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Graph substitutions

JOSEPH P. PREVITE

School of Science, Penn State Erie, The Behrend College, Erie, PA 16563, USA
(e-mail: jpp@vortex.bd.psu.edu)

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Abstract. In 1984, Gromov (see [**4**] and [**6**]) introduced the idea of subdividing a 'branching' polyhedron into smaller cells and replacing these cells by more complex objects, reminiscent of the growth of multicellular organisms in biology. The simplest situation of this kind is a graph substitution which replaces certain subgraphs in a graph G by bigger finite graphs. The most basic graph substitution is a vertex replacement rule which replaces certain vertices of G with finite graphs. This paper develops a framework for studying vertex replacements and discusses the asymptotic behavior of iterated vertex replacements, the limit objects, and the induced dynamics on the space of infinite graphs from the viewpoint of geometry and dynamical systems.

1. *Introduction*

This work was motivated by studying the horospheres of the geodesic flow on a two-dimensional singular space X of non-positive curvature. Since X has non-positive curvature, any two points x and y can be connected by a unique distance minimizing unit speed geodesic. In a singular space X, geodesics branch, i.e. there may be many possible extensions of a given geodesic γ. Let x be a point in X and γ a geodesic ray containing x with $\gamma(0) = x$. The *horosphere* $S^{\infty}(\gamma, x)$ in the γ direction at x is the limit as $R \to \infty$ of spheres $S(\gamma(R), R)$ of radius R centered at $\gamma(R)$. For two dimensional singular spaces of non-positive curvature, the horospheres of X are graphs.

For $0 < t_1 < t_2$ and any two points $\gamma(t_1)$ and $\gamma(t_2)$ on the ray γ, there is a natural radial projection map from $S^{\infty}(\gamma, \gamma(t_1))$ onto $S^{\infty}(\gamma, \gamma(t_2))$. If $\gamma(t_2)$ is a singular point and γ contains no singularities between t_1 and t_2, these horospheres differ topologically by a vertex replacement rule, as determined by the link at $\gamma(t_2)$.

In 1968, Lindenmayer (see [**9**] and [**10**]) introduced a class of iterative systems which he used to model the growth of plants and simple multicellular organisms. He theorized that the development of a complex object, such as a plant, must be governed by a relatively simple set of production rules. His approach created a new branch of biomathematics. Lindenmayer systems were later used in the areas of data and image compression.

A simple Lindenmayer system (or L-system) is a substitution system with geometric procedures assigned to each symbol. Simple geometric objects can be described by

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a sequence of procedures $p_1, p_2, \ldots, p_k$, which one might think of as commands to a pencil (e.g. draw a segment of length l, turn right at a 90 degree angle, etc). An L-system replaces certain procedures p with a finite sequence of procedures $L(p) = q_1, q_2, \ldots, q_m$. Complex geometric objects are created after relatively few iterations.

The work on L-systems amounted to producing examples and displaying pictures to model complex geometric objects. The asymptotic behavior of objects resulting from iterated applications of a Lindenmayer system was never studied although it has biological implications. Since the systems introduced in this paper are more natural and geometric, they promise applications in the same fields that Lindenmayer impacted. Also, the asymptotic behavior of the objects that are produced by iterative vertex replacements is studied.

In §2, the concept of a vertex replacement rule is introduced and examples are given. An application of a vertex replacement rule $\mathcal{R}$ to a graph G produces a graph $\mathcal{R}(G)$ by replacing certain vertices v in G with a finite graph $H(v)$ taken from a finite list $\{H_1, \ldots, H_m\}$. Each graph H_i has an associated set of boundary vertices ∂H_i and the cardinalities of the ∂H_i differ. A vertex v whose degree is equal to the number of boundary vertices for some $H_i = H(v)$ is called *replaceable*. $\mathcal{R}(G)$ is obtained by replacing each replaceable vertex in G with a copy of $H(v)$ by attaching the edges that were adjacent to v to the boundary vertices of $H(v)$.

In §3, we show that for a vertex replacement given by a graph H, the sequence of graphs $(\mathcal{R}^n(G), 1)$, normalized to have diameter one, converges in the Gromov–Hausdorff metric except for special cases. The box (and in certain cases Hausdorff) dimension of the limit spaces are calculated in §4. §5 gives examples of vertex replacement rules $\mathcal{R}$ that lead to convergent and divergent sequences $(\mathcal{R}^n(G), 1)$.

In §6, we introduce how a vertex replacement rule $\mathcal{R}$ acts on marked graphs. If (G, p) is a *marked graph* (i.e. a graph with a marked vertex p) and p is replaceable, define $\mathcal{R}(G, p)$ to be the set of marked graphs $(\mathcal{R}(G), q)$, where q is a non-boundary replaceable vertex in $H(p)$. A sequence (G_i, p_i) with $(G_{i+1}, p_{i+1}) \in \mathcal{R}(G_i, p_i)$ is called an *$\mathcal{R}$-orbit* of (G_0, p_0). Every $\mathcal{R}$-orbit has limit points in the space of marked graphs. The map π which reverses replacement, $\pi(\mathcal{R}(G), q) = (G, p)$, is continuous on the space $\mathcal{R}^\infty$ of all limit points for $\mathcal{R}$. The dynamical system $(\mathcal{R}^\infty, \pi)$ is shown to be a subshift of finite type and its dynamical properties (e.g. topological mixing) are determined by the combinatorial structure of the replacements.

2. *Vertex replacement rules*

Unless otherwise stated, all graphs are connected, locally finite, unit metric graphs, i.e. each graph is a metric space and the length of every edge is one. For any (possibly infinite) graph G, let $V(G)$ and $E(G)$ be the set of vertices and edges of G, respectively.

A graph H with a selected set of vertices $\{v_1, \ldots, v_k\} \subset V(H)$ is called *symmetric* about $\{v_1, \ldots, v_k\}$ if every permutation of $\partial H := \{v_1, \ldots, v_k\}$ can be realized by an isometry of H. Such vertices are called *boundary vertices*.

A *vertex replacement rule* $\mathcal{R}$ consists of a finite list of finite graphs (called *replacement graphs*) $\{H_1, \ldots, H_m\}$, each with a set ∂H_i of boundary vertices, so that $|\partial H_i| \neq |\partial H_j|$ for $i \neq j$, where $|\cdot|$ denotes the cardinality of a set.

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Example 1. Define a vertex replacement rule $\mathcal{R}$ as shown (topologically) in Figure 1. Each graph H_i is symmetric about the set of boundary vertices ∂H_i which are indicated by open circles.

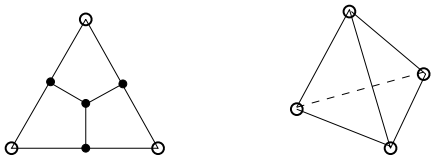


FIGURE 1.

A vertex $v \in G$ is called *replaceable* (with respect to $\mathcal{R}$) if $\deg(v) = |\partial H_i|$ for some i. The rule $\mathcal{R}$ acts on a (finite or infinite) graph G as follows: to obtain the graph $\mathcal{R}(G)$ from G, replace each replaceable vertex v in G ($|\partial H_i| = \deg(v)$) with a copy of the graph H_i so that the $\deg(v)$ edges previously adjacent to v are attached to the $|\partial H_i|$ boundary vertices of H_i. Since the graphs H_i are symmetric about ∂H_i this procedure is well defined.

Example 2. Let H define a vertex replacement rule $\mathcal{R}$ with boundary vertices indicated by open circles in Figure 2.

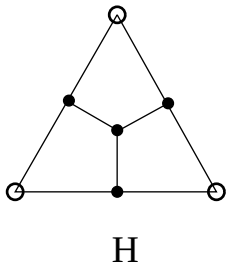


FIGURE 2.

Let G be as in Figure 3. Only the vertex v in G is replaceable (only v has degree 3). The construction of $\mathcal{R}(G)$ is shown in Figure 3. Note that the boundary vertices in this example are themselves replaceable, i.e. after inserting the graph H into G, the boundary vertices become replaceable in $\mathcal{R}(G)$.

In general, a boundary vertex $v \in \partial H_i$ is called *replaceable* if $\deg(H_i) + 1 = |\partial H_j|$ for some j.

Example 3. Let H define a vertex replacement rule $\mathcal{R}$ with boundary vertices as shown in Figure 4. The graph $\mathcal{R}(G)$ is shown for the given graph G in Figure 5. Note that the boundary vertices are non-replaceable after the graph is inserted.

Example 4. Let H define a vertex replacement rule $\mathcal{R}$ as shown in Figure 6. Let G be as in Figure 7. The graphs $\mathcal{R}(G)$ and $\mathcal{R}^2(G)$ are also shown in Figure 7.

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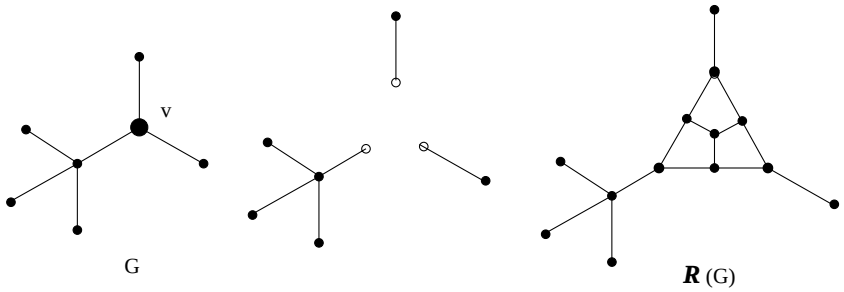


FIGURE 3.

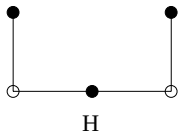


FIGURE 4.

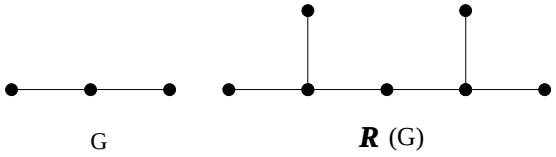


FIGURE 5.

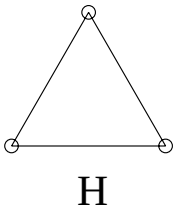


FIGURE 6.

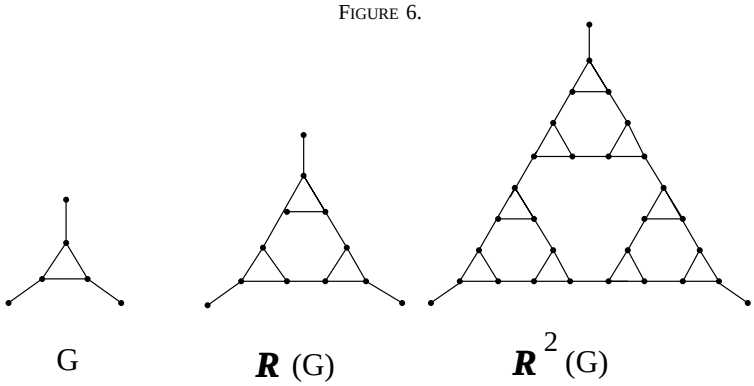


FIGURE 7.

