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A semigroup embedding problem and an arithmetical function

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1. *Semigroup embeddings*

For unexplained terms in semigroup theory see [**1**] or [**4**].

Let $\mathbf{C}, \mathbf{D}$ be classes of semigroups such that every finite semigroup in the class $\mathbf{C}$ is embeddable in a finite semigroup in the class $\mathbf{D}$. If $n \geqslant 2$ then k is said to be a $\mathbf{C}-\mathbf{D}$ *cover of* n if every semigroup of order n in the class $\mathbf{C}$ is embeddable in a semigroup in the class $\mathbf{D}$ of order not greater than k. Let $\mathbf{K}_{\mathbf{C}}^{\mathbf{D}}(n)$ be the least $\mathbf{C}-\mathbf{D}$ cover of n.

This concept was introduced in [**6**], and a summary of results known up to 1988 appears in [**7**]. We shall be concerned here with the case where $\mathbf{D} = \mathbf{IG}$, the class of idempotent-generated semigroups. This was the class considered in [**6**], where it was shown that if $\mathbf{Sg}$ is the class of all semigroups then

$$\mathbf{K}_{\mathbf{Sg}}^{\mathbf{IG}}(n) \leqslant 2n^2 + 4n + 1. \tag{1}$$

In the investigation of $\mathbf{K}_{\mathbf{Gp}}^{\mathbf{IG}}(n)$ (where $\mathbf{Gp}$ denotes the class of all groups) a key role was played by the function

$$q(n) = \min\{(r+1)(s+1) : rs \geqslant n\}. \tag{2}$$

Norman R. Reilly (private communication) pointed out that the function $q(n)$ has a wider applicability, and used it to produce an improved bound for $\mathbf{K}_{\mathbf{Sg}}^{\mathbf{IG}}(n)$.

THEOREM 1 (Reilly). *With the above definitions,*

$$\mathbf{K}_{\mathbf{Sg}}^{\mathbf{IG}}(n) \leqslant (n+1)\, q(n-1).$$

Proof. Let S be a semigroup of order n, and let $T = M[S^1; r+1, s+1; P]$, an $(r+1) \times (s+1)$ Rees matrix semigroup over S^1. The elements of T are triples (i, a, j), where $1 \leqslant i \leqslant r+1$, $1 \leqslant j \leqslant s+1$, $a \in S^1$. $P = (p_{ij})$ is an $(s+1) \times (r+1)$ matrix with entries in S^1, and multiplication in T takes place according to the rule

$$(i, a, j)(k, b, l) = (i, ap_{jk}b, l).$$

We choose $P = (p_{ij})$ so that

$$p_{i1} = p_{1j} = 1 \ (i = 1, \ldots, s+1; j = 1, \ldots, r+1)$$

and the entries $p_{ij} (i, j \geqslant 2)$ include a set of generators for the semigroup S. The elements $(i, 1, 1) (1 \leqslant i \leqslant s+1)$ and $(1, 1, j) (1 \leqslant j \leqslant r+1)$ are easily seen to be idempotent. In fact T is idempotent-generated, since every a in S has an expression

$$a = p_{i_1 j_1} p_{i_2 j_2} \cdots p_{i_m j_m}$$

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as a product of entries in P, and

$$(i,a,j) = (i,1,1)(1,1,i_1)(j_1,1,1)(1,1,i_2)(j_2,1,1)\ldots \quad \ldots(1,1,i_m)(j_m,1,1)(1,1,j).$$

The map $a \mapsto (1,a,1)\,(a \in S)$ embeds S in T, and T has order at most $(n+1)(r+1)(s+1)$. To minimize the order of T we must choose r and s such that $(r+1)(s+1)$ is as small as possible subject to the condition that $rs \geqslant \operatorname{rank} S$. (By rank S we mean as usual the cardinality of a set of generators for S of minimal size.) Now we certainly have $\operatorname{rank} S \leqslant n$. Moreover, it is clear (and see also [3, 2]) that if $\operatorname{rank} S = n$ then every element of S is idempotent and so S itself is trivially idempotent-generated. So we may assume that $\operatorname{rank} S \leqslant n-1$. Thus we may take T of order $(n+1)\,q(n-1)$.

