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THE BRAUER–SIEGEL THEOREM

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Abstract

Explicit bounds are given for the residues at $s=1$ of the Dedekind zeta functions of number fields. As a consequence, a simple proof of the Brauer–Siegel theorem and explicit lower bounds for class numbers of number fields are obtained. Compared with Stark's original approach, the paper is explicit and more satisfactory for number fields containing quadratic subfields. Examples are given of fully explicit lower bounds for class numbers of various types of number fields, for example normal and non-normal number fields of odd degree, with an emphasis on cubic fields, real cyclic quartic number fields, and non-normal quartic number fields containing an imaginary quadratic subfield.

1. *Bounds on residues of zeta functions*

Let $c>0$ be given (to be selected below). It has long been known that Hecke's integral representations of Dedekind zeta functions $\zeta_K(s)$ of number fields K can be used to obtain lower bounds for their residues κ_K at $s=1$ of the type

$$1-(c/\log d_K)\leqslant\beta<1 \text{ and } \zeta_K(\beta)\leqslant 0 \text{ imply that } \kappa_K\geqslant(1-\beta)d_K^{(\beta-1)/2}(1+o(1)),$$

where d_K is the absolute value of the discriminant of K and $o(1)$ is an error term which approaches zero as $d_K\to\infty$ provided that K ranges over number fields of a given degree. (See for example [**16**, Proposition A]. See also [**6**, Lemma 4; **9**, Chapter XVI, Section 2, Lemma 3, p. 323; **30**, Lemma 4] for weaker results). Notice that the best lower bound that one can deduce (for $\beta=1-(2/\log d_K)$) is of the type

$$\zeta_K(1-(2/\log d_K))\leqslant 0 \text{ implies that } \kappa_K\geqslant\frac{2}{e\log d_K}(1+o(1)).$$

In this paper, we prove Theorem 1 (generalizing [**26**, Theorem 1]), which provides a nice treatment of this error term by simply getting rid of it!

Throughout this paper, we use the following notation. We let K be a number field of degree $n\geqslant 2$ and N be its normal closure, and set

$$\alpha_K=(\log d_K)/\log d_N.$$

Hence $0<\alpha_K\leqslant 1, \alpha_K=1$ if and only if K is normal, and

$$\alpha_K\in[1/((n-1)[N:K]),1/[N:K]]\subseteq[1/((n-1)\cdot(n-1)!),1]. \tag{1}$$

(Since $N=K_1\dots K_{n-1}$ is the compositum of any $n-1$ of the n complex embeddings of K, we obtain d_N divides $d_K^{(n-1)[N:K]}$, by [**30**, Lemma 7]). We set

$$c_K=\begin{cases} c_3:=2 & K \text{ is abelian}\\ c_2:=2(\sqrt{3}-1)^2=1.07\dots & K \text{ is normal but not abelian}\\ c_1:=2(\sqrt{2}-1)^2=0.343\dots & K \text{ is not normal.}\end{cases}$$

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TABLE 1.1.

$m=$	2	3	4	5	6	7	8
$\rho_m(c_3)=$	2 245	454	222	150	118	101	90
$\rho_m(c_2)=$	772	227	134	101	85	76	70
$\rho_m(c_1)=$	330	131	89	73	65	61	58
$\rho'_m(c_3)=$	26 514	1 120	334	184	131	106	92
$\rho'_m(c_2)=$	33 990	1 325	381	205	144	115	99
$\rho'_m(c_1)=$	201 947	4 325	916	410	254	186	150

TABLE 1.2.

$m=$	2	4	6	8
$\rho_m(c_3)=$	855	83	43	33
$\rho_m(c_2)=$	288	49	31	26
$\rho_m(c_1)=$	121	33	24	21
$\rho'_m(c_3)=$	7 618	98	40	29
$\rho'_m(c_2)=$	9 577	110	44	31
$\rho'_m(c_1)=$	58 761	271	78	48

We let $\gamma = 0.577\,215\ldots$ denote Euler's constant.

THEOREM 1. *Fix c and c' with $0 < c' \leqslant c \leqslant 2$. Let $m \geqslant 2$ be a positive integer. Assume that $d_K > e^4$, which implies that $\frac{1}{2} < 1 - (c/\log d_K) < 1$. There exist $\rho_m(c)$ (increasing with c) and $\rho_m(c,c')$ effective such that, for any number field K of degree $n \geqslant m$ and root discriminant $\rho_K := d_K^{1/n}$, we have*

$$\kappa_K \geqslant \frac{1}{2}(1-\beta)d_K^{(\beta-1)/2} \geqslant \frac{1-\beta}{2e^{c/2}} \tag{2}$$

if $\rho_K \geqslant \rho_m(c)$ and $\zeta_K(\beta) \leqslant 0$ for some $\beta \in [1-(c/\log d_K), 1)$, and

$$\kappa_K \geqslant (1-\beta)d_K^{(\beta-1)/2} \geqslant \frac{c'}{e^{c'/2}\log d_K} \tag{3}$$

if $\rho_K \geqslant \rho_m(c,c')$ and $\zeta_K(\beta) \leqslant 0$ for $\beta \in [1-(c/\log d_K), 1-(c'/\log d_K)]$.

For $c \in \{c_1, c_2, c_3\}$, we may take $\rho_m(c)$ and $\rho'_m(c) := \rho_m(c,c)$ as shown in Table 1.1. Moreover, for totally imaginary number fields K, we may take $\rho_m(c)$ and $\rho'_m(c) := \rho_m(c,c)$, as shown in Table 1.2.