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Example 5. Define a vertex replacement rule $\mathcal{R}$ as shown in Figure 8. The graph $\mathcal{R}(G)$ for a graph G is shown in Figure 9.

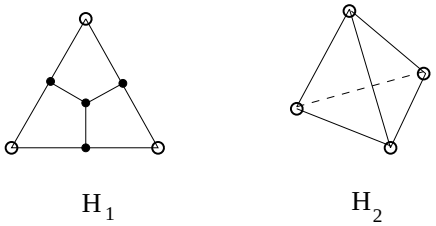


FIGURE 8.

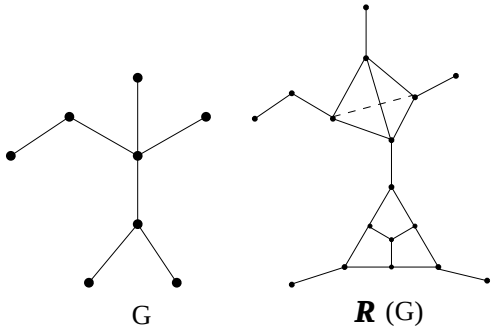


FIGURE 9.

If v is a replaceable vertex in G, let $\mathcal{R}(v)$ be the copy of the graph H_i in $\mathcal{R}(G)$ that replaced the vertex v. There is a pointwise map π from $\mathcal{R}(G)$ to G which undoes replacement by crushing the inserted copies of the H_i's to the vertices that they replaced. In general, for any set F in G, let $\mathcal{R}(F)$ be the points in $\mathcal{R}(G)$ whose image under π is in F. If $F \subset G$ contains no replaceable vertices, then $\mathcal{R}^n(F)$ can be identified with F. In this case, label $\mathcal{R}^n(F)$ as $F \subset \mathcal{R}^n(G)$. If $z \in G$ is non-replaceable, label $\mathcal{R}^n(z)$ as $z \in \mathcal{R}^n(G)$.

3. *Convergence and divergence results*

For any metric space X, denote the metric by dist_X. We recall some facts about the Gromov–Hausdorff metric (see [**5**]). Let Z be a metric space with A and B closed subsets of Z. For any ϵ, let $A_\epsilon = \{z \in Z : \text{dist}_Z(z, A) \leq \epsilon\}$ and $B_\epsilon = \{z \in Z : \text{dist}_Z(z, B) \leq \epsilon\}$. The *Hausdorff distance* is defined as

$$\text{dist}_Z^{\text{Haus}}(A, B) = \inf\{\epsilon : A \subseteq B_\epsilon, B \subseteq A_\epsilon\}.$$

This defines a metric when restricted to closed subsets of Z. Denote by $\mathcal{M}$ the collection of all isometry classes of compact metric spaces. The *Gromov–Hausdorff distance* between two compact metric spaces X and Y is

$$\text{dist}_{\mathcal{M}}^{\text{GH}}(X, Y) = \inf_{Z \in \mathcal{M}} \{\epsilon : \text{dist}_Z^{\text{Haus}}(I(X), J(Y)) < \epsilon\},$$

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where I and J are isometric embeddings of X and Y into Z, respectively. In particular, we have that $\text{dist}_{\mathcal{M}}^{\text{GH}}(X, Y) = 0$ iff X is isometric to Y. The space $(\mathcal{M}, \text{dist}_{\mathcal{M}}^{\text{GH}})$ is a complete metric space.

If X and Y are two compact spaces such that $\text{dist}_{\mathcal{M}}^{GH}(X, Y) < \epsilon$, then for any ϵ-net $\{x_1, \ldots, x_n\}$ of X (i.e. for all $x \in X$, $\text{dist}_X(x, \{x_i\}) \leq \epsilon$) there is a 2ϵ-net of Y $\{y_1, \ldots, y_n\}$ with $|\text{dist}_X(x_i, x_j) - \text{dist}_Y(y_i, y_j)| \leq 2\epsilon$ for all i, j. Conversely, if there are ϵ-nets $\{x_1, \ldots, x_n\}$ of X and $\{y_1, \ldots, y_n\}$ of Y with $|\text{dist}_X(x_i, x_j) - \text{dist}_Y(y_i, y_j)| < \epsilon$ for all i, j, then $\text{dist}_{\mathcal{M}}^{\text{GH}}(X, Y) \leq 2\epsilon$.

For any compact metric space X, define $\text{Cov}(X, \epsilon)$ to be the smallest number of closed balls of radius ϵ which cover X.

THEOREM 3.1. (Gromov compactness theorem.) *Let $\mathcal{C} \subset \mathcal{M}$ be a collection of compact metric spaces. The space $\mathcal{C}$ is precompact, i.e. every sequence in $\mathcal{C}$ contains a convergent subsequence in $\mathcal{M}$ if and only if* $\sup_{X \in \mathcal{C}} \text{Cov}(X, \epsilon)$ *is finite for every $\epsilon > 0$.*

For a finite graph G, let $(\mathcal{R}^n(G), 1)$ be the metric space $\mathcal{R}^n(G)$ normalized to have diameter 1, i.e. every edge in $(\mathcal{R}^n(G), 1)$ has length $(\text{diam}(\mathcal{R}^n(G)))^{-1}$. Let H define a vertex replacement rule $\mathcal{R}$. In this section, we classify the behavior of the sequence of metric spaces $(\mathcal{R}^n(G), 1)$ in the Gromov–Hausdorff metric.

If either G or H have no replaceable vertices, $(\mathcal{R}^n(G), 1)$ is constant for $n \geq 1$. Hence, throughout the rest of this section, we assume that both G and H each have at least one replaceable vertex.

The remainder of this section consists of the proof of the following.

THEOREM 3.2. *Let H define a vertex replacement rule $\mathcal{R}$ and let G be a finite graph with at least one replaceable vertex. Then the normalized sequence $(\mathcal{R}^n(G), 1)$ converges in the Gromov–Hausdorff metric if and only if one of the following hold:*

1. *H contains exactly one replaceable vertex;*
2. *$|\partial H| > 1$ and every path in H connecting two different boundary vertices contains at least two replaceable vertices.*

For any path η let $L(\eta)$ be the length of η. A path $\eta \subset G$ is called *distance minimizing* if $L(\eta) = \text{dist}_G(x, y)$, where x and y are the endpoints of η. A path ξ in G *realizes the diameter* of a graph G if ξ is distance minimizing and $L(\xi) = \text{diam}(G)$.

LEMMA 3.1. *Suppose that $|\partial H| = 1$, then* $\text{diam}(\mathcal{R}^n(G))$ *grows at most linearly with n, i.e. there is a number A depending on H so that*

$$\text{diam}(\mathcal{R}^n(G)) \leq An$$

for large n.

Proof. Since $|\partial H| = 1$, a replaceable vertex has degree 1. Thus, any simple path η in $\mathcal{R}^{n-1}(G)$ contains at most two replaceable vertices (the endpoints of η). So,

$$\text{diam}(\mathcal{R}^n(G)) \leq \text{diam}(\mathcal{R}^{n-1}(G)) + 2\,\text{diam}(H)$$

or

$$\text{diam}(\mathcal{R}^n(G)) \leq \text{diam}(G) + 2n\,\text{diam}(H). \qquad \square$$

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Suppose that $|\partial H| > 1$. For any set F, let $N(F)$ be the number of replaceable vertices contained in F. A path η in a graph Γ is called *preminimizing* if for any path η' connecting the endpoints of η:

1. $N(\eta') \geq N(\eta)$;
2. if $N(\eta') = N(\eta)$, then $L(\eta') \geq L(\eta)$.

If η is preminimizing, then any subpath of η is preminimizing.

For two different boundary vertices $b_1, b_2 \in \partial H$, define σ to be a preminimizing path from b_1 to b_2. By symmetry, for any other pair of boundary vertices b_1' and b_2' there is a preminimizing path σ' connecting b_1' to b_2' with $L(\sigma) = L(\sigma')$ and $N(\sigma) = N(\sigma')$. Thus, the numbers $L(\sigma)$ and $N(\sigma)$ only depend on H.

LEMMA 3.2. *If $N(\sigma) = 0$, then* $\operatorname{diam}(\mathcal{R}^n(G))$ *grows at most linearly with n.*

Proof. Let η be a simple path in $\mathcal{R}^n(G)$ for $n \geq 1$. Consider the path $\pi(\eta)$ in $\mathcal{R}^{n-1}(G)$. For any interior replaceable vertex u on $\pi(\eta)$, we can create a new path η' by concatenating the paths $\eta \cap (\mathcal{R}^n(G) \setminus \mathcal{R}(u))$ with $\sigma \cap \mathcal{R}(u)$. Thus every pair of vertices in $\mathcal{R}^n(G)$ can be connected by a path which passes through at most $2N(H)$ replaceable vertices. (The path $\pi(\eta) \subset \mathcal{R}^{n-1}(G)$ may have replaceable endpoints.) Hence,

$$\operatorname{diam}(\mathcal{R}^{n+1}(G)) \leq \operatorname{diam}(\mathcal{R}^n(G)) + 2N(H)\operatorname{diam}(H)$$

for $n \geq 1$. □

PROPOSITION 3.1. *Let H define a replacement rule $\mathcal{R}$. Suppose that $N(H) > 1$ and either $|\partial H| = 1$ or $N(\sigma) = 0$. Then the normalized sequence $(\mathcal{R}^n(G), 1)$ diverges in the Gromov–Hausdorff metric for any finite G with at least one replaceable vertex.*

Proof. Since $\mathcal{R}^n(G)$ contains at least $N(H)^n$ replaceable vertices, the number of edges in $\mathcal{R}^n(G)$ is at least $N(H)^n - 1$. A ball of radius ϵ in $(\mathcal{R}^n(G), 1)$ corresponds to a ball of radius $\epsilon \operatorname{diam}(\mathcal{R}^n(G))$ in $\mathcal{R}^n(G)$. Let D be the maximum degree of any vertex in G or H. A ball of radius $\epsilon \operatorname{diam}(\mathcal{R}^n(G))$ in $\mathcal{R}^n(G)$ can contain at most $D^{\epsilon \operatorname{diam}(\mathcal{R}^n(G))+2}$ edges. By Lemma 3.1 (or 3.2), there is a number $A > 0$ (which depends only on H and G) so that $\operatorname{diam}(\mathcal{R}^n(G)) \leq An$ for large n. Thus,

$$D^{\epsilon \operatorname{diam}(\mathcal{R}^n(G))+2} \leq D^{\epsilon An+2}.$$

The number of balls needed to cover $\mathcal{R}^n(G)$ is at least

$$\frac{N(H)^n - 1}{D^{\epsilon An} D^2} = \left(\frac{N(H)}{D^{\epsilon A}}\right)^n \frac{1}{D^2} - \frac{1}{D^{\epsilon An+2}},$$

which implies that for ϵ small enough, the number of balls needed to cover $N(H)^n - 1$ edges tends to infinity as $n \to \infty$. Thus,

$$\sup_{n\to\infty} (\operatorname{Cov}((\mathcal{R}^n(G), 1), \epsilon)) = \infty.$$

Hence, by the Gromov compactness theorem, $(\mathcal{R}^n(G), 1)$ diverges. □

PROPOSITION 3.2. *Let H define a replacement rule $\mathcal{R}$. Suppose that $N(H) = 1$ and either $|\partial H| = 1$ or $N(\sigma) = 0$. Then the normalized sequence $(\mathcal{R}^n(G), 1)$ converges in the Gromov–Hausdorff metric.*

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Proof. Let w be the replaceable vertex in H. Suppose that $|\partial H| = 1$, then the vertex $b \in \partial H$ is not replaceable (to be replaceable $\deg(b) + 1 = 1$, i.e. $\deg(b) = 0$ which makes the replacement trivial). Thus, $\text{dist}_H(w, \partial H) \geq 1$. Now suppose that $N(\sigma) = 0$, then $\text{dist}_H(w, \partial H) \geq 1$.