It is not of course immediately clear that this bound is better than the one given by formula (1). In Section 2, however, we obtain an upper bound for the function $q(n)$ which implies an upper bound for $\mathbf{K}^{\mathrm{IG}}_{\mathrm{Sg}}(n)$ with leading term n^2. The lower bound for $\mathbf{K}^{\mathrm{IG}}_{\mathrm{Sg}}(n)$ obtained in [6], which is still the best available, has leading term $n \log_2 n$.

First, however, we investigate a more special algebraic situation where it is possible to give an exact value for $\mathbf{K}^{\mathrm{D}}_{\mathrm{C}}(n)$. The *null* semigroup Z_n of order n is given by

$$Z_n = \{x_1, \ldots, x_{n-1}, 0\},$$

where all products are equal to 0. Let **Nu** be the class of all null semigroups. Notice that $\operatorname{rank} Z_n = n-1$, so that Theorem 1 would give an upper bound of $(n+1)\,q(n-1)$ for $\mathbf{K}^{\mathrm{IG}}_{\mathrm{Nu}}(n)$. In fact we can do much better than this.

Before stating the theorem it is convenient to prove two preliminary lemmas. The first is algebraic in character.

LEMMA 1. *Let $S = M^0[G; I, \Lambda; P]$ be a finite completely 0-simple idempotent-generated semigroup. Then P has at least $|I| + |\Lambda| - 1$ non-zero entries.*

Proof. We may suppose that $I \cap \Lambda = \varnothing$. Let $\Gamma(S)$ be the graph whose set of vertices is $I \cup \Lambda$, and where there is an edge between $i\,(\in I)$ and $\lambda\,(\in \Lambda)$ if and only if $p_{\lambda i} \neq 0$. It is known (see [5]) that if S is idempotent-generated then $\Gamma(S)$ is connected, in the usual sense that there is a path from every element of $I \cup \Lambda$ to every other element.

Choose and fix i in I, and let $C_m(i)\,(m \geqslant 1)$ be the subset of $I \cup \Lambda$ consisting of all elements x in $I \cup \Lambda$ for which there is a path of length m from i to x but no shorter path. (In fact $C_m(i) \subseteq I$ if m is even, $C_m(i) \subseteq \Lambda$ if m is odd.) Let $C_0(i) = \{i\}$. The sets $C_0(i), C_1(i), \ldots$ are disjoint, and since S is connected

$$\bigcup_{m \geqslant 0} C_m(i) = I \cup \Lambda.$$

Since each element of $C_m(i)\,(m \geqslant 1)$ is connected to some element of $C_{m-1}(i)$ by an edge, the total number of edges, which is also the total number of non-zero entries in P, is at least

$$\sum_{m \geqslant 1} |C_m(i)| = |I| + |\Lambda| - 1.$$

The second, essentially trivial lemma concerns the function $q(n)$ defined by formula (2).

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LEMMA 2. *For all* $k, m \geqslant 1$,

$$q(km) \leqslant kq(m).$$

Proof. Suppose that $q(m) = (r+1)(s+1)$, with $rs \geqslant m$. Then $(kr)s \geqslant km$ and so

$$q(km) \leqslant (kr+1)(s+1) \leqslant k(r+1)(s+1) = kq(m).$$

Let **COSIG** be the class of completely 0-simple idempotent-generated semigroups. Then

