sigma_{q_3}(1) = \{1(10^3)^k 01(10)^\infty, 1(10^3)^\infty\}.$$ □

Figure 2 demonstrates the construction of $\Sigma_{q_3}(1)$.

THEOREM 4.2. *Let* $q_1$ ($\approx 1.64541$) *be the positive solution of the equation* $x^6 - x^4 - x^3 - 2x^2 - x - 1 = 0$*. Then* 1 *has* $2^{\aleph_0}$ $q_1$*-expansions.*

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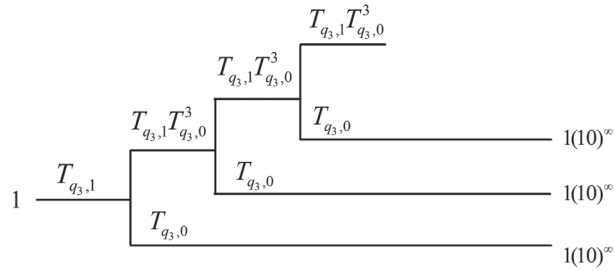



Figure 2: The construction of $\Sigma_{q_3}(1)$.

*Proof.* We proceed via a proof by contradiction. We assume that 1 has $\aleph_0$ $q_1$-expansions and obtain a contradiction. It is a simple calculation to show that

$$w := (T^5_{q_1,0} \circ T^2_{q_1,1})(1) \in S_{q_1}.$$

Therefore, $w$ has either $\aleph_0$ $q_1$-expansions or $2^{\aleph_0}$ $q_1$-expansions. Since we have assumed that 1 has $\aleph_0$ $q_1$-expansions, $w$ must also have $\aleph_0$ $q_1$-expansions. Therefore, $w$ can be mapped to a $q_1$ null infinite point and, by Lemma C, it can be mapped to a point with a periodic $q_1$-expansion. The above implies that 1 has a $q_1$ expansion that begins $(1\,100\,000(\delta_i)_{i=1}^{\infty})$, where $(\delta_i)_{i=1}^{\infty}$ is eventually periodic. This is obviously equivalent to

$$1 = \frac{1}{q_1} + \frac{1}{q_1^2} + \frac{1}{q_1^7}\sum_{i=1}^{\infty}\frac{\delta_i}{q_1^i}. \tag{19}$$

Since $(\delta_i)_{i=1}^{\infty}$ is eventually periodic, we may use properties of geometric series to deduce that (19) is equivalent to

$$1 = \frac{1}{q_1} + \frac{1}{q_1^2} + \frac{f(q_1)}{g(q_1)}, \tag{20}$$

where $f(x), g(x) \in \mathbb{Z}[x]$. Equation (20) is just an algebraic relation and so must also be satisfied by the conjugates of $q_1$, that is, the other roots of $x^6 - x^4 - x^3 - 2x^2 - x - 1 = 0$. We now show that this cannot be the case for a particular choice of conjugate, namely $q_1^* \approx -1.20458$. Equation (20) is equivalent to (19), so (19) must also hold with $q_1$ replaced by $q_1^*$. We observe the following:

$$\begin{aligned} 1 &= \frac{1}{q_1^*} + \frac{1}{(q_1^*)^2} + \frac{1}{(q_1^*)^7}\sum_{i=1}^{\infty}\frac{\delta_i}{(q_1^*)^i} \\ &\leqslant \frac{1}{q_1^*} + \frac{1}{(q_1^*)^2} + \frac{1}{(q_1^*)^7}\frac{q_1^*}{(q_1^*)^2 - 1} \\ &< 1, \end{aligned}$$

where the final strict inequality follows from a simple calculation. Thus, we have our desired contradiction. □

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The proof that $q_2 \notin \mathcal{B}_{1,\aleph_0}$ is done analogously. In this case we similarly use a conjugate of $q_2$, namely the number $q_2^* \approx -1.26493$.

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