PROPOSITION 2. (1) *Let K be a number field of degree $n \geqslant 2$. Then*

$$\kappa_K \leqslant \left(\frac{e\log d_K}{2(n-1)}\right)^{n-1} \leqslant 2(e/2)^n \log^{n-1}\rho_K. \tag{4}$$

(2) *Let L be a quadratic number field. Then*

$$\kappa_L \leqslant \tfrac{1}{2}(\log d_L + \mu_L), \tag{5}$$

where

$$\mu_L = \begin{cases} 2+\gamma-\log(4\pi) = 0.046\ldots & L \text{ is real} \\ 2+\gamma-\log\pi = 1.432\ldots & L \text{ is imaginary.} \end{cases}$$

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Moreover, if $\zeta_L(\beta)=0$ *for some* $\beta\in(0,1)$, *then*

$$\kappa_L \leqslant \frac{1-\beta}{8}\log^2 d_L. \tag{6}$$

Proof. For (4), see [**22**, Theorem 1; **23**, Theorem 1]. For (5), see [**15**, **19**]. For (6), see [**23** Theorem 7; **28**], in which a uniform proof of (5) and (6) is given. However, we may take

$$\mu_L=\begin{cases}0 & \text{L } \textit{is real}\\ 5-2\log 6=1.416\ldots & \text{L } \textit{is imaginary}\end{cases}$$

by [**29**, Corollary 1]. □

PROPOSITION 3. *Let* K *be a number field and let* N *be its normal closure. Assume that* $d_K>e^2$. *Either* (i) K *contains a quadratic subfield* L *for which* $\zeta_L(\beta)=\zeta_K(\beta)=0$ *for some* $\beta\in[1-(c_K/\log d_N),1)=[1-(c_K\alpha_K/\log d_K),1)$, *or* (ii) $\zeta_K(1-(c_K/\log d_N))=\zeta_K(1-(c_K\alpha_K/\log d_K))\leqslant 0$.

Proof. Let $m\geqslant 1$ be a given positive integer, set $c_m=2(\sqrt{m+1}-1)^2$ and let E be a number field of discriminant $d_E>\exp(2(\sqrt{m+1}-1))$. Then $\zeta_E(s)$ has at most m real zeros in the range $1-(c_m/\log d_E)\leqslant s<1$ (this generalisation of [**30**, Lemma 3] is proved in [**13**, Lemma 15]).

First, the abelian case is easy to deal with by using the factorization of $\zeta_K(s)$ as a product of the Dirichlet L-series. Second, assume that K is normal and that we are not in case (ii). Then $N=K$, $c_K=c_2$ and $\zeta_K(1-(c_2/\log d_K))>0$. Now, κ_K is positive and $\zeta_K(s)$ has a simple pole at $s=1$. Hence $\lim_{s\uparrow 1}\zeta_K(s)=-\infty$ and $\zeta_K(s)$ has a real zero $\beta\in(1-(c_2/\log d_K),1)$ of odd multiplicity $n_\beta\geqslant 1$. In this range we have $n_\beta\leqslant 2$. Hence $n_\beta=1$ and there exists some quadratic subfield L of K such that for $E\subseteq K$ we have $\zeta_E(\beta)=0$ if and only if $L\subseteq E$ (see [**30**, Theorem 3]). In particular, $\zeta_L(\beta)=0$, and $\zeta_K(\beta)=0$ implies that $L\subseteq K$, and we are in case (i). Third, assume that K is not normal and that we are not in case (ii). Then $c_K=c_1$ and $\zeta_K(1-(c_1/\log d_N))>0$. Since $\lim_{s\uparrow 1}\zeta_K(s)=-\infty$, there exists some real zero $\beta\in(1-(c_1/\log d_N),1)$ of $\zeta_K(s)$. Since N/K is normal, $\zeta_K(s)$ divides $\zeta_N(s)$ (see [**3**, Corollary 2] for a short proof of the Aramata–Brauer theorem based on Stark's ideas). Hence $\zeta_N(\beta)=0$ and β is a simple zero of $\zeta_N(s)$, for $1-(c_1/\log d_N)<\beta<1$. According to [**30**, Theorem 3], there exists a quadratic subfield $L\subseteq N$ such that $E\subseteq N$ and $\zeta_E(\beta)=0$ if and only if $L\subseteq E$. In particular, $\zeta_L(\beta)=0$, and $\zeta_K(\beta)=0$ implies that $L\subseteq K$, and we are in case (i). □

2. *The Brauer–Siegel theorem*

Recall that (see [**9**, Chapter XIII, Section 3, Theorem 2])

$$\frac{\sqrt{d_K}}{3^n}\kappa_K\leqslant h_K\mathrm{Reg}_K=\frac{w_K\sqrt{d_K}}{2^{r_1}(2\pi)^{r_2}}\kappa_K\leqslant\frac{4\sqrt{d_K}}{\pi}\kappa_K \tag{7}$$

(for the upper bound, notice that $\sqrt{w_K/2}\leqslant\phi(w_K)=[\mathbf{Q}(\zeta_{w_K}):\mathbf{Q}]\leqslant[K:\mathbf{Q}]=n$ implies that $w_K\leqslant 2n^2$, and $w_K=2$ if $r_1\neq 0$).

THEOREM 4. *Let* K *be a number field of degree* $n\geqslant 2$. *Then the following hold.*

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(1) *If K contains a quadratic subfield L for which $\zeta_L(\beta) = \zeta_K(\beta) = 0$ for some β in the range $1 - (c_K/\log d_N) = 1 - (c_K\alpha_K/\log d_K) \leqslant \beta < 1$, then*

$$\kappa_K/\kappa_L \geqslant \frac{4e^{-c_K\alpha_K/2}}{\log^2 d_L} \geqslant \frac{2ne^{-c_K\alpha_K/2}}{(\log d_L + \mu_L)\log d_K} \tag{8}$$

for $\rho_K \geqslant \rho_n(c_K\alpha_K)$, and hence for $\rho_K \geqslant \rho_n(c_K)$.

(2) *Otherwise (in particular if K contains no quadratic subfield),*

$$\kappa_K \geqslant \begin{cases} \dfrac{c_K e^{-c_K/2}}{\log d_K} & K \text{ is normal and } \rho_K \geqslant \rho_n'(c_K) \\ \dfrac{c_K\alpha_K e^{-c_K\alpha_K/2}}{2\log d_K} & K \text{ is not normal and } \rho_K \geqslant \rho_n(c_K\alpha_K), \end{cases} \tag{9}$$

and for any quadratic subfield L of K it holds that

$$\kappa_K/\kappa_L \geqslant \begin{cases} \dfrac{2c_K e^{-c_K/2}}{(\log d_L + \mu_L)\log d_K} & K \text{ is normal and } \rho_K \geqslant \rho_n'(c_K), \\ \dfrac{c_K\alpha_K e^{-c_K\alpha_K/2}}{(\log d_L + \mu_L)\log d_K} & K \text{ is not normal and } \rho_K \geqslant \rho_n(c_K\alpha_K). \end{cases} \tag{10}$$

(Also $c_K\alpha_K e^{-c_K\alpha_K}$ decreases as a function of $c_K \in [0,2]$ and $\alpha_K \in [0,1]$).