THEOREM 2. *With the above definitions*

$$\mathbf{K}_{\mathbf{Nu}}^{\mathbf{COSIG}}(n) = q(n-1)+1.$$

Proof. Choose r and s so that $rs \geqslant n-1$ and $(r+1)(s+1) = q(n-1)$. Let S be the completely 0-simple semigroup $M^0[1; r+1, s+1; P]$, where P is the $(s+1)\times(r+1)$ matrix

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

We may denote the non-zero elements of S by (i,j), where $1 \leqslant i \leqslant r+1$, $1 \leqslant j \leqslant s+1$. The elements $(i,1)$ and $(1,j)$ are all idempotent, and since

$$(i,j) = (i,1)(1,j)$$

it is clear that S is idempotent-generated. If we choose $n-1$ elements from $\{2,\dots,r+1\}\times\{2,\dots,s+1\}$ then the product of any two such elements is 0; so S contains the null semigroup of order n as a subsemigroup. The order of S is $q(n-1)+1$, and so we have proved that

$$\mathbf{K}_{\mathbf{Nu}}^{\mathbf{COSIG}}(n) \leqslant q(n-1)+1.$$

Conversely, suppose that $S = M^0[G; r+1, s+1; P]$ is a completely 0-simple idempotent-generated semigroup containing a null subsemigroup of order n. We shall show that the order of S is at least $q(n-1)+1$.

Let us denote the order of G by g. The semigroup S contains $n-1$ distinct elements

$$(i_1, a_1, j_1), \dots, (i_{n-1}, a_{n-1}, j_{n-1})$$

which together with 0 form a null subsemigroup. The entries $p_{j_1 i_1}, \dots, p_{j_{n-1} i_{n-1}}$ of P are all zero, and since no more than g of the pairs $(i_1,j_1), \dots, (i_{n-1},j_{n-1})$ can be identical there must be at least $\lceil (n-1)/g \rceil$ zero entries in P. (The symbol $\lceil x \rceil$ denotes the smallest integer l such that $l \geqslant x$.)

By Lemma 1, P must also have $r+s+1$ non-zero entries, and so

$$(r+1)(s+1) \geqslant r+s+1+\lceil (n-1)/g \rceil.$$

Hence $rs \geqslant \lceil (n-1)/g \rceil$ and so

$$(r+1)(s+1) \geqslant q(\lceil (n-1)/g \rceil).$$

It follows that $|S| \geqslant g.q(\lceil (n-1)/g \rceil)+1$.

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Table 1

n	$q(n)$	$\mathbf{K}_{\mathrm{Nu}}^{\mathrm{COSIG}}(n)$	$(n+1)q(n-1)$
2	6	5	12
3	8	7	24
4	9	9	40
5	12	10	54
6	12	13	84
7	15	13	96
8	15	16	135
9	16	16	150
10	18	17	176
11	20	19	216
12	20	21	260
13	24	21	280
14	24	25	360
15	24	25	400
16	25	25	425

Finally, by Lemma 2

$$g.q(\lceil (n-1)/g \rceil) \geqslant q(n-1)$$

and so we obtain

$$|S| \geqslant q(n-1)+1$$

as required.

The function q behaves in a somewhat erratic manner – though it is an increasing function. A brief table of values (Table 1) may be of interest.

The exact value of $\mathbf{K}_{\mathrm{Sg}}^{\mathrm{IG}}(n)$ remains something of a mystery. The lower bound

$$n+n(\log_2 n-2)+2n\sqrt{\tfrac{1}{2}(\log_2 n-3)}$$

(for $n \geqslant 8$) obtained in [**6**] takes the value 80 when $n = 16$, and between 80 and 425 there is a disconcertingly wide gap. A computer search does not appear very promising, since the number of semigroups of even a small order such as 16 is very large.

2. *The estimation of* $q(n)$

In [**6**] it was shown that

$$(\sqrt{n}+1)^2 \leqslant q(n) \leqslant n+3\sqrt{n}+1$$

and it was also remarked that the lower bound is best possible. Here we determine an improved upper bound.