In either case, we see that $\text{diam}(\mathcal{R}^n(G)) \geq n$. Since $N(H) = 1$, the number of replaceable vertices in $\mathcal{R}^n(G)$ is $N(G)$ for all G. Moreover, if x and y are non-replaceable in $\mathcal{R}^{m-1}(G)$, then there is a path connecting $x = \mathcal{R}(x)$ and $y = \mathcal{R}(y)$ in $\mathcal{R}^m(G)$ which passes through no replaceable vertices. Thus, $\text{dist}_{\mathcal{R}^{n+m}(G)}(x, y)$ is bounded, so

$$\lim_{n\to\infty} \frac{\text{dist}_{\mathcal{R}^{n+m}(G)}(x, y)}{\text{diam}(\mathcal{R}^{n+m}(G))} = 0.$$

Let v be a replaceable vertex in G. For all n there is only one replaceable vertex in $\mathcal{R}^n(v)$, call this vertex v. Let x be non-replaceable in $\mathcal{R}^m(G)$. Then

$$\text{dist}_{\mathcal{R}^{n+m}(G)}(v, x) = n\,\text{dist}_H(v, \partial H) + \text{dist}_{\mathcal{R}^m(G)}(v, x). \tag{1}$$

If $w \neq v$ are two replaceable vertices in $\mathcal{R}^m(G)$, then

$$\text{dist}_{\mathcal{R}^{n+m}(G)}(v, w) = 2n\,\text{dist}_H(v, \partial H) + \text{dist}_{\mathcal{R}^m(G)}(v, w). \tag{2}$$

It is clear that if ξ realizes the diameter in $\mathcal{R}^n(G)$, then $\pi^n(\xi)$ is not a single point in G. Thus if $N(G) = 1$, the diameter grows like (1), otherwise the diameter grows like (2).

Using these facts, one can show that $(\mathcal{R}^n(G), 1)$ converges in the Gromov–Hausdorff metric. For $N(G) = 1$, $(\mathcal{R}^n(G), 1)$ either converges to the unit interval $[0, 1]$, or S^1 depending on the combinatorics of G. For $N(G) > 1$, since

$$\lim_{n\to\infty} \text{dist}_{(\mathcal{R}^n(G),1)}(\mathcal{R}^n(v), \mathcal{R}^n(w)) = 0$$

for all replaceable v, w, the sequence $(\mathcal{R}^n(G), 1)$ converges to $N(G)$ copies of an interval attached according to the combinatorics of G. □

Henceforth, assume that $N(\sigma) \geq 1$ and $|\partial H| > 1$. Let v be a replaceable vertex in G. Consider the graph $\mathcal{R}^n(v) \subset \mathcal{R}^n(G)$. Define the set $\partial\mathcal{R}^n(v)$ to be the vertices $w \in \mathcal{R}^n(v)$ that are adjacent to one of the $\deg(v)$ edges outside of $\mathcal{R}^n(v)$ that were adjacent to v. Note that $|\partial\mathcal{R}^n(v)| = |\partial H|$. Define the two functions

$$a(n) = \text{dist}_{\mathcal{R}^n(v)}(u, u'); \quad \text{where } u, u' \in \partial\mathcal{R}^n(v) \text{ for } u \neq u'$$

and

$$b(n) = \sup_{z\in\mathcal{R}^n(v)} \{\text{dist}_{\mathcal{R}^n(v)}(u, z) \mid u \in \partial\mathcal{R}^n(v)\}.$$

By the symmetry of H about ∂H, the above definitions are independent of the choices of $u, u' \in \partial\mathcal{R}^n(v)$.

Since all graphs are unit, a point z which realizes $b(n)$ must either be a vertex or the midpoint of an edge. Clearly, $a(n) \leq b(n)$. Moreover, by the triangle inequality, $\text{diam}(\mathcal{R}^n(v)) \leq 2b(n)$. If G is a graph with at least two replaceable vertices, then $\text{diam}(\mathcal{R}^n(G)) \geq 2b(n)$.

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Since any path which realizes $a(n-1)$ must contain at least $N(\sigma)^{n-1}$ replaceable vertices, we have that

$$a(n) \geq a(1)(1 + N(\sigma) + \cdots + N(\sigma)^{n-1}).$$

LEMMA 3.3. *Let Γ be a finite graph. There is a number $\kappa(\Gamma)$ so that if ξ is a distance minimizing path in $\mathcal{R}^n(\Gamma)$ and $n > \kappa(\Gamma)$, then $\pi^n(\xi)$ is preminimizing in Γ.*

Proof. Suppose ξ is a minimizing path in $\mathcal{R}^n(\Gamma)$ such that the endpoints w_1 and w_2 of $\pi^n(\xi) \subset \Gamma$ are non-replaceable (or not vertices). Assume $\pi^n(\xi)$ is not preminimizing. Let $\eta \subset \Gamma$ be a preminimizing path with endpoints w_1 and w_2. Then $N(\eta) < N(\pi^n(\xi))$. Note that $L(\eta) + N(\eta)a(n)$ and $L(\pi^n(\xi)) + N(\pi^n(\xi))a(n)$ are the lengths of shortest paths in $\mathcal{R}^n(\eta) \subset \mathcal{R}^n(\Gamma)$ and $\mathcal{R}^n(\pi^n(\xi)) \subset \mathcal{R}^n(\Gamma)$ from w_1 to w_2, respectively. There exists κ so that for all $n > \kappa$,

$$a(n) > \frac{L(\eta) - L(\pi^n(\xi))}{N(\pi^n(\xi)) - N(\eta)}.$$

In particular, κ can be chosen depending only on $L(\eta)$. Since Γ is a finite graph, $\kappa(\Gamma)$ can be chosen as $\max_{\eta \subset \Gamma} L(\eta)$, where η is simple. But this implies that for $n > \kappa(\Gamma)$

$$L(\eta) + N(\eta)a(n) < L(\pi^n(\xi)) + N(\pi^n(\xi))a(n) = L(\xi),$$

which contradicts the fact that ξ is minimizing.

If the endpoints of $\pi^n(\xi)$ are replaceable, apply the above argument to the path $\pi^n(\xi)$ with the endpoints removed. The resulting path is preminimizing. Hence the entire path $\pi^n(\xi)$ is preminimizing. □

PROPOSITION 3.3. *There is a number K (depending on H) so that for $n > K$,*

$$a(n) = L(\sigma) + N(\sigma)a(n-1).$$

In particular:

1. *if $N(\sigma) > 1$, then*

$$a(n) = N(\sigma)^{n-K}\left(\frac{L(\sigma)}{N(\sigma)-1} + a(K)\right) - \frac{L(\sigma)}{N(\sigma)-1},$$

and

$$\lim_{n\to\infty} \frac{a(n)}{a(n+m)} = \frac{1}{N(\sigma)^m};$$

2. *if $N(\sigma) = 1$,*

$$a(n) = L(\sigma)(n-K) + a(K).$$

Proof. Let G be a graph with a replaceable vertex v. Let $\Gamma \subset \mathcal{R}(G)$ be the graph consisting of $\mathcal{R}(v)$ together with the $\deg(v)$ edges that were adjacent to v in G, i.e. $\Gamma = \mathcal{R}(\mathrm{Star}(v))$. By Lemma 3.3, there is a fixed number $\kappa(\Gamma)$ (which really depends on H) so that for all $n - 1 > \kappa(\Gamma)$, $\pi^{n-1}(\xi)$ is preminimizing whenever ξ is a minimizing path in $\mathcal{R}^{n-1}(\Gamma) \subset \mathcal{R}^n(G)$. In particular, take ξ to realize $a(n)$. Recall that the path $\sigma \subset H$ is preminimizing. Hence, for all $n > K$, where $K = \kappa(\Gamma) + 1$, we have that

$$a(n) = L(\sigma) + N(\sigma)a(n-1).$$

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This implies that

$$a(n) = L(\sigma)(1 + N(\sigma) + \cdots + N(\sigma)^{n-K-1}) + N(\sigma)^{n-K}a(K).$$

For $N(\sigma) > 1$,

$$a(n) = N(\sigma)^{n-K}\left(\frac{L(\sigma)}{N(\sigma)-1} + a(K)\right) - \frac{L(\sigma)}{N(\sigma)-1}.$$

For $N(\sigma) = 1$,

$$a(n) = L(\sigma)(n-K) + a(K). \qquad \square$$

For any boundary vertex $b \in \partial H$, let M be the maximum of $N(\eta)$, where $\eta \subset H$ is a preminimizing path with b as one endpoint. Again, by symmetry M is independent of b. Note that a path ν which realizes $b(n)$ in $\mathcal{R}^n(\upsilon)$ must have one of its endpoints $w \in \partial\mathcal{R}^n(\upsilon)$, with $\pi^{n-1}(w) \in \partial\mathcal{R}(\upsilon)$.