Proof. (8) follows from (2) and (6); (9) follows from Proposition 3(ii), and (3) if K is normal but (2) if K is not normal. For (10), use (9) and (5). □

COROLLARY 5 (the Brauer–Siegel theorem). *If K ranges over an infinite family of abelian fields, over an infinite family of normal fields for which $\rho_K \to \infty$, or over an infinite family of non-normal number fields of a given degree, then $\log(h_K\mathrm{Reg}_K)$ is asymptotic to $\frac{1}{2}\log d_K$. Moreover, if the fields of these families contain no quadratic subfield, this asymptotic is explicit and effective.*

Proof. Start from (7), and use (4), (8), (9) and the Brauer–Siegel result for the quadratic number fields, a simple proof of this being given in [**4**] (see also [**2**]). Notice how simple this proof is compared with the one given in [**9**, Chapter XVI]. □

3. *Some explicit bounds on class numbers*

In this section, we give several examples showing that our lower bounds in Theorem 4 yield better explicit lower bounds for class numbers than the ones previously published in the literature.

3.1. *Normal number fields of odd degree*

To begin with, the following corollary provides us with a 24-fold improvement on the lower bound

$$h_K\mathrm{Reg}_K \geqslant \frac{\sqrt{d_K}}{33\cdot 2^{n-1}\log(4d_K)}$$

for real abelian number fields of odd degree given in [**12**] (and it could be, for example, applied to the dihedral fields of degree 10 studied in [**7**]).

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COROLLARY 6. *Let K be a normal number field of odd degree $n \geqslant 3$. Set $c_K = c_3 := 2$ if K is abelian and $c_K = c_2 := 2(\sqrt{3}-2)^2 = 1.07\ldots$ otherwise. Assume that $\rho_K \geqslant \rho'_n(c_K)$. Then $w_K = 2$ and*

$$h_K \mathrm{Reg}_K \geqslant \frac{\sqrt{d_K}}{\lambda_K \log d_K}, \tag{11}$$

where $\lambda_K = 2^{n-1} e^{c_K/2}/c_K$.

Proof. Use (7) and the first lower bound in (9). □

In particular, if $K = K_m$ is a simplest cyclic cubic field associated with the polynomial $P_m(x) = x^3 - mx^2 - (m+3)x - 1$, $m \geqslant -1$, of discriminant $d_m = \Delta_m^2$ (where $\Delta_m = m^2 + 3m + 9$ is assumed to be square-free), we have $\mathrm{Reg}_{K_m} \leqslant \frac{1}{4}\log^2 \Delta_m$ and we obtain $h_{K_m} > 1$ for $m \geqslant 47$, and $h_{K_m} = 1$ if and only if $m \in \{-1, 1, 2, 4, 7, 8, 10\}$ (see [**25**], in which a more general problem is in fact solved). Notice that (11) is much better than the lower bounds used in [**8**] and [**14**].

3.2. *Non-normal number fields of odd degree*

Now, Le in [**12**] made the mistake of applying his lower bound which is only valid for abelian number fields to the case of non-normal pure cubic fields $\mathbf{Q}(\sqrt[3]{D})$. Hence we provide in the following corollary a correct lower bound for non-normal number fields.

COROLLARY 7. *Let K be a non-normal number field of odd degree $n \geqslant 3$. Let N denote its normal closure. Set $c_K = c_1 := 2(\sqrt{2}-1)^2 = 0.343\ldots$ and $\alpha_K = (\log d_K)/\log d_N$. Assume that $\rho_K \geqslant \rho_n(c_K \alpha_K)$ (and notice that $\rho_n(c_K\alpha_K) \leqslant \rho_n(c_1/2) \leqslant \rho_n(c_1)$). Then $w_K = 2$ and*

$$h_K \mathrm{Reg}_K \geqslant \frac{\sqrt{d_K}}{\lambda_K \log d_K}, \tag{12}$$

where $\lambda_K = 2^{r_1}(2\pi)^{r_2} e^{c_K \alpha_K/2}/(c_K \alpha_K)$.

Proof. Use (7) and the second lower bound in (9). □

3.3. *Non-normal cubic fields*

In particular, let K be a non-normal cubic field. Then d_N divides d_K^3 (see below), $\alpha_K \in [1/3, 1/2]$ (which improves upon (1)), $\rho_K \geqslant \rho_3(c_1/2) = 115$ if and only if $d_K \geqslant 1\,520\,875$, and

$$\lambda_K \leqslant 2^{r_1}(2\pi)^{r_2} e^{c_1/6}/(c_1/3) = \begin{cases} 24e^{c_1/6}/c_1 = 74.057\ldots & K \text{ is totally real} \\ 12\pi e^{c_1/6}/c_1 = 116.329\ldots & \text{otherwise} \end{cases}$$

(to be compared with [**18**, Theorem 1]). However, a more clever use of the tools developed here yields a better lower bound for $h_K \mathrm{Reg}_K$. Indeed, the normal closure $N = KL$ of K is a dihedral sextic field, where $L = \mathbf{Q}(\sqrt{D_K})$, with D_K being the discriminant of K (notice that N and L are real if K is totally real, and that N and L are imaginary otherwise). Hence d_L divides d_K. Since

$$\zeta_N(s) = (\zeta_K(s)/\zeta(s))^2 \zeta_L(s), \tag{13}$$

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we have $\kappa_K = \sqrt{\kappa_N/\kappa_L}$, $d_N = d_L d_K^2$, and d_N divides d_K^3. Now, assume that $\rho_N \geqslant \max(\rho_6(c_3), \rho_6'(c_3))$, that is, that $d_K = \rho_K^3 \geqslant \max(118, 131)^3 = 118^3 = 1\,643\,032$ if K is totally real and that $d_K = \rho_K^3 \geqslant \max(43, 40)^3 = 43^3 = 79\,507$ otherwise (and notice that $\rho_N \geqslant \rho_K$).

(1) If there exists β in the range $1 - (2/\log d_N) \leqslant \beta < 1$ such that $\zeta_L(\beta) = 0$, then $\zeta_N(\beta) = 0$ and

$$\kappa_N/\kappa_L \geqslant \frac{8}{e \log^2 d_L} \geqslant \frac{24}{e(\log d_L + \mu_L)\log d_N},$$

by (3) and (6).

