ON THE SOLUTION OF CERTAIN INTEGRAL EQUATIONS BY GENERALIZED FUNCTIONS

By J. B. MILLER

Communicated by J. H. Williamson

Received 1 November 1960

1. *Introduction.* This note is concerned with a method by which generalized solutions can be shown to exist for certain types of integral equations

$$f(x) = \mathscr{L}\{g, x\} = \int_0^\infty l(xy)\, g(y)\, dy, \tag{1}$$

where $f(x) \in L^2(0, \infty)$. The method is briefly this. An extended meaning is given to such equations by using generalized functions of a particular type. Then, if (1) denotes a transformation which has no everywhere-defined inverse in the usual sense, it may be possible to define in the extended sense an inverse transformation

$$g(x) = \mathscr{K}\{f, x\} = \int_0^\infty k(xy) f(y)\, dy, \tag{2}$$

so that if f is given, a generalized function $g = \mathscr{K}f$ can be determined as a solution of (1). A number of examples are given in illustration, but the present purpose is to show the existence of unique solutions rather than to identify them in particular cases. One example takes $\mathscr{L}$ as the Laplace transform.

The analysis starts from the space $L^2(0, \infty)$. The particular space of generalized functions required is determined (with some latitude) by the integral equation to be described; it is an extension of L^2. The extended interpretation of (2) involves using kernels k which are themselves generalized functions.

To describe a space of generalized functions appropriate to a given integral equation (1), we first choose a suitable *extending transformation* $\mathscr{A}$ and define a norm $\|\cdot\|_{\mathscr{A}}$ on real L^2 by writing $\|f\|_{\mathscr{A}} = \|\mathscr{A}f\|$, $\|\cdot\|$ being the norm of L^2. With this norm, L^2 is incomplete; its completion is the required space of generalized functions, $L^2_{\mathscr{A}}$. This we call the *extension space*. Thus the generalized functions are defined as strong limits, and a space of test functions is not required (though one may be defined, by using the adjoint transformation of $\mathscr{A}$).

We use Mellin transforms as an auxiliary device, classifying kernels according to the behaviour of their transforms. We begin with a summary of some standard results.

2. *Preliminaries.* We follow the notation of (2), and write the Mellin transform $\mathfrak{F}$ of f as

$$\mathfrak{F}(s) = \int_0^\infty f(x)\, x^{s-1} dx; \tag{3}$$

https://doi.org/10.1017/S030500410003591X Published online by Cambridge University Press

the inverse transformation is

$$f(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathfrak{F}(s)\, x^{-s}\, ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathfrak{F}(\tfrac{1}{2}+it)\, x^{-\frac{1}{2}-it}\, dt. \tag{4}$$

The corresponding Parseval relations are

$$\int_0^\infty f(x)\, g(x)\, dx = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathfrak{F}(s)\, \mathfrak{G}(1-s)\, ds, \quad \int_0^\infty \{f(x)\}^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathfrak{F}(\tfrac{1}{2}+it)|^2\, dt. \tag{5}$$

(The second assumes f real.) Throughout the paper, variables s and t will be related as in (4): $s = \frac{1}{2}+it$, $-\infty < t < \infty$. The following standard results are given for reference; for the proofs, see (2), § 3·17, §§ 8·5–8·7.

LEMMA α. *Let $f(x) \in L^2(0,\infty)$; or let $\mathfrak{F}(\frac{1}{2}+it) \in L^2(-\infty,\infty)$. Then* (3) *and* (4) *hold, f determining $\mathfrak{F}$ or vice versa, the integrals both converging in mean square:*

$$\mathfrak{F}(\tfrac{1}{2}+it) = \underset{\substack{-\infty<\lambda<\infty \\ X\to\infty}}{\text{l.i.m.}} \int_{1/X}^{X} f(x)\, x^{-\frac{1}{2}+it}\, dx, \quad f(x) = \underset{\substack{0<x<\infty \\ T\to\infty}}{\text{l.i.m.}} \frac{1}{2\pi}\int_{-T}^{T} \mathfrak{F}(\tfrac{1}{2}+it)\, x^{-\frac{1}{2}-it}\, dt. \tag{6}$$

If also $g(x) \in L^2(0,\infty)$, then (5) *holds, the integrals converging absolutely.*

(We shall call f and $\mathfrak{F}$ 'Mellin transforms of L^2' in this sense, not showing the limit-in-mean explicitly. The distinction between $L^2(0,\infty)$ and $L^2(-\infty,\infty)$ will be clear from the context.)

LEMMA β. *Let the function $\mathfrak{L}(s)$ be such that $|\mathfrak{L}(s)|$ is bounded, and let $(1-s)^{-1}\mathfrak{L}(s)$ have inverse Mellin transform $x^{-1}l_1(x)$ in L^2. Then the transformation $f = \mathscr{L}g$ defined by*

$$\int_0^x f(u)\, du = \int_0^\infty u^{-1} l_1(xu)\, g(u)\, du \tag{7}$$

maps L^2 into L^2, and the Mellin transforms satisfy

$$\mathfrak{F}(s) = \mathfrak{L}(s)\, \mathfrak{G}(1-s); \tag{8}$$

conversely, (8) *with $\mathfrak{G} \in L^2$ implies* (7).