LEMMA 3.4. *Suppose that $M > N(\sigma)$. Then the following hold:*

1. *there is $K' > 0$ (depending on H) so that if ν realizes $b(n)$ in $\mathcal{R}^n(G)$, with $n > K'$, then one endpoint of $\pi^{n-1}(\nu)$ is not in $\partial\mathcal{R}(\upsilon)$ and is replaceable;*
2. *if G is a finite graph with at least two replaceable vertices, then there is $K'(G) > 0$ so that for $n > K'(G)$, any path which realizes the diameter in $\mathcal{R}^n(G)$ must project to a path in G with replaceable endpoints.*

Proof. Let $n - 1 > K$ as in Lemma 3.3. Hence, $\pi^{n-1}(\nu)$ is preminimizing in $\mathcal{R}(\upsilon)$. Since ν realizes $b(n)$, $N(\nu) = M$. Thus one endpoint of $\pi^{n-1}(\nu)$ is not in $\partial\mathcal{R}(\upsilon)$. Call this endpoint z. Suppose z is non-replaceable, then

$$L(\nu) = L(\pi^{n-1}(\nu)) + N(\pi^{n-1}(\nu))a(n-1).$$

This implies that $b(n-1) \le a(n-1) + \operatorname{diam}(H)$, or $b(n-1) - a(n-1) \le \operatorname{diam}(H)$.

Now $b(n-1) \ge Ma(n-2)$ and $a(n-1) = L(\sigma) + N(\sigma)a(n-2)$, so

$$b(n-1) - a(n-1) \ge [M - N(\sigma)]a(n-2) - L(\sigma).$$

Since $M > N(\sigma)$, there is a constant K' (which depends only on H) so that $[M - N(\sigma)]a(n-2) - L(\sigma) > \operatorname{diam}(H)$ for $n > K'$. Thus, for $n > K'$, $b(n-1) - a(n-1) > \operatorname{diam}(H)$. So the point z is replaceable. The proof of 2 is identical, except that K' depends on G. $\square$

LEMMA 3.5. *Suppose that $N(\sigma) = 1$ and $M = 1$. Then for any $m > 0$,*

$$\liminf_{n\to\infty} \frac{a(n)}{b(n+m)} \ge \frac{1}{\operatorname{diam}(H)}.$$

Proof. Let $\Gamma \subset \mathcal{R}(G)$ and K be as in Proposition 3.3. Then for $n > K$, any path ν realizing $b(n)$ projects to a preminimizing path in Γ. Since $M = 1$,

$$b(n) \le \operatorname{diam}(H) + b(n-1).$$

Hence

$$b(n) \le (n-K)\operatorname{diam}(H) + b(K).$$

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Since $a(n) \geq n$, for any $m > 0$,

$$\liminf_{n\to\infty} \frac{a(n)}{b(n+m)} \geq \frac{1}{\operatorname{diam}(H)}. \qquad \square$$

LEMMA 3.6. *Suppose that $N(\sigma) = 1$ and $M > N(\sigma)$. Then for any $m > 0$,*

$$\lim_{n\to\infty} \frac{b(n)}{b(n+m)} = 1.$$

Proof. Let $\bar{K}$ be greater than K and K', the numbers given in Proposition 3.3 and Lemma 3.4.

Let $\gamma \subset H$ be a longest preminimizing path connecting $b \in \partial H$ to a replaceable vertex with $N(\gamma) = M$. Let ν realize $b(n)$ for $n > \bar{K}$. Consider $\pi^{n-1}(\nu) \subset \mathcal{R}(G)$. By Lemma 3.3, $\pi^{n-1}(\nu)$ is preminimizing, so $N(\pi^{n-1}(\nu)) = M$. By Lemma 3.4, $\pi^{n-1}(\nu)$ has an endpoint z which is not in $\partial\mathcal{R}(\upsilon)$ and is replaceable. Thus, by the construction of γ, $L(\gamma) = L(\pi^{n-1}(\nu))$. Hence,

$$b(n) = L(\gamma) + (M-1)a(n-1) + b(n-1).$$

This implies that

$$b(n) = (n - \bar{K})L(\gamma) + (M-1)(a(n-1) + \cdots + a(\bar{K})) + b(\bar{K}).$$

By Proposition 3.3, we have that

$$\begin{aligned}\sum_{j=\bar{K}}^{j=n-1} a(j) &= L(\sigma)\left(\sum_{j=\bar{K}}^{j=n-1} j\right) + (n-\bar{K})(a(\bar{K}) - L(\sigma)(\bar{K}+1)) \\ &= L(\sigma)\left(\frac{n(n-1) - \bar{K}(\bar{K}-1)}{2}\right) + (n-\bar{K})(a(\bar{K}) - L(\sigma)(\bar{K}+1)).\end{aligned}$$

This implies that for any $m \geq 0$,

$$\lim_{n\to\infty} \frac{b(n)}{b(n+m)} = 1. \qquad \square$$

PROPOSITION 3.4. *Let H define a replacement rule $\mathcal{R}$. Suppose that $N(\sigma) = 1$ and $N(H) > 1$. Then the normalized sequence $(\mathcal{R}^n(G), 1)$ diverges in the Gromov–Hausdorff metric for any finite G with at least one replaceable vertex.*

Proof. Without loss of generality, let G be finite with at least two replaceable vertices. The graph $\mathcal{R}^m(G)$ contains at least 2^m replaceable vertices. Let V_m be the set of replaceable vertices in $\mathcal{R}^m(G)$.

First suppose that $M = N(\sigma)$. For each $w \in V_m$ choose w_n to be the midpoint of a path which realizes $a(n)$ in $\mathcal{R}^n(w) \subset \mathcal{R}^{n+m}(G)$, and let $\beta(w_n, a(n)/2)$ be the ball of radius $a(n)/2$ centered at w_n. Let S_n be the set of all w_n. Note that $|S_n| = |V_m|$.

By the construction of $a(n)$, the balls $\beta(w_n, a(n)/2)$ are contained in $\mathcal{R}^n(w)$, thus for $w_n \neq w_n' \in S_m$, the balls $\beta(w_n, a(n)/2)$ and $\beta(w_n', a(n)/2)$ are disjoint. By Lemma 3.5,

$$\liminf_{n\to\infty} \frac{a(n)}{b(n+m)} \geq \frac{1}{\operatorname{diam}(H)}.$$

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Since $\operatorname{diam}(\mathcal{R}^{n+m}(G)) \leq N(G)b(n+m) + \operatorname{diam}(G)$,

$$\liminf_{n\to\infty} \frac{a(n)/2}{\operatorname{diam}(\mathcal{R}^{n+m}(G))} \geq \frac{1}{2N(G)\operatorname{diam}(H)}.$$

Thus, for large enough n and $\epsilon < 1/2N(G)\operatorname{diam}(H)$, the scaled balls

$$\beta(w_n, a(n)/2) \subset (\mathcal{R}^{n+m}(G), 1)$$

have radius at least ϵ.

Therefore any element of a cover of $(\mathcal{R}^{n+m}(G), 1)$ by $\epsilon/2$ balls can contain at most one element of S_n. So the number of balls in such a cover is at least $|S_n| = |V_m| \geq 2^m$.

Since m is independent of ϵ, this implies that

$$\sup_m\{\operatorname{Cov}((\mathcal{R}^{n+m}(G), 1), \epsilon/2)\} = \infty.$$

Hence the sequence $(\mathcal{R}^n(G), 1)$ diverges by the Gromov compactness theorem.

Now suppose that $M > N(\sigma)$. By Lemma 3.6,

$$\lim_{n\to\infty} \frac{b(n)}{b(n+m)} = 1$$

$b(n)$ grows at least quadratically, and $a(n)$ grows linearly. Hence,

$$\lim_{n\to\infty} \frac{a(n)}{b(n+m)} = 0$$

for all $m \geq 0$. In particular,

$$\lim_{n\to\infty} \frac{b(n) - a(n)}{b(n+m)} = 1$$

for all $m \geq 0$.

For each $w \in V_m$, choose a point w_n in $\mathcal{R}^n(w) \subset \mathcal{R}^{n+m}(G)$ which is the endpoint of a path that realizes $b(n)$ and is not the endpoint that is in $\partial\mathcal{R}^n(w)$. Let S_n be the set of all such points. Note that $|S_n| = |V_m|$. Let $\beta(w_n, b(n) - a(n)) \subset \mathcal{R}^{n+m}(G)$ be the ball of radius $b(n) - a(n)$ centered at w_n.

By the triangle inequality and the construction of $b(n)$ and $a(n)$, we have that the ball $\beta(w_n, b(n) - a(n))$ is contained in $\mathcal{R}^n(w)$. Thus, for $w \neq w' \in V_m$, the balls $\beta(w_n, b(n) - a(n))$ and $\beta(w'_n, b(n) - a(n))$ are disjoint. Recall that $\operatorname{diam}(\mathcal{R}^{n+m}(G)) \leq N(G)b(n+m) + \operatorname{diam}(G)$ and

$$\liminf_{n\to\infty} \frac{b(n) - a(n)}{\operatorname{diam}(\mathcal{R}^{n+m}(G))} \geq \frac{1}{N(G)}.$$

Thus, for $\epsilon < 1/N(G)$ and large n, the scaled balls $\beta(w_n, b(n) - a(n))$ in $(\mathcal{R}^{n+m}(G), 1)$ have radius at least ϵ.

Again, this implies that the number of $\epsilon/2$ balls needed to cover $(\mathcal{R}^{n+m}(G), 1)$ is at least 2^m, where m is independent of ϵ. So we have that

$$\sup_m\{\operatorname{Cov}((\mathcal{R}^{n+m}(G), 1), \epsilon/2)\} = \infty.$$

Hence, the sequence $(\mathcal{R}^n(G), 1)$ diverges by the Gromov compactness theorem. □

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Now assume that $N(\sigma) > 1$. By Proposition 3.3, the sequence $a(n)$ (and $b(n)$) grows faster than 2^n for large n, and

$$\lim_{n\to\infty} \frac{a(n)}{a(n+m)} = \frac{1}{N(\sigma)^m}.$$

PROPOSITION 3.5. *If* $N(\sigma) > 1$, *then*

$$\lim_{n\to\infty} \frac{a(n)}{b(n)} = \frac{N(\sigma)-1}{M-1}.$$

Proof. Let $\bar{K}$ be bigger than K and K' as in Proposition 3.3 and Lemma 3.4. Let ν be a path in $\mathcal{R}^n(G)$ which realizes $b(n)$ and $n > \bar{K}$. Consider $\pi^{n-1}(\nu) \subset \mathcal{R}(G)$. By Lemma 3.3, $\pi^{n-1}(\nu)$ is preminimizing and $N(\pi^{n-1}(\nu)) = M$. Moreover,

$$b(n) \le \operatorname{diam}(H) + (M-1)a(n-1) + b(n-1).$$

This implies that

$$b(n) \le (n-\bar{K})\operatorname{diam}(H) + (M-1)(a(n-1) + \cdots + a(\bar{K})) + b(\bar{K}).$$

By Proposition 3.3,

$$\begin{aligned}
\sum_{j=\bar{K}}^{j=n-1} a(j) &= \sum_{j=\bar{K}}^{j=n-1} N(\sigma)^{j-\bar{K}} \left(\frac{L(\sigma)}{N(\sigma)-1} + a(\bar{K}) \right) - \frac{L(\sigma)}{N(\sigma)-1} \\
&= \left[\frac{L(\sigma)}{N(\sigma)-1} + a(\bar{K}) \right] \left[\frac{N(\sigma)^{n-\bar{K}} - 1}{N(\sigma)-1} \right] - (n-\bar{K}) \frac{L(\sigma)}{N(\sigma)-1} \\
&= \frac{a(n) - a(\bar{K}) - (n-\bar{K})L(\sigma)}{N(\sigma)-1}.
\end{aligned}$$

If $M = N(\sigma)$. Then

$$b(n) \le (n-\bar{K})(\operatorname{diam}(H) - L(\sigma)) + a(n) + b(\bar{K}) - a(\bar{K})$$

and hence,

$$\lim_{n\to\infty} \frac{a(n)}{b(n)} = 1.$$

If $M > N(\sigma)$. Then the endpoint of $\pi^{n-1}(\nu)$ that is not in $\mathcal{R}(v)$ is a replaceable vertex.

