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Lyapunov functions and a control problem

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1. One of the most important techniques in the study of non-linear differential systems is the Lyapunov's second method and its extensions. One of the extensions of the method depends upon the fact that the function satisfying a differential inequality can be majorized by the maximal solution of the corresponding differential equation. This method was used extensively by V. Lakshmikantham and others for obtaining results, in a unified way, on stability and boundedness of systems of differential equations.

In this paper we use this principle to study a type of control problem. Let X denote a Hilbert space, and F and ϕ denote non-linear operators. Consider the equation

$$dx/dt = F(t,x)+\phi(u(t),t), \tag{1·1}$$

where for $t \in I$: $[0,\infty)$, x and u are elements in X. Let, for $t \in I$, $y = y(t)$ be any arbitrary given element in X. For given F, u and y Liberman (6) has shown that under suitable conditions solutions $x(t)$ of (1·1) can be approximated so as to be arbitrarily close to $y(t)$ for all $t \in I$. His results are generalizations of the results of Bellman (1, 2), Barbasin (3) and Liberman (5). Using the comparison principles, we obtain some interesting theorems regarding the control problem stated above, which include Liberman's results as special cases.

2. Let $x \in X$, with the following substitutions

$$z = x-y(t), \quad \psi(y,t) = -F(t,y)+(dy/dt),$$

$$f(t,z) = F(t,z+y)-F(t,y), \quad \phi(u(t),t)-\psi(y(t),t) = \xi(t),$$

the equation (1·1) reduces to

$$dz/dt = f(t,z)+\xi(t), \tag{2·1}$$

where $f(t,0) \equiv 0$. Along with the equation (2·1), consider the equation

$$d\tilde{z}/dt = f(t,\tilde{z}). \tag{2·2}$$

3. Let a scalar functional $V = V(t,z,\tilde{z})$ be defined and continuous on $I \times X \times X$ satisfying the following conditions:

$$V(t,z,\tilde{z}) \geqslant 0 \quad \text{for all} \quad (t,z,\tilde{z}) \in I \times X \times X, \tag{3·1}$$

$$|V(t,z,\tilde{z})-V(t,z_1,\tilde{z}_1)| \leqslant k(t)[\|z-z_1\|+\|\tilde{z}-\tilde{z}_1\|] \tag{3·2}$$

for each $t \in I$ and $z, z_1, \tilde{z}, \tilde{z}_1 \in X$ where $k(t)$ is a positive continuous function on I.

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Define

$$V^*(t,z,\tilde{z}) = \limsup_{h\to 0+} 1/h[V(t+h, z+hf(t,z), \tilde{z}+hf(t,\tilde{z})) - V(t,z,\tilde{z})]. \tag{3·3}$$

With respect to the functions defined above, we state the following lemma.

LEMMA. *Let there exist a function* $V = V(t,z,\tilde{z})$ *satisfying the conditions* (3·1) *and* (3·2) *such that*

$$V^*(t,z,\tilde{z}) \leqslant W(t, V(t,z,\tilde{z})), \tag{3·4}$$

where $W(t,r)$ *is defined and continuous on* $I\times I$. *Then* $V(t,z(t),\tilde{z}(t)) \leqslant r(t)$ $(t \geqslant t_0)$, *whenever* $v(t_0, z(t_0), \tilde{z}(t_0)) \leqslant r(t_0)$, *where* $z(t)$ *and* $\tilde{z}(t)$ *are solutions of* (2·1) *and* (2·2) *respectively*, $r(t)$ *being the maximal solution of the equation*

$$r' = W(t,r) + k(t)\|\xi(t)\|, r(t_0) = r_0. \tag{3·5}$$

Proof. Let $m(t) = v(t, z(t), \tilde{z}(t))$.

Then $m(t+h) - m(t) = V(t+h, z(t+h), \tilde{z}(t+h)) - V(t, z(t), \tilde{z}(t))$.

