Embeddings in Euclidean space

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Let X be a compact differentiable manifold without boundary. A differentiable regular map of X into Euclidean N-space, $\dim X < N$, is called an *immersion* (written $X \subseteq N$). An immersion which is also one-one is called an *embedding* (written $X \subset N$). A non-embedding theorem of the form: if X is a manifold with the property P then $X \nsubseteq N-1$, is *best possible* if there exists a manifold X with property P and an embedding $X \subset N$. The purpose of this note is to show that two frequently used non-embedding theorems are best possible.

The embedding $X \subset N$ is constructed by induction, using the fact that the normal vector bundle of an embedding can be identified with a tubular neighbourhood of the embedded manifold. The same method has been used by Epstein and Schwarzenberger and by Sanderson in two recent papers to which this is a footnote (2), (4). The convenient notation for embeddings and immersions is that of Sanderson.

If $\dim X = N$ let $X \subseteq N$ mean that $X \subseteq N+1$ with a normal vector field. If $\dim X = 2n$ let $\mathrm{ch}(X)$ be the set of Chern characters of complex vector bundles on X, and let $s(z)/n!$ be the component of degree $2n$ of $z \in \mathrm{ch}(X)$. Let $\alpha(n)$ be the number of non-zero terms in the dyadic expansion of n.

The following conditions can be satisfied by a manifold X:

$a(n,r)$: $\dim X = n$ and the normal Stiefel–Whitney class $\overline{w}_{n-r}(X)$ is non-zero,

$b(n,t)$: $\dim X = 2n$, $t-\alpha(n) = r \geqslant 0$, n is divisible by 2^r, the integral Stiefel–Whitney class $W_3(X) = 0$ and there is a $z \in \mathrm{ch}(X)$ for which $s(z)/2^r$ is an odd integer.

A standard application of Stiefel–Whitney classes, together with a theorem of Massey ((3), § 2), gives:

$$a(n,r) \Rightarrow \alpha(n) \leqslant r \leqslant n \quad \text{and} \quad X \nsubseteq 2n-r, \quad X \nsubseteq 2n-r-1.$$

A theorem of Atiyah and Hirzebruch ((1), § 3) gives:

$$b(n,t) \Rightarrow \alpha(n) \leqslant t \leqslant n \quad \text{and} \quad X \nsubseteq 4n-2t.$$

To show that these two theorems are best possible it is sufficient to construct manifolds X which satisfy one of the more stringent conditions:

$$A(n,r)\text{: } X \subset 2n-r+1, \quad X \subseteq 2n-r \quad \text{and} \quad X \text{ satisfies } a(n,r),$$
$$B(n,t)\text{: } X \subset 4n-2t+1, \quad X \subseteq 4n-2t \quad \text{and} \quad X \text{ satisfies } b(n,t).$$

Lemma A. *If X_1 satisfies $A(n_1,r_1)$ and X_2 satisfies $A(n_2,r_2)$ then $X_1 \times X_2$ satisfies $A(n_1+n_2, r_1+r_2)$.*

Proof. The fact that $X_1 \subseteq 2n_1-r_1$, $X_2 \subseteq 2n_2-r_2$ will be used in three ways. First,

$$\overline{w}_{i_1}(X_1) = \overline{w}_{i_2}(X_2) = 0 \quad \text{for} \quad i_1 > n_1-r_1,\ i_2 > n_2-r_2$$

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and hence $\overline{w}_{n_1+n_2-r_1-r_2}(X_1 \times X_2) = \overline{w}_{n_1-r_1}(X_1) \times \overline{w}_{n_2-r_2}(X_2)$
is non-zero. Therefore $X_1 \times X_2$ satisfies $a(n_1+n_2, r_1+r_2)$. Secondly,

$$X_1 \times X_2 \subseteq 2n_1 + 2n_2 - r_1 - r_2.$$

Thirdly, if $s > r_2$, $X_2 \subseteq 2n_2 + s - r_2$ with s linearly independent normal vector fields, and the normal bundle is independent of the choice of immersion. Since $s > r_2$ there is an embedding $X_2 \subset 2n_2 + s - r_2$. There are s linearly independent normal vector fields and therefore, if D^s is the closed disk of dimension s, $X_2 \times D^s \subset 2n_2 + s - r_2$. We may suppose that $n_1 \geqslant n_2$ and take $s = 2n_1 - r_1 + 1$. Since $X_1 \subset s$ there is now an embedding $X_1 \times X_2 \subset 2(n_1+n_2) - (r_1+r_2) + 1$.