If also $\mathfrak{L}(s)$ belongs to L^2, and so has an inverse Mellin transform $l(x)$, then

$$l_1(x) = \int_0^x l(v)\, dv, \tag{9}$$

and the transformation $\mathscr{L}$ can be written

$$f(x) = \int_0^\infty l(xu)\, g(u)\, du. \tag{10}$$

3. *The extension space $L^2_{\mathscr{A}}$.* Let $\mathscr{A}$ be a linear transformation mapping real $L^2(0,\infty)$ into itself; for example, of the form

$$g(x) = \mathscr{A}\{f,x\} = \int_0^\infty a(x,y) f(y)\, dy. \tag{11}$$

Writing $\|\,.\,\|$ for the norm of $L^2(0,\infty)$ and

$$\|f\|_{\mathscr{A}} = \|\mathscr{A}f\| = \left(\int_0^\infty \left|\int_0^\infty a(x,y) f(y)\, dy\right|^2 dx\right)^{\frac{1}{2}}, \tag{12}$$

https://doi.org/10.1017/S030500410003591X Published online by Cambridge University Press

we take the extension space $L^2_{\mathscr{A}}$ to be the extension of $L^2(0,\infty)$ isomorphic with the space of equivalence classes of sequences (f_n) which are Cauchy in $\mathscr{A}$-norm, that is, sequences with the property

$$\|f_n - f_m\|_{\mathscr{A}} \to 0 \quad \text{as} \quad \min(n,m) \to \infty, \quad f_n \in L^2 \text{ for all } n.$$

We find that the following conditions on the transformation are sufficient to ensure that the extension is a proper one and serviceable:

$E(a)$ $\mathscr{A}$ is a bounded transformation mapping L^2 into L^2;

$E(b)$ $\mathscr{A}f = 0$ implies $f = 0$;

$E(c)$ the delta sequence

$$\delta_n(y,\xi) = \begin{cases} \frac{1}{2}n & (|y-\xi| < 1/n), \\ 0 & (|y-\xi| > 1/n) \end{cases}$$

is Cauchy in $\mathscr{A}$-norm for all (or almost all) $\xi > 0$ (defining the delta 'function' Δ_ξ).

The linearity of $\mathscr{A}$ and $E(b)$ together ensure that $\|\,.\,\|_{\mathscr{A}}$ is a norm. $E(a)$ ensures that $\|f_n - f\| \to 0$ implies $\|f_n - f\|_{\mathscr{A}} \to 0$, so that strong convergence in L^2 is preserved in $L^2_{\mathscr{A}}$; and $E(c)$ that $L^2_{\mathscr{A}}$ is a proper extension of L^2.

DEFINITION. *A transformation $\mathscr{A}$ satisfying conditions $E(a)$, (b) and (c) will be called an extending transformation, and the corresponding a an extending kernel.*

A sufficient condition for $E(c)$ is that

$$\alpha_n^2 = \int_0^\infty \left| \int_{-1}^1 \{a(x,\xi + y/n) - a(x,\xi)\}\, dy \right|^2 dx \to 0 \quad \text{as} \quad n \to \infty \tag{13}$$

for all (or almost all) ξ; this may be verified by considering the expression

$$\|\mathscr{A}\{\delta_n, x\} - a(x,\xi)\|.$$

4. *Integral equations with kernels $k(xy)$.* Watson kernels provide a well-known class of transformations $f = \mathscr{L}g$ mapping L^2 into L^2, which take the form (7). A Watson kernel l (defined, if at all, by the relation (9)) is characterized by the property that its Mellin transform $\mathfrak{L}$ (in the general case of asymmetric transforms) is bounded in the sense

$$\alpha^{-1} \leqslant |\mathfrak{L}(\tfrac{1}{2} + it)| \leqslant \beta \quad (\alpha, \beta \text{ positive constants}) \tag{14}$$

in which case the transformation has an inverse of the same kind, whose kernel k is determined through its Mellin transform $\mathfrak{K}$ by

$$\mathfrak{L}(\tfrac{1}{2} + it)\,\mathfrak{K}(\tfrac{1}{2} - it) = 1: \tag{15}$$

this is a consequence of (8). Here it is the finite upper bound in (14) which ensures that $\mathscr{L}$ maps L^2 into L^2 (cf. Lemma β, above), and the positive lower bound which ensures that the inverse transformation is definable from (15). Since that lower bound must be zero if, for example, $x^{-\frac{1}{2}}l(x) \in L(0,\infty)$ (by the Riemann–Lebesgue theorem), the number of functions l giving rise to invertible equations of the type (7) is noticeably restricted; there are, on the other hand, any number of simple functions l whose Mellin transforms satisfy

$$0 < |\mathfrak{L}(\tfrac{1}{2} + it)| \leqslant \beta \ (t \text{ finite}), \quad |\mathfrak{L}(\tfrac{1}{2} + it)| \to 0 \text{ as } t \to \pm\infty, \tag{16}$$

https://doi.org/10.1017/S030500410003591X Published online by Cambridge University Press

and hence which define transformations (7) whose inverses, if they exist, are not defined throughout L^2. These transformations can be dealt with by extending the class of Watson kernels to cover kernels k for which $|\mathfrak{K}(\frac{1}{2}+it)| \to \infty$ as $t \to \pm\infty$.

We shall show how, for such a $\mathfrak{K}$, an extension space of L^2 can be found which contains a unique $\mathscr{K}$-transform $g = \mathscr{K}f$ for every function f of L^2. Then if $\mathfrak{L}$, satisfying (16), defines $\mathfrak{K}$ by (15), we shall have found a space of generalized functions containing a unique solution g of $f = \mathscr{L}g$ for every $f \in L^2$.

We construct different extension spaces to cater for different rates of growth of $\mathfrak{K}$. and hence start with the assumption

I. *$\mathfrak{K}(\frac{1}{2}+it)$ is measurable and bounded in every finite interval,*

$$\overline{\mathfrak{K}(\tfrac{1}{2}+it)} = \mathfrak{K}(\tfrac{1}{2}-it),$$

and

$$0 < c \leqslant |\mathfrak{K}(\tfrac{1}{2}+it)| \leqslant \omega(|t|) \tag{17}$$

for an increasing function ω and some constant c.

The condition on the conjugate of $\mathfrak{K}$ is necessary and sufficient for the corresponding k to be real. We shall consider two types of extending kernels.

5. *Homogeneous extending kernels.* A function $h(x,y)$ will here be called homogeneous if it is homogeneous of degree -1; i.e. if $h(xt,yt) = t^{-1}h(x,y)$ identically $(t \neq 0)$. Given a function $\phi(y)$, one and only one homogeneous function $h(x,y)$ $(x \neq 0)$ can be constructed so that $h(1,y) = \phi(y)$.

