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Pairing theorems about parabolas through duality

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1. *Introduction*

We investigate the pairing of theorems about parabolas through a dual transformation. Theorems and constructions concerning a parabola in a two-dimensional space can be in one-to-one correspondence with theorems and constructions concerning a parabola in the two-dimensional dual space. These theorems are called *dual theorems*.

As an example, we pair the well-known triangle construction method for tangent lines of a parabola with a new construction method for points of a parabola. Then we express this new result in a very general manner without referencing duality.

This might remind some readers of duality in projective geometry, where one theorem leads to a dual theorem by interchanging points and lines, collinearity and concurrency, and so forth. All conic sections are equivalent in projective geometry and, often, particular transformations are secondary. The foundation of the current results is more specific. We present a methodology for finding new, or perhaps long-forgotten, theorems about curves, in this case parabolas. Although our scope is limited to theorems about parabolas, the methodology may be adapted to other curves.

2. *The duality transformation*

There are several definitions of dual transformations between two two-dimensional spaces. The original space has rectangular coordinates x and y, and the dual space has rectangular coordinates u and v. All of the definitions require that lines and points in x,y-space correspond to points and lines, respectively, in the dual u,v-space. Because theorems about parabolas in the original space are to be paired with theorems about parabolas in the dual space, it is required that the image of a given parabola is a parabola.

We shall select the dual transformation that is based upon the dot product in which

the line $ax + by = 1$ corresponds to the point (a, b) in u,v-space

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and

the point (a, b) in u,v-space corresponds to the line $au + bv = 1$.

(See [1, p. 23], [2, pp. 73-90].) A failure of this transformation is that lines through either origin do not transform, because they cannot be written as $ax + by = 1$ or $au + bv = 1$. Another failure is that the origin $(0, 0)$ has no transformation. Those lines and the origin can be avoided. All duality transformations have a similar shortcoming in that they fail to transform all points and lines.

Since the correspondence is reversible, the dual of the dual of a point or line is the original point or line (excluding the origin and lines through the origin, which have no dual). The dual transformation of an already transformed space produces a space that is essentially the same as the original space (see [1, pp. 24-25], [3, p. 143]). This property is called *reflexivity*. Thus the dual transformation of an item B, which itself is the dual of item A, is identical to the original item A, subsequent to a change in the names of the coordinates. Simply put, *the dual of the dual is the original*.

In the original space, use

$$C : y = \frac{x^2}{4p} \tag{1}$$

as the equation of a parabola. This is a *standard form of the parabola*, whose focus is $F(0, p)$. No generality is lost by choosing (1), because in rectangular coordinates any parabola can be put into that form with rigid motions (see [4, pp. 122-141], [5, pp. 666-673]).

For the parabola (1), the tangent line at its point of tangency

$$P : \left(a, \frac{a^2}{4p}\right) \tag{2}$$

is

$$L : y = \frac{a}{2p}x - \frac{a^2}{4p},$$

i.e.

$$L : \frac{2}{a}x - \frac{4p}{a^2}y = 1. \tag{3}$$

Dual lines and points are designated by the same alphabetical symbol, and subscript if any, with a prime. (As a result, we express all derivatives using Leibniz notation.) The line that is dual to the point (2) for $a \neq 0$ is

$$P' : au + \frac{a^2}{4p}v = 1,$$

i.e.

$$P' : v = -\frac{4p}{a}u + \frac{4p}{a^2}. \tag{4}$$

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The point that is dual to the tangent line (3) is

$$L' : \left(\frac{2}{a}, -\frac{4p}{a^2}\right), \tag{5}$$

which is on the line (4). Because

$$(-p)\left(\frac{2}{a}\right)^2 = -\frac{4p}{a^2},$$

point (5) satisfies the equation of the parabola

$$C' : v = -pu^2 \tag{6}$$

in u, v-space, which is said to be the *dual curve* to C in x,y-space and conversely. Parabola (6) can be changed to the format of the parabola (1), as

$$C' : v = \frac{u^2}{4\left(-\frac{1}{4p}\right)}.$$

Thus the focus of the parabola C' in (6) is $\left(0, -\frac{1}{4p}\right)$.