(2) Otherwise, $\zeta_L(1 - (2/\log d_N)) \leqslant 0$; hence $\zeta_N(1 - (2/\log d_N)) \leqslant 0$, by (13), and

$$\kappa_N/\kappa_L \geqslant \frac{4}{e(\log d_L + \mu_L)\log d_N},$$

by (2) and (5). Since the right-hand side of the latter inequality is less than the right-hand side of the former inequality, the latter inequality always holds. Using $(\log d_L + \mu_L)\log d_N \leqslant 3(\log d_K + \mu_L)\log d_K \leqslant 3(\log d_K + \mu_L/2)^2$, we obtain $\kappa_K = \sqrt{\kappa_N/\kappa_L} \geqslant 2/(\sqrt{3e}(\log d_K + \mu_L/2))$ and, by (7), we finally obtain the following corollary (to be compared with [**24**, Theorem 7]).

COROLLARY 8. *Let K be a non-normal cubic field. Set*

$$\lambda_K = 2^{r_1}(2\pi)^{r_2}\sqrt{3e}/4 = \begin{cases} 2\sqrt{3e} = 5.711\ldots & K \text{ is totally real} \\ \pi\sqrt{3e} = 8.971\ldots & \text{otherwise.} \end{cases}$$

Assume that $d_K \geqslant 1\,643\,032$ if K is totally real and that $d_K \geqslant 79\,507$ otherwise. Then, it holds that

$$h_K \mathrm{Reg}_K \geqslant \frac{\sqrt{d_K}}{\lambda_K(\log d_K + \mu_L/2)}. \tag{14}$$

3.4. *Pure cubic fields*

In the case of pure cubic fields $K = \mathbf{Q}(\sqrt[3]{m})$, $m \geqslant 2$ cube-free, we have an even better lower bound on class numbers (see also [**1**, Corollaire 4]).

COROLLARY 9. *Set*

$$\lambda = 2\pi\sqrt{\pi e/3\sqrt{3}} = 8.0549\ldots.$$

Let $K = \mathbf{Q}(\sqrt[3]{m}), m \geqslant 2$ cube-free, be a pure cubic field. Assume that $d_K \geqslant 36\,951$. Then

$$h_K \mathrm{Reg}_K \geqslant \frac{1}{\lambda}\sqrt{\frac{d_K}{\log(\sqrt{3}d_K)}}. \tag{15}$$

Proof. The normal closure $N = KL$ of K is an imaginary dihedral sextic field, where $L = \mathbf{Q}(\sqrt{-3})$. Since $\zeta_N(s) = (\zeta_K(s)/\zeta(s))^2\zeta_L(s)$ and $\zeta_L(s) < 0$ for $0 < s < 1$, we have $d_N = d_L d_K^2 = 3d_K^2$, $\zeta_N(1 - (2/\log d_N)) \leqslant 0$ and

$$\kappa_N = \kappa_L \kappa_K^2 = \frac{\pi}{3\sqrt{3}}\kappa_K^2 \geqslant \frac{2}{e\log d_N} = \frac{1}{e\log(\sqrt{3}d_K)}$$

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for $d_N = 3d_K^2 \geqslant \rho_6'(c_3)^6 = 40^6$ (recall that N is totally imaginary), and hence for $d_K \geqslant 36\,951$. Since

$$h_K \mathrm{Reg}_K = \frac{\sqrt{d_K}}{2\pi}\kappa_K,$$

we obtain (15). □

In particular, if $K = K_m = \mathbf{Q}(\sqrt[3]{m^3+1})$, $m \geqslant 1$ and m^3+1 cube-free, then $\epsilon_m = \omega_m^2 + m\omega_m + m^2 \leqslant 3\omega_m^2 \leqslant (\sqrt{3}d_{K_m})^{2/3}$ is a unit in K_m (where $\omega_m = \sqrt[3]{m^3+1} \geqslant m$), for $(\omega_m - m)\alpha_m = 1$, $\mathrm{Reg}_{K_m} \leqslant \log \epsilon_m \leqslant \frac{2}{3}\log(\sqrt{3}d_{K_m})$ and

$$h_{K_m} \geqslant \frac{1}{\lambda'}\sqrt{\frac{d_{K_m}}{\log^3(\sqrt{3}d_{K_m})}}$$

(for all $m \geqslant 1$ since the right-hand side of this lower bound is less than 1 for $d_{K_m} < 36\,951$), where $\lambda' = 2\lambda/3 = 5.369\,94\ldots$, and the exponent of the ideal class group of K_m is at most 3 if and only if $m \in \{1, 2, 3, 5\}$ (see [**20**, Theorem 9]).

3.5. *Real cyclic quartic fields*

We now give a more interesting example, a case in which K contains a quadratic subfield. Let K be a real cyclic quartic field and let L be its real quadratic subfield. Let f_K and U_K (respectively f_L and U_L) be the conductor and unit group of K (respectively L), where $\epsilon_L > 1$ is the fundamental unit of L. Let σ be a generator of the Galois group of K. Finally, let $U_K^* = \{\epsilon \in U_K;\ N_{K/L}(\epsilon) \in \{\pm 1\}\}$ denote the so-called group of *relative units* of K. If $\pm 1 \neq \epsilon \in U_K^*$, then $\epsilon^\sigma \in U_K^*$ and

$$\mathrm{Reg}(\epsilon_L, \epsilon, \epsilon^\sigma) = 2\mathrm{Reg}_L \mathrm{Reg}_\epsilon^*,$$

where

$$\mathrm{Reg}_\epsilon^* := \log^2|\epsilon| + \log^2|\epsilon^\sigma| > 0.$$

It is known that there exists some so-called *generating relative unit* $\epsilon_* \in U_K^*$ such that $\{-1, \epsilon_*, \epsilon_*^\sigma\}$ generate U_K^* (see [**5**, Satz 22]), and we set

$$\mathrm{Reg}_K^* := \mathrm{Reg}_{\epsilon^*}^* = \log^2|\epsilon_*| + \log^2|\epsilon_*^\sigma| > 0.$$

This does not depend on the choice of the generating relative unit ϵ_* and

$$\mathrm{Reg}(\epsilon_L, \epsilon_*, \epsilon_*^\sigma) = 2\mathrm{Reg}_L \mathrm{Reg}_K^* = Q_K \mathrm{Reg}_K \tag{16}$$

for some $Q_K \in \{1, 2\}$ (see [**5**, Satz 16]). Since K/L is ramified at at least one finite place of L, h_L divides h_K and $h_K^* := h_K/h_L$ is a positive integer that divides h_K, which we call the *relative class number* of K. According to (7) (applied to L and K) and (16), we have

$$h_K^* = \frac{Q_K f_K}{8\mathrm{Reg}_K^*}\frac{\kappa_K}{\kappa_L}. \tag{17}$$

Noticing that the lower bound (8) with $n = 4$, $c_K = 2$ and $\alpha_K = 1$ is better than the first lower bound (10) with $c_K = 2$, we see that this latter lower bound always holds true and we obtain the following corollary. (Use $d_K = f_L f_K^2 \leqslant f_K^3$ and $d_L = f_L \leqslant f_K$, and notice that $f_K \geqslant 4\,9890$ implies that $\rho_K = d_K^{1/4} = (f_L f_K^2)^{1/4} \geqslant (5f_K^2)^{1/4} \geqslant 334 = \max(\rho_4(c_3), \rho_4'(c_3)) = \max(\rho_K(c_K), \rho_K'(c_K))$).

