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RESEARCH ARTICLE

Asymptotic properties of wavelet estimators in a semi-parametric EV models with ANA errors

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In this paper, when the errors in the semi-parametric errors-in-variables model are asymptotic negatively associated (or ρ^-, for short) random variables, the estimators of parameter, non-parameter, and error variances in the model are $\widehat{\beta}_n$, $\widehat{g}_n(t)$, and $\widehat{\sigma}_n^2$, respectively, by using wavelet smoothing and least square method. Under some general assumptions, we also establish some results on the strong consistency of the estimators. Furthermore, simulations are conducted to assess the finite sample behavior of the estimators and confirm the validity of the theoretical results.

1. Introduction

Consider the following semi-parametric errors-in-variables (EV) model

$$\begin{cases} y_i = x_i\beta + g\left(t_i\right) + \varepsilon_i, \\ X_i = x_i + \mu_i, \end{cases} \tag{1}$$

where (t_i, X_i, y_i) for the sample observations, $\beta \in \mathbb{R}$ for the unknown parameters, $g(\cdot)$ is unknown functions defined on compact set $[0,1]$, random errors $\{\varepsilon_i\}$ are asymptotically negatively associated (ANA) random variables with $E\varepsilon_i = 0$ and $E\varepsilon_i^2 = 1$, and the variables $\{x_i\}$ are measured with errors and are not directly observable. However, $\{x_i\}$ are observed through $X_i = x_i + \mu_i$, where $\{\mu_i\}$ are the measurement errors with $E\mu_i = 0$, $E\mu_i^2 = \sigma_\mu^2 < \infty$. Suppose $\{\mu_i\}$ are independent and identically distributed random variables, which are assumed to be independent of $\{\varepsilon_i\}$.

In various applications, errors in the measurement of covariates are frequently encountered. For instance, existing literature demonstrates that covariates such as blood pressure, urinary sodium chloride levels, and exposure to pollutants are often subject to measurement errors. These errors can pose challenges and complications in the statistical analysis process. Therefore, EV models are more practical than ordinary regression models. In recent years, there has been a proliferation of research into semi-parametric EV models. In the case of independence, Cui [5] proved the asymptotic normality of M-estimators in the EV model; Liu and Chen [14] demonstrated the consistency for the least square (LS) estimators $\widehat{\beta}_n$ and $\widehat{g}_n(t)$ and concluded that strong convergence and weak convergence are equivalent; Miao *et al.* [15] and Miao *et al.* [16] established the central limit theorem and some limit behaviors for the LS estimators in simple linear EV regression models, respectively. Nevertheless, the assumption

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of independence regarding the errors $\{\varepsilon_i\}$ in model (1) is not always applicable in practice, particularly in the context of sequentially collected economic data, which frequently demonstrates evident dependencies in the errors. Scholars have expanded their horizons to research on EV models under dependent errors. For example, Zhang *et al.* [24] established the asymptotic normality of estimators in model (1) under assumptions of stationary strong mixing and independence; Hu *et al.* [9] obtained asymptotic representation of parametric estimators as well as asymptotic distributions and weak convergence rates of parametric and nonparametric estimators under long-memory errors using wavelet method; Xi *et al.* [22] obtained the asymptotic properties of LS estimators and weighted least squares (WLS) estimators under both known and unknown variances based on α-mixing errors, respectively; Wu *et al.* [21] investigated the strong consistency and convergence rates of the LS estimators and WLS estimators with widely orthant dependent samples, and so on.

As pointed out by Zhang and Wang [26], ANA random variables include ρ^*-mixing random variables and NA random variables as special cases. On the one hand, NA sequence is an important dependent series that has been widely used in reliability theory, probabilistic process, stochastic process, multivariate statistics, and so on. On the other hand, ρ^*-mixing sequence is a more common mixed form of time series, and it has a wide range of applications in risk management, economic decision-making, and variable forecasting. They also gave an example to show that ANA sequences are not necessarily NA or ρ^*-mixing. Hence, the study of the convergence properties for ANA random variables is of more interest.

Now, let us recall some dependence structures. The concept of ANA random variables was introduced by Zhang and Wang [26].

Definition 1.1. *A sequence* $\{X_n, n \geq 1\}$ *of random variables is said to be ANA (or* ρ^-*, for short) if*

$$\rho^-(s) = \sup\{\rho^-(S,T) : S, T \subset \mathbb{N}, dist(S,T) \geq s\} \to 0, \quad as \quad s \to \infty,$$

where

$$\rho^-(S,T) = 0 \vee \left\{ \frac{Cov\left(f_1(X_i, i \in S), f_2(X_j, j \in T)\right)}{\sqrt{Var(f_1(X_i, i \in S)) \cdot Var(f_2(X_j, j \in T))}} : f_1, f_2 \in \mathcal{C} \right\},$$

and $\mathcal{C}$ *is the set of nondecreasing functions.*

Since Zhang and Wang [26] proposed the concept of ANA random variables, many results on ANA random variables have been established. For the probabilistic limit theory of ANA random variables, one can refer to [26], [18], and [23] for some moment inequalities and some limiting behavior; [25] for some central limit theorems; [12] for the Hájek–Rényi inequality and the strong law of large numbers; [20] for almost sure convergence of ANA random variables; and [4] for complete and complete moment convergence. Based on the probability inequality of ANA random variables, researchers have further extended its application to various statistical models. For example, Tang *et al.* [17] derived asymptotic normality of wavelet estimator for nonparametric fixed design regression model with ANA errors, Zhang and Liu [27] studied the consistency of estimators in heteroscedasticity partially linear model with ANA errors. So far, there is no literature studying the asymptotic properties of model (1) under ANA random error. This research contributes to the existing body of knowledge by extending the scope of ANA random variable studies.

As is well known, following the introduction of the wavelet method by [1], numerous researchers have begun to explore its applications in the field of statistics. For example, Antoniadis *et al.* [1] and Donoho *et al.* [7] estimated regression function and density function with wavelet technique, respectively; Chai and Liu [3] discussed the weak convergence, strong convergence, and convergence rate of wavelet estimators in nonparametric regression models with ρ-mixing sequence; Liu *et al.* [13] obtained consistency and uniform asymptotic normality of wavelet estimators in regression models with associated samples;

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Zhou and Lin [28] studied the asymptotic properties of wavelet estimator in a semi-parametric regression model with the mixing dependent errors; Ding and Chen [6] studied the asymptotic normality of wavelet estimators of slope parameters and nonparametric components in semi-parametric regression models with known error variance and so on.

It can be seen from the above literature that the wavelet method has a great adaptability to the local features of unknown curves. In the event that the errors in the semi-parametric EV model are ANA sequences, we are prepared to employ wavelet smoothing and LS methods to estimate the parameters, non-parameters, and variance functions in the model, respectively. In addition, under certain standard assumptions, it has been demonstrated that they exhibit strong consistency.

The work is organized as follows. In Section 2, we introduce the estimators of β, $g(t)$, and σ^2 in the semi-parametric EV model. It also describes some assumptions and remarks and presents the main results. Some numerical simulations are provided in Section 3. Proofs of the main results and the preliminary lemmas are provided in Sections 4 and A, respectively.

Throughout this article, the symbol C represents some positive constant which may vary in different places. The symbol $\xrightarrow{a.s.}$ represent almost sure convergence. Let $I(B)$ be the indicator function of the set B and $\lfloor x \rfloor$ denote the integer part of x. Denote $x^+ = xI(x \geq 0)$ and $x^- = -xI(x < 0)$. $a_n = O(b_n)$ means $|a_n| \leq C|b_n|$, while $a_n = o(b_n)$ means $a_n/b_n \to 0$. All limits are taken as the sample size n tends to ∞, unless it is specially mentioned.

2. Assumptions and main results

2.1. Construction of estimators

In order to avoid symbol abuse, we adopt the symbols and definitions set forth by [1] for the introduction of the wavelet estimator.

Definition 2.1. *A function space S_q ($q \in \mathbb{R}$) is said to be Sobolev space with order* q, *if* $\mathrm{S_q}$ *is continuous differentiable for* $\mathrm{S_q}$*-times function space, and the function in* $\mathrm{S_q}$ *rapidly decreasing at infinity, that is for $h \in S_q$, there exists a constant $C_{pq} > 0$ such that*

$$\left|h^{(k)}(t)\right| \leq C_{pk}(1+|t|)^{-p}, \quad k = 0, 1, \ldots, q, \quad p \in \mathbb{N}, \quad t \in \mathbb{R}.$$

Definition 2.2. *A function space H^ν ($\nu \in \mathbb{R}$) is said to be Sobolev space with order ν, that is, if $h \in H^\nu$ then*

$$\int |\hat{h}(\omega)|^2 (1+\omega^2)^\nu d\omega < \infty,$$

where $\hat{h}$ is the Fourier transform of h.