Let γ be a longest preminimizing path from $b \in \partial H$ to a replaceable vertex with $N(\gamma) = M$. By the construction of γ, we have that $L(\pi^{n-1}(\nu)) = L(\gamma)$ and $N(\pi^{n-1}(\nu)) = N(\gamma) = M$. Thus,

$$\begin{aligned}
b(n) &= L(\gamma) + [N(\gamma)-1]a(n-1) + b(n-1) \\
&= (n-\bar{K})L(\gamma) + (M-1)\left(\sum_{j=\bar{K}}^{j=n-1} a(j) \right) + b(\bar{K}) \\
&= (n-\bar{K})L(\gamma) + (M-1)\left[\frac{a(n) - a(\bar{K}) - (n-\bar{K})L(\sigma)}{N(\sigma)-1} \right] + b(\bar{K}).
\end{aligned}$$

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Since $a(n)$ and $b(n)$ grow faster than 2^n,

$$\lim_{n\to\infty}\frac{a(n)}{b(n)}=\frac{N(\sigma)-1}{M-1}. \qquad \square$$

Since

$$\lim_{n\to\infty}\frac{a(n)}{b(n)}=\frac{N(\sigma)-1}{M-1}$$

and

$$\lim_{n\to\infty}\frac{a(n)}{a(n+m)}=\frac{1}{N(\sigma)^m},$$

we have that

$$\lim_{n\to\infty}\frac{b(n)}{b(n+m)}=\frac{1}{N(\sigma)^m},$$

and

$$\lim_{n\to\infty}\frac{a(n)}{b(n+m)}=\frac{1}{N(\sigma)^m}\frac{N(\sigma)-1}{M-1}.$$

For any G, let

$$\partial_n(G)=\bigcup_w \partial\mathcal{R}(w),$$

where w is replaceable in $\mathcal{R}^{n-1}(G)$. Let

$$D(n,G)=\max_{x\in\mathcal{R}^n(G)}\{\mathrm{dist}_{\mathcal{R}^n(G)}(x,\partial_n(G))\}.$$

PROPOSITION 3.6. $D(n,G)\le \mathrm{diam}(H)n+\mathrm{diam}(G)$.

Proof. Clearly, if w is replaceable, $\mathrm{dist}_{\mathcal{R}^n(G)}(w,\partial(n))\le \mathrm{diam}(H)$. Let x be any non-replaceable point in $\mathcal{R}^{n-1}(G)$. Then, $\mathrm{dist}_{\mathcal{R}^n(G)}(x,\partial_n(G))\le \mathrm{diam}(H)+\mathrm{dist}_{\mathcal{R}^{n-1}(G)}(x,\partial_{n-1}(G))$. $\square$

PROPOSITION 3.7. *Let H define a vertex replacement rule $\mathcal{R}$. Suppose that $N(\sigma)\ge 2$. Then the sequence $(\mathcal{R}^n(G),1)$ converges in the Gromov–Hausdorff metric for any G.*

Proof. Note that if either G or H has no replaceable vertices, then the result holds trivially. Let $\epsilon>0$ be given. There is m so that $1/N(\sigma)^{m-1}\le\epsilon$ and (by Proposition 3.6) so that $\partial_m(G)$ is an ϵ-net in $(\mathcal{R}^m(G),1)$. Choose $\bar{K}$ larger than $\kappa(\mathcal{R}^m(G))$, as given by Lemma 3.3 for the finite graph $\mathcal{R}^m(G)$, and larger than $K'(\mathcal{R}^m(G))$, as given in the second part of Lemma 3.4.

Consider $\partial_m(G)\subset\mathcal{R}^m(G)$. For each $x\in\partial_m(G)$, and all $n\ge 0$, let x_n be the unique vertex in $\partial\mathcal{R}^{n+m-1}(x)$ that is adjacent to the edge that was adjacent to x $(=x_1)$.

Let $n>\bar{K}$ and $x,y\in\partial_m(G)$. Any distance realizing path connecting x_n to y_n must project to a preminimizing path in $\mathcal{R}^m(G)$ from x to y. Let η be a preminimizing path in $\mathcal{R}^m(G)$ from x to y. Then

$$\mathrm{dist}_{\mathcal{R}^{n+m-1}(G)}(x_n,y_n)=L(\eta)+N(\eta)a(n-1)$$

for $n>\bar{K}$.

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Also, any path realizing the diameter in $\mathcal{R}^{m+n-1}(G)$ must project to a preminimizing path in $\mathcal{R}^m(G)$ which contains a fixed number W of replaceable vertices, determined by $\mathcal{R}^m(G)$. As before, if $M > N(\sigma)$, then the endpoints of the projection of a diameter are replaceable. Thus

$$(W-2)a(n-1) + 2b(n-1) \leq \operatorname{diam}(R^{m+n-1}(G))$$

and

$$\operatorname{diam}(R^{m+n-1}(G)) \leq \operatorname{diam}(\mathcal{R}^m(G)) + (W-2)a(n-1) + 2b(n-1).$$

By Proposition 3.5,

$$\lim_{n\to\infty} \frac{a(n-1)}{b(n-1)} = \frac{N(\sigma)-1}{M-1}.$$

Thus

$$\lim_{n\to\infty} \frac{\operatorname{dist}_{\mathcal{R}^{n+m-1}(G)}(x_n, y_n)}{\operatorname{diam}(R^{m+n-1}(G))} = \frac{N(\eta)}{(W-2) + 2(M-1)/(N(\sigma)-1)}.$$

If $M = N(\sigma)$, then

$$Wa(n-1) \leq \operatorname{diam}(\mathcal{R}^{m+n-1}(G))$$

and

$$\operatorname{diam}(\mathcal{R}^{m+n-1}(G)) \leq \operatorname{diam}(\mathcal{R}^m(G)) + (W-2)a(n-1) + 2b(n-1).$$

By Proposition 3.5,

$$\lim_{n\to\infty} \frac{a(n-1)}{b(n-1)} = 1,$$

which implies that

$$\lim_{n\to\infty} \frac{\operatorname{dist}_{\mathcal{R}^{m+n-1}(G)}(x_n, y_n)}{\operatorname{diam}(\mathcal{R}^{m+n-1}(G))} = \frac{N(\eta)}{W}.$$

In either case, we have that

$$\frac{\operatorname{dist}_{\mathcal{R}^{m+n-1}(G)}(x_n, y_n)}{\operatorname{diam}(R^{m+n-1}(G))}$$

converges as $n \to \infty$. So $\operatorname{dist}_{(\mathcal{R}^{n+m-1}(G),1)}(x_n, y_n)$ converges as $n \to \infty$, for any $x, y \in \partial_m(G)$.

For each $x_n \in \mathcal{R}^{m+n-1}(G)$, consider the ball $B(x_n, b(n))$ of radius $b(n)$ centered at x_n. Since

$$\lim_{n\to\infty} \frac{b(n)}{b(n+m-1)} = \frac{1}{N(\sigma)^{m-1}},$$

for large enough n we have that

$$\left| \frac{b(n)}{b(n+m-1)} - \frac{1}{N(\sigma)^{m-1}} \right| < \epsilon. \tag{3}$$

Since $2b(j) \geq \operatorname{diam}(\mathcal{R}^j(v))$ for all j, the balls $B(x_n, b(n))$, where $x_n \in \partial\mathcal{R}(\pi(x))$ for $x \in \partial_m(G)$, cover $\mathcal{R}^n(\pi(x)) \subset \mathcal{R}^{n+m-1}(G)$ for all $n \geq 0$. Moreover, since $\partial_m(G)$ was an ϵ-net in $\mathcal{R}^m(G)$ and

$$\mathcal{R}^{n+m-1}(G) \setminus \bigcup_{x\in\partial_m(G)} \mathcal{R}^n(\pi(x))$$

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contains only non-replaceable vertices for all n, we have that the balls $B(x_n, b(n))$ cover all of $(\mathcal{R}^{n+m-1}(G), 1)$.

Since $\text{diam}(\mathcal{R}^{n+m-1}(G)) \geq 2b(n+m-1)$ and by condition (3), these balls have radius smaller than 2ϵ in $(\mathcal{R}^{n+m-1}(G), 1)$ for large n. Using this and the fact that $\text{dist}_{(\mathcal{R}^{n+m-1}(G),1)}(x_n, y_n)$ converges as $n \to \infty$, we see that there exists $L > 0$ so that

$$\text{dist}_{\mathcal{M}}^{\text{GH}}[(\mathcal{R}^l(G), 1), (\mathcal{R}^{l'}(G), 1)] \leq 3\epsilon$$

for $l, l' \geq L$. Since $(\mathcal{M}, \text{dist}_{\mathcal{M}}^{\text{GH}})$ is complete, there is a compact metric space X to which $(\mathcal{R}^n(G), 1)$ converges as $n \to \infty$. □

LEMMA 3.7. *Suppose that* $N(\sigma) = 1$ *and* $N(H) = 1$. *Then* $a(n) = b(n)$, *and* $D(n, G) \leq (L(\sigma)/2)n + \text{diam}(G)$.

Proof. Let w be the replaceable vertex in H. Let b_1, b_2 be two different vertices in ∂H. Then $\text{dist}_H(b_1, b_2) = L(\sigma)$. Since H is symmetric about ∂H, we have $\text{dist}_H(b_i, w) = L(\sigma)/2$. Also by symmetry, $H \setminus \{w\}$ consists of exactly $|\partial H|$ isometric pieces. In particular, $a(n) = b(n) = L(\sigma)n$.