The tangent line to the parabola C' in (6) at the point L' in (5) is the point P' in (4), which illustrates the following property of duality for differentiable curves [6].

Theorem 1 (see Figure 1): Consider a point of a curve C, which is defined by the differentiable function $y = f(x)$ with derivative $\frac{df(x)}{dx}$. There is a smooth dual curve C', such that the dual transformation of the point is the tangent line to curve C' at the point that is the dual transformation of C's tangent line (except tangent lines through the origin, as they have no dual transformations).

Proof: At the point

$$(a, f(a)) \tag{7}$$

of C in x, y-space, the tangent line is

$$y = \frac{df(a)}{dx}(x - a) + f(a),$$

i.e.

$$\frac{\frac{df(a)}{dx}}{a\frac{df(a)}{dx} - f(a)}x - \frac{1}{a\frac{df(a)}{dx} - f(a)}y = 1. \tag{8}$$

(Note: the denominators in (8) would be zero if the tangent line were at the origin.)

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The dual transformations of the point (7) and the line (8) are

$$au + f(a)\,v = 1 \tag{9}$$

and

$$(u, v) = \left(\frac{\dfrac{df(a)}{dx}}{a\dfrac{df(a)}{dx} - f(a)}, \frac{-1}{a\dfrac{df(a)}{dx} - f(a)} \right), \tag{10}$$

respectively. Equation (10) is the parametric representation of a smooth curve C'. Substitution shows that the point (10) is on the line (9).

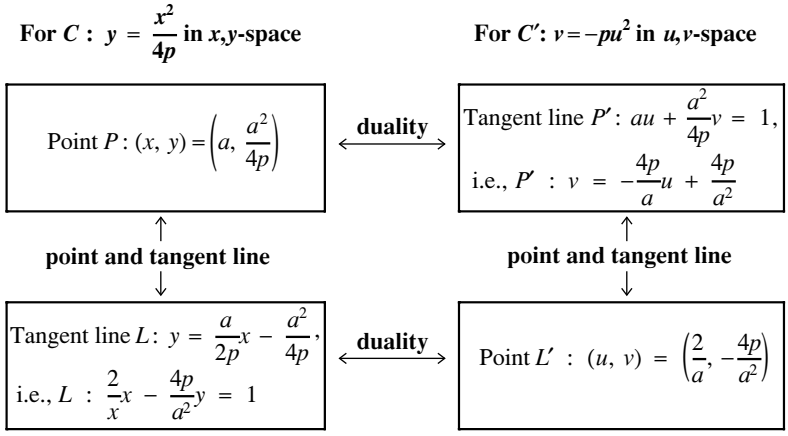


FIGURE 1: Schematic diagram showing dual relationships between points and tangent lines of differentiable curves, using the dual parabolas (1) and (6) as the example

Theorem 2 contains another result that is cited in the example in Section 3.

Theorem 2: If the points P_1 and P_2 are on a line L in one space, then the dual point L' of the line L is the point of intersection of the dual lines P_1' and P_2' in the dual space. Similarly, if the lines L_1 and L_2 meet in a point P in one space, then the dual points L_1' and L_2' of the lines L_1 and L_2 lie on the dual line P' in the dual space.

Proof: For the first statement, express P_1, P_2 and L as (x_1, y_1), (x_2, y_2) and $ax + by = 1$, so that

$$ax_1 + by_1 = 1 \text{ and } ax_2 + by_2 = 1. \tag{11}$$

The dual lines P_1' and P_2' have equations

$$x_1u + y_1v = 1 \qquad \text{and} \qquad x_2u + y_2v = 1, \tag{12}$$

and the dual point L' of L is $(u, v) = (a, b)$. To see that the dual point (a, b) is the point of intersection of P_1' and P_2', substitute $(u, v) = (a, b)$ into (12) which results in (11). The second statement is shown similarly.

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3. *An example of dual theorems that contain parabolas*

In the following pair of dual theorems, Theorem 3 concludes that constructed lines are tangent to a parabola, and Theorem 4 concludes that constructed points are on a parabola. The dual transformations of the tangent lines are the points, and vice versa, as described by Theorem 1.