Let $\varphi(x)$ be a given scaling function in the Schwarz space with order l. A multiresolution analysis of $L^2(\mathbb{R})$ consists of an increasing sequence of the closed subspace $\{V_m,\ m \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the integer set and $L^2(\mathbb{R})$ is a set of square integral functions over the real line. Since $\{\varphi(x-k),\ k \in \mathbb{Z}\}$ is an orthogonal family of $L^2(\mathbb{R})$ and V_0 is the subspace spanned, if we define

$$\varphi_{mk}(x) = 2^{m/2}\varphi(2^m x - k), \quad k \in \mathbb{Z},$$

then $\{\varphi_{0k},\ k \in \mathbb{Z}\}$ is an orthogonal basis of V_0 , and $\{\varphi_{mk},\ k \in \mathbb{Z}\}$ is an orthogonal basis of V_m. The associated integral kernel of V_m is given by

$$E_m(t,s) = 2^m E_0\left(2^m t, 2^m s\right) = 2^m \sum_{k \in \mathbb{Z}} \varphi\left(2^m t - k\right) \varphi\left(2^m s - k\right).$$

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The following wavelet method similar to [9] is used to estimate β, $g(t)$, and σ^2. From model (1), it can be obtained

$$y_i = X_i\beta + g(t_i) + \varepsilon_i - \mu_i\beta. \tag{2}$$

Let $\xi_i := \varepsilon_i - \mu_i\beta$. Then the model (2) becomes

$$y_i = X_i\beta + g(t_i) + \xi_i. \tag{3}$$

Denote

$$x_i = f(t_i) + \eta_i, \quad i = 1, 2, \ldots, n, \tag{4}$$

where $f(\cdot)$ is a function defined on interval $[0, 1]$, $\{\eta_i\}$ is independent and identically distributed with $E\eta_i = 0$ and $\mathrm{Var}(\eta_i) = \sigma_\eta^2 < \infty$. Moreover, $\{\eta_i\}$ are independent of $\{(\varepsilon_i, \mu_i)\}$.

Let $A_i = [s_{i-1}, s_i)$ denote intervals that partition $[0, 1]$ with $t_i \in A_i$ for $1 \le i \le n$. First, we assume that β is known and that $g(t_i) = E(y_i - X_i\beta)$ can be obtained from $E\xi_i = 0$. Therefore, the estimate of $g(\cdot)$ is defined as

$$\widehat{g}_0(t, \beta) = \sum_{i=1}^{n} (y_i - X_i\beta) \int_{A_i} E_m(t, s)\mathrm{d}s. \tag{5}$$

On this basis, the estimator of β can be defined as

$$\widehat{\beta}_n = \arg\min_{\beta} \sum_{i=1}^{n} \left[(y_i - X_i\beta - \widehat{g}_0(t, \beta))^2 - \sigma_\mu^2\beta^2 \right] = \left(\widetilde{X}^T\widetilde{X} - n\sigma_\mu^2 \right)^{-1} \widetilde{X}^T\widetilde{Y}, \tag{6}$$

where $X = (X_1, \ldots, X_n)^T$, $Y = (y_1, \ldots, y_n)^T$, $S = (S_{ij})_{n\times n}$, $S_{ij} = \int_{A_j} E_m(t_i, s)\,\mathrm{d}s$, $\widetilde{X} = (I - S)X$, $\widetilde{Y} = (I - S)Y$. Therefore, the estimator of $g(\cdot)$ is defined as

$$\widehat{g}_n(t) = \widehat{g}_0\left(t, \widehat{\beta}_n\right) = \sum_{i=1}^{n} \left(y_i - X_i\widehat{\beta}_n \right) \int_{A_i} E_m(t, s)\mathrm{d}s. \tag{7}$$

Since $\sigma^2 = \mathrm{E}(y_i - x_i\beta - g(t_i))^2$, using the method in [2], the estimator of σ^2 can be defined as

$$\widehat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^{n} \left(\widetilde{y}_i - \widetilde{X}_i\widehat{\beta}_n \right)^2 - \sigma_\mu^2\widehat{\beta}_n^2. \tag{8}$$

Remark 2.3. If σ_μ^2 is unknown, we can estimate σ_μ^2 by repeatedly measuring X. The specific method can be found in [2].

2.2. Assumptions

Now we list the following assumptions.

(**H1**) (i) Functions $g(\cdot)$ and $f(\cdot)$ satisfy the Lipschitz condition of order $\gamma > 0$;

(ii) $g(\cdot) \in H^\alpha$ and $f(\cdot) \in H^\alpha$, $\alpha > \frac{1}{2}$.

(**H2**) $\varphi(x)$ belongs to S_l, which is a Schwartz space for $l \ge \alpha$. $\varphi(x)$ is a Lipschitz function of order 1 and have compact support, in addition to $|\hat{\varphi}(\varrho) - 1| = O(\varrho)$ as $\varrho \to 0$, where $\hat{\varphi}$ is the Fourier transform of φ.

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(**H3**) (i) $\max\limits_{1\le i\le n}|s_i - s_{i-1}| = O\left(n^{-1}\right)$;
(ii) $\frac{2^m}{n} \to 0$ and $\frac{2^{3m}}{n} \to \infty$ as $n \to \infty$.

Remark 2.4. Assumptions (**H1**)–(**H3**) represent standard conditions for the asymptotic normality of wavelet estimation. Additional insights can be found in [1], [13], [28], [17], [6] among others. Thus, the conditions outlined in this paper are deemed appropriate and rational.

2.3. Main results

Our main results can be given below.

Theorem 2.5. *Suppose that assumptions* (**H1**)–(**H3**) *hold. Let* $\{\varepsilon_i,\ i \ge 1\}$ *be a sequence of identically distributed ANA random variables with* $E\varepsilon_i = 0$ *and* $E|\varepsilon_i|^p < \infty$ *for some* $\mathrm{p} > 2$. *Assume that* $\rho^-(N) \le r$ *for* $N \ge 1$ *and* $0 \le r < (\frac{1}{6p})^{p/2}$. *Then*
(i) *as* $n \to \infty$,

$$\widehat{\beta}_n \xrightarrow{a.s.} \beta;$$

(ii) *for any* $t \in [0,1]$,

$$\widehat{g}_n(t) \xrightarrow{a.s.} g(t).$$

Theorem 2.6. *Suppose that assumptions* (**H1**)–(**H3**) *hold. Let* $\{\varepsilon_i,\ i \ge 1\}$ *be a sequence of identically distributed ANA random variables with* $E\varepsilon_i = 0$ *and* $E|\varepsilon_i|^{2p} < \infty$ *for some* $\mathrm{p} > 2$. *Assume that* $\rho^-(N) \le r$ *for* $N \ge 1$ *and* $0 \le r < (\frac{1}{6p})^{p/2}$. *Then*

$$\widehat{\sigma}_n^2 \xrightarrow{a.s.} \sigma^2.$$

Remark 2.7. The above theorems show that the estimators $\widehat{\beta}_n$, $\widehat{g}_n(t)$, and $\widehat{\sigma}_n^2$ have strong consistency. Compared with the corresponding ones in [19], we extend the model errors from NA sequences to ANA sequences. It is worth noting that, to the best of our knowledge, there is no literature on the strong consistency for the estimators of the parametric, nonparametric, and error variances in model (1) based on ANA random errors.

3. Simulation

In this section, we carry out a simulation to study the finite sample performance of the proposed estimators. We will carry out a numerical simulation to study the consistency and asymptotic normality of wavelet estimation for $\widehat{\beta}_n$, $\widehat{g}_n(t)$, $\widehat{\sigma}_n^2$, respectively.