To define the kernel $a(x,y)$, we start with its Mellin transform.

II. *Let $\mathfrak{A}(\frac{1}{2}+it)$ be a measurable function which*

(i) *is bounded,* $|\mathfrak{A}(\frac{1}{2}+it)| \leqslant M < \infty$;

(ii) *belongs to* $L^2(-\infty,\infty)$;

(iii) *satisfies* $\overline{\mathfrak{A}(\frac{1}{2}+it)} = \mathfrak{A}(\frac{1}{2}-it)$;

(iv) *is non-zero, except possibly on a set of measure zero.*

By (ii) and Lemma α, there is a unique homogeneous function $a(x,y)$ for which

$$\mathfrak{A}(s) = \int_0^\infty a(1,y)\,y^{s-1}\,dy, \quad a(1,y) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathfrak{A}(s)\,y^{-s}\,ds. \tag{18}$$

It is easily seen that $a(x,1)$ has Mellin transform $\mathfrak{A}(1-s)$; and that, while

$$a(x,y) \in L^2 \quad (0 < y < \infty)$$

for $x \neq 0$ and *vice versa*, a is not of finite double norm in L^2. Condition (iii) ensures that a is real.

LEMMA 1. *The real homogeneous kernel $a(x,y)$ defined by* (18) *determines a mapping*

$$h(x) = \mathscr{A}\{f,x\} = \int_0^\infty a(x,y)f(y)\,dy \tag{19}$$

of $L^2(0,\infty)$ *into itself; and if f (and so h) belongs to L^2, the Mellin transforms satisfy*

$$\mathfrak{H}(s) = \mathfrak{A}(1-s)\,\mathfrak{F}(s). \tag{20}$$

Proof. The integral in (19) converges absolutely. Lemma β with $\mathfrak{L}(s) \equiv \mathfrak{A}(s)$ shows that $\mathfrak{A}$ gives rise to a mapping of L^2 into itself represented by (10), which is (19) with $h(x) = x^{-1}f(x^{-1})$.

https://doi.org/10.1017/S030500410003591X Published online by Cambridge University Press

LEMMA 2. *The transformation $\mathscr{A}$ of* (19) *is an extending transformation, and defines an extension space $L^2_{\mathscr{A}}$ containing delta functions Δ_ξ for all $\xi > 0$.*

Proof. By (5) and Lemma α, if $f \in L^2$ and $g = \mathscr{A}f$ then

$$\|\mathscr{A}f\|^2 = (2\pi)^{-1}\int_{-\infty}^{\infty}|\mathfrak{G}(s)|^2\,dt = (2\pi)^{-1}\int_{-\infty}^{\infty}|\mathfrak{A}(1-s)\,\mathfrak{F}(s)|^2\,dt$$
$$\leqslant M^2(2\pi)^{-1}\int_{-\infty}^{\infty}|\mathfrak{F}(s)|^2\,dt = M^2\|f\|^2.$$

Hence $E(a)$ holds. Moreover, if $g = 0$ then $\mathfrak{A}(1-s)\,\mathfrak{F}(s) = \mathfrak{G}(s) = 0$ and hence $f = 0$, by II (iv); thus (b) holds. We now prove (c), by proving (13). From the homogeneity of $a(x, y)$ and from the Mellin Parseval relation (5), we can deduce

$$a(x,y) = \frac{x^{-1}}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\mathfrak{A}(s)\left(\frac{y}{x}\right)^{-s}ds,$$
$$\int_{-1}^{1}a\left(x,\xi+\frac{y}{n}\right)dy = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\mathfrak{A}(s)\,x^{s-1}ds\int_{-1}^{1}\left(\xi+\frac{y}{n}\right)^{-s}dy.$$

Thus the expression within the modulus signs in (13) has Mellin transform

$$\mathfrak{A}(1-s).\int_{-1}^{1}\{(\xi+y/n)^{s-1}-\xi^{s-1}\}\,dy = \mathfrak{A}(1-s)\,\mathfrak{J}(1-s,n,\xi) \quad \text{say,} \tag{21}$$

and the Mellin Parseval relation applied to (13) gives

$$\alpha_n^2 = (2\pi)^{-1}\int_{-\infty}^{\infty}|\mathfrak{A}(\tfrac{1}{2}+it)\,\mathfrak{J}(\tfrac{1}{2}+it,n,\xi)|^2\,dt. \tag{22}$$

Now for large n,

$$|\mathfrak{J}(s,n,\xi)| \leqslant \int_{-1}^{1}\{(\xi+y/n)^{-\frac{1}{2}}+\xi^{-\frac{1}{2}}\}\,dy \leqslant C\xi^{-\frac{1}{2}},$$

a bound independent of t and n. Again, if $\xi > 0$, the mean-value theorem can be applied to the real and imaginary parts of the integrand of $\mathfrak{J}(s,n,\xi)$ to give another order condition on $\mathfrak{J}$: the contribution of the real part, for example, is

$$-\int_{-1}^{1}\frac{y}{n}\left(\xi+\frac{\theta y}{n}\right)^{-\frac{3}{2}}\left\{\tfrac{1}{2}\cos\left[t\left(\xi+\frac{\theta y}{n}\right)\right]+t\sin\left[t\left(\xi+\frac{\theta y}{n}\right)\right]\right\}dy,$$

where $0 < \theta(y,n,\xi) < 1$; and from this and a similar expression we get, for large enough n,

$$|\mathfrak{J}(s,n,\xi)| \leqslant \frac{2}{n}\left(\xi-\frac{1}{n}\right)^{-\frac{3}{2}}(\tfrac{1}{2}+|t|).$$

In (22) write

$$\int_{-\infty}^{\infty} = \int_{-\infty}^{-T}+\int_{-T}^{T}+\int_{T}^{\infty}.$$