If w is replaceable, $\text{dist}_{\mathcal{R}^n(G)}(w, \partial(n)) \leq L(\sigma)/2$. Let x be any non-replaceable point in $\mathcal{R}^{n-1}(G)$. Then

$$\text{dist}_{\mathcal{R}^n(G)}(x, \partial_n(G)) \leq \frac{L(\sigma)}{2} + \text{dist}_{\mathcal{R}^{n-1}(G)}(x, \partial_{n-1}(G)).$$ □

PROPOSITION 3.8. *Suppose that* $N(\sigma) = 1$ *and* $N(H) = 1$. *Then the normalized sequence* $(\mathcal{R}^n(G), 1)$ *converges in the Gromov–Hausdorff metric for any* G.

Proof. Using Lemma 3.7, we see that $a(n)$ grows faster than $D(n, G)$. Using similar arguments to those in the proof of Proposition 3.7, one can use the balls $B(x_n, b(n))$ to show that $(\mathcal{R}^l(G), 1)$ is a Cauchy sequence in l. In particular, one sees that the limit space X is $N(G)$ copies of $\text{Star}(v)$, where v is a replaceable vertex in G, attached according to the combinatorics of G and scaled to have diameter one. □

Propositions 3.1, 3.2, 3.4, 3.7 and 3.8 prove Theorem 3.2.

4. *The box and Hausdorff dimension of limit spaces*

Let H define a vertex replacement rule $\mathcal{R}$. For all convergent sequences $(\mathcal{R}^n(G), 1)$ we calculate the box dimension of the limit space X, and show that the box dimension and the Hausdorff dimension agree for a certain class of replacements rules.

The *box dimension* (or *box-counting dimension*) of a compact metric space X is defined to be

$$\dim_{\text{Box}}(X) = \limsup_{\epsilon \to 0} \frac{\ln(\text{Cov}(X, \epsilon))}{-\ln \epsilon}$$

(see [**2**]). Recall that σ is a preminimizing path between a pair of boundary vertices (of H).

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THEOREM 4.1. *Let H define a vertex replacement rule $\mathcal{R}$. Suppose that the sequence $(\mathcal{R}^n(G), 1)$ converges in the Gromov–Hausdorff metric to a metric space X. If $N(\sigma) \geq 2$, then*

$$\dim_{\text{Box}}(X) = \frac{\ln N(H)}{\ln N(\sigma)}.$$

Otherwise, the box dimension (and Hausdorff dimension) of X is 1.

Proof. In the case $N(H) = 1$, the limit objects (see Propositions 3.2 and 3.8) are completely described and are one-dimensional cellular complexes, which have box (and Hausdorff) dimension 1.

Now suppose that $N(\sigma) \geq 2$. We first show that $\dim_{\text{Box}}(X) \leq \ln N(H)/\ln N(\sigma)$. Let $\epsilon > 0$ be given. As in the proof of Proposition 3.7, choose m so that $1/N(\sigma)^{m-1} < \epsilon \leq 1/N(\sigma)^{m-2}$ and recall that each $x \in \partial_m(G) \subset \mathcal{R}^m(G)$ has a unique associated point $x_n \in \mathcal{R}^{n+m-1}(G)$. Let $S_n \subset \mathcal{R}^{n+m-1}(G)$ be the set of all such points.

One can show (as in Proposition 3.7) that for large n, the closed balls $B(x_n, b(n)) \subset \mathcal{R}^{n+m-1}(G)$ of radius $b(n)$ centered at x_n cover $\mathcal{R}^{n+m-1}(G)$ and have radius less than 2ϵ. The set S_n is a 2ϵ-net of $\mathcal{R}^{n+m-1}(G)$. The number of points in S_n is

$$|\partial_m(G)| = N(G)N(H)^{m-1}|\partial H|.$$

For large enough n, $\text{dist}^{\text{GH}}_{\mathcal{M}}(X, (\mathcal{R}^{n+m-1}(G), 1)) \leq 2\epsilon$. Thus the set of points $S_n \subset \mathcal{R}^{n+m-1}(G)$ produce a 4ϵ-net in X (see the remark before Theorem 3.2).

So $\text{Cov}(X, 4\epsilon) \leq N(G)|\partial H|N(H)^{m-1}$. Thus

$$\frac{\ln(\text{Cov}(X, 4\epsilon))}{\ln(4\epsilon)^{-1}} \leq \frac{\ln[N(G)|\partial H|N(H)^{m-1}]}{\ln[N(\sigma)^{m-2}/4]}.$$

When $\epsilon \to 0$, we have that $m \to \infty$, so

$$\limsup_{\epsilon \to 0} \frac{\ln(\text{Cov}(X, 4\epsilon))}{\ln(4\epsilon)^{-1}} \leq \frac{\ln N(H)}{\ln N(\sigma)}.$$

Hence $\dim_{\text{Box}}(X) \leq \ln N(H)/\ln N(\sigma)$.

We now prove that $\dim_{\text{Box}}(X) \geq \ln N(H)/\ln N(\sigma)$. Let ϵ be given, fix m so that

$$\frac{N(\sigma) - 1}{N(G)N(\sigma)^{m+1}(M-1)} < \epsilon \leq \frac{N(\sigma) - 1}{N(G)N(\sigma)^m(M-1)}.$$

Let V be the set of all replaceable vertices in $\mathcal{R}^m(G)$. For each vertex $w \in V$, let w_n be the midpoint of a path in $\mathcal{R}^n(w) \subset \mathcal{R}^{n+m}(G)$ that realizes $a(n)$. Let $S_n \subset \mathcal{R}^{n+m}(G)$ be the set of all such points. Let $B(w_n, a(n)/2)$ be the closed ball of radius $a(n)/2$ centered at w_n. By construction, the balls $B(w_n, a(n)/2)$ are disjoint. Recall that

$$\lim_{n \to \infty} \frac{a(n)}{b(n+m)} = \frac{1}{N(\sigma)^m} \frac{N(\sigma) - 1}{M - 1}.$$

So for large n

$$\left| \frac{a(n)/2}{N(G)b(n+m) + \text{diam}(G)} - \frac{N(\sigma) - 1}{2N(G)N(\sigma)^m(M-1)} \right| < \frac{\epsilon}{4}. \tag{4}$$

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The choice m, the fact that $\operatorname{diam}(\mathcal{R}^{n+m}(G)) \leq N(G)b(n+m) + \operatorname{diam}(G)$, and (4) imply that the scaled balls $B(w_n, a(n)/2)$ in $(\mathcal{R}^{n+m}(G), 1)$ have radius at least $\epsilon/4$.

The points in S_n are more than $(\epsilon/2)$-separated in $(\mathcal{R}^{n+m}(G), 1)$. Choose n large enough so that $\operatorname{dist}_{\mathcal{M}}^{\mathrm{GH}}(X, (\mathcal{R}^{n+m}(G), 1)) < \epsilon/8$. Then for each $w_n \in S_n$, one can choose an associated point $x(w_n)$ in X and create an $(\epsilon/4)$-separated set S in X.

Thus any element of a cover of X by $\epsilon/8$ balls can contain at most one element of S, so the number of balls in such a cover is at least $|S| = |V| = N(G)N(H)^m$. So

$$\operatorname{Cov}(X, \epsilon/8) \geq N(G)N(H)^m$$

or

$$\frac{\ln(\operatorname{Cov}(X, \epsilon/8))}{\ln(8/\epsilon)} \geq \frac{\ln[N(G)N(H)^m]}{\ln[8N(G)N(\sigma)^{m+1}(M-1)/(N(\sigma)-1)]}.$$

As $\epsilon \to 0$, we have that $m \to \infty$, so

$$\limsup_{\epsilon \to 0} \frac{\ln(\operatorname{Cov}(X, \epsilon))}{\ln \epsilon^{-1}} \geq \frac{\ln N(H)}{\ln N(\sigma)}. \qquad \square$$

We now recall the definition of the Hausdorff dimension of a metric space X. We say a cover $\mathcal{U}$ is a *δ-cover* of X if $X \subset \bigcup_{U \in \mathcal{U}} U$ and $0 < \operatorname{diam} U \leq \delta$ for all $U \in \mathcal{U}$.

Let X be a metric space and let s be a non-negative number. For $\delta > 0$, define

$$\mathcal{H}_\delta^s(X) = \inf \sum_{i=1}^{\infty} (\operatorname{diam}\ U_i)^s,$$

where the infimum is over all (countable) δ-covers of X. Let

$$\mathcal{H}^s(X) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(X) = \sup_{\delta > 0} \mathcal{H}_\delta^s(X).$$

The *Hausdorff dimension* of X, $\dim_H(X)$, is the unique value s' such that $\mathcal{H}^s(X) = \infty$ for $0 \leq s < s'$ and $\mathcal{H}^s(X) = 0$ for $s > s'$.

It follows directly from the definitions that for any X,

$$\dim_{\mathrm{H}}(X) \leq \dim_{\mathrm{Box}}(X).$$

There are simple examples where this inequality is strict. The above inequality gives the following.

COROLLARY 4.1. *If X is the limit space of a sequence $(\mathcal{R}^n(G), 1)$, then* $\dim_{\mathrm{H}}(X) \leq \ln N(H)/\ln N(\sigma)$.

For any edge e and any replaceable vertex v in any finite graph G, let $Q(e, v)$ be the maximum number of (different) edges adjacent to v which can be connected to e by a non-replaceable path (containing no interior replaceable vertices). Let $Q(e) = \sum_{v \in V} Q(e, v)$, where V is the set of replaceable vertices in G.

If the boundary vertices of H are replaceable (i.e. have degree $|\partial H| - 1$), then we have that $Q(\mathcal{R}(e)) = Q(e)$ for $\mathcal{R}(e) \subset \mathcal{R}(G)$. Let $Q(G) = \max_{e \in E(G)} Q(e)$. Thus we have the following.

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LEMMA 4.1. *Let H define a vertex replacement rule $\mathcal{R}$. Suppose that the boundary vertices of H are replaceable (have degree $|\partial H| - 1$). Then for all $n \geq 0$, $Q(\mathcal{R}^n(G)) = Q(\mathcal{R}(G))$.*

Let H define a vertex replacement rule $\mathcal{R}$. If the boundary vertices of H are replaceable (have degree $|\partial H| - 1$), then $|\partial H| > 1$ (otherwise replaceable vertices have degree zero, and the replacement is trivial). Thus assume that $N(\sigma) \geq 2$.

THEOREM 4.2. *Let H define a vertex replacement rule $\mathcal{R}$. Suppose that the boundary vertices of H are replaceable, and that G is a graph containing replaceable vertices with $(\mathcal{R}^n(G), 1) \to X$ in the Gromov–Hausdorff metric as $n \to \infty$. Then* $\dim_{\mathrm{H}}(X) = \ln N(H)/\ln N(\sigma)$.