Observations are generated from

$$\begin{cases} Y_i = x_i\beta + g\left(t_i\right) + \varepsilon_i, \\ X_i = x_i + \mu_i, \end{cases} \tag{9}$$

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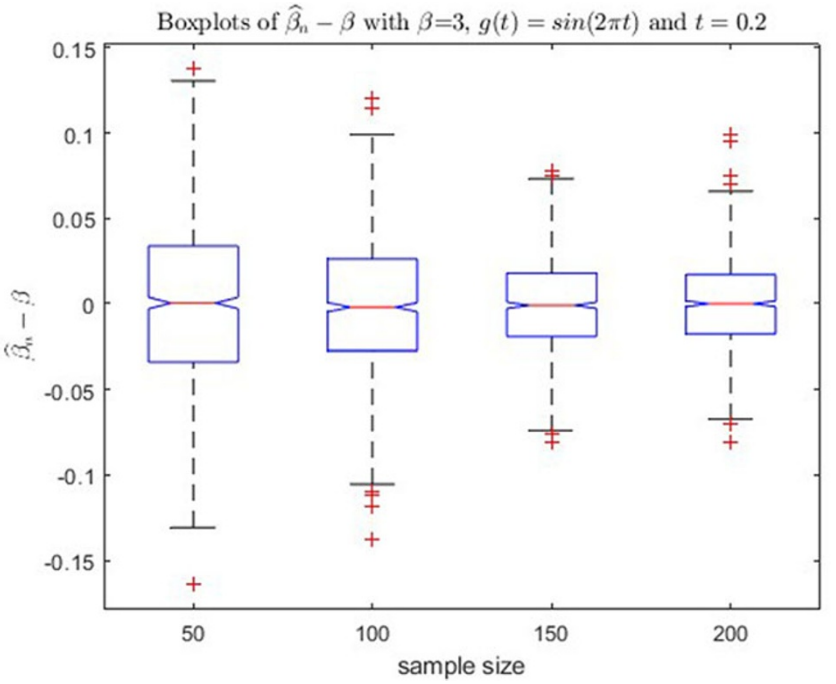


Figure 1. *Boxplots of* $\widehat{\beta}_n - \beta$ *with* $\beta = 3$, $g(t) = \sin(2\pi t)$, *and* $t = 0.2$.

where $\beta = 3$, $g(t) = \sin(2\pi t)$, $t_i = i/n$, $\mu_i \overset{i.i.d.}{\sim} N(0, 0.1^2)$, and $x_i = 2t_i^2 + \eta_i$ with $\eta_i \overset{i.i.d.}{\sim} U(-1, 1)$ for $1 \le i \le n$. For any $n \ge 3$, assume $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \sim N_n(\mathbf{0}, \mathbf{\Sigma})$, where $\mathbf{0}$ is a zero vector and

$$\mathbf{\Sigma} = \begin{pmatrix} 1+\theta^2 & -\theta & 0 & \cdots & 0 & 0 & 0 \\ -\theta & 1+\theta^2 & -\theta & \cdots & 0 & 0 & 0 \\ 0 & -\theta & 1+\theta^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\theta^2 & -\theta & 0 \\ 0 & 0 & 0 & \cdots & -\theta & 1+\theta^2 & -\theta \\ 0 & 0 & 0 & \cdots & 0 & -\theta & 1+\theta^2 \end{pmatrix}_{n\times n}, \quad 0 < \theta < 1.$$

According to [11], it is obvious that $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ is an NA vector for each $n \ge 5$ with finite moment of any order, and thus is an ANA vector with $\rho^-(1) = 0$. In order to verify the consistency of the estimators, we choose the scale function $\varphi(x) = I(0 \le x \le 1)$. Set the sample sizes n as $n = 50,\ 100,\ 150$, and 200. For a given sample size n, taking $2^m = (n \log n)^{1/3}$. We compute the values of $\widehat{\beta}_n - \beta$, $\widehat{g}_n(t) - g(t)$, and $\widehat{\sigma}_n^2 - \sigma^2$ for 1,000 times under $t = 0.2$ and 0.5. The boxplots of the estimators under different sample sizes are shown in Figures 1–6. The corresponding values of bias, SD, and RMSE for $\widehat{\beta}_n$, $\widehat{g}_n(t)$, and $\widehat{\sigma}_n^2$ are listed in Table 1.

Figures 1–6 show that $\widehat{\beta}_n - \beta$, $\widehat{g}_n(t) - g(t)$, and $\widehat{\sigma}_n^2 - \sigma^2$, regardless of $t = 0.2$ or 0.5, fluctuate around zero and the variation ranges decrease substantially as the increasing of sample size n. From Table 1, we can see that the estimators of β, $g(t)$, and σ^2 have small SD and RMSE, and they decrease as the increasing of sample size n. It demonstrates that $\widehat{\beta}_n$, $\widehat{g}_n(t)$, and $\widehat{\sigma}_n^2$ are close to the true values as the increasing of sample size n. Hence, numerical results here are sufficient to verify the consistency of $\widehat{\beta}_n$, $\widehat{g}_n(t)$, and $\widehat{\sigma}_n^2$ based on finite samples.

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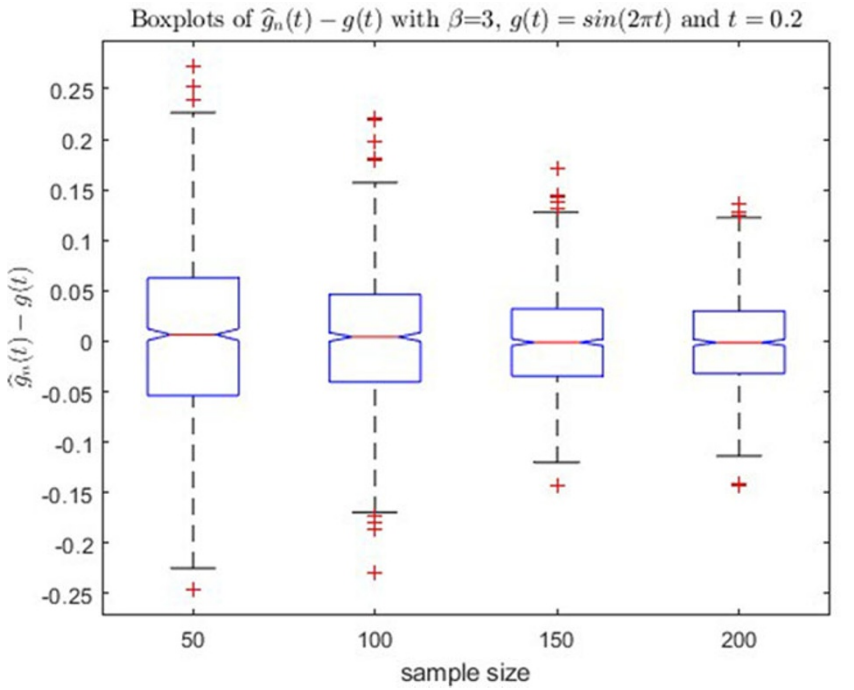


Figure 2. *Boxplots of* $\widehat{g}_n(t) - g(t)$ *with* $\beta = 3$, $g(t) = \sin(2\pi t)$, *and* $t = 0.2$.

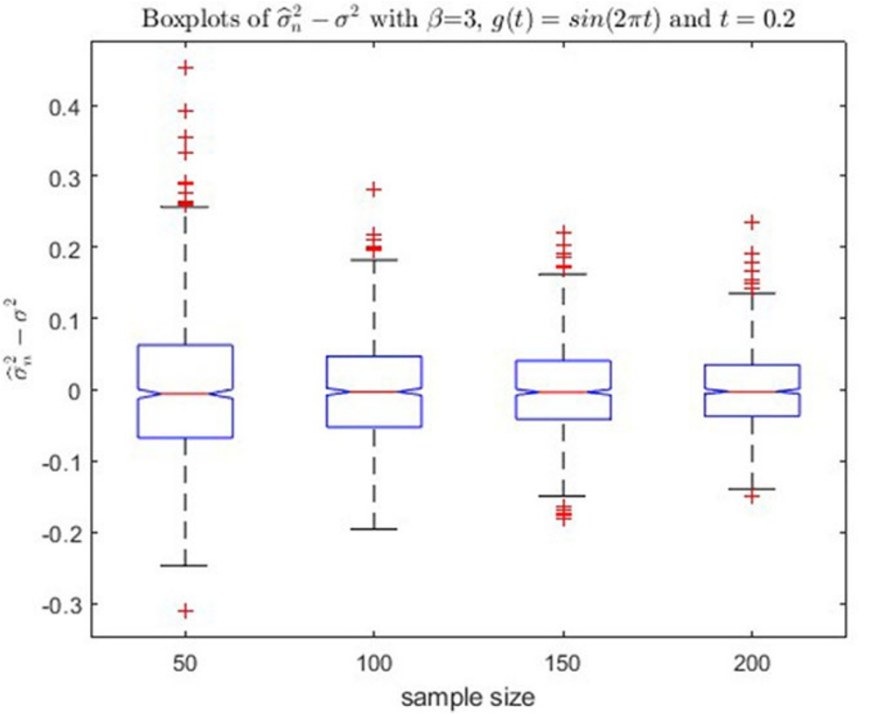


Figure 3. *Boxplots of* $\widehat{\sigma}_n^2 - \sigma^2$ *with* $\beta = 3$, $g(t) = \sin(2\pi t)$, *and* $t = 0.2$.