Since, for example,

$$\int_{T}^{\infty} \leqslant C^2\xi^{-1}\int_{T}^{\infty}|\mathfrak{A}(\tfrac{1}{2}+it)|^2\,dt,$$

we can by choice of sufficiently large T make the first and third integrals arbitrarily small; and with this fixed T,

$$\int_{-T}^{T} \leqslant \frac{4}{n^2}\left(\xi-\frac{1}{n}\right)^{-3}(\tfrac{1}{2}+T)^2\int_{-T}^{T}|\mathfrak{A}(\tfrac{1}{2}+it)|^2\,dt = o(1)$$

for large n. Thus $\alpha_n \to 0$ as $n \to \infty$, and $E(c)$ holds for all $\xi > 0$. This proves the lemma.

https://doi.org/10.1017/S030500410003591X Published online by Cambridge University Press

We remark that distinct kernels a_1, a_2 do not necessarily determine distinct extension spaces. In fact, if $\mathfrak{A}_1(s)$ and $\mathfrak{A}_2(s)$ both satisfy II and

$$0 < c_1 \leqslant |\mathfrak{A}_1(s)/\mathfrak{A}_2(s)| \leqslant c_2 < \infty \tag{23}$$

for some constants c_1 and c_2, the corresponding spaces $L^2_{\mathscr{A}_1}$ and $L^2_{\mathscr{A}_2}$ may be identified. For

$$\|f\|^2_{\mathscr{A}_1} = (2\pi)^{-1}\int_{-\infty}^{\infty} |\mathfrak{A}_1(1-s)\,\mathfrak{F}(s)|^2\,dt \leqslant c_2^2(2\pi)^{-1}\int_{-\infty}^{\infty} |\mathfrak{A}_2(1-s)\,\mathfrak{F}(s)|^2\,dt = c_2^2\,\|f\|^2_{\mathscr{A}_2},$$

and similarly $\|f\|_{\mathscr{A}_2} \leqslant c_1^{-1}\|f\|_{\mathscr{A}_1}$. It follows that a sequence is Cauchy in one norm if and only if it is Cauchy in the other.

The space L^2 is embedded in $L^2_{\mathscr{A}}$ by identifying $\phi \in L^2$ with the repeating sequence $(\phi, \phi, \ldots)$. It follows that *a generalized function g belongs to L^2 if and only if it possesses at least one sequence which is Cauchy in L^2.* (This is not to say that *every* sequence for g which is composed of L^2 functions necessarily converges to g in L^2.)

A Mellin transform can be defined for a generalized function f, as follows. If (f_n), $f_n \in L^2$, is a Cauchy sequence for f, the relation

$$\begin{aligned}\|f_n - f_m\|^2_{\mathscr{A}} &= \int_0^{\infty} |\mathscr{A}\{f_n, x\} - \mathscr{A}\{f_m, x\}|^2\,dx \\ &= (2\pi)^{-1}\int_{-\infty}^{\infty} |(\mathfrak{F}_n(s) - \mathfrak{F}_m(s))\,\mathfrak{A}(1-s)|^2\,dt\end{aligned}$$

shows that $\mathfrak{F}_n(s)\,\mathfrak{A}(1-s)$ converges in L^2, to (say) $\widetilde{\mathfrak{F}}(s)\,\mathfrak{A}(1-s)$. We define the Mellin transform of f to be $\widetilde{\mathfrak{F}}(s)$; it is determined for almost all t, by II (iv). It is easy to show that the definition is independent of the particular Cauchy sequence used for f (for if $(f_n') \sim (f_n)$ in $L^2_{\mathscr{A}}$, i.e. $\|f_n' - f_n\|_{\mathscr{A}} \to 0$, then $(\mathfrak{F}_n'(s) - \mathfrak{F}_n(s))\,\mathfrak{A}(1-s)$ converges to zero), and that when f belongs to L^2 and has an ordinary Mellin transform $\mathfrak{F}(\bar{s})$, then $\widetilde{\mathfrak{F}}(s) \equiv \mathfrak{F}(s)$ (the proof uses II (i)).

6. *Definition of the transformation $\mathscr{K}$.* Up to this point we have left $\mathfrak{A}$ and $\mathfrak{K}$ unrelated. Suppose that $\mathfrak{K}$ is given, satisfying Condition I for a known function $\omega(|t|)$, and now let $\mathfrak{A}$ be chosen so as to satisfy II (i)–(iv) and also

II (v) $$|\mathfrak{A}(\tfrac{1}{2}+it)| \leqslant \frac{M}{\omega(|t|)\,(1+|t|^{\eta})} \quad (M,\ \eta \text{ constant},\ \eta > \tfrac{1}{2}).$$

Then $$|\mathfrak{K}(\tfrac{1}{2}+it)\,\mathfrak{A}(\tfrac{1}{2}-it)| \leqslant M(1+|t|^{\eta})^{-1},$$

and $\mathfrak{K}(s)\,\mathfrak{A}(1-s)$ belongs to L^2.

The kernel k corresponding to $\mathfrak{K}$ (formally, its inverse Mellin transform) is taken to be the unique generalized function of $L^2_{\mathscr{A}}$ defined as follows. Write

$$\mathfrak{K}_n(\tfrac{1}{2}+it) = \begin{cases} \mathfrak{K}(\tfrac{1}{2}+it) & (|t| < n), \\ 0 & (|t| > n); \end{cases}$$

https://doi.org/10.1017/S030500410003591X Published online by Cambridge University Press

and let k_n be the inverse Mellin transform of $\mathfrak{K}_n$. Then $k_n \in L^2$ by I, and its $\mathscr{A}$-transform $\mathscr{A}\{k_n, x\}$ has Mellin transform $\mathfrak{K}_n(s)\,\mathfrak{A}(1-s)$ in L^2. Hence, if $n > m$ say,

$$\begin{aligned}\|k_n - k_m\|_{\mathscr{A}}^2 &= \int_0^\infty |\mathscr{A}\{k_n, x\} - \mathscr{A}\{k_m, x\}|^2\,dx \\ &= (2\pi)^{-1}\int_{-\infty}^{\infty} |(\mathfrak{K}_n(s) - \mathfrak{K}_m(s))\,\mathfrak{A}(1-s)|^2\,dt \\ &= (2\pi)^{-1}\left(\int_{-n}^{-m} + \int_m^n\right)|\mathfrak{K}(s)\,\mathfrak{A}(1-s)|^2\,dt \\ &\leqslant \pi^{-1}M^2\int_m^n (1+|t|^\eta)^{-2}\,dt \\ &\to 0 \quad \text{as} \quad m \to \infty.\end{aligned}$$