The triangle construction method for a parabola in Theorem 3 is one of the most popular constructions of a set of tangent lines to an arc of a parabola in demonstrations on YouTube (see [7, 8, 9]) and with string art (see [10, 11, 12]). It can be used to characterise the parabola (see [13, pp. 423-425]). (See Figure 2.)

Tangent lines to a parabola that are obtained in this construction are determined by three points that coincide with the vertices Q_1, Q_2 and Q_3 of an isosceles triangle. Vertices Q_1 and Q_2 lie at the ends of a parabolic arc, while the parabola's vertex lies at the midpoint of the perpendicular bisector of the third vertex Q_3 and its opposing side between Q_1 and Q_2. For each point of the line through Q_1 and Q_3, a tangent line is found following the theorem's criterion in Step 7.

We select the three points so that the parabola's vertex is at the origin and the equation of the parabola takes the form (1). The slopes of the triangle's sides between Q_1 and Q_3 and between Q_2 and Q_3 are labelled $\pm m$.

Theorem 3 (Triangle construction method): The following eight-step process, which is illustrated in Figure 2, produces a tangent line to parabola (1). Select values for p, $m > 0$ and t. The values of p and m are fixed, and each value of t yields a different tangent line.

1. Find
$$Q_1 : (-2pm, pm^2), \tag{13}$$
which is on the parabola (1).
2. Find
$$Q_2 : (2pm, pm^2), \tag{14}$$
which is on the parabola (1).
3. Find
$$Q_3 : (0, -pm^2). \tag{15}$$
4. Construct (draw)
$$L_1 : y = -mx - pm^2, \text{ i.e., } -\frac{1}{pm}x - \frac{1}{pm^2}y = 1, \tag{16}$$
which contains Q_1 and Q_3 and is tangent to the parabola (1) at Q_1.
5. Construct (draw)
$$L_2 : y = mx - pm^2, \text{ i.e., } \frac{1}{pm}x - \frac{1}{pm^2}y = 1, \tag{17}$$
which contains Q_2 and Q_3 and is tangent to the parabola (1) at Q_2.

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6. For a given value of t, find
$$P_1 \ : \ \big(t, \ -m\,(t \ + \ pm)\big), \tag{18}$$
which is on L_1.
7. For the same value of t, find
$$P_2 \ : \ \big(t \ + \ 2pm, \ m\,(t \ + \ pm)\big), \tag{19}$$
which is on L_2 and is selected so that the Euclidean distances $D\big(Q_1, \ P_1\big)$ and $D\big(Q_3, \ P_2\big)$ are equal.
8. For the same value of t, construct (draw)
$$L_3 : y = \frac{t+pm}{p}x - \frac{(t+pm)^2}{p}, \text{ i.e., } \frac{1}{t+pm}x - \frac{p}{(t+pm)^2}y = 1. \tag{20}$$
which contains P_1 and P_2.

Then line L_3 is tangent to the parabola (1) for each value of t.

Proof: In Step 7, consider the square of the distance $D\big(Q_1, \ P_1\big)$:

$$D^2\big(Q_1, P_1\big) = (-2pm - t)^2 + \big(pm^2 - \big(-mt - pm^2\big)\big)^2 = (t + 2pm)^2\big(1 + m^2\big).$$

Designate the coordinates of P_2 by $(x, \ mx \ - \ pm^2)$, so that

$$D^2\big(Q_3, \ P_2\big) \ = \ x^2 \ + \ \big(-pm^2 \ - \ \big(mx \ - \ pm^2\big)\big)^2 \ = \ x^2\big(1 \ + \ m^2\big).$$

The squared distances (and thus the distances) are equal when $x \ = \ t \ + \ 2pm$, which is the x-coordinate of P_2 chosen in (19).