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COROLLARY 10. *Let K be a real cyclic quartic field of conductor $f_K \geqslant 49\,890$. Let L denote its real quadratic subfield. Then*

$$h_K^* \mathrm{Reg}_K^* \geqslant \frac{Q_K \sqrt{d_K/d_L}}{2e(\log d_L + \mu_L)\log d_K} \geqslant \frac{f_K}{6e(\log f_K + \mu_L/2)^2}. \tag{18}$$

In particular, if $K = K_m$ is the simplest cyclic quartic field associated with the polynomial $P_m(x) = x^4 - mx^3 - 6x^2 + mx + 1$, $m \geqslant 1$, $m \neq 3$, of discriminant $d_m = 4\Delta_m^3$ (where the odd part of $\Delta_m = m^2 + 16$ is assumed to be square-free), then we have $\mathrm{Reg}_{K_m} \leqslant \frac{1}{4}\log^2 \Delta_m$ and we obtain $h_{K_m} > 1$ for $m \geqslant 382$, and $h_{K_m} = 1$ if and only if $m \in \{1, 2, 4, 5, 6, 8, 9, 10, 11, 15, 24\}$ (see [**27**], in which a more general problem is in fact solved). Notice that because of much weaker lower bounds for h_{K_m}, Lazarus could not solve in [**10**] and [**11**] this class number 1 problem for the simplest quartic fields.

3.6. *Non-normal quartic number fields containing an imaginary quadratic subfield*

Finally, we give an even more interesting example, a case in which K is not normal and contains a quadratic subfield. (See also [**17**, Theorem 2], which yields $h_K \mathrm{Reg}_K \geqslant \sqrt{d_K}/(136\log(d_K/16))$ for $d_K \geqslant 32 \cdot 10^9$, and [**21**, Corollary 15] (note the misprint there: it should read $h_K^- \geqslant \sqrt{d_K/d_{K^+}}/(12(\log(d_K/d_{K^+}) + 0.052)^2)$).

COROLLARY 11. *Let K/L be a quadratic extension of an imaginary quadratic number field L. Assume that the quartic extension $K/\mathbf{Q}$ is not normal and that $d_K \geqslant 12 \cdot 10^5$. Then*

$$h_K \mathrm{Reg}_K \geqslant \frac{(w_K/w_L)\sqrt{d_K/d_L}}{\lambda(\log(d_K/d_L) + \mu)^2} h_L, \tag{19}$$

where $\lambda = \pi\sqrt{e/27} = 0.996\,81\ldots$ and $\mu = 1.750\,46\ldots$. Moreover, if $\zeta_L(s) < 0$ for $s \in (0, 1)$ (for example if $L = \mathbf{Q}(\sqrt{-1})$) and $d_K \geqslant 2.1 \cdot 10^{12}$, then we have the better lower bound

$$h_K \mathrm{Reg}_K \geqslant \frac{w_K \sqrt{d_K}}{\lambda' \log d_K}, \tag{20}$$

where $\lambda' = 16\pi^2 e^{c_1/8}/c_1 = 480.362\ldots$, with $c_1 = 2(\sqrt{2} - 1)^2 = 0.343\,14\ldots$.

Proof. Let K' denote the conjugate of K. Then $N = KK'$ is an imaginary normal dihedral octic field. Let M be the bicyclic biquadratic subfield of N and let $L_0 = L$, L_1 and L_2 be the three quadratic subfields of M. Since K' is isomorphic to K and N/L is bicyclic biquadratic, we have $\zeta_N(s)/\zeta_M(s) = (\zeta_K(s)/\zeta_L(s))^2$. Hence $\kappa_N/\kappa_M = (\kappa_K/\kappa_L)^2$ and $d_N/d_M = (d_K/d_L)^2$. Now, there are two cases to consider.

(1) Assume that $\zeta_M(1 - (2/\log d_N)) > 0$. Then $\zeta_M(\beta) = 0$ for some $\beta \in [1 - (2/\log d_N), 1)$. Hence, $\zeta_{L_i}(\beta) = 0$ for some $i \in \{1, 2, 3\}$ and, by (5) and (6), it holds that

$$\kappa_M \leqslant \frac{1-\beta}{8}(\log d_{L_i}) \prod_{j=1}^{3} \frac{1}{2}(\log d_{L_j} + \mu_{L_j}) \leqslant \frac{1-\beta}{2^8 \cdot 3^3}(\log d_M + \mu')^3 \log d_N$$

(the geometric mean is less than or equal to the arithmetic mean, $d_M = d_{L_0} d_{L_1} d_{L_2}$ and $d_{L_i}^4 \leqslant d_N$), where $\mu' = \mu_{L_0} + \mu_{L_1} + \mu_{L_2} = 2(2 + \gamma - \log\pi) + (2 + \gamma - \log(4\pi)) =$