LEMMA B. *If X_1 satisfies $B(n_1, t_1)$, X_2 satisfies $B(n_2, t_2)$, and n_1+n_2 is divisible by 2^r where $r = t_1+t_2-\alpha(n_1+n_2)$, then $X_1 \times X_2$ satisfies $B(n_1+n_2, t_1+t_2)$.*

Proof. Since X_1, X_2 are both orientable, the conditions $W_3(X_1) = W_3(X_2) = 0$ imply that $W_3(X_1 \times X_2) = 0$. We have

$$s(z_1 \otimes z_2) = \binom{n_1+n_2}{n_1} s(z_1)\, s(z_2)$$

and so $s(z_1 \otimes z_2)$ contains r powers of two. Therefore $X_1 \times X_2$ satisfies $b(n_1+n_2, t_1+t_2)$. The rest of the proof follows as for Lemma A.

EXAMPLE A. *Let (n, r) be integers for which $\alpha(n) \leqslant r \leqslant n$. There is a manifold X which satisfies $A(n, r)$.*

Proof. Let P_q be real projective space of dimension q. Then if q is a power of two, P_q is known to satisfy $A(q, 1)$. Since $\alpha(n) \leqslant r \leqslant n$, it is possible to express n as a sum of r powers of two: $n = q_1 + \ldots + q_r$, $q_i = 2^{p_i}$ $(p_i \geqslant 0)$.

Let $X = P_{q_1} \times \ldots \times P_{q_r}$. By Lemma A, X satisfies $A(n, r)$.

Remark. The manifold X is non-orientable except in the extreme case $r = n$ when $X = (S^1)^n$ is a torus. If $A(n, r)$ is amended to include the condition $w_1(X) = 0$, then a manifold satisfying it cannot always be constructed (3).

EXAMPLE B. *Let (n, t) be integers for which $\alpha(n) \leqslant t \leqslant n$, $t = \alpha(n) + r$ and n is divisible by 2^r. There is a manifold X which satisfies $B(n, t)$.*

Proof. Let P_q be complex projective space of dimension q. Then P_q is an orientable manifold of (real) dimension $2q$, and so is $(P_q) \sharp (-P_q) = K_q$ where $\sharp$ denotes connected sum. Alternatively, K_q is obtained from P_q by blowing up a point. Now $P_q \subset 4q-1$ and hence $K_q \subset 4q-1$ and $K_q \subseteq 4q-2$ ((4), Theorem 4·6). Moreover $W_3(K_q) = 0$ and the Hopf bundle on one copy of P_q induces a complex line bundle on K_q with $s(z) = 1$. Therefore if q is a power of two, K_q satisfies $B(q, 1)$. Since $\alpha(n) \leqslant t \leqslant n$, it is possible to express n as a sum of $t = \alpha(n) + r$ powers of two:

$$n = q_1 + \ldots + q_{\alpha(n)-1} + 2^s, \quad q_i = 2^{p_i}, \quad p_i > s \geqslant r,$$
$$2^s = (u_1 + \ldots + u_r) + u_r, \quad u_j = 2^{s-j}.$$

Let $X = X_{q_1} \times \ldots \times K_{q_{\alpha(n)-1}} \times K_{u_1} \times \ldots \times K_{u_r} \times K_{u_r}$. By Lemma B, X satisfies $B(n, t)$.

Remark. The manifold X has $w_2(X) \neq 0$ except in the extreme case $n = t = 1$. If $B(n, t)$ is amended to include the condition $w_2(X) = 0$, then a manifold X satisfying it cannot always be constructed (1).

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Simple proof of a formula due to Racah

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After some manipulation from the general formula, Racah (1, 2) obtained a closed formula for his $\overline{V}$ coefficient in the special case of all three m_j components zero. He also showed that the integral of the product of three Legendre polynomials followed at once from these special $\overline{V}$ coefficients. In this case the general formula ((2), eq. 10·14) becomes

$$\overline{V}\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix} = \Delta(a,b,c) \sum_p \frac{(-1)^{c+p}\, a!\, b!\, c!}{p!\,(a+b-c-p)!\,(a-p)!\,(c-b+p)!\,(b-p)!\,(c-a+p)!}, \quad (1)$$

where

$$\Delta(a,b,c) = \left[\frac{(a+b-c)!\,(b+c-a)!\,(c+a-b)!}{(a+b+c+1)!}\right]^{\frac{1}{2}}$$

and a, b, c are all non-negative integers. By changing the parameter of summation to $(a+b-c-p)$ it follows that $\overline{V}$ is zero unless $2g = a+b+c$ is even (2).

It is the purpose of this note to remark that Racah's closed formula follows at once from a formula for summing products of three binomial coefficients due to Fjeldstad (3) and proved succinctly by Good (4). If we take $2g$ even and set $s = p+c-g$, equation (1) becomes

$$\overline{V}\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix} = \Delta(a,b,c)\,(-1)^g \sum_s (-1)^s \binom{a}{g-c+s}\binom{b}{g-a+s}\binom{c}{g-b+s}$$

$$= \Delta(a,b,c)\,\frac{(-1)^g\, g!}{(g-a)!\,(g-b)!\,(g-c)!}$$

which is Racah's result.

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