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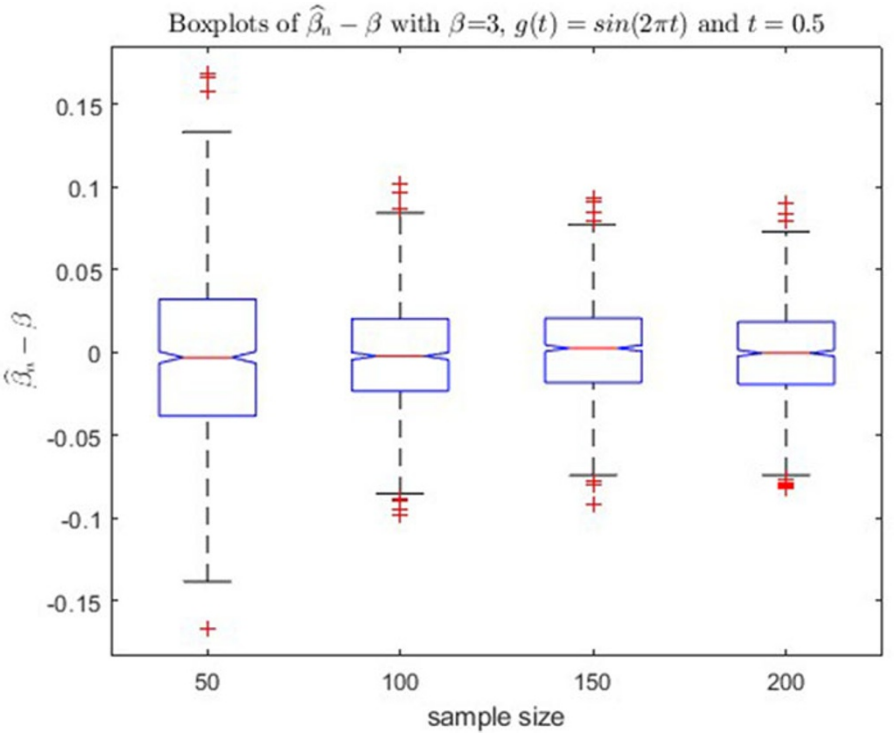


Figure 4. *Boxplots of* $\widehat{\beta}_n - \beta$ *with* $\beta = 3$, $g(t) = \sin(2\pi t)$, *and* $t = 0.5$.

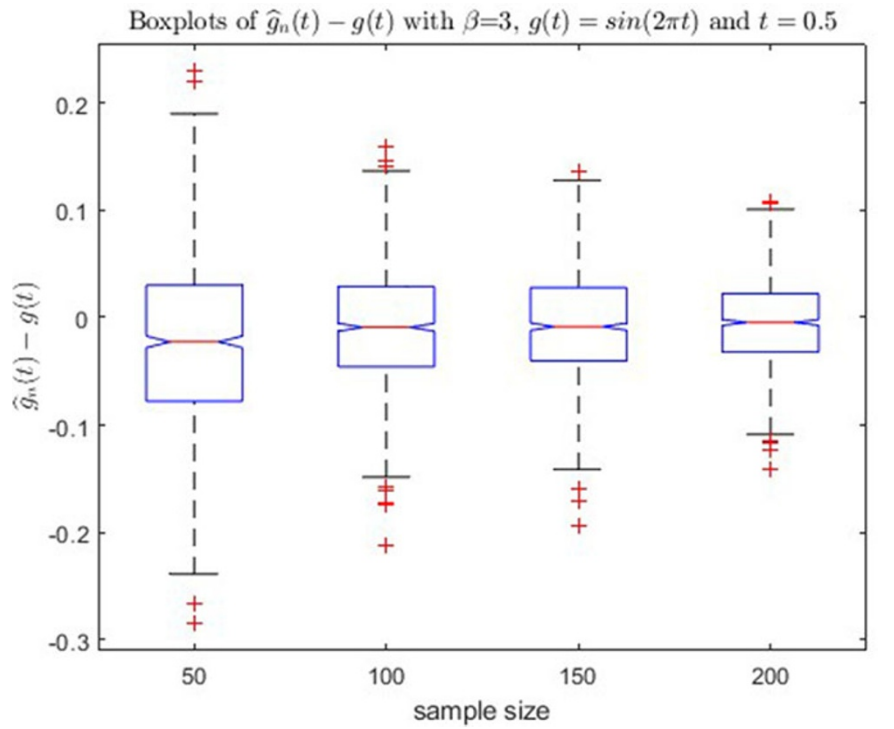


Figure 5. *Boxplots of* $\widehat{g}_n(t) - g(t)$ *with* $\beta = 3$, $g(t) = \sin(2\pi t)$, *and* $t = 0.5$.

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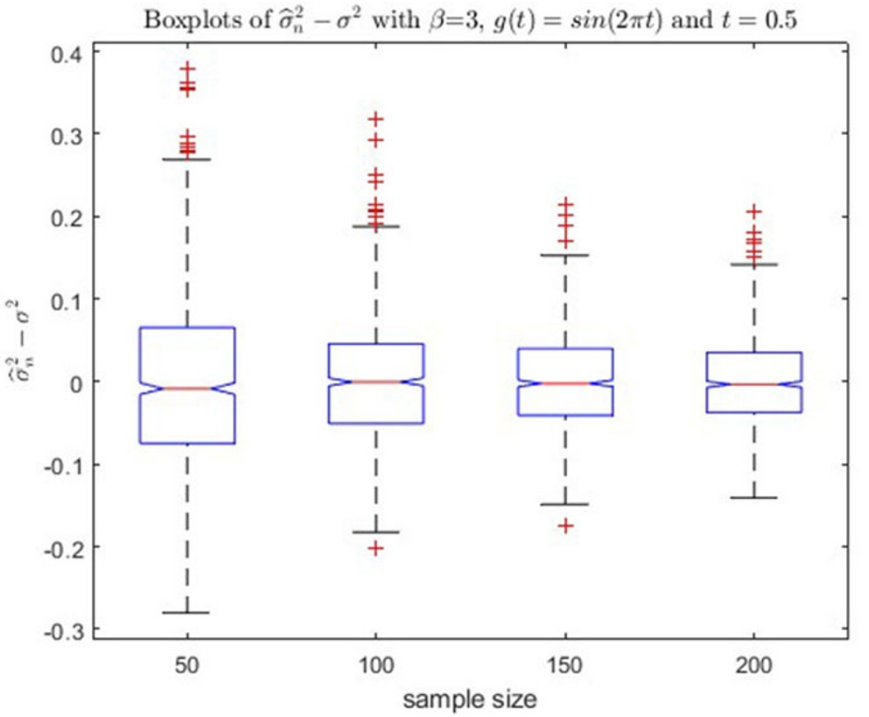


Figure 6. *Boxplots of* $\widehat{\sigma}_n^2 - \sigma^2$ *with* $\beta = 3$, $g(t) = \sin(2\pi t)$, *and* $t = 0.5$.

4. Proofs of main results

Proof of Theorem 2.5. To facilitate the notations, write

$$\tilde{\xi} = (I - S)(\xi_1, \ldots, \xi_n)^T := (I - S)\xi, \qquad \widetilde{g} = (I - S)(g(t_1), \ldots, g(t_n))^T := (I - S)g.$$

By (6), we obtain that

$$\widehat{\beta}_n - \beta = \left(n^{-1}\widetilde{X}^T\widetilde{X} - \sigma_\mu^2\right)^{-1}\left(n^{-1}\widetilde{X}^T\widetilde{g} + n^{-1}\widetilde{X}^T\widetilde{\xi} + \sigma_\mu^2\beta\right). \tag{10}$$

We just need to prove $n^{-1}\widetilde{X}^T\widetilde{X} - \sigma_\mu^2 \xrightarrow{a.s.} \sigma_\eta^2$, $n^{-1}\widetilde{X}^T\widetilde{g} \xrightarrow{a.s.} 0$, and $n^{-1}\widetilde{X}^T\widetilde{\xi} \xrightarrow{a.s.} -\sigma_\mu^2\beta$ as $n \to \infty$.