The tangent line to the parabola (1) at any of its points $\big(b, \ b^2/(4p)\big)$ is

$$y \ = \ \frac{b}{2p}x \ - \ \frac{b^2}{4p}. \tag{21}$$

Setting $b \ = \ 2\,(t \ + \ pm)$ in (21) shows that the line with equation (20) is tangent to the parabola (1) at the point

$$P_3 \ : \ \left(2\,(t \ + \ pm), \ \frac{(t \ + \ pm)^2}{p}\right), \tag{22}$$

which is on the parabola (1) and tangent line (20).

The triangle in the triangle construction method has vertices Q_1, Q_2 and Q_3 and is used for all values of t. Using any three non-collinear points for the triangle's vertices would produce a parabola, but not in the standard form (1). Often the slopes $\pm m$ of the triangle's sides between Q_1 and Q_3 and between Q_2 and Q_3 are taken to be ± 1. If t is in the interval $(-2pm, \ 0)$, then P_1 is between Q_1 and Q_3 and the point of tangency is on the parabola's arc between Q_1 and Q_2, as in the example in Figure 2. The point of tangency P_3 in (22) is found incidentally during the proof of the theorem.

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All of a parabola's tangent lines are generated through the construction by using all points of L_1 in (16). Usually Theorem 3 is employed to draw many tangent lines, in order to visualise the parabola as the envelope of the lines. On paper, the tangent lines can be used to sketch the parabola.

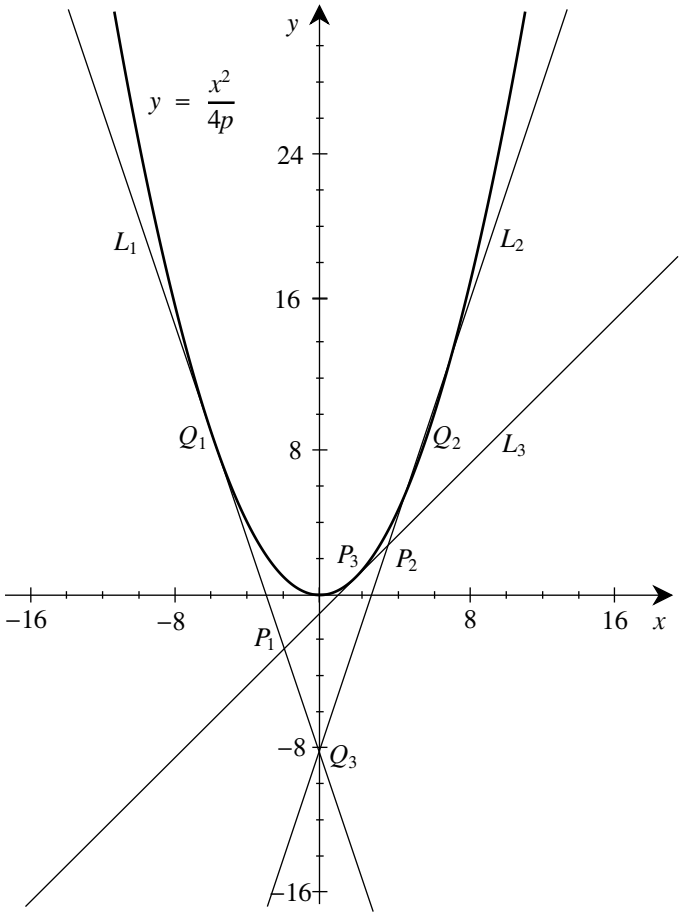


FIGURE 2: Triangle construction method of the tangent line L_3 to (1) with $p = 1$ for $m = 3$ and $t = -2$ (Theorem 3)

In string art, we can construct a whole family of tangent lines that reveal the shape of the parabola in a particularly beautiful and pleasing way (see [10, 11, 12]). An example is illustrated in Figure 3, where the bold lines represent the rigid wood frame, points N_i and P_i, $i = 1, 2, \ldots, 11$, indicate nails or pegs, and the line segments between the nails or pegs with matching indices are pieces of string. Points Q_1, Q_2 and Q_3 are present to correlate with Figure 2.