Thus (k_n) is a Cauchy sequence in $L^2_{\mathscr{A}}$; we take this to define the kernel k. It is clear that $\mathfrak{K}$ is the Mellin transform of the generalized function k, in the sense previously described. We write (formally)

$$g(x) = \mathscr{K}\{f, x\} = \int_0^\infty k(xy) f(y)\,dy \tag{24}$$

for the corresponding transformation on L^2: this is to be understood in the sense

$$g = \lim_{\mathscr{A}} g_n, \quad g_n(x) = \mathscr{K}_n\{f, x\} = \int_0^\infty k_n(xy) f(y)\,dy, \tag{25}$$

that is, g is a generalized function, the limit in $\mathscr{A}$-norm of the sequence (g_n) of functions of L^2. For the Mellin transform of $g_n(x)$ is $\mathfrak{G}_n(s) = \mathfrak{K}_n(s)\,\mathfrak{F}(1-s)$, and thus

$$\begin{aligned}\|g_n - g_m\|_{\mathscr{A}}^2 &= (2\pi)^{-1}\left(\int_{-n}^{-m} + \int_m^n\right)|\mathfrak{F}(1-s)\,\mathfrak{K}(s)\,\mathfrak{A}(1-s)|^2\,dt \\ &\leqslant (2\pi)^{-1}M^2\left(\int_{-n}^{-m} + \int_m^n\right)|\mathfrak{F}(\tfrac{1}{2} - it)|^2\,dt \to 0 \quad \text{as} \quad \min(n, m) \to \infty.\end{aligned}$$

This proves that $\mathscr{K}f$ exists in $L^2_{\mathscr{A}}$ if $f \in L^2$, and is unique; but the definition depends upon the particular sequence (k_n) used for k. Suppose that (k'_n) is any other L^2 sequence defining k (so that $\|k_n - k'_n\|_{\mathscr{A}} \to 0$) and such that $|\mathfrak{K}'_n(s)\,\mathfrak{A}(1-s)| \leqslant M'$: we show that $(\mathscr{K}_n f)$ and $(\mathscr{K}'_n f)$ define the same generalized function. For arbitrary $f \in L^2$ let f_p be that function for which $\mathfrak{F}_p(s) = \mathfrak{F}(s)$ for $|\mathfrak{F}(s)| \leqslant p$, $= p$ otherwise. By using Mellin transforms, we can show that

$$\begin{aligned}\|\mathscr{K}_n f - \mathscr{K}'_n f\|_{\mathscr{A}} &\leqslant \|\mathscr{K}_n(f - f_p)\|_{\mathscr{A}} + \|(\mathscr{K}_n - \mathscr{K}'_n)f_p\|_{\mathscr{A}} + \|\mathscr{K}'_n(f_p - f)\|_{\mathscr{A}} \\ &\leqslant M\,\|f - f_p\| + p\,\|k_n - k'_n\|_{\mathscr{A}} + M'\,\|f_p - f\|,\end{aligned}$$

and so, by choice first of p and then of n, that $\|\mathscr{K}_n f - \mathscr{K}'_n f\|_{\mathscr{A}} \to 0$ as $n \to \infty$, which was to be proved. Thus $\mathscr{K}$ maps L^2 into $L^2_{\mathscr{A}}$.

Now let $\mathfrak{L}$ be the function defined by (15), so that $|\mathfrak{L}(s)| \leqslant c^{-1}$ (by I), and let $\mathscr{L}$ be the corresponding transformation (7), mapping L^2 into L^2. We can extend the domain of $\mathscr{L}$ to $L^2_{\mathscr{A}}$ by continuity, defining $\mathscr{L}g$ when $g \in L^2_{\mathscr{A}}$ by

$$\mathscr{L}g = \lim_{\mathscr{A}} \mathscr{L}g_n, \quad \text{if} \quad g = \lim_{\mathscr{A}} g_n,\ g_n \in L^2.$$

https://doi.org/10.1017/S030500410003591X Published online by Cambridge University Press

The limit always exists; for

$$\|\mathscr{L}g_n - \mathscr{L}g_m\|^2_{\mathscr{A}} = (2\pi)^{-1}\int_{-\infty}^{\infty} |(\mathfrak{G}_n(1-s) - \mathfrak{G}_m(1-s))\,\mathfrak{L}(s)\,\mathfrak{A}(1-s)|^2\,dt$$

$$\leqslant (2\pi c^2)^{-1}\int_{-\infty}^{\infty} |(\mathfrak{G}_n(1-s) - \mathfrak{G}_m(1-s))\,\mathfrak{A}(s)|^2 \left|\frac{\mathfrak{A}(1-s)}{\mathfrak{A}(s)}\right|^2 dt$$

$$= c^{-2}\|g_n - g_m\|^2_{\mathscr{A}}, \quad \text{by II (iii).}$$

A similar argument shows that the limit is independent of the particular Cauchy sequence (g_n) used to define g, and that $\mathscr{L}g$ has its ordinary meaning when $g \in L^2$. Thus $\mathscr{L}$ maps $L^2_{\mathscr{A}}$ into $L^2_{\mathscr{A}}$, and L^2 into L^2.

We now prove a succession of results, which show the sense in which $\mathscr{K}$ and $\mathscr{L}$ are inverse transformations, and we shall deduce that the equation $f = \mathscr{L}g$ has a unique solution g in $L^2_{\mathscr{A}}$ when $f \in L^2$, thus establishing the result foreshadowed in § 4.