First, let's prove

$$n^{-1}\widetilde{X}^T\widetilde{X} - \sigma_\mu^2 \xrightarrow{a.s.} \sigma_\eta^2. \tag{11}$$

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Table 1. *The bias, SD, and RMSE of $\widehat{\beta}_n$, $\widehat{g}_n(t)$, and $\widehat{\sigma}_n^2$.*

		$\widehat{\beta}_n$			$\widehat{g}_n(t)$			$\widehat{\sigma}_n^2$		
t	n	Bias	SD	GMSE	Bias	SD	GMSE	Bias	SD	GMSE
0.2	50	0.001204	0.003841	0.002462	−0.093263	0.009399	0.014599	0.009165	0.000213	0.000296
	100	−0.000694	0.001381	0.001383	−0.060718	0.005262	0.008961	0.005262	0.000052	0.000070
	150	−0.000376	0.000742	0.000743	−0.041306	0.003048	0.004753	0.002782	0.000026	0.000033
	200	0.000314	0.000551	0.000552	−0.031618	0.001982	0.002983	0.001461	0.000012	0.000014
0.5	50	−0.002772	0.002994	0.002748	−0.001584	0.007898	0.007906	0.008329	0.000165	0.000234
	100	−0.002213	0.001342	0.001346	−0.001290	0.004693	0.004706	0.004207	0.000042	0.000059
	150	−0.001275	0.000741	0.000922	−0.000742	0.002445	0.002441	0.002536	0.000022	0.000029
	200	−0.000094	0.000568	0.000552	0.000567	0.001643	0.001641	0.001693	0.000012	0.000015

It is easily seen that

$$
\begin{aligned}
n^{-1}\widetilde{X}^T\widetilde{X} =& \frac{1}{n}\sum_{i=1}^{n}\left\{f(t_i)+\eta_i+\mu_i-\sum_{j=1}^{n}S_{ij}\left[f(t_j)+\eta_j+\mu_j\right]\right\}^2 \\
=& \frac{1}{n}\sum_{i=1}^{n}\left[f(t_i)-\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\,\mathrm{d}sf(t_j)\right]^2 \\
&+2\cdot\frac{1}{n}\sum_{i=1}^{n}\left[f(t_i)-\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\,\mathrm{d}sf(t_j)\right]\left[\eta_i-\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\,\mathrm{d}s\eta_j\right] \\
&+2\cdot\frac{1}{n}\sum_{i=1}^{n}\left[f(t_i)-\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\,\mathrm{d}sf(t_j)\right]\left[\mu_i-\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\,\mathrm{d}s\mu_j\right] \\
&+2\cdot\frac{1}{n}\sum_{i=1}^{n}\left[\eta_i-\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\,\mathrm{d}s\eta_j\right]\left[\mu_i-\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\,\mathrm{d}s\mu_j\right] \\
&+\frac{1}{n}\sum_{i=1}^{n}\left[\eta_i-\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\,\mathrm{d}s\eta_j\right]^2+\frac{1}{n}\sum_{i=1}^{n}\left[\mu_i-\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\,\mathrm{d}s\mu_j\right]^2 \\
:=& F_1+2F_2+2F_3+2F_4+F_5+F_6. \qquad (12)
\end{aligned}
$$

According to Lemma A.2, it can be concluded that

$$
|F_1|\le\max_{1\le i\le n}\left|f(t_i)-\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\,\mathrm{d}sf(t_j)\right|^2\xrightarrow{a.s.}0. \qquad (13)
$$

It is known from Lemma A.6 that

$$
\max_{1\le i\le n}\left|\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\mathrm{d}s\eta_j\right|\xrightarrow{a.s.}0 \qquad (14)
$$

and

$$
\max_{1\le i\le n}\left|\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\mathrm{d}s\mu_j\right|\xrightarrow{a.s.}0 \qquad (15)
$$

By the Kolmogorov's strong law of numbers, (14), and Lemma A.2, we have

$$
\begin{aligned}
|F_2|\le& \left[\max_{1\le i\le n}\left|f(t_i)-\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\,\mathrm{d}sf(t_j)\right|\right]\cdot\left[\frac{1}{n}\sum_{i=1}^{n}|\eta_i|+\max_{1\le i\le n}\left|\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\,\mathrm{d}s\eta_j\right|\right] \\
=& o(1)\left[E|\eta_i|+o(1)\right]\xrightarrow{a.s.}0. \qquad (16)
\end{aligned}
$$

Similar to the proof process in (16), it is easy to obtain

$$
|F_3|=o(1)\left[E|\mu_i|+o(1)\right]\xrightarrow{a.s.}0. \qquad (17)
$$

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Based on the Kolmogorov's strong law of large numbers, (14), and (15), it can be inferred that

$$\begin{aligned}|F_4| &= \left|\frac{1}{n}\sum_{i=1}^{n}\left[\eta_i - o(1)\right]\left[\mu_i - o(1)\right]\right| \\ &\le \left|\frac{1}{n}\sum_{i=1}^{n}\eta_i\mu_i\right| + o(1)\left|\frac{1}{n}\sum_{i=1}^{n}(\mu_i + \eta_i)\right| + o^2(1) \\ &= [o(1) + o(1)E|\mu_i| + o(1)E|\eta_i| + o^2(1)] \xrightarrow{a.s.} 0. \end{aligned} \tag{18}$$

By the Kolmogorov's strong law of large numbers, (14), $E\eta_i = 0$, and $\mathrm{Var}\,(\eta_i) = \sigma_\eta^2$, we find

$$\begin{aligned}&|F_5 - \sigma_\eta^2| \\ &= \left|\frac{1}{n}\sum_{i=1}^{n}\left[\eta_i - \sum_{j=1}^{n}\int_{A_j} E_m(t_i, s)\,\mathrm{d}s\eta_j\right]^2 - \sigma_\eta^2\right| \\ &\le \left|\frac{1}{n}\sum_{i=1}^{n}\eta_i^2 - \sigma_\eta^2\right| + 2\frac{1}{n}\left|\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\int_{A_j} E_m(t_i, s)\,\mathrm{d}s\eta_j\right| + \frac{1}{n}\left|\sum_{i=1}^{n}\left[\sum_{j=1}^{n}\int_{A_j} E_m(t_i, s)\,\mathrm{d}s\eta_j\right]^2\right| \\ &\le \left|\frac{1}{n}\sum_{i=1}^{n}\eta_i^2 - \sigma_\eta^2\right| + 2\frac{1}{n}\sum_{i=1}^{n}|\eta_i|\max_{1\le i\le n}\left|\sum_{j=1}^{n}\int_{A_j} E_m(t_i, s)\mathrm{d}s\eta_i\right| + \max_{1\le i\le n}\left|\sum_{j=1}^{n}\int_{A_j} E_m(t_i, s)\mathrm{d}s\eta_j\right|^2 \\ &= [o(1) + 2o(1)E|\eta_i| + o^2(1)] \xrightarrow{a.s.} 0. \end{aligned} \tag{19}$$

Likewise, it can be inferred that

$$F_6 \xrightarrow{a.s.} \sigma_\mu^2. \tag{20}$$

By using (12), (13), (16), (17), (18), (19), and (20), we can determine that

$$n^{-1}\widetilde{X}^T\tilde{X} \xrightarrow{a.s.} \sigma_\eta^2 + \sigma_\mu^2. \tag{21}$$

The (11) is derived from (21).