THEOREM 3. *Let* $q(n) = \min\{(r+1)(s+1): rs \geqslant n\}$. *Then, except for* $n = 73$, 601, 1261 *and* 4033,

$$q(n) < n+2n^{\frac{1}{2}}+2n^{\frac{1}{4}}.$$

Proof. Let $m = \lceil 2n^{\frac{1}{2}} \rceil$; thus m is the unique integer such that

$$(m-1)^2 < 4n \leqslant m^2.$$

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It is easy to verify that for all $k \geqslant 1$

$$k(k-1) < n \leqslant k^2 \quad \text{implies} \quad m = 2k, \tag{3}$$

$$k^2 < n \leqslant k(k+1) \quad \text{implies} \quad m = 2k+1. \tag{4}$$

Let r, s be such that $(r+1)(s+1) = q(n)$. Then $rs-n \geqslant 0$ by definition, and

$$r+s \geqslant 2\sqrt{(rs)} \geqslant 2n^{\frac{1}{2}},$$

from which we deduce that $r+s-m \geqslant 0$ since $r+s$ is integral. We may thus define

$$\begin{aligned} E(n) &= \min\{(rs-n)+(r+s-m) : rs \geqslant n\} \\ &= q(n)-n-m-1 \end{aligned}$$

and be sure that $E(n) \geqslant 0$. Our aim will be to show that with finitely many exceptions

$$E(n) < 2(n^{\frac{1}{4}}-1). \tag{5}$$

From this it will follow that

$$\begin{aligned} q(n) &< n+2n^{\frac{1}{2}}+2n^{\frac{1}{4}}+(m-2n^{\frac{1}{2}}-1) \\ &< n+2n^{\frac{1}{2}}+2n^{\frac{1}{4}} \end{aligned}$$

(since $m < 2n^{\frac{1}{2}}+1$).

For certain 'landmark' numbers we have $E(n) = 0$. First, if $n = k^2$ for some k then $m = 2k$; so if we take $r = s = k$ we get

$$(rs-n)+(r+s-m) = 0.$$

Similarly, if $n = k(k+1)$ for some k then $m = 2k+1$; so if we take $r = k, s = k+1$ then

$$(rs-n)+(r+s-m) = 0.$$

Indeed we can say something a bit more general.

LEMMA 4. (i) *If* $n = k^2-t^2$, *where* $0 \leqslant t \leqslant \sqrt{k}$, *then* $E(n) = 0$.
(ii) *If* $n = k(k+1)-t(t-1)$, *where* $0 \leqslant t(t-1) < k$, *then* $E(n) = 0$.

Proof. In case (i), where $m = 2k$, take $r = k+t, s = k-t$. In case (ii), where $m = 2k+1$, take $r = k+t, s = k-t+1$.

It will make later arguments easier if we remark at this stage that the zeros of E given by Lemma 4 are in fact the only zeros of the function. If $E(n) = 0$ then we have a factorization $n = rs$ with $r+s = m$. If $m = 2k$ is even then we may assume that $r = k+t, s = k-t$ for some $t \geqslant 0$ and so $n = k^2-t^2$. If $m = 2k+1$ is odd then we may assume that $r = k+1-t, s = k+t$ for some $t \geqslant 0$ and so

$$n = k(k+1)-t(t-1).$$

By *successive zeros* of E we shall mean numbers p, q such that

$$p < q, \quad E(p) = E(q) = 0, \quad E(n) > 0 \quad (p < n < q).$$

LEMMA 5. *Let* p, q *be successive zeros of* E. *Then* $E(n) \leqslant q-p-1$ *for all* n *such that* $p < n < q$.

Proof. Let n be such that $p < n < q$. Then $\lceil 2n^{\frac{1}{2}} \rceil = \lceil 2q^{\frac{1}{2}} \rceil$, $= m$ (say); for otherwise (by (3) and (4)) there is a number l of the form k^2 or $k(k+1)$ such that $n \leqslant l < q$. Since $E(l) = 0$ this contradicts the assumption that p, q are successive zeros.