Proof. For $s = \ln N(H)/\ln N(\sigma)$, we will show that

$$\mathcal{H}^s_\delta(X) = \inf \sum_{U \in \mathcal{U}} (\operatorname{diam} U)^s \geq A > 0,$$

where the infimum is over all δ-covers $\mathcal{U}$ of X.

Consider a δ-cover $\mathcal{U}$ of X. Create an open cover $\widetilde{\mathcal{U}}$ of X as follows: for each $U \in \mathcal{U}$, define $\widetilde{U}$ to be all points in X which are less than $\operatorname{diam} U$ away from a point in U. Then clearly $\operatorname{diam} \widetilde{U} \leq 3 \operatorname{diam} U$. Thus

$$\frac{1}{3^s} \sum_{U \in \widetilde{\mathcal{U}}} (\operatorname{diam} \widetilde{U})^s \leq \sum_{U \in \mathcal{U}} (\operatorname{diam} U)^s.$$

Since X is compact, we may assume that $\widetilde{\mathcal{U}}$ is a finite cover.

Let $\widetilde{U}_1$ have the smallest diameter d of elements in $\widetilde{\mathcal{U}}$. Choose a $(d/16)$-net $\{x_1, \ldots, x_k\}$ for X. For large enough n, $\operatorname{dist}^{\mathrm{GH}}_{\mathcal{M}}(X, (\mathcal{R}^n(G), 1)) < d/16$. There is an associated $(d/8)$-net $\{y_1, \ldots, y_k\}$ for $(\mathcal{R}^n(G), 1)$.

For each $\widetilde{U}_i$, set

$$V_i := \bigcup_{x_j \in \widetilde{\mathcal{U}}_i} B(y_j, d/8),$$

where the above balls are open. The V_i cover $(\mathcal{R}^n(G), 1)$ by construction.

By the triangle inequality, $\operatorname{diam} V_i \leq \operatorname{diam} \widetilde{U}_i + d/2 \leq \frac{3}{2} \operatorname{diam} \widetilde{U}_i$. So

$$\frac{2}{9^s} \sum_{i=1}^{|\widetilde{\mathcal{U}}|} (\operatorname{diam}\ V_i)^s \leq \sum_{U \in \mathcal{U}} (\operatorname{diam} U)^s. \tag{5}$$

For each V_i, choose m_i so that $1/N(\sigma)^{m_i+1} \leq \operatorname{diam} V_i \leq 1/N(\sigma)^{m_i}$. Since

$$\lim_{n \to \infty} \frac{a(n-m)}{b(n)} = \frac{1}{N(\sigma)^m} \frac{N(\sigma) - 1}{M - 1},$$

and $\operatorname{diam} \mathcal{R}^n(G) \leq N(G) b(n) + \operatorname{diam}(G)$, one can fix a constant $C > 0$ such that, for large enough n, $a(n - m_i)/\operatorname{diam} \mathcal{R}^n(G) \geq C(1/N(\sigma)^{m_i})$ for all m_i (C is independent of $\mathcal{U}$.) Suppose that V_i contains a replaceable vertex v'. Then the projection of V_i in $\mathcal{R}^{m_i}(G)$ must contain a replaceable vertex; call it v. Consider the number $K(v)$ of replaceable

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vertices that can be connected to v by paths containing less than $[1/C]+1$ replaceable vertices ($[1/C]$ is the integer part of $1/C$). This number $K(v)$ (by Lemma 4.1) is less than $(|\partial H|Q(\mathcal{R}(G)))^{[1/C]+1}$. Since $a(n-m_i)/C \operatorname{diam}\mathcal{R}^n(G) \geq \operatorname{diam}(V_i)$, V_i can intersect at most $C' := (|\partial H|Q(\mathcal{R}(G)))^{[1/C]+1}$ copies of $\mathcal{R}^{n-m_i}(w)$, where w is replaceable in $\mathcal{R}^{m_i}(G)$. Thus V_i can contain at most $C'N(H)^{n-m_i}$ replaceable vertices.

However,

$$(\operatorname{diam} V_i)^{\ln N(H)/\ln N(\sigma)} \geq \left(\frac{1}{N(\sigma)^{m_i+1}}\right)^{\ln N(H)/\ln N(\sigma)} = \frac{1}{N(\sigma)N(H)^{m_i}} = \frac{N(H)^{n-m_i}}{N(\sigma)N(H)^n}.$$

Thus

$$\sum_i (\operatorname{diam}\ V_i)^{\ln N(H)/\ln N(\sigma)} \geq \frac{1}{N(\sigma)N(H)^n}\sum_i N(H)^{n-m_i}.$$

The total number of replaceable vertices in $\mathcal{R}^n(G)$ is $N(G)N(H)^n$. Since the V_i form a cover of $(\mathcal{R}^n(G),1)$, and each V_i can contain at most $C'N(H)^{n-m_i}$ replaceable vertices, we have that

$$\sum_i C'N(H)^{n-m_i} \geq N(G)N(H)^n.$$

So

$$\sum_i (\operatorname{diam} V_i)^{\ln N(H)/\ln N(\sigma)} \geq \frac{1}{N(\sigma)N(H)^n}\sum_i N(H)^{n-m_i} \geq \frac{N(G)}{C'N(\sigma)} > 0.$$

By (5), for $A := 2N(G)/9^s C'N(\sigma)$,

$$\sum_{U\in\mathcal{U}} (\operatorname{diam} U)^s \geq A > 0$$

for all δ-covers $\mathcal{U}$ of X. Hence

$$\dim_H(X) \geq \frac{\ln N(H)}{\ln N(\sigma)}.$$

By Corollary (4.1), $\dim_H(X) \leq \ln N(H)/\ln N(\sigma)$. □

5. *Examples*

In this section we give examples of convergent and divergent sequences $(\mathcal{R}^n(G),1)$.

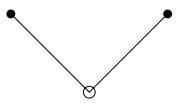
Example 6. Let H define a replacement rule $\mathcal{R}$ as in Figure 10, and let G be as in Figure 11. Notice that $|\partial H| = 1$ and $N(H) = 2$. By Theorem 3.2, the sequence $(\mathcal{R}^n(G),1)$ diverges. Clearly, $\mathcal{R}^n(G)$ approaches an infinite binary tree as $n \to \infty$.

Example 7. Let H define a replacement rule $\mathcal{R}$ as in Figure 12 and let G be as in Figure 13. Notice that $N(\sigma) = 2$ and $N(H) = 3$. Thus, by Theorem 3.2, the sequence $(\mathcal{R}^n(G),1)$ converges. The limit object is the Sierpinski gasket (Figure 14). The Hausdorff (box) dimension of the Sierpinski gasket is $\ln[N(H)]/\ln[N(\sigma)] = \ln 3/\ln 2$.

Example 8. Let H define a replacement rule $\mathcal{R}$ as in Figure 15. Notice that $N(\sigma) = 1$ and $N(H) = 1$. Thus, by Theorem 3.2, the sequence $(\mathcal{R}^n(G),1)$ converges. Several iterates of $\mathcal{R}$ on a graph G are shown in Figure 16.

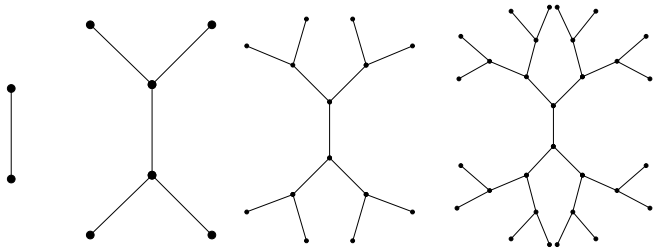
The limit space X is the unit interval $[0,1]$. Let G be a graph with three replaceable vertices. Then the limit of $(\mathcal{R}^n(G),1)$ is three copies of the interval $[0,\frac{1}{2}]$ attached according to the combinatorics of G.

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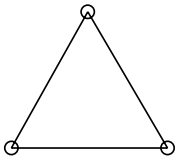
H

FIGURE 10.



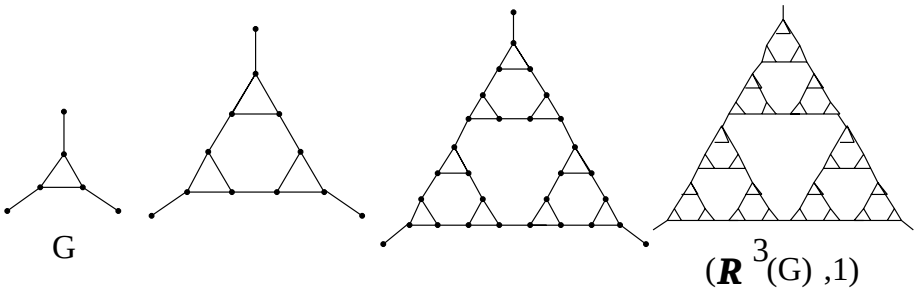
G $(\boldsymbol{R}^{3}(G),1)$

FIGURE 11.



H

FIGURE 12.



G $(\boldsymbol{R}^{3}(G),1)$

FIGURE 13.

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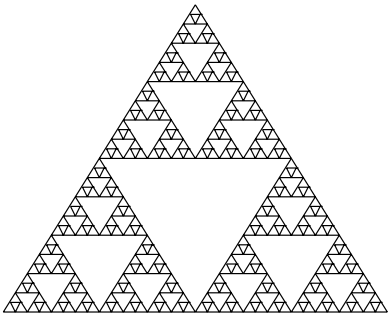


FIGURE 14.

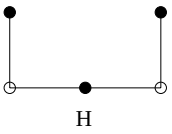


FIGURE 15.

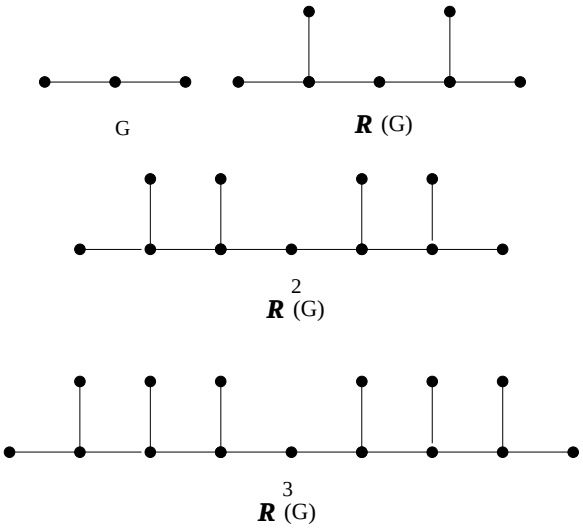


FIGURE 16.