Adding and subtracting

$$V(t+h, z(t)+hf(t,z(t)), \tilde{z}(t)+hf(t,\tilde{z}(t)))$$

and applying the condition (3·2), we get

$$\begin{aligned} m(t+h)-m(t) &\leqslant k(t)[\|z(t+h)-z(t)-hf(t,z(t))\| + \|\tilde{z}(t+h)-\tilde{z}(t)-hf(t,\tilde{z}(t))\|] \\ &\quad + V(t+h, z(t)+hf(t,z(t)), \tilde{z}(t)+hf(t,\tilde{z}(t))) - V(t,z(t),\tilde{z}(t)), \end{aligned}$$

so that

$$\begin{aligned} \limsup_{h\to 0+} \frac{m(t+h)-m(t)}{h} &\leqslant k(t)\left[\left\|\frac{dz(t)}{dt} - f(t,z(t))\right\| + \left\|\frac{d\tilde{z}(t)}{dt} - f(t,\tilde{z}(t))\right\|\right] + V^*(t,z(t),\tilde{z}(t)) \\ &\leqslant W(t, V(t,z,\tilde{z})) + k(t)\|\xi(t)\| \\ &= W(t, m(t)) + k(t)\|\xi(t)\|. \end{aligned}$$

Thus $m(t)$ can be majorized by the maximal solution of $r' = W(t,r) + k(t)\|\xi(t)\|$. Hence the result.

Remark 1. Let $f(t,z)$ satisfy the condition

$$\|f(t,z) - f(t,\tilde{z})\| \leqslant L\|z-\tilde{z}\|,$$

L being a positive constant. Consider

$$V(t,z,\tilde{z}) = +\|z-\tilde{z}\|.$$

Then

$$\begin{aligned} V^*(t,z,\tilde{z}) &= \limsup_{h\to 0+} 1/h\,[\|z+hf(t,z) - \tilde{z} - hf(t,\tilde{z})\| - \|z-\tilde{z}\|] \\ &\leqslant \limsup_{h\to 0+} 1/h[\|z-\tilde{z}\| + h\|f(t,z) - f(t,\tilde{z})\| - \|z-\tilde{z}\|] \\ &\leqslant \|f(t,z) - f(t,\tilde{z})\| \\ &\leqslant L\|z-\tilde{z}\|. \end{aligned}$$

Thus $V^*(t,z,\tilde{z}) \leqslant LV$.

Also $|\|z-\tilde{z}\| - \|z_1-\tilde{z}_1\|| \leqslant \|z-\tilde{z}-z_1+\tilde{z}_1\| \leqslant \|z-z_1\| + \|\tilde{z}-\tilde{z}_1\|$.

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Thus from (3·2) we have $k(t) \equiv 1$. Hence the equation (3·5) reduces to

$$r' = Lr + \|\xi(t)\|,$$

which is the fundamental equation of Liberman (6). Thus his Theorem 1 is a deduction of the above lemma.

4. Liberman's problem demands that the solution $x(t)$ of (1·1) should be close to $y(t)$ for all $t \in I$. In this section we introduce control conditions similar to the notions of eventual stability (4) and quasi-asymptotic stability of the null solution of differential systems. In the following conditions $z(t)$ and $\tilde{z}(t)$ are any solutions of (2·1) and (2·2) respectively.

(i) Given $\epsilon > 0$, there exist a positive number $T = T(\epsilon)$ and a positive function $\delta = \delta(\epsilon)$, such that

$$\|z(t) - \tilde{z}(t)\| < \epsilon \quad (t \geqslant t_0 \geqslant T),$$

whenever

$$\|z(t_0) - \tilde{z}(t_0)\| \leqslant \delta.$$

(ii) Given $\epsilon > 0$, $\alpha \geqslant 0$ and $t_0 \geqslant 0$ there exists a positive number $T = T(t_0, \epsilon, \alpha)$, such that

$$\|z(t) - \tilde{z}(t)\| < \epsilon \quad (t \geqslant t_0 + T)$$

provided

$$\|z(t_0) - \tilde{z}(t_0)\| \leqslant \alpha.$$

Null solution exists for the equation (2·2). Conditions obtained by substituting $\tilde{z}(t) \equiv 0$ in (i) and (ii) above, will be denoted by (i*) and (ii*) respectively.