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Now by assumption we have a factorization rs of q with $rs-q = r+s-m = 0$. At worst we can use this factorization for n; so

$$E(n) \leqslant (rs-n)+(r+s-m) = q-n \leqslant q-p-1.$$

We now define the *gap functions* G_1 and G_2 as follows:

$$G_1(k) = \max\{q-p : E(q) = E(p) = 0, k(k-1) \leqslant q, p \leqslant k^2\} \quad (k \geqslant 2),$$

$$G_2(k) = \max\{q-p : E(q) = E(p) = 0, k^2 \leqslant q, p \leqslant k(k+1)\} \quad (k \geqslant 1).$$

Then we have

LEMMA 6. (i) *For all* $k \geqslant 3$, $G_1(k-1) \leqslant G_1(k)$.
(ii) *For all* $k \geqslant 2$, $G_2(k-1) \leqslant G_2(k)$.

Proof. (i) Between k^2 and $k(k-1)$ the function E has zeros at

$$k^2, k^2-1, \ldots, k^2-t^2, k^2-k, \tag{5}$$

where $t^2 \leqslant k < (t+1)^2$. Suppose first that $t^2 \leqslant k-1$. Then the gaps are

$$1, 3, \ldots, 2t-1, k-t^2$$

and $G_1(k) = \max\{2t-1, k-t^2\}$. Between $(k-1)^2$ and $(k-1)(k-2)$ the function E has zeros at

$$(k-1)^2, (k-1)^2-1, \ldots, (k-1)^2-t^2, (k-1)^2-(k-1) \tag{6}$$

with gaps $$1, 3, \ldots, 2t-1, k-t^2-1.$$

Thus $G_1(k-1) = \max\{2t-1, k-t^2-1\}$ and so certainly $G_1(k-1) \leqslant G_1(k)$. In the remaining case, where $t^2 = k$, the zeros (5) become

$$k^2, k^2-1, \ldots, k^2-(t-1)^2, k^2-k$$

and $G_1(k) = 2t-1$. The zeros (6) become

$$(k-1)^2, (k-1)^2-1, \ldots, (k-1)^2-(t-1)^2, (k-1)^2-(k-1),$$

giving $G_1(k-1) = 2t-2$.

This completes the proof of (i). The proof of (ii) follows a closely similar pattern and is omitted.

The same technique, of listing zeros and counting gaps, leads to the following formulae, which will be crucial in establishing the theorem:

$$G_1(v^2-1) = G_2(v^2-1) = 2v-2 \quad (v \geqslant 2); \tag{7}$$

$$G_1(v^2+v) = G_2(v^2+v-1) = 2v-1 \quad (v \geqslant 1). \tag{8}$$

We now obtain

LEMMA 7. (i) *Let* $v \geqslant 2$. *If* $n \leqslant v^2(v^2-1)$ *then* $E(n) < 2(v-1)$.
(ii) *Let* $v \geqslant 1$. *If* $n \leqslant (v^2+v)^2$ *then* $E(n) < 2v-1$.

Proof. (i) By Lemma 5,

$$E(n) < \max\{G_2(v^2-1), G_1(v^2-1), G_2(v^2-2), G_1(v^2-2), \ldots, G_2(2), G_1(2), G_1(1)\}.$$

Hence, by formula (7) and Lemma 6, $E(n) < 2(v-1)$. The proof of (ii) is closely similar.