Second, we will prove

$$n^{-1}\widetilde{X}^T\widetilde{g} \xrightarrow{a.s.} 0. \tag{22}$$

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It is evident that

$$\begin{aligned} n^{-1}\widetilde{X}^T\widetilde{g} =& \frac{1}{n}\sum_{i=1}^{n}\left(X_i-\sum_{j=1}^{n}S_{ij}X_j\right)\left[g\left(t_i\right)-\sum_{k=1}^{n}S_{ik}g\left(t_k\right)\right] \\ =& \frac{1}{n}\sum_{i=1}^{n}\left(\eta_i-\sum_{j=1}^{n}S_{ij}\eta_j\right)\left[g\left(t_i\right)-\sum_{k=1}^{n}S_{ik}g\left(t_k\right)\right] \\ &+\frac{1}{n}\sum_{i=1}^{n}\left[f\left(t_i\right)-\sum_{j=1}^{n}S_{ij}f\left(t_j\right)\right]\left[g\left(t_i\right)-\sum_{k=1}^{n}S_{ik}g\left(t_k\right)\right] \\ &+\frac{1}{n}\sum_{i=1}^{n}\left(\mu_i-\sum_{j=1}^{n}S_{ij}\mu_j\right)\left[g\left(t_i\right)-\sum_{k=1}^{n}S_{ik}g\left(t_k\right)\right] \\ :=& G_1+G_2+G_3. \end{aligned} \tag{23}$$

Similar to the proof of (16), one can easily show that

$$G_1 \xrightarrow{a.s.} 0 \quad \text{and} \quad G_3 \xrightarrow{a.s.} 0. \tag{24}$$

By Lemma A.2, it can be inferred that

$$|G_2| \le \max_{1\le i\le n}\left|\left[f\left(t_i\right)-\sum_{j=1}^{n}S_{ij}f\left(t_j\right)\right]\left[g\left(t_i\right)-\sum_{k=1}^{n}S_{ik}g\left(t_k\right)\right]\right| = O\left(n^{-2\gamma}\right)+O\left(\tau_m^2\right) \xrightarrow{a.s.} 0. \tag{25}$$

Hence, (22) is derived from (23) to (25).

Finally, we will prove

$$n^{-1}\tilde{X}^T\tilde{\xi}+\sigma_\mu^2\beta \xrightarrow{a.s.} 0. \tag{26}$$

It is easily seen that

$$\begin{aligned} n^{-1}\widetilde{X}^T\widetilde{\xi}+\sigma_\mu^2\beta =& \frac{1}{n}\sum_{i=1}^{n}\left(X_i-\sum_{j=1}^{n}S_{ij}X_j\right)\left(\xi_i-\sum_{k=1}^{n}S_{ik}\xi_k\right)+\sigma_\mu^2\beta \\ =& \frac{1}{n}\sum_{i=1}^{n}\left(\eta_i-\sum_{j=1}^{n}S_{ij}\eta_j\right)\left(\xi_i-\sum_{k=1}^{n}S_{ik}\xi_k\right) \\ &+\frac{1}{n}\sum_{i=1}^{n}\left[f\left(t_i\right)-\sum_{j=1}^{n}S_{ij}f\left(t_j\right)\right]\left(\xi_i-\sum_{k=1}^{n}S_{ik}\xi_k\right) \\ &+\left[\frac{1}{n}\sum_{i=1}^{n}\left(\mu_i-\sum_{j=1}^{n}S_{ij}\mu_j\right)\left(\xi_i-\sum_{k=1}^{n}S_{ik}\xi_k\right)+\sigma_\mu^2\beta\right] \\ :=& J_1+J_2+J_3. \end{aligned} \tag{27}$$

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Clearly, we obtain

$$J_1 = \frac{1}{n}\sum_{i=1}^{n}\eta_i\xi_i - \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}S_{ij}\eta_j\right)\xi_i - \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{k=1}^{n}S_{ik}\xi_k\right)\eta_i + \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}S_{ij}\eta_j\right)\left(\sum_{k=1}^{n}S_{ik}\xi_k\right)$$
$$:= J_{11} - J_{12} - J_{13} + J_{14}. \tag{28}$$

According to Lemma A.3, it can be seen that $\{\varepsilon_i^+,\ 1 \le i \le n\}$ and $\{\varepsilon_i^-,\ 1 \le i \le n\}$ are both ANA sequences and their mixing coefficients are not greater than $\rho^-(N_\varepsilon)$. Since $\{\eta_i\}$ and $\{\varepsilon_i\}$ are independent, it can be inferred from Lemma A.4 that $\{\eta_i^+\varepsilon_i^+, 1 \le i \le n\}$, $\{\eta_i^+\varepsilon_i^-, 1 \le i \le n\}$, $\{\eta_i^-\varepsilon_i^+, 1 \le i \le n\}$, and $\{\eta_i^-\varepsilon_i^-, 1 \le i \le n\}$ are all ANA random variables with mixing coefficients $\rho^-(N_{\eta\varepsilon}) < r$ for $0 \le r < (\frac{1}{6p})^{p/2}$. Because $E|\eta_i\varepsilon_i|^2 < \infty$, it can be inferred from Lemma A.6 and the Kolmogorov's strong law of numbers that

$$\begin{aligned}
&|J_{11}|\\
\le&\left|\frac{1}{n}\sum_{i=1}^{n}(\eta_i\varepsilon_i - E\eta_i\varepsilon_i)\right| + \left|\beta\frac{1}{n}\sum_{i=1}^{n}\eta_i\mu_i\right|\\
=&\left|\frac{1}{n}\sum_{i=1}^{n}\left\{\left[\eta_i^+\varepsilon_i^+ - E(\eta_i^+\varepsilon_i^+)\right] + \left[\eta_i^-\varepsilon_i^- - E(\eta_i^-\varepsilon_i^-)\right] - \left[\eta_i^+\varepsilon_i^- - E(\eta_i^+\varepsilon_i^-)\right] - \left[\eta_i^-\varepsilon_i^+ - E(\eta_i^-\varepsilon_i^+)\right]\right\}\right|\\
&+\left|\beta\frac{1}{n}\sum_{i=1}^{n}\eta_i\mu_i\right|\\
\le&\frac{1}{n}\left|\sum_{i=1}^{n}\left[\eta_i^+\varepsilon_i^+ - E(\eta_i^+\varepsilon_i^+)\right]\right| + \frac{1}{n}\left|\sum_{i=1}^{n}\left[\eta_i^-\varepsilon_i^- - E(\eta_i^-\varepsilon_i^-)\right]\right| + \frac{1}{n}\left|\sum_{i=1}^{n}\left[\eta_i^+\varepsilon_i^- - E(\eta_i^+\varepsilon_i^-)\right]\right|\\
&+\frac{1}{n}\left|\sum_{i=1}^{n}\left[\eta_i^-\varepsilon_i^+ - E(\eta_i^-\varepsilon_i^+)\right]\right| + \beta\left|\frac{1}{n}\sum_{i=1}^{n}\eta_i\mu_i\right|\\
=&[4o(1) + \beta o(1)] \xrightarrow{a.s.} 0.
\end{aligned} \tag{29}$$

Furthermore, it can be concluded from Lemma A.6 that

$$\frac{1}{n}\left|\sum_{i=1}^{n}\varepsilon_i\right| \xrightarrow{a.s.} 0. \tag{30}$$

Based on the (14), (30), and Kolmogorov's strong law of large numbers, it can be inferred that

$$\begin{aligned}
|J_{12}| &= \left|\frac{1}{n}\sum_{i=1}^{n}\left[\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\,\mathrm{d}s\eta_j\right](\varepsilon_i - \mu_i\beta)\right|\\
&\le \left|\frac{1}{n}\sum_{i=1}^{n}\left[\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\,\mathrm{d}s\eta_j\right]\varepsilon_i\right| + \left|\beta\cdot\frac{1}{n}\sum_{i=1}^{n}\left[\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\,\mathrm{d}s\eta_j\right]\mu_i\right|\\
&\le \max_{1\le i\le n}\left|\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\,\mathrm{d}s\eta_j\right|\cdot\left|\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\right| + \beta\cdot\max_{1\le i\le n}\left|\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\,\mathrm{d}s\eta_j\right|\cdot\left(\frac{1}{n}\sum_{i=1}^{n}|\mu_i|\right)\\
&= o(1)\,[o(1) + \beta\cdot\mathrm{E}\,|\mu_i|] \xrightarrow{a.s.} 0.
\end{aligned} \tag{31}$$