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$6+3\gamma-\log(4\pi^3)$ (for precisely one of the L_i is real), and

$$\kappa_N \geqslant \frac{1-\beta}{2e}$$

for $\rho_N \geqslant \rho_8(c_N)=\rho_8(c_3)=33$ (recall that N is totally imaginary), by (2). Hence,

$$\kappa_N/\kappa_M \geqslant \frac{2^7\cdot 3^3}{e(\log d_M+\mu')^3\log d_N}.$$

(2) Assume that $\zeta_M(1-(2/\log d_N))\leqslant 0$. Then $\zeta_N(1-(2/\log d_N))\leqslant 0$ for $\rho_N\geqslant \rho'_8(c_N)=\rho'_8(c_3)=29$ (recall that N is totally imaginary), for $\zeta_N(s)/\zeta_M(s)=(\zeta_K(s)/\zeta_L(s))^2$. Hence $\kappa_N\geqslant 2/(e\log d_N)$, by (3), and

$$\kappa_M \leqslant \prod_{j=1}^{3}\frac{1}{2}(\log d_{L_j}+\mu_{L_j})\leqslant \frac{1}{2^3\cdot 3^3}(\log d_M+\mu')^3,$$

by (5). Hence

$$\kappa_N/\kappa_M \geqslant \frac{2^4\cdot 3^3}{e(\log d_M+\mu')^3\log d_N}.$$

Since this latter lower bound is worse than the previous one, it always holds true. Using $\kappa_N/\kappa_M=(\kappa_K/\kappa_L)^2$, $d_M\leqslant\sqrt{d_N/d_M}=d_K/d_L$ and $d_N\leqslant(d_N/d_M)^2=(d_K/d_L)^4$, we finally obtain

$$\kappa_K/\kappa_L \geqslant \frac{\sqrt{2^2\cdot 3^3/e}}{(\log(d_K/d_L)+3\mu'/4)^2},$$

which in using (7) yields (19) (for $\mu:=3\mu'/4=3(6+3\gamma-\log(4\pi^3))/4=1.750\,46\ldots$, and $d_K\geqslant 1\,185\,921$ implies $\rho_N\geqslant\rho_K\geqslant\max(33,29)$).

Now, assume that $\zeta_L(s)<0$ for $0<s<1$. According to the factorization $\zeta_N(s)=(\zeta_K(s)/\zeta_L(s))^2\zeta_M(s)$, any real zero $\beta\in(0,1)$ of $\zeta_K(s)$ is at least a double zero of $\zeta_N(s)$ and hence is less than $1-(c_1/\log d_N)$ (see the beginning of the proof of Proposition 3). It follows that $\zeta_K(1-(c_1/\log d_N))\leqslant 0$. Now, d_K^2 divides d_N and d_N divides d_K^4 (by [**30**, Lemma 7], for $N=KK'$). Hence $1-(c_1/\log d_N)\in[1-(c_1/2\log d_K),1-(c_1/4\log d_K)]$ and $\rho_K\geqslant\rho'_4(c_1/2,c_1/4)=1192$ (recall that K is totally imaginary and use the second point of Proposition 15 below with $C=2\pi^2$ and $n=4$ to compute $\rho'_4(c_1/2,c_1/4)$) implies that

$$\kappa_K \geqslant \frac{c_1}{4e^{c_1/8}\log d_K},$$

by (3). We finally use (7) to obtain (20). □

4. *Proof of Theorem* 1

Write $n=r_1+2r_2\geqslant 2$, and set $A_K=\sqrt{d_K/4^{r_2}\pi^n}$ and

$$\Gamma_K(s)=\Gamma^{r_1}(s/2)\Gamma^{r_2}(s)=\left(\frac{2^{s-1}}{\sqrt{\pi}}\right)^{r_2}\Gamma^{r_1+r_2}(s/2)\Gamma^{r_2}((s+1)/2). \tag{21}$$

According to the proof of [**26**, Lemma 2], we have the following.

LEMMA 12. *Fix $\alpha>1$. Assume that $\zeta_K(\beta)\leqslant 0$ for some $\beta\in(0,1)$. Then*

$$\kappa_K \geqslant \frac{\beta(1-\beta)}{2\pi i}\int_{\Re(z)=\alpha}\frac{\Gamma_K(z)}{\Gamma_K(1)}A_K^{z-1}\left(\frac{1}{z-\beta}+\frac{1}{z-(1-\beta)}\right)dz.$$

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From now on, we assume that $\frac{1}{2} < \beta < 1$. We set

$$f_K(z) = z\frac{\Gamma_K(z)}{\Gamma_K(1)}(4^{r_2}\pi^n)^{(1-z)/2} \tag{22}$$

(hence $f_K(1) = 1$), and

$$M(\beta) = \sup_{\Re(z)=\frac{1}{2}} \Big|\frac{1}{z-\beta} + \frac{1}{z-(1-\beta)}\Big| = \sup_{-\infty<t<\infty} \frac{2|t|}{(\beta-\frac{1}{2})^2+t^2} = \frac{2}{2\beta-1}. \tag{23}$$

LEMMA 13. *It holds that* $|\Gamma(1+it)|^2 = \pi t/\sinh(\pi t)$. *Moreover, for* $0 < a \leqslant b$, *we have*

$$|\Gamma(a+it)| \leqslant \frac{\Gamma(a)}{\Gamma(b)}|\Gamma(b+it)|. \tag{24}$$

In particular,

$$I_K := \frac{1}{2\pi}\int_{-\infty}^{\infty} |\Gamma_K(\frac{1}{2}+it)|/\Gamma_K(1)\,dt \leqslant \frac{\pi^{r_2/2}\Gamma^{r_1}(1/4)}{\pi^{r_1/2}} J_n$$

and

$$A_K^{-1/2} I_K = (4^{r_2}\pi^n/\rho_K^n)^{1/4} I_K \leqslant \left(\frac{C}{\rho_K}\right)^{n/4} J_n,$$

where $C = \Gamma^4(1/4)/\pi = 55.001\ldots$ *in general and* $C = 2\pi^2 = 19.739\,20\ldots$ *if* K *is totally imaginary, and*

$$J_n = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{\pi t/2}{\sinh(\pi t/2)}\right)^{n/2} dt = \frac{2}{\pi^2}\int_0^{\infty} \left(\frac{t}{\sinh t}\right)^{n/2} dt$$

decreases towards zero as $n \geqslant 1$ *goes to infinity. Moreover,* $J_2 = 1/2$ *and* $J_4 = 1/3$.