(1′) *If $f \in L^2$, then $\mathscr{L}\mathscr{K}f \in L^2$ and $\mathscr{L}\mathscr{K}f = f$.*

Let $f \in L^2$. Then $(\mathscr{K}_n f)$ is a Cauchy sequence of L^2 functions defining $\mathscr{K}f$, and so $\mathscr{L}\mathscr{K}f = \lim_{\mathscr{A}} \mathscr{L}(\mathscr{K}_n f)$. But $\mathscr{L}\mathscr{K}_n f$ has the Mellin transform

$$\mathfrak{L}(s)\,\mathfrak{K}_n(1-s)\,\mathfrak{F}(s) = \begin{cases} \mathfrak{F}(s) & (|t| < n), \\ 0 & (|t| > n), \end{cases}$$

and this converges in L^2, to $\mathfrak{F}(s)$. Therefore $\mathscr{L}\mathscr{K}_n f$ converges in L^2, to f. The result follows from the remarks on the embedding $L^2 \subset L^2_{\mathscr{A}}$ at the end of § 5.

(2′) *If $f \in L^2$ and $\mathscr{K}f = 0$, then $f = 0$.*

By (1′), $$f = \mathscr{L}(\mathscr{K}f) = \mathscr{L}0 = 0.$$

(3′) *If g, belonging to $L^2_{\mathscr{A}}$, has Mellin transform $\mathfrak{G}(s)$, then $h = \mathscr{L}g$ has Mellin transform $\mathfrak{L}(s)\,\mathfrak{G}(1-s)$.*

If $g \in L^2$, this is result (8). Let $g \in L^2_{\mathscr{A}}$, and let (g_n), $g_n \in L^2$, be a defining sequence. Then $\mathfrak{G}_n(s)\,\mathfrak{A}(1-s)$ converges to $\mathfrak{G}(s)\,\mathfrak{A}(1-s)$; and $\mathfrak{L}(s)\,\mathfrak{G}_n(1-s)\,\mathfrak{A}(1-s)$ converges to $\mathfrak{H}(s)\,\mathfrak{A}(1-s)$. Since $|\mathfrak{A}(1-s)/\mathfrak{A}(s)| = 1$ and $|\mathfrak{L}(s)| \leqslant c^{-1}$, it follows that

$$\mathfrak{L}(s)\,\mathfrak{G}_n(1-s)\,\mathfrak{A}(1-s)$$

also converges to $\mathfrak{L}(s)\,\mathfrak{G}(1-s)\,\mathfrak{A}(1-s)$. The result follows.

(4′) *If $g \in L^2_{\mathscr{A}}$ and $\mathscr{L}g = 0$, then $g = 0$.*

Let (h_n), $h_n \in L^2$, be a Cauchy sequence defining $\mathscr{L}g$, and let $\mathfrak{H}(s)$ be the Mellin transform of $\mathscr{L}g$. Then

$$(2\pi)^{-1}\int_{-\infty}^{\infty} |\mathfrak{H}(s)\,\mathfrak{A}(1-s)|^2\,dt = \lim_{n\to\infty} (2\pi)^{-1}\int_{-\infty}^{\infty} |\mathfrak{H}_n(s)\,\mathfrak{A}(1-s)|^2\,dt$$

$$= \lim_{n\to\infty} \|h_n\|^2_{\mathscr{A}} = 0,$$

and so, by II (iv), $\mathfrak{H}(s) = 0$ almost everywhere. Therefore by (3′), $\mathfrak{L}(s)\,\mathfrak{G}(1-s) = 0$ almost everywhere. Since $\mathfrak{K}(s)$ is bounded in every finite interval, $\mathfrak{L}(s)$ is zero nowhere; hence $\mathfrak{G}(1-s) = 0$ almost everywhere, and $g = 0$.

(5′) *If $g \in L^2_{\mathscr{A}}$ and $\mathscr{L}g \in L^2$, then $\mathscr{K}\mathscr{L}g = g$.*

Write $f = \mathscr{L}g \in L^2$. Then $\mathscr{K}f$ exists, and by (1′), $\mathscr{L}\mathscr{K}f = f$. Thus $\mathscr{L}\mathscr{K}f = \mathscr{L}g$, $\mathscr{L}(\mathscr{K}f - g) = 0$, and by (4′), $\mathscr{K}f = g$, i.e. $\mathscr{K}\mathscr{L}g = g$.

https://doi.org/10.1017/S030500410003591X Published online by Cambridge University Press

From (1′) and (5′) we see that $\mathscr{L}\mathscr{K}$ and $\mathscr{K}\mathscr{L}$ are both, on their domains, equal to the identity transformation. Consider now the solution, for g, of the integral equation $f = \mathscr{L}g$, supposing f given in L^2. By (1′), one solution is $\mathscr{K}f$. If h is another, then $\mathscr{L}(g-h) = 0$, and (4′) shows that $g = h$. Thus the solution is unique. Likewise, the solution f of the equation $g = \mathscr{K}f$ with $g \in L^2_{\mathscr{A}}$, if it exists in L^2, is unique, by (2′) and equals $\mathscr{L}g$, by (5′). Thus we have

THEOREM. *Let $\mathscr{L}$ be the transformation of Lemma β, determined by a function $\mathfrak{L}$ for which $\mathfrak{K}(s) = 1/\mathfrak{L}(1-s)$ satisfies* I *of* § 4. *Then $\mathscr{L}$ has an extension to $L^2_{\mathscr{A}}$ if $\mathscr{A}$ is a transformation determined by a function $\mathfrak{A}$ satisfying* II (i)–(v); *and if $f \in L^2$, the equation*

$$f = \mathscr{L}g$$

has a unique solution g in $L^2_{\mathscr{A}}$. The solution is

$$g = \mathscr{K}f,$$

where $\mathscr{K}$ is the transformation determined by $\mathfrak{K}$.

If $g' \in L^2_{\mathscr{A}}$ and if $g' = \mathscr{K}f'$ has a solution f' in L^2, then that solution is $\mathscr{L}g'$, and is unique.