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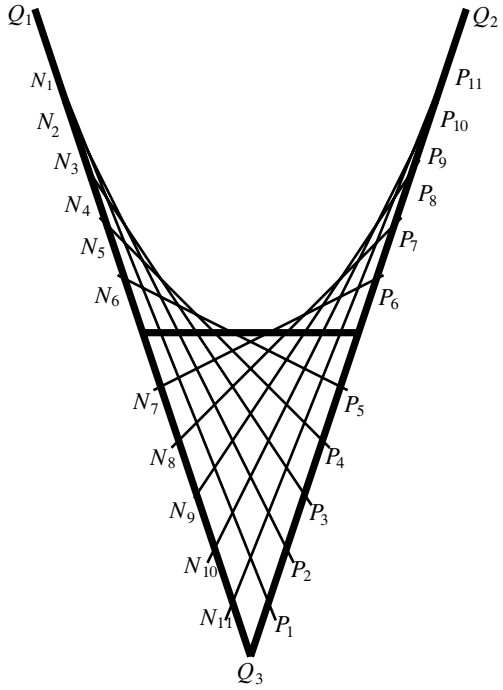


FIGURE 3: A representation of string art that constructs numerous tangent lines, which give the illusion of a parabolic arc (Theorem 3)

Theorem 4 contains the step-by-step dual transformation of the triangle construction method in Theorem 3 and generates all points of the dual parabola (6), $v = -pu^2$ with parameter p. In the first three steps of Theorem 4, the dual transformation of the triangle's vertices Q_1, Q_2 and Q_3 of Theorem 3 are the lines Q_1', Q_2' and Q_3', which form a triangle. In the remaining steps of Theorem 4, the dual transformations of P_1, P_2, L_1, L_2 and L_3 of Theorem 3 are found in order to find the point L_3', which is dual to a tangent line in Theorem 3, and thus a point of the parabola (6), which is dual the parabola (1) in Theorem 3. See Figure 4.

Theorem 5 is the outcome of Theorem 4 written without the terminology of duality.

Theorem 4 (Dual theorem of the triangle construction method): The dual transformations of items (13) – (20), whose outcomes are in Steps 1' – 8', are a construction for a point of the parabola (6), C' : $v = -pu^2$, which is the dual curve of the parabola (1).

1'. Construct (draw) the line

$$Q_1' : -2pmu + pm^2v = 1, \text{ i.e., } v = \frac{2}{m}u + \frac{1}{pm^2},$$

which is dual to the vertex Q_1.

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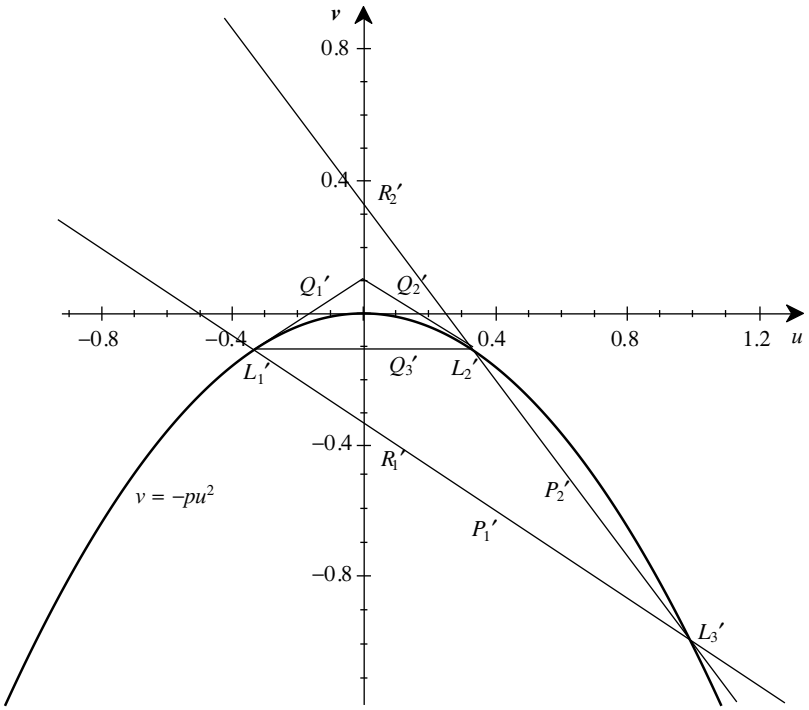


FIGURE 4: The construction of points of a parabola, which is dual to the triangle construction method in Theorem 3 for tangent lines to a parabola, with $p = 1$ for $m = 3$ and $t = -2$ (Theorem 4)

2'. Construct (draw) the line

$$Q_2' : 2pmu + pm^2v = 1, \text{ i.e., } v = -\frac{2}{m}u + \frac{1}{pm^2},$$

which is dual to the vertex Q_2.