Proof. For the first assertion, use the infinite products of the Γ and sin functions. For (24), use the infinite product of the Γ-function to obtain

$$\frac{|\Gamma(a+it)|^2\Gamma^2(b)}{|\Gamma(b+it)|^2\Gamma^2(a)} = \prod_{n\geqslant 0} \frac{(n+a)^2}{(n+a)^2+t^2} \Big/ \frac{(n+b)^2}{(n+b)^2+t^2} \leqslant 1,$$

for $x \longmapsto x/(x+t^2)$ decreases for $x > 0$. According to (21) and (24) (with $b = 1$, and $a = 1/4$ or $a = 3/4$), we have

$$\begin{aligned}\left|\Gamma_K\left(\frac{1}{2}+it\right)\right| &= \frac{1}{(2\pi)^{r_2/2}}\left|\Gamma\left(\frac{1}{4}+i\frac{t}{2}\right)\right|^{r_1+r_2}\left|\Gamma\left(\frac{3}{4}+i\frac{t}{2}\right)\right|^{r_2} \\ &\leqslant \frac{\Gamma^{r_1+r_2}(1/4)\Gamma^{r_2}(3/4)}{(2\pi)^{r_2/2}}\left|\Gamma\left(1+i\frac{t}{2}\right)\right|^{n} \\ &= \pi^{r_2/2}\Gamma^{r_1}(1/4)\left(\frac{\pi t/2}{\sinh(\pi t/2)}\right)^{n/2}\end{aligned}$$

(by (21) for $s = 1/2$). Using $\Gamma_K(1) = \pi^{r_1/2}$, we obtain the desired bound on I_K, and

$$A_K^{-1/2} I_K \leqslant \left(\Gamma^4(1/4)/\pi\right)^{r_1/4}\left(2\pi^2\right)^{2r_2/4} J_n/\rho_K^{n/4} \leqslant \left(\frac{C}{\rho_K}\right)^{n/4} J_n,$$

where $C = \max(\Gamma^4(1/4)/\pi, 2\pi^2) = \Gamma^4(1/4)/\pi = 55.001\ldots$ in general and $C = 2\pi^2 = 19.739\ldots$ if K is totally imaginary.

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Finally, noticing that $(-e^{-2t}/(1-e^{-2t}))' = 2e^{-2t}/(1-e^{-2t})^2$, we have

$$\frac{\pi^2}{2}J_4 = \int_0^\infty \frac{4t^2e^{-2t}}{(1-e^{-2t})^2}\,dt = \int_0^\infty \frac{4te^{-2t}}{1-e^{-2t}}\,dt = 4\sum_{n\geqslant 1}\int_0^\infty te^{-2nt}\,dt = \sum_{n\geqslant 1}\frac{1}{n^2}$$

and $J_4 = 1/3$. In the same way,

$$\frac{\pi^2}{2}J_2 = \int_0^\infty \frac{te^{-t}}{1-e^{-2t}}\,dt = \sum_{n\geqslant 0}\int_0^\infty 2te^{-(2n+1)t}\,dt = 2\sum_{n\geqslant 0}\frac{1}{(2n+1)^2} = \frac{\pi^2}{4}$$

and $J_2 = 1/2$. □

In Lemma 12, we shift the vertical line of integration $\Re(z) = \alpha > 1$ leftwards to the vertical line $\Re(z) = 1/2$. We pick up only one residue, at $z = \beta$ (for we assume that $1/2 < \beta < 1$), and using (23) and Lemma 13, we obtain

$$\begin{aligned}\kappa_K &\geqslant \beta(1-\beta)\left(\frac{\Gamma_K(\beta)}{\Gamma_K(1)}A_K^{\beta-1} - M(\beta)A_K^{-1/2}I_K\right)\\ &= (1-\beta)d_K^{(\beta-1)/2}\left(f_K(\beta) - \frac{2\beta d_K^{(1-\beta)/2}}{2\beta-1}A_K^{-1/2}I_K\right)\\ &\geqslant (1-\beta)d_K^{(\beta-1)/2}\left(f_K(\beta) - \frac{2\beta d_K^{(1-\beta)/2}}{2\beta-1}\left(\frac{C}{\rho_K}\right)^{n/4}J_n\right).\end{aligned} \tag{25}$$

LEMMA 14. *Let $f_K(z)$ be as in (22). Assume that $n \geqslant 2$. Then*

$$f_K'(1) = 1 - \frac{r_1}{2}(\gamma + \log(4\pi)) - r_2(\gamma + \log(2\pi)) \leqslant 1 - n\frac{\gamma + \log(2\pi)}{2} \leqslant 0,$$

and in the range $0 < \beta \leqslant 1$, it holds that

$$f_K(\beta) \geqslant 1 - (1-\beta)f_K'(1) \geqslant 1 + n(1-\beta)\left(\frac{\gamma+\log(2\pi)}{2} - \frac{1}{n}\right) \geqslant 1.$$

Proof. Use the infinite product of $\Gamma(s)$ to obtain $(\Gamma'/\Gamma)(1) = -\gamma$ and $(\Gamma'/\Gamma)(1/2) = -(\gamma + \log 4)$ and notice that $f_K'(1) = (f_K'/f_K)(1)$. Since $f_K(\beta)$ is positive and log-convex in the range $\beta > 0$ (use the infinite product of the Γ-function), $f_K(s)$ is convex in this range and the last assertion follows. □

Using (25) and Lemma 14, and noticing that $1/2 < 1 - (c/\log d_K) \leqslant \beta < 1$ implies that

$$\frac{\beta d_K^{(1-\beta)/2}}{2\beta-1} \leqslant \frac{n\log\rho_K - c}{n\log\rho_K - 2c}e^{c/2}$$

and $1 - (c/\log d_K) \leqslant \beta \leqslant 1 - (c'/\log d_K)$ implies that

$$\frac{\beta}{2\beta-1}\frac{d_K^{(1-\beta)/2}}{1-\beta} \leqslant n\frac{n\log\rho_K - c}{n\log\rho_K - 2c}\frac{e^{c'/2}}{c'},$$

we finally obtain the following proposition. (Compare with [**26**, Proposition 4]).

PROPOSITION 15. *Fix c and c' with $0 < c' \leqslant c \leqslant 2$. Let K be a number field of degree $n \geqslant 2$. Assume that $d_K \geqslant e^4$. Set $C = \Gamma^4(1/4)/\pi = 55.001\,48\ldots$, and $C = 2\pi^2 = 19.739\,20\ldots$ if K is totally imaginary.*

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(1) *Assume that* $\zeta_K(\beta) \leqslant 0$ *for some* $\beta \in [1-(c/\log d_K), 1)$. *Then*

$$\kappa_K \geqslant (1-\beta) d_K^{(\beta-1)/2} \bigl(1 - S_K(c)\bigr), \tag{26}$$

where

$$S_K(c) = 2e^{c/2} J_n \frac{n\log\rho_K - c}{n\log\rho_K - 2c} \left(\frac{C}{\rho_K}\right)^{n/4}$$

decreases towards 0 *as* ρ_K *goes to infinity, and hence is less than or equal to* $1/2$ *for* ρ_K *large enough.*