If the inverse Mellin transform of $\mathfrak{K}$ is a known function, κ say ($\kappa \notin L^2$), it may in some cases be possible to identify k and κ, and (24) with κ written for k may hold in some more usual sense when f belongs to a subset of L^2.

7. *Extending kernels for particular functions $\omega(t)$.*

(1″) $|\mathfrak{K}(\frac{1}{2}+it)| \leqslant \omega_0 |t|^\rho$ (ω_0 constant).

Let $\mathfrak{P}_\lambda(s) = \Gamma(s)/\Gamma(s+\lambda)$; since

$$|\Gamma(\sigma+it)| \sim |t|^{\sigma-\frac{1}{2}} e^{-\frac{1}{2}\pi|t|}\sqrt{(2\pi)} \quad \text{as} \quad t \to \pm\infty$$

we have $|\mathfrak{P}_\lambda(\frac{1}{2}+it)| \sim |t|^{-\lambda}$. For $\lambda > \frac{1}{2}$, $\mathfrak{P}_\lambda$ satisfies the conditions II (i)–(iv). Since $\mathfrak{P}_\lambda$ is the Mellin transform of the homogeneous kernel

$$p_\lambda(x,y) = \begin{cases} x^{-\lambda}(x-y)^{\lambda-1}/\Gamma(\lambda) & (0 < y < x), \\ 0 & (x < y), \end{cases}$$

we have $\mathscr{P}_\lambda f = x^{-\lambda} f^{(-\lambda)}(x)$, where $f^{(-\lambda)}$ is the Riemann–Liouville fractional integral of f, of order λ. The corresponding space $L^2_{\mathscr{P}_\lambda}$ is discussed in (1). For given ρ, a space sufficient to contain the $\mathscr{K}$-transforms of all L^2 functions is found by taking $\lambda > \rho + \frac{1}{2}$.

Alternatively, let $\mathfrak{Q}_\lambda(s) = \Gamma(\lambda) s^{-\lambda}$; for $\lambda > \frac{1}{2}$, $\mathfrak{Q}_\lambda(s)$ satisfies II (i)–(iv), and $|\mathfrak{Q}_\lambda(\frac{1}{2}+it)| \sim \Gamma(\lambda)|t|^{-\lambda}$. The corresponding homogeneous kernel is

$$q_\lambda(x,y) = \begin{cases} x^{-1}\{\log(xy^{-1})\}^{\lambda-1} & (0 < y < x), \\ 0 & (x < y). \end{cases}$$

The spaces $L^2_{\mathscr{P}_\lambda}$ and $L^2_{\mathscr{Q}_\lambda}$ are identical.

A kernel sufficient for $\rho < \frac{3}{2}$ is given by

$$\mathfrak{R}(s) = \frac{1}{1+s^2}, \quad r(x,y) = \begin{cases} x^{-1}\sin\{\log(xy^{-1})\} & (0 < y < x), \\ 0 & (x < y). \end{cases}$$

https://doi.org/10.1017/S030500410003591X Published online by Cambridge University Press

Examples. (i) The kernel $l(x) = J_\mu(ax)\,J_\nu(ax)$ $(a > 0)$ has Mellin transform

$$\mathfrak{L}(s) = \frac{2^{s-1}a^{-s}\Gamma(1-s)\,\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}\mu+1-\frac{1}{2}s)\,\Gamma(\frac{1}{2}\mu-\frac{1}{2}\nu+1-\frac{1}{2}s)\,\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+1-\frac{1}{2}s)} \quad (\mu+\nu > -\tfrac{1}{2}),$$

and $|\mathfrak{L}(\frac{1}{2}+it)| \sim (a\pi)^{-\frac{1}{2}}\,|t|^{-1}$ as $t \to \pm\infty$. Hence given $f \in L^2$, the equation

$$f(x) = \int_0^\infty J_\mu(axy)\,J_\nu(axy)\,g(y)\,dy$$

has a unique solution g in $L^2_{\mathscr{P}_2}$; or in $L^2_{\mathscr{R}}$.

(ii) Write $f \in L^2(\mu)$ if $x^\mu f(x) \in L^2(0,\infty)$. For $-\frac{1}{2} < \mu < \frac{1}{2}$, the Fourier cosine transformation

$$g(x) = \mathscr{C}\{f,x\} = 2\int_0^\infty \cos 2\pi xy f(y)\,dy$$

is easily seen to be equivalent formally to a transformation on $L^2(\mu)$ by the kernel $c_\mu(x) = 2x^{-\mu}\cos 2\pi x$, for which

$$\mathfrak{C}_\mu(s) = 2(2\pi)^{\mu-s}\,\Gamma(s-\mu)\cos\tfrac{1}{2}(s-\mu)\pi, \quad |\mathfrak{C}_\mu(\tfrac{1}{2}+it)| \sim (2\pi)^\mu\,|t|^{-\mu}.$$

Thus $\mathscr{C}_\mu$ maps $L^2(\mu)$ into $L^2(-\mu)$ if $0 \leqslant \mu < \frac{1}{2}$, and into (say) the space of generalized functions g, where $x^{-\mu}g(x) \in L^2_{\mathscr{P}_\lambda}$ $(\lambda > \frac{1}{2}-\mu)$ if $-\frac{1}{2} < \mu < 0$.

(2″) $|\mathfrak{R}(\frac{1}{2}+it)| \leqslant \omega_0\,e^{\rho|t|}$ $(\rho > 0)$.

Let $\mathfrak{U}_\alpha(s) = \alpha\,\mathrm{cosec}\,\alpha s$, where $2\pi n < \alpha < 2\pi(n+1)$ for some integer n. Then $\mathfrak{U}_\alpha$ satisfies II (i)–(iv); $|\mathfrak{U}_\alpha(\frac{1}{2}+it)| \sim 2\alpha\,e^{-\alpha|t|}$ as $t \to \pm\infty$ and $\mathfrak{U}_\alpha(s)$ is the Mellin transform of the homogeneous kernel

$$u_\alpha(x,y) = (-1)^n\frac{x^{\pi(n+1)/\alpha-1}y^{-\pi n/\alpha}}{x^{\pi/\alpha}+y^{\pi/\alpha}}.$$

Given ρ, it is sufficient to use $L^2_{\mathscr{U}_\alpha}$ with $\alpha > \rho$. If $\alpha = \pi$, $\mathscr{U}_\alpha$ is the Stieltjes transformation.