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Based on the Kolmogorov's strong law of large numbers, Lemma A.1, Lemma A.7, and (15), it can be inferred that

$$\begin{aligned}
|J_{13}| &= \left| \frac{1}{n} \sum_{i=1}^{n} \left[\sum_{k=1}^{n} \int_{A_k} E_m(t_i, s)\, \mathrm{d}s\xi_k \right] \eta_i \right| \\
&\le \left| \frac{1}{n} \sum_{i=1}^{n} \left[\sum_{k=1}^{n} \int_{A_k} E_m(t_i, s)\, \mathrm{d}s\varepsilon_k \right] \eta_i \right| + \left| \beta \cdot \frac{1}{n} \sum_{i=1}^{n} \left[\sum_{k=1}^{n} \int_{A_k} E_m(t_i, s)\, \mathrm{d}s\mu_k \right] \eta_i \right| \\
&\le \max_{1 \le i \le n} \left| \sum_{k=1}^{n} \int_{A_k} E_m(t_i, s)\, \mathrm{d}s\varepsilon_k \right| \cdot \left(\frac{1}{n} \sum_{i=1}^{n} |\eta_i| \right) \\
&\quad + \beta \cdot \max_{1 \le i \le n} \left| \sum_{k=1}^{n} \int_{A_k} E_m(t_i, s)\, \mathrm{d}s\mu_k \right| \cdot \left(\frac{1}{n} \sum_{i=1}^{n} |\eta_i| \right) \\
&= o(1) \left[\mathrm{E}\,|\eta_i| + \beta \cdot \mathrm{E}\,|\eta_i| \right] \xrightarrow{a.s.} 0.
\end{aligned} \tag{32}$$

Based on the Kolmogorov's strong law of large numbers, Lemma A.1, Lemma A.7, (14), and (15), it can be inferred that

$$\begin{aligned}
|J_{14}| &= \left| \frac{1}{n} \sum_{i=1}^{n} \left[\sum_{j=1}^{n} \int_{A_j} E_m(t_i, s)\, \mathrm{d}s\eta_j \right] \left[\sum_{k=1}^{n} \int_{A_k} E_m(t_i, s)\, \mathrm{d}s\xi_k \right] \right| \\
&\le \left| \frac{1}{n} \sum_{i=1}^{n} \left[\sum_{j=1}^{n} \int_{A_j} E_m(t_i, s)\, \mathrm{d}s\eta_j \right] \left[\sum_{k=1}^{n} \int_{A_k} E_m(t_i, s)\, \mathrm{d}s\varepsilon_k \right] \right| \\
&\quad + \left| \beta \cdot \frac{1}{n} \sum_{i=1}^{n} \left[\sum_{j=1}^{n} \int_{A_j} E_m(t_i, s)\, \mathrm{d}s\eta_j \right] \left[\sum_{k=1}^{n} \int_{A_k} E_m(t_i, s)\, \mathrm{d}s\mu_k \right] \right| \\
&\le \max_{1 \le i \le n} \left| \sum_{k=1}^{n} \int_{A_k} E_m(t_i, s)\, \mathrm{d}s\varepsilon_k \right| \cdot \max_{1 \le i \le n} \left| \sum_{j=1}^{n} \int_{A_j} E_m(t_i, s)\, \mathrm{d}s\eta_j \right| \\
&\quad + \beta \cdot \max_{1 \le i \le n} \left| \sum_{k=1}^{n} \int_{A_k} E_m(t_i, s)\, \mathrm{d}s\mu_k \right| \cdot \max_{1 \le i \le n} \left| \sum_{j=1}^{n} \int_{A_j} E_m(t_i, s)\, \mathrm{d}s\eta_j \right| \\
&= o(1) \left[o(1) + \beta \cdot o(1) \right] \xrightarrow{a.s.} 0.
\end{aligned} \tag{33}$$

Therefore, from (28), (29), (31), (32), and (33), it can be concluded that

$$J_1 \xrightarrow{a.s.} 0. \tag{34}$$

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For J_2, it can be inferred from (30), Lemmas A.1, A.2, and A.7 that

$$\begin{aligned}
|J_2| &= \left|\frac{1}{n}\sum_{i=1}^{n}\left[f(t_i)-\sum_{j=1}^{n}S_{ij}f(t_j)\right]\left(\varepsilon_i-\sum_{k=1}^{n}S_{ik}\varepsilon_k\right)\right| \\
&\quad+\left|\beta\cdot\frac{1}{n}\sum_{i=1}^{n}\left[f(t_i)-\sum_{j=1}^{n}S_{ij}f(t_j)\right]\left(\mu_i-\sum_{k=1}^{n}S_{ik}\mu_k\right)\right| \\
&\le \max_{1\le i\le n}\left|f(t_i)-\sum_{j=1}^{n}S_{ij}f(t_j)\right|\cdot\left(\frac{1}{n}\left|\sum_{i=1}^{n}\varepsilon_i\right|+\max_{1\le i\le n}\left|\sum_{k=1}^{n}S_{ik}\varepsilon_k\right|\right) \\
&\quad+\beta\cdot\max_{1\le i\le n}\left|f(t_i)-\sum_{j=1}^{n}S_{ij}f(t_j)\right|\left(\frac{1}{n}\sum_{i=1}^{n}|\mu_i|+\max_{1\le i\le n}\left|\sum_{k=1}^{n}S_{ik}\mu_k\right|\right) \\
&=[O(n^{-\gamma})+O(\tau_m)]\cdot\{2o(1)+\beta[E|\mu_i|+o(1)]\}\xrightarrow{a.s.}0. \qquad (35)
\end{aligned}$$

From the Kolmogorov's strong law of large numbers, Lemma A.1, Lemma A.7, (15), and (30), it follows that

$$\begin{aligned}
|J_3| &= \left|\frac{1}{n}\sum_{i=1}^{n}\left(\mu_i-\sum_{j=1}^{n}S_{ij}\mu_j\right)\left[(\varepsilon_i-\mu_i\beta)-\sum_{k=1}^{n}S_{ik}(\varepsilon_k-\mu_k\beta)\right]+\sigma_\mu^2\beta\right| \\
&=\left|\frac{1}{n}\sum_{i=1}^{n}\left(\mu_i-\sum_{j=1}^{n}S_{ij}\mu_j\right)\left(\varepsilon_i-\sum_{k=1}^{n}S_{ik}\varepsilon_k\right)-\beta\cdot\frac{1}{n}\sum_{i=1}^{n}\left(\mu_i-\sum_{j=1}^{n}S_{ij}\mu_j\right)\left(\mu_i-\sum_{k=1}^{n}S_{ik}\mu_k\right)+\sigma_\mu^2\beta\right| \\
&\le\left[\frac{1}{n}\sum_{i=1}^{n}|\mu_i|+\max_{1\le i\le n}\left|\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\,\mathrm{d}s\mu_j\right|\right]\cdot\left[\frac{1}{n}\left|\sum_{i=1}^{n}\varepsilon_i\right|+\max_{1\le i\le n}\left|\sum_{k=1}^{n}\int_{A_k}E_m(t_i,s)\,\mathrm{d}s\varepsilon_k\right|\right] \\
&\quad+\beta\cdot\left|\frac{1}{n}\sum_{i=1}^{n}\left(\mu_i-\sum_{j=1}^{n}S_{ij}\mu_j\right)\left(\mu_i-\sum_{k=1}^{n}S_{ik}\mu_k\right)-\sigma_\mu^2\right| \\
&\le[\mathrm{E}|\mu_i|+o(1)]\,o(1)+\beta\cdot\left|\frac{1}{n}\sum_{i=1}^{n}\left(\mu_i^2-\sum_{j=1}^{n}S_{ij}\mu_j\mu_i-\sum_{k=1}^{n}S_{ik}\mu_k\mu_i+\sum_{j=1}^{n}S_{ij}\mu_j\sum_{k=1}^{n}S_{ik}\mu_k\right)-\sigma_\mu^2\right| \\
&\le o(1)+\beta\left|\frac{1}{n}\sum_{i=1}^{n}\mu_i^2-\sigma_\mu^2\right|+\beta\max_{1\le i\le n}\left|\sum_{j=1}^{n}S_{ij}\mu_j\right|\frac{1}{n}\sum_{i=1}^{n}|\mu_i|+\beta\max_{1\le i\le n}\left|\sum_{k=1}^{n}S_{ik}\mu_k\right|\frac{1}{n}\sum_{i=1}^{n}|\mu_i| \\
&\quad+\beta\max_{1\le i\le n}\left|\sum_{j=1}^{n}S_{ij}\mu_j\right|\cdot\max_{1\le i\le n}\left|\sum_{k=1}^{n}S_{ik}\mu_k\right| \\
&=\{o(1)+\beta[o(1)+o(1)E|\mu_i|+o(1)E|\mu_i|+o(1)]\}\xrightarrow{a.s.}0. \qquad (36)
\end{aligned}$$

From (27) and (34)–(36), we get (26).

Theorem 2.5(i) is proved by (10), (11), (22), and (26).