6. *Marked Graphs*

A *marked graph* (G, p) is a graph with one marked vertex $p \in G$. The set $\mathcal{G}$ of isometry classes of all marked graphs is a metric space with $\text{dist}_{\mathcal{G}}((G_1, p_1), (G_2, p_2)) = 2^{-M}$, where M is the largest integer such that the balls of radius M centered at $p_i \in G_i$ are isometric.

A vertex replacement rule $\mathcal{R}$ is *expanding* if each H_i contains a replaceable, non-

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boundary vertex. For a marked graph (G, p) with p replaceable, let $\mathcal{R}(G, p)$ be the set of marked graphs $(\mathcal{R}(G), q)$, where q is a non-boundary replaceable vertex inside the copy of H_i that replaced p. Note that one could weaken the definition of $\mathcal{R}(G, p)$ for arbitrary vertex replacement rules $\mathcal{R}$, but for our purposes we restrict to expanding rules $\mathcal{R}$ with the above constraint on the marked vertex.

A sequence satisfying $(G_i, p_i) \in \mathcal{R}(G_{i-1}, p_{i-1})$ for all $i \geq 1$ is called an *$\mathcal{R}$-orbit* of (G_0, p_0).

PROPOSITION 6.1. *Every $\mathcal{R}$-orbit has limit points in $\mathcal{G}$, and the space $\mathcal{R}^\infty$ of the limit points of all $\mathcal{R}$-orbits is compact. Moreover, all of $\mathcal{R}^\infty$ can be realized as the limit points of the set of marked graphs (H_j, p), where the H_j's are replacement graphs.*

Proof. Let (G_i, p_i) be an $\mathcal{R}$-orbit of (G_0, p_0). Let Y be the complement of the set of all replaceable vertices in G_0. Since $\mathcal{R}$ is expanding,

$$\text{dist}_{(G, p_i)}(p_i, \mathcal{R}^i(Y)) \geq i.$$

Since the replacement graphs are finite, for all $n \geq i$, there are only a finite number of non-isometric balls of radius i centered at the marked point p_n in G_n. Hence, by a diagonalization process, one obtains a limit point. Moreover, all such limit points can be obtained from the set of marked graphs (H, p) where H is one of the H_j's and p is a replaceable non-boundary vertex of H_j. □

A vertex replacement rule $\mathcal{R}$ is called *invertible* if for any $(\tilde{G}, \tilde{p}) \in \mathcal{R}^\infty$ there is a unique $(G, p) \in \mathcal{R}^\infty$ such that $(\tilde{G}, \tilde{p}) \in \mathcal{R}(G, p)$. For $\mathcal{R}$ expanding and invertible, let π be the map on $\mathcal{R}^\infty$ which undoes replacement, i.e. $\pi[(\tilde{G}, \tilde{p})] = (G, p)$ for all $(\tilde{G}, \tilde{p}) \in \mathcal{R}(G, p)$. Since $\mathcal{R}$ is invertible, π is well defined and continuous on $\mathcal{R}^\infty$.

We state an invertibility result without proof (see [**8**]).

THEOREM 6.1. *An expanding vertex replacement rule $\mathcal{R}$ defined by a single graph H with ∂H non-replaceable is invertible.*

We state the main theorem of this section.

THEOREM 6.2. *Let $\mathcal{R}$ be expanding and invertible. Then the topological dynamical system $(\mathcal{R}^\infty, \pi)$ is a one-sided subshift of finite type. Moreover, the system $(\mathcal{R}^\infty, \pi)$ is topologically transitive if for any $k > 0$ and any $H_j, H_{j'}$, there is a number $n \geq k$ so that for some replaceable, non-boundary vertex $v \in H_j$ we have that $\mathcal{R}^n(v)$ contains a replaceable, non-boundary vertex w with $\mathcal{R}(w) = H_{j'}$. The system $(\mathcal{R}^\infty, \pi)$ is topologically mixing if there is $k > 0$ so that for any $H_j, H_{j'}$ and all $n \geq k$, there is some replaceable, non-boundary vertex $v \in H_j$, so that $\mathcal{R}^n(v)$ contains a replaceable, non-boundary vertex w with $\mathcal{R}(w) = H_{j'}$.*

We prove this by proving the following two lemmas. For any metric space X with $x \in X$, let $B_X(x, n)$ be the closed ball of radius n centered at x.

LEMMA 6.1. *Let $\mathcal{R}$ be expanding and invertible. Then, the map π is an expanding map with constant $\epsilon = 2^{-J}$, and expansion constant 2, where J is a number larger than the diameter of all replacement graphs.*

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Proof. Suppose that

$$0 < \text{dist}_{\mathcal{G}}[(G_1, p_1), (G_2, p_2)] < 2^{-J}.$$

Since $\text{dist}_{\mathcal{G}}[(G_1, p_1), (G_2, p_2)] > 0$, there is an integer $\bar{N}$ such that $(B_{G_1}(p_1, \bar{N}), p_1)$ is isometric to $(B_{G_2}(p_2, \bar{N}), p_2)$ but $(B_{G_1}(p_1, \bar{N}+1), p_1)$ is not isometric to $(B_{G_2}(p_2, \bar{N}+1), p_2)$. Since $\mathcal{R}$ is expanding, the marked point p_i is at least a distance of 1 away from the image of everything non-replaceable in $\pi(G_i, p_i)$, then $B_{G_i}(p_i, \bar{N}+1) \subseteq \mathcal{R}(B_{\pi(G_i)}(\pi(p_i), \bar{N}))$. Thus $\pi[B_{G_i}(p_i, \bar{N}+1)] \subseteq B_{\pi(G_i)}(\pi(p_i), \bar{N})$.

Suppose that $(\pi[B_{G_1}(p_1, \bar{N}+1)], \pi(p_1))$ is isometric (in $\mathcal{G}$) to $(\pi[B_{G_2}(p_2, \bar{N}+1)], \pi(p_2))$. Then clearly, as graphs, $\mathcal{R}(\pi[B_{G_i}(p_i, \bar{N}+1)])$ are isometric for $i = 1$ and $i = 2$. Since

$$\text{dist}_{\mathcal{G}}[(G_1, p_1), (G_2, p_2)] < 2^{-J},$$

where J is larger than the diameter of all replacement graphs, the marked graph $(\mathcal{R}(\pi(p_1)), p_1)$ is isometric (in $\mathcal{G}$) to $(\mathcal{R}(\pi(p_2)), p_2)$. Hence the marked graphs $(\mathcal{R}(B_{\pi(G_i)}(\pi(p_i), \bar{N})), p_i)$ are isometric. This contradicts the fact that $(B_{G_1}(p_1, \bar{N}+1), p_1)$ is not isometric (in $\mathcal{G}$) to $(B_{G_2}(p_2, \bar{N}+1), p_2)$. Thus $(\pi[B_{G_1}(p_1, \bar{N}+1)], \pi(p_1))$ is not isometric (in $\mathcal{G}$) to $(\pi[B_{G_2}(p_2, \bar{N}+1)], \pi(p_2))$. So $B_{\pi(G_1)}(\pi(p_1), \bar{N})$ is not isometric to $B_{\pi(G_2)}(\pi(p_2), \bar{N})$ (as marked graphs centered at $\pi(p_i)$).

Hence

$$\text{dist}_{\mathcal{G}}(\pi(G_1, p_1), \pi(G_2, p_2)) \geq 2^{-(\bar{N}-1)} \geq 2\,\text{dist}_{\mathcal{G}}((G_1, p_1), (G_2, p_2)). \qquad \square$$

LEMMA 6.2. *Let $\mathcal{R}$ be expanding and invertible. If $\pi(G, p)$ is isometric to (G', p'), then there is a marked graph in $\mathcal{R}(G', p')$ isometric to (G, p). In particular,*

$$B_{\mathcal{R}^\infty}[\pi(G, p), 2^{-J}] \subseteq \pi(B_{\mathcal{R}^\infty}[(G, p), 2^{-J}]),$$

i.e. the balls of radius 2^{-J} form a Markov partition for π.

Proof. Suppose that $\pi(G, p)$ is isometric as a marked graph to (G', p'); in particular, the graphs G' and $\pi(G)$ are isometric as graphs, which implies that $\mathcal{R}(G')$ and G are isometric. The vertices $\pi(p)$ and p' have the same degree, so $\mathcal{R}(p')$ and $\mathcal{R}(\pi(p))$ are isometric as graphs. In particular, choose the marked point in $\mathcal{R}(p')$ so that $(\mathcal{R}(p'), p')$ is isometric to $(\mathcal{R}(\pi(p)), p)$. $\square$

Lemmas 6.1 and 6.2 prove that $(\mathcal{R}^\infty, \pi)$ is a one-sided subshift of finite type. The conditions for topological transitivity and topological mixing are obvious. Thus Theorem 6.2 is proven. Note that mixing implies that every graph (G, p) in $\mathcal{R}^\infty$ contains replaceable vertices of all types, i.e. there is a vertex $v \in G$ so that $\mathcal{R}(v) = H_i(v)$ for all i.

Define an equivalence class $\sim$ on R^∞ by $(G_1, p_1) \sim (G_2, p_2)$ if (G_2, p_2) can be obtained from (G_1, p_1) by moving the marked point p_1, i.e. G_1 is isometric to G_2 as a graph. Note that $(G_1, p_1) \sim (G_2, p_2)$ if and only if $\pi^N(G_1, p_1)$ is isometric to $\pi^N(G_2, p_2)$ as a marked graph for large enough N.

THEOREM 6.3. *Let $\mathcal{R}$ be expanding and invertible. If $(\mathcal{R}^\infty, \pi)$ is topologically mixing, then each equivalence class of $\sim$ is dense in R^∞.*

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Proof. Consider a ball of radius 2^{-n} in $\mathcal{R}^\infty$ centered about a graph $(G, p) \in \mathcal{R}^\infty$. This consists of all graphs (G', p') in $\mathcal{R}^\infty$ whose ball $B_{G'}(p', n)$ is isometric to the ball $B_G(p, n)$. Clearly, $\pi^n[B_G(p, n)]$ is one vertex in $(\pi^n(G, p), \pi^n(p))$.

Consider the $\sim$ equivalence class of $(G_1, p_1) \in \mathcal{R}^\infty$. Since $(\mathcal{R}, \pi)$ is mixing there is a vertex q in $\pi^n(G_1, p_1)$ with the same degree as $\pi^n[B_G(p, n)]$. Consider the marked graph $(\pi^n(G_1, p_1), q)$. Choose the marked point q' in $\mathcal{R}(q)$ so that $(\mathcal{R}(q), q')$ is isometric to $(\pi^{n-1}(G, p), \pi^{n-1}(p))$. Continuing in this fashion we see that inside the graph G_1 sits an isometrically embedded copy of $B_G(p, n)$. Since n is arbitrary, the equivalence class of (G_1, p_1) is dense in $\mathcal{R}^\infty$. □

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