5. Now corresponding to the conditions (i) and (ii) stated above, if we say that the differential equation (3·5) has the property (ia), we mean the following condition is satisfied.

(ia) Given $\eta > 0$ there exist a positive number $T = T(\eta)$ and a positive function $d = d(\eta)$, such that

$$r(t) < \eta \quad (t \geqslant t_0 \geqslant T(\eta)),$$

whenever $r(t_0) \leqslant d$.

Property (ii$a$) can be similarly formulated. Assume that there exists a continuous, non-decreasing function $b = b(r)$, $r \in I$, $b(r) > 0$ for $r > 0$ and

$$b(\|z - \tilde{z}\|) \leqslant V(t, z, \tilde{z}), \quad \text{for all} \quad (t, z, \tilde{z}) \in I \times X \times X, \tag{5·1}$$

$$V(t, z, \tilde{z}) \to 0 \quad \text{as} \quad \|z - \tilde{z}\| \to 0 \quad \text{uniformly in } t. \tag{5·2}$$

We state and prove the following:

THEOREM 1. *Let the assumptions of the lemma hold together with* (5·1) *and* (5·2). *Then condition* (ia) *implies condition* (i).

Proof. Let $\epsilon > 0$ be given. Then due to (5·1) $b(\epsilon) > 0$ and

$$\|z - \tilde{z}\| = \epsilon \quad \text{implies} \quad b(\epsilon) \leqslant V(t, z, \tilde{z}). \tag{5·3}$$

Since (ia) holds, given $b(\epsilon) > 0$ there exist a positive number $T = T(b(\epsilon)) = T(\epsilon)$ and a positive number $d = d(\epsilon)$, such that

$$r(t) < b(\epsilon) \quad (t \geqslant t_0 \geqslant T), \tag{5·4}$$

whenever

$$r(t_0) \leqslant d. \tag{5·5}$$

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Because of (5·2), there exists a $\delta = \delta(d)$ such that

$$\sup_{\|z(t_0)-\tilde{z}(t_0)\| \leqslant \delta} V(t_0, z(t_0), \tilde{z}(t_0)) \leqslant d. \tag{5·6}$$

From (5·6) and (5·5) it is obvious that we can choose $r(t_0)$, whenever $\|z(t_0)-\tilde{z}(t_0)\| \leqslant \delta$, to satisfy the inequality

$$V(t_0, z(t_0), \tilde{z}(t_0)) \leqslant r(t_0) \leqslant d. \tag{5·7}$$

We now assert that if $t \geqslant t_0 \geqslant T$ and $\|z(t_0)-\tilde{z}(t_0)\| \leqslant \delta$ then $\|z(t)-\tilde{z}(t)\| < \epsilon$. For otherwise there exists a $t_1 \geqslant t_0 \geqslant T$ such that $\|z(t_1)-\tilde{z}(t_1)\| = \epsilon$. But this leads to the contradiction

$$b(\epsilon) \leqslant V(t_1, z(t_1), \tilde{z}(t_1)) \leqslant r(t_1) < b(\epsilon),$$

due to (5·1), (5·6), (5·7), the lemma, (5·5) and (5·4). This proves the theorem.

We state the following theorem for condition (ii), omitting the proof as it follows closely the proof given above.

THEOREM 2. *Let the assumptions of the lemma hold together with* (5·1) *and* (5·2). *Then condition* (ii$a$) *implies condition* (ii).

Remark. (i) It is clear that the condition (i) and (ii) in the theorems can be replaced by conditions (i*) and (ii*) respectively, due to the existence of the null solution of (2·2).

(ii) The scalar differential equation (3·5) will possess the property (ia) under the following set of conditions

$$W \leqslant 0 \quad \text{and} \quad \lim_{t \to \infty} \left[\sup_{a > 0} \int_{t_0}^{t_0+a} K(t) \|\xi(t)\| \, dt \right] = 0.$$

Thus if $W \leqslant 0$, the control function $\xi(t)$ can be so chosen that the conditions (i) and (i*) hold.

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