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Suppose now that n is such that $E(n) = 2(v-1)$, an even number, with $v \geqslant 2$. The first n for which this can happen is $v^2(v^2-1)+1$. (It *does* in fact happen for $n = 13$ and $n = 73$: $E(13) = 2, E(73) = 4$.) Between $v^2(v^2-1)$ and $v^2(v^2+1)$ E has zeros (in descending order) at

$$v^2(v^2+1), v^2(v^2+1)-2, \ldots, v^2(v^2+1)-v(v-1), v^4,$$
$$v^4-1, v^4-4, \ldots, v^4-(v-1)^2, v^2(v^2-1)$$

with gaps $2, 4, \ldots, 2v-2, v, 1, 3, \ldots, 2v-1$. So, by Lemma 5, $E(n) < 2v-2$ for all n such that $v^2(v^2-1)+1 < n \leqslant v^2(v^2+1)$. Not until $n = v^2(v^2+1)+1$ (at the very least) can there be another n with $E(n)$ as large as $2v-2$.

Similarly, if we suppose that $E(n) = 2v-1$, an odd number, with $v \geqslant 1$, then the smallest n for which this can happen is $(v^2+v)^2+1$. Between $(v^2+v)^2$ and $(v^2+v+1)^2$ the function E has zeros at

$$(v^2+v+1)^2, (v^2+v+1)^2-1, \ldots, (v^2+v+1)^2-v^2,$$
$$(v^2+v)(v^2+v+1), (v^2+v)(v^2+v+1)-2, \ldots,$$
$$(v^2+v)(v^2+v+1)-v(v-1), (v^2+v)^2,$$

with gaps $1, 3, \ldots, 2v-3, v+1, 2, 4, \ldots, 2v$. So not until $n = (v^2+v+1)^2+1$ (at the very least) can there be another n with $E(n)$ as large as $2v-1$.

We can now prove the inequality (5) for all but certain exceptional numbers:

LEMMA 8. *Suppose that n does not belong to either of the sequences $(v^2(v^2-1)+1)$, $((v^2+v)^2+1)$. Then $E(n) < 2n^{\frac{1}{4}}-2$.*

Proof. Suppose first that $E(n)$ is even. By the argument above we may suppose that if $E(n) = 2(v-1)$ then $n > v^2(v^2+1) > v^4$. Hence

$$E(n) = 2v-2 < 2n^{\frac{1}{4}}-2.$$

On the other hand, if we suppose that $E(n)$ is odd, say $E(n) = 2v-1$, then again by the argument above $n > (v^2+v+1)^2 > (v+\frac{1}{2})^4$. Hence

$$E(n) = 2(v+\tfrac{1}{2})-2 < 2n^{\frac{1}{4}}-2.$$

It remains to consider the two exceptional sequences. Taking the easier one first, consider $n = (v^2+v)^2+1$ $(v \geqslant 1)$. Then

$$m = 2v^2+2v+1$$

by (4). Since $n+2 = rs$, where

$$r = v^2+3v+3, s = v^2-v+1$$

we have $$E(n) \leqslant (rs-n)+(r+s-m) = 2+3 = 5.$$

For $v \geqslant 4$ (i.e. $n \geqslant 401$) we have $2n^{\frac{1}{4}}-2 > 5$ and so the inequality $E(n) \leqslant 5$ is certainly enough to establish that $E(n) < 2n^{\frac{1}{4}}-2$. For $v = 1, 2, 3$ we obtain

v	n	$E(n)$	$2n^{\frac{1}{4}}-2$
1	5	1	0·99
2	37	3	2·93
3	145	5	4·94

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Table 2

v	n	r	s	$E(n)$	$2n^{\frac{1}{4}}-2$
2	13	7	2	2	1·80
3	73	11	7	4	3·85
4	241	22	11	2	5·88
5	601	29	21	8	7·90
6	1261	41	31	10	9·92
7	2353	62	38	5	11·93
8	4033	71	57	14	13·94
9	6481	94	69	6	15·94
10	9901	127	78	10	17·95
11	14521	137	106	2	19·95
12	20593	161	128	16	21·96

and so the inequality fails for $n = 5, 37, 145$. However, if we return to the inequality we are trying to establish, namely

$$q(n) < n + 2n^{\frac{1}{2}} + 2n^{\frac{1}{4}}$$

we find

n	$q(n)$	$n+2n^{\frac{1}{2}}+2n^{\frac{1}{4}}$
5	12	12·463
37	54	54·098
145	176	176·023.