(2) *Assume that* $\zeta_K(\beta) \leqslant 0$ *for some* $\beta \in [1-(c/\log d_K), 1-(c'/\log d_K)]$. *Then*

$$\kappa_K \geqslant (1-\beta) d_K^{(\beta-1)/2} \bigl(1 + n(1-\beta) S_K(c,c')\bigr) \tag{27}$$

where

$$S_K(c,c') = \frac{\gamma + \log(2\pi)}{2} - \frac{1}{n} - \frac{2e^{c'/2}}{c'} J_n \frac{n\log\rho_K - c}{n\log\rho_K - 2c} \left(\frac{C}{\rho_K}\right)^{n/4} \log\rho_K$$

increases with ρ_K *in the range* $\rho_K \geqslant e^{4/n}$, *that is, in the range* $d_K \geqslant e^4$, *and hence is greater than or equal to* 0 *for* ρ_K *large enough.*

To complete the proof of Theorem 1, we use the following numerical values: $J_2 = 1/2$, $J_3 = 0.392\,927\ldots$, $J_4 = 1/3$, $J_5 = 0.294\,312\ldots$, $J_6 = 0.266\,299\ldots$, $J_7 = 0.244\,960\ldots$ and $J_8 = 0.228\,017\ldots$.

References

1. P. Barrucand and S. Louboutin, 'Majoration et minoration du nombre de classes d'idéaux des corps réels purs de degré premier', *Bull. London Math. Soc.* 25 (1993) 533–540.

2. H. Davenport, *Multiplicative number theory*, 3rd edn, Graduate Texts in Mathematics 74 (Springer, New York, 2000) Chapter 21: Siegel's Theorem.

3. R. Foote and V. K. Murty, 'Zeros and poles of Artin *L*-series', *Math. Proc. Cambridge Philos. Soc.* 105 (1989) 5–11.

4. D. M. Goldfeld, 'A simple proof of Siegel's theorem', *Proc. Nat. Acad. Sci. U.S.A.* 71 (1974) 1055.

5. H. Hasse, 'Arithmetische Bestimmung von Grundeinheit und Klassenzahl in zyklischen kubischen und biquadratischen Zahlkörpern', Mathematische Abhandlungen 3 (Walter de Gruyter, Berlin, 1975) 289–379.

6. J. Hoffstein, 'Some analytic bounds for zeta functions and class numbers', *Invent. Math.* 55 (1979) 37–47.

7. O. Kihel, 'Groupe des unités pour des extensions diédrales complexes de degré 10 sur $\mathbf{Q}$', *J. Théor. Nombres Bordeaux* 13 (2001) 469–482.

8. Y. Kishi, 'A family of cyclic cubic polynomials whose roots are systems of fundamental units', *J. Number Theory* 102 (2003) 90–106, erratum, *J. Number Theory* 103 (2003) 132–133.

9. S. Lang, *Algebraic number theory*, 2nd edn, Graduate Texts in Mathematics 110 (Springer, New York, 1994).

10. A. J. Lazarus, 'Class numbers of simplest quartic fields', *Number theory* (*Banff, AB,* 1988) (Walter de Gruyter, Berlin, 1990) 313–323.

11. A. J. Lazarus, 'On the class number and unit index of simplest quartic fields', *Nagoya Math. J.* 121 (1991) 1–13.

12. M. H. Le, 'A lower bound for the class numbers of abelian algebraic number fields with odd degree', *Proc. Amer. Math. Soc.* 123 (1995) 1347–1350.

13. F. Lemmermeyer, S. Louboutin and R. Okazaki, 'The class number one problem for some non-abelian normal CM-fields of degree 24', *J. Théor. Nombres Bordeaux* 11 (1999) 387–406.

14. G. Lettl, 'A lower bound for the class number of certain cubic number fields', *Math. Comp.* 46 (1986) 659–666.

15. S. Louboutin, 'Majorations explicites de $|L(1,\chi)|$', *C. R. Acad. Sci. Paris Sér.* I *Math.* 316 (1993) 11–14.

16. S. Louboutin, 'Lower bounds for relative class numbers of CM-fields', *Proc. Amer. Math. Soc.* 120 (1994) 425–434.

https://doi.org/10.1112/S0024610705006654 Published online by Cambridge University Press

17. S. Louboutin, 'The exponent 2 class-group problem for non Galois over **Q** quartic fields', *J. Number Theory* 49 (1994) 133–141.
18. S. Louboutin, 'Class number problems for cubic number fields', *Nagoya Math. J.* 138 (1995) 199–208.
19. S. Louboutin, 'Majorations explicites de $|L(1,\chi)|$ (suite)', *C. R. Acad. Sci. Paris Sér.* I *Math.* 323 (1996) 443–446.
20. S. Louboutin, 'Class group problems for cubic number fields', *Japan. J. Math.* 23 (1997) 365–378.
21. S. Louboutin, 'The class number one problem for the non-abelian normal CM-fields of degree 16', *Acta Arith.* 82 (1997) 173–196.
22. S. Louboutin, 'Explicit bounds for residues of Dedekind zeta functions, values of L-functions at $s = 1$, and relative class numbers', *J. Number Theory* 85 (2000) 263–282.
23. S. Louboutin, 'Explicit upper bounds for residues of Dedekind zeta functions and values of L-functions at $s = 1$, and explicit lower bounds for relative class numbers of CM-fields', *Canad. J. Math.* 53 (2001) 1194–1222.
24. S. Louboutin, 'Class number and class group problems for some non-normal totally real cubic number fields', *Manuscripta Math.* 106 (2001) 411–427.
25. S. Louboutin, 'The exponent three class group problem for some real cyclic cubic number fields', *Proc. Amer. Math. Soc.* 130 (2002) 353–361.
26. S. Louboutin, 'Explicit lower bounds for residues at $s = 1$ of Dedekind zeta functions and relative class numbers of CM-fields', *Trans. Amer. Math. Soc.* 355 (2003) 3079–3098.
27. S. Louboutin, 'The simplest quartic fields with ideal class groups of exponents $\leqslant 2$', *J. Math. Soc. Japan* 56 (2004) 717–727.
28. S. Louboutin, 'Lower bounds for relative class numbers of imaginary abelian number fields and CM-fields', submitted for publication.
29. O. Ramaré, 'Approximate formulae for $L(1,\chi)$', *Acta Arith.* 100 (2001) 245–256.
30. H. M. Stark, 'Some effective cases of the Brauer–Siegel Theorem', *Invent. Math.* 23 (1974) 135–152.

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