Examples. (i) The kernel e^{-x} has Mellin transform $\Gamma(s)$ for which

$$|\Gamma(\tfrac{1}{2}+it)| \sim \sqrt{(2\pi)}\,e^{-\frac{1}{2}\pi|t|}.$$

Thus if $f \in L^2$, the equation

$$f(x) = \int_0^\infty e^{-xy}\,g(y)\,dy$$

has a unique solution in (for example) $L^2_{\mathscr{U}_\pi}$, generated by the Stieltjes transform. Since, formally,

$$\int_0^\infty e^{-zx}f(x)\,dx = \int_0^\infty \frac{g(y)}{y+z}\,dy,$$

$L^2_{\mathscr{U}_\pi}$ in fact contains a solution g for every f whose Laplace transform belongs to L^2.

(ii) The kernel $l(x) = {}_2F_1(\alpha,\beta;\gamma;-x)$ has Mellin transform

$$\mathfrak{L}(s) = \frac{\Gamma(\gamma)\,\Gamma(s)\,\Gamma(\alpha-s)\,\Gamma(\beta-s)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma-s)} \quad (\alpha,\beta > \tfrac{1}{2}, \gamma \neq \tfrac{1}{2})$$

and $|\mathfrak{L}(\frac{1}{2}+it)| \sim$ constant $|t|^{\alpha+\beta-\gamma-1}\,e^{-\pi|t|}$. Thus if $\alpha+\beta > \gamma+\frac{3}{2}$, a solving extension space for $f = \mathscr{L}g$ is generated by the Stieltjes transform.

(iii) The kernel $\kappa(x) = \frac{1}{2}(\pi c)^{-\frac{1}{2}}\cos\frac{1}{4}(c^{-1}(\log x)^2-\pi)$ $(c > 0)$ has Mellin transform $\mathfrak{K}(s) = \cos(cs^2)$ (the corresponding integral (3) converging conditionally), for which $|\mathfrak{K}(\frac{1}{2}+it)| \sim \frac{1}{2}e^{c|t|}$. A function of L^2 has a unique $\mathscr{K}$-transform in $L^2_{\mathscr{U}_\alpha}$, for $\alpha > c$.

https://doi.org/10.1017/S030500410003591X Published online by Cambridge University Press

8. *Extending kernels of the form $a(xy)$.* A very slight modification of the work of §§ 5, 6 allows us to consider extending kernels of the form $a(xy)$, rather than homogeneous kernels of degree -1. We assume that $\mathfrak{A}$ is subject to Condition II as before, and instead of (18) define $a(x)$ directly as the inverse Mellin transform of $\mathfrak{A}(s)$, as in (3). In place of (19) and (20) we have

$$g(x) = \mathscr{A}\{f, x\} = \int_0^\infty a(xy) f(y)\, dy, \quad \mathfrak{G}(s) = \mathfrak{A}(s)\,\mathfrak{F}(1-s);$$

and by much the same reasoning as before, $\mathscr{A}$ is shown to satisfy $E(a)$–(c), so that a is an extending kernel; again $L^2_{\mathscr{A}}$ contains delta functions Δ_ξ for all $\xi > 0$.

The determination of $\mathfrak{A}$ for given $\mathfrak{K}$, and the definition of k by the sequence (k_n) proceeds exactly as in § 6, and the theorem is valid without change in wording, for the present interpretation of $\mathscr{A}$.

9. *Further extending kernels.* The kernels of § 7 provide examples of the present type, by the obvious device of writing $a(xy)$ in place of $a(1, xy)$.

$(3'')$ $|\mathfrak{K}(\frac{1}{2}+it)| \leqslant \omega_0 e^{\rho|t|}$.

Let $$\mathfrak{B}_\beta(s) = \beta^{-1}\Gamma(\beta^{-1}s) \quad (\beta > 0).$$

Then $v(x) = e^{-x^\beta}$, and $|\mathfrak{B}_\beta(\frac{1}{2}+it)| \sim \text{constant}\,|t|^{(1-\beta)/2\beta} e^{-\pi|t|/2\beta}$. For given ρ it is sufficient to take $\beta < \pi/2\rho$.

$(4'')$ $|\mathfrak{K}(\frac{1}{2}+it)| \leqslant \omega_0 e^{|t|^\sigma}$, $(\sigma > 0)$.

Let $\mathfrak{W}(s) = e^{\frac{1}{2}s^2}$, so that $w(x) = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}(\log x)^2}$, and $|\mathfrak{W}(\frac{1}{2}+it)| = e^{\frac{1}{2}(\frac{1}{4}-t^2)}$. Then $L^2_{\mathscr{W}}$ contains $\mathscr{K}$-transforms for all L^2 functions, if $0 < \sigma < 2$.

10. The present method can be applied to integral equations (1) with homogeneous kernels $l(x, y)$ in place of kernels of the type $l(xy)$, by making modifications of the same kind as those made in § 9 for a.

The author is indebted to his referee, whose comments enabled him to make a number of improvements to the paper.

REFERENCES

(1) Miller, J. B. Hilbert spaces of generalized functions extending L^2, (I). *J. Australian Math. Soc.* 1 (1960), 281–98.

(2) Titchmarsh, E. C. *Introduction to the theory of Fourier integrals*, 2nd ed. (Oxford, 1948).

(3) Bateman Manuscript Project. *Tables of integral transforms* (McGraw-Hill, 1954).

Australian National University
Canberra

https://doi.org/10.1017/S030500410003591X Published online by Cambridge University Press