So the numbers are not in fact exceptions.

Now consider $n = v^2(v^2-1)+1$ (where $v \geqslant 2$). Here $m = 2v^2$ (by (3)) and $n+15 = rs$, where

$$r = v^2+3v+4, \quad s = v^2-3v+4.$$

Thus $$E(n) \leqslant (rs-n)+(r+s-m) = 15+8 = 23.$$

For $v = 13$ (i.e. $n \geqslant 28393$) we have $2n^{\frac{1}{4}}-2 > 23$ and so the inequality $E(n) \leqslant 23$ is certainly enough to establish that $E(n) < 2n^{\frac{1}{4}}-2$. For $v < 13$ we obtain the results in Table 2, in which r and s are 'best possible' factors.

The inequality $E(n) < 2n^{\frac{1}{4}}-2$ thus fails when $n = 13, 73, 601, 1261, 4033$. If we examine the value of $q(n)$ for these possibly exceptional numbers we obtain Table 3.

Thus 13 is not actually an exception, and we conclude that the complete list of exceptions is 73, 601, 1261 and 4033.

Remark 1. Since the number of exceptions is so small and since from Table 3 it is clear that they are 'near misses', it is easy to obtain an inequality that holds for all n. The inequalities

$$\left.\begin{aligned} q(n) &< n+\lceil 2n^{\frac{1}{2}}\rceil + 2n^{\frac{1}{4}}, \\ q(n) &< n+2n^{\frac{1}{2}}+2n^{\frac{1}{4}}+0{\cdot}07 \\ q(n) &< n+2n^{\frac{1}{2}}+(2{\cdot}0226)\,n^{\frac{1}{4}} \end{aligned}\right\} \tag{9}$$

are all universally valid.

Remark 2. $E(n)$ can easily be found on a programmable calculator or microcomputer. The main loop has a dozen or so instructions and there are fewer than $n^{\frac{1}{4}}$ iterations of this loop involved in obtaining $E(n)$.

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Table 3

n	$q(n)$	$n+2n^{\frac{1}{2}}+2n^{\frac{1}{4}}$
13	24	24·009
73	96	95·934
601	660	659·933
1261	1344	1343·939
4033	4176	4175·950

Table 4

n	$E(n)$	$E(n)/n^{\frac{1}{4}}$	$r+s-m$
5	1	0·669	
13	2	1·053	
37	3	1·216	
73	4	1·368	
145	5	1·441	
487	7	1·490	
601	8	1·616	
1261	10	1·678	
2761	12	1·655	
3727	13	1·664	
4033	14	1·757	
9811	17	1·708	
29,416	23	1·756	4
70,001	28	1·721	
77,881	29	1·736	
80,401	30	1·782	
89,113	32	1·852	
314,777	43	1·815	
954,637	57	1·824	
2,388,601	71	1·806	
2,391,691	72	1·831	35
6,888,025	94	1·835	

We made a search for large values of $E(n)$. Table 4 gives numbers n for which $E(n)/n^{\frac{1}{4}}$ attains a new maximum or for which $E(n)$ attains a new maximum and $E(n)/n^{\frac{1}{4}}$ is close (within 0·05) to the previously attained maximum. When $r+s-m>0$ it is listed.

Remark 3. Using (9) we obtain a corollary to Theorem 1 to the effect that

$$\mathbf{K}_{\mathbf{Sg}}^{\mathbf{IG}}(n) \leqslant (n+1)\,(n-1+\lceil 2(n-1)^{\frac{1}{2}}\rceil+2(n-1)^{\frac{1}{4}}).$$

This bound, with leading term n^2, is better than the bound given by formula (1).

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