3'. Construct (draw) the line

$$0u - pm^2v = 1, \text{ i.e., } v = -\frac{1}{pm^2},$$

which is dual to the vertex Q_3.

4'. Find the point

$$L_1' : \left(-\frac{1}{pm}, -\frac{1}{pm^2}\right),$$

which is the intersection of Q_1' and Q_3' (by Theorem 2).

5'. Find the point

$$L_2' : \left(\frac{1}{pm}, -\frac{1}{pm^2}\right),$$

which is the intersection of Q_2' and Q_3' (by Theorem 2).

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6'. Construct (draw) the line

$$P_1' : tu - m(t+pm)v = 1, \text{i.e., } v = \frac{t}{m(t+pm)}u - \frac{1}{m(t+pm)}, \quad (23)$$

which intersects Q_1' and Q_3' at L_1' (by Theorem 2).

7'. Construct (draw) the line

$$P_2' : (t + 2pm)u + m(t + pm)v = 1,$$

$$\text{i.e., } v = -\frac{t + 2pm}{m(t + pm)}u + \frac{1}{m(t + pm)}, \quad (24)$$

which intersects Q_2' and Q_3' at L_2' (by Theorem 2).

8'. Find

$$L_3' : \left(\frac{1}{t + pm}, -\frac{p}{(t + pm)^2}\right), \quad (25)$$

which is the intersection of P_1' and P_2' (by Theorem 2).

Then the point L_3' is a point of the parabola (6) for each value of t.

Proof: The point (25) satisfies $C' : v = -pu^2$.

Additionally, as Theorem 3 says, the dual transformation of P_3 in (22) is

$$P_3' : 2(t+pm)u + \frac{(t+pm)^2}{p}v = 1, \text{i.e., } v = -\frac{2p}{t+pm}u + \frac{p}{(t+pm)^2},$$

which is tangent to the parabola (6) at the point L_3' in (25).

Observe that the v-intercept of P_1' in (23) is the negation of the v-intercept of P_2' in (24). These intersections are labelled R_1' and R_2', respectively, in Figure 4.

In Theorem 5, which is illustrated in Figure 5, the result of Theorem 4 is more simply stated and adapted to produce a parabola in the standard form (1) in x,y-space. By taking advantage of the observation made in the previous paragraph, Theorem 5 gives a more natural description of a point-by-point construction method of a parabola, which is paired with the triangular construction method through duality.

Theorem 5: Select two points $A(a, b)$ and $B(-a, b)$ with $a > 0$ and $b > 0$. These points remain unchanged during the construction. For any value of c, find the pair of points $C(0, c)$ and $D(0, -c)$ and construct the line containing A and D and the line containing B and C. Then, as c varies over $\mathbb{R}$, the intersections $\left(\frac{ac}{b}, \frac{c^2}{b}\right)$ of the lines AD and BC produce all the points of the parabola $y = \frac{x^2}{a^2/b}$; and if we then write $p = \frac{a^2}{4b}$, the equation of the parabola is just $y = \frac{x^2}{4p}$ (as in (1)).

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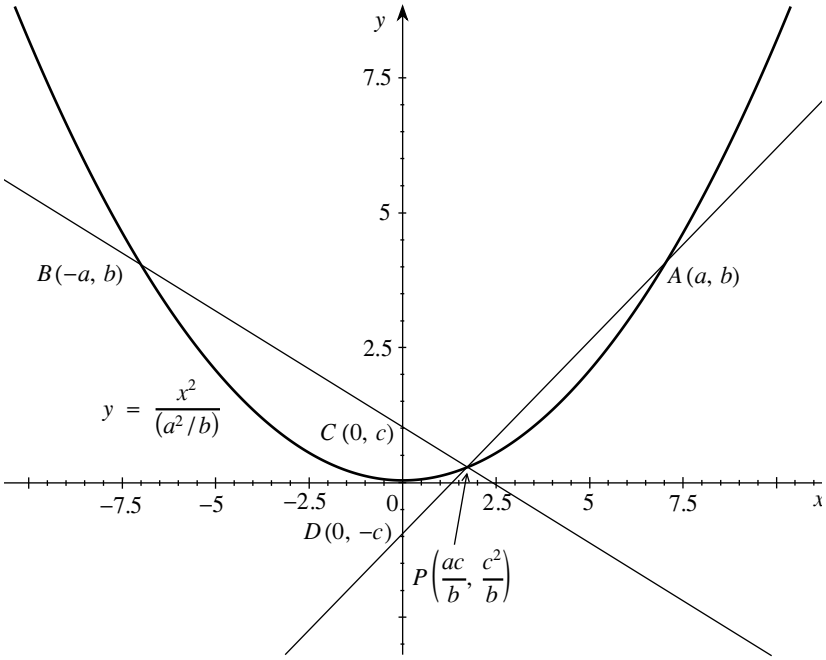


FIGURE 5: The new construction for the points of a parabola with $a = 7, b = 4$ and $c = 1$ (Theorem 5)

Acknowledgement

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References

1. P. R. Halmos, *Finite-dimensional vector spaces* (2nd edn.), Dover (2017).
2. D. C. Lay, *Linear algebra and its applications* (3rd edn.), Pearson/ Addison-Wesley (2006).
3. S. R. Lay, *Convex sets and their applications*, Dover (2007).
4. C. H. Sisam, *Analytic geometry*, Henry Holt (1936).
5. F. Demana, B. K. Waits, G. D. Foley and D. Kennedy, *Precalculus: functions and graphs* (5th edn.), Pearson/Addison-Wesley (2004).
6. E. Maor, Line equations of curves: Duality in analytic geometry, *International Journal of Mathematics Education in Science and Technology* **9** (1978) pp. 311-322.
7. https://www.youtube.com/watch?v=xZ0ur4y5SS0
8. https://www.youtube.com/watch?v=lwNt-SdihVY

https://doi.org/10.1017/mag.2021.105 Published online by Cambridge University Press

9. https://www.youtube.com/watch?v=hEbqHPkn264
10. Wikipedia, String art, accessed April 2021 at https://en.wikipedia.org/wiki/String_art
11. R. Gross, Bridges, string art, and Bézier curves, chapter in M. Pitici, editor, *The best writing in mathematics 2013,* Princeton University Press (2014) pp. 77-89.
12. G. Quenell, Envelopes and string art, *Mathematics Magazine* **82** (2009) pp. 174-185.
13. S. J. Kilner and D. L. Farnsworth, Characterisations of the parabola, *Math. Gaz*. **103** (November 2019) pp. 416-430.

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The answers to the *Nemo* page from July 2021 on mathematics and everyday life were:

1.	Francis Bacon	Of Studies	
2.	George Eliot	Scenes of Clerical Life: Amos Barton	Chapter 4
3.	Norman Maclean	A River runs through it	
4.	Richard Sheridan	The Rivals	Act 1, Scene 2
5.	Graham Greene	The Comedians	Part 2, Chapter 4
6.	Lewis Carroll	Echoes	

Congratulations to Martin Lukarewski and Henry Ricardo on tracking all of these down. This issue we return to a theme already explored in July 2015: spirals and helices. Quotations are to be identified by reference to author and work. Solutions are invited to the Editor by 23rd January 2022

1. What is art,
But life upon the larger scale, the higher,
When, graduating up in a spiral line
Of still expanding and ascending gyres,
It pushes toward the intense significance
Of all things, hungry for the Infinite?
Art's life,—and where we live, we suffer and toil.

Continued on page 415.

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