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For the proof of Theorem 2.5(ii), it is easily seen that

$$\begin{aligned}\sup_{t\in[0,1]}|\widehat{g}_n(t)-g(t)| &= \sup_{t\in[0,1]}\left|\widehat{g}_0(t,\beta)-g(t)+\sum_{j=1}^{n}X_j\left(\beta-\widehat{\beta}_n\right)\int_{A_j}E_m(t,s)\mathrm{d}s\right| \\ &\le \sup_{t\in[0,1]}\left|\sum_{j=1}^{n}g\left(t_j\right)\int_{A_j}E_m(t,s)\mathrm{d}s-g(t)\right|+\sup_{t\in[0,1]}\left|\sum_{j=1}^{n}\xi_j\int_{A_j}E_m(t,s)\mathrm{d}s\right| \\ &\quad+\left|\widehat{\beta}_n-\beta\right|\sup_{t\in[0,1]}\left|\sum_{j=1}^{n}\int_{A_j}E_m(t,s)\mathrm{d}s\eta_j\right|+\left|\widehat{\beta}_n-\beta\right|\sup_{t\in[0,1]}\left|\sum_{j=1}^{n}\int_{A_j}E_m(t,s)\mathrm{d}s\mu_j\right| \\ &\quad+\left|\widehat{\beta}_n-\beta\right|\sup_{t\in[0,1]}\left|\sum_{j=1}^{n}f\left(t_j\right)\int_{A_j}E_m(t,s)\mathrm{d}s\right| \\ &:=K_1+K_2+K_3+K_4+K_5. \end{aligned} \tag{37}$$

By Lemma A.2, it follows that

$$K_1=O\left(n^{-\gamma}\right)+O\left(\tau_m\right)\xrightarrow{a.s.}0. \tag{38}$$

From Lemma A.1, Lemma A.7, and (15), it follows that

$$\begin{aligned}|K_2| &\le \beta\max_{1\le i\le n}\left|\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\mathrm{d}s\mu_j\right|+\max_{1\le i\le n}\left|\sum_{j=1}^{n}\int_{A_j}E_m(t,s)\mathrm{d}s\varepsilon_j\right| \\ &= [o(1)+\beta o(1)]\xrightarrow{a.s.}0. \end{aligned} \tag{39}$$

By Theorem 2.5(i), (14), and (15), we have that

$$K_3\xrightarrow{a.s.}0\ \text{ and }\ K_4\xrightarrow{a.s.}0. \tag{40}$$

From Theorem 2.5(i) and Lemma A.1, we know that

$$|K_5|\le\left|\hat{\beta}_n-\beta\right|\cdot\sup_{t\in[0,1]}|f(t)|\cdot\sup_{t\in[0,1]}\int_0^1|E_m(t,s)|\,\mathrm{d}s\le C\left|\hat{\beta}_n-\beta\right|\xrightarrow{a.s.}0. \tag{41}$$

Therefore, the result of the Theorem 2.5(ii) can be obtained by (37)–(41). □

Proof of Theorem 2.6. According to (8) and $\widetilde{y}_i=\widetilde{X}_i\beta+\widetilde{g}\left(t_i\right)+\widetilde{\xi}_i$, we have that

$$\begin{aligned}\widehat{\sigma}_n^2-\sigma^2 &=\frac{1}{n}\sum_{i=1}^{n}\left[\widetilde{X}_i\left(\beta-\widehat{\beta}_n\right)\right]^2+\frac{1}{n}\sum_{i=1}^{n}\widetilde{g}^2\left(t_i\right)+\frac{1}{n}\sum_{i=1}^{n}\widetilde{\xi}_i^2+2\frac{1}{n}\sum_{i=1}^{n}\widetilde{X}_i\left(\beta-\widehat{\beta}_n\right)\widetilde{g}\left(t_i\right) \\ &\quad+2\frac{1}{n}\sum_{i=1}^{n}\widetilde{X}_i\left(\beta-\widehat{\beta}_n\right)\widetilde{\xi}_i+2\frac{1}{n}\sum_{i=1}^{n}\widetilde{g}\left(t_i\right)\widetilde{\xi}_i-\sigma^2-\sigma_\mu^2\widehat{\beta}_n^2 \\ &:=L_1+L_2+L_3+2L_4+2L_5+2L_6-\sigma^2-\sigma_\mu^2\widehat{\beta}_n^2. \end{aligned} \tag{42}$$

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Let $\widetilde{\vartheta}_i = \vartheta_i - \sum_{j=1}^{n} \int_{A_j} E_m(t_i, s)\mathrm{d}s\vartheta_j$, where $\widetilde{\vartheta}_i = \widetilde{\varepsilon}_i$, $\widetilde{\mu}_i$, or $\widetilde{\eta}_i$. By the Kolmogorov's strong law of large numbers and (14), we have

$$\begin{aligned}
&\left|\frac{1}{n}\sum_{i=1}^{n}\widetilde{\eta}_i^2 - \sigma_\eta\right| \\
\le&\left|\frac{1}{n}\sum_{i=1}^{n}\eta_i^2 - \sigma_\eta\right| + 2\max_{1\le i\le n}\left|\sum_{i=1}^{n}\int_{A_i} E_m(t_i, s)\mathrm{d}s\eta_i\right|\left|\frac{1}{n}\sum_{i=1}^{n}\eta_i\right| + \max_{1\le i\le n}\left|\sum_{i=1}^{n}\int_{A_i} E_m(t_i, s)\mathrm{d}s\eta_i\right|^2 \\
\le&[o(1) + 2o(1)E|\eta_i| + o^2(1)] \xrightarrow{a.s.} 0.
\end{aligned}$$

Likewise, it can be inferred that

$$\frac{1}{n}\sum_{i=1}^{n}\widetilde{\mu}_i^2 \xrightarrow{a.s.} \sigma_\mu, \quad \frac{1}{n}\sum_{i=1}^{n}\widetilde{\eta}_i\widetilde{\mu}_i \xrightarrow{a.s.} 0.$$

Through the Kolmogorov's strong law of large numbers, (14), (15), and Theorem 2.5(i), we can prove

$$\begin{aligned}
|L_1| \le&\left|\beta - \widehat{\beta}_n\right|^2 \cdot \left\{\frac{1}{n}\sum_{i=1}^{n}\left[f(t_i) - \sum_{j=1}^{n}\left(\int_{A_j} E_m(t_i, s)\mathrm{d}s\right)f(t_j)\right]^2 + \frac{1}{n}\left|\sum_{i=1}^{n}(\widetilde{\eta}_i^2 + \widetilde{\mu}_i^2)\right|\right\} \\
&+ 2\left|\beta - \widehat{\beta}_n\right|^2 \cdot \left\{\left|\frac{1}{n}\sum_{i=1}^{n}(\widetilde{\eta}_i + \widetilde{\mu}_i) \cdot \left[f(t_i) - \sum_{j=1}^{n}\left(\int_{A_j} E_m(t_i, s)\mathrm{d}s\right)f(t_j)\right]\right| + \left|\frac{1}{n}\sum_{i=1}^{n}\widetilde{\eta}_i\widetilde{\mu}_i\right|\right\} \\
\le&o^2(1) \cdot \sup_{t\in[0,1]}\left|f(t) - \sum_{i=1}^{n}\left(\int_{A_i} E_m(t_i, s)\mathrm{d}s\right)f(t_i)\right|^2 + o^2(1)\left|\frac{1}{n}\sum_{i=1}^{n}\{\widetilde{\eta}_i^2 + \widetilde{\mu}_i^2\}\right| \\
&+ 2o^2(1) \cdot \sup_{t\in[0,1]}\left|f(t) - \sum_{i=1}^{n}\left(\int_{A_i} E_m(t_i, s)\mathrm{d}s\right)f(t_i)\right|\left|\frac{1}{n}\sum_{i=1}^{n}(\widetilde{\eta}_i + \widetilde{\mu}_i)\right| + 2o^2(1) \cdot \left|\frac{1}{n}\sum_{i=1}^{n}\widetilde{\eta}_i\widetilde{\mu}_i\right| \\
\le&o^4(1) + o^2(1) \cdot (\sigma_\eta + \sigma_\mu) + 2o^3(1) \cdot \left|\frac{1}{n}\sum_{i=1}^{n}(\eta_i + \mu_i)\right| + 2o^3(1) \cdot \max_{1\le i\le n}\left|\sum_{i=1}^{n}\int_{A_i} E_m(t_i, s)\mathrm{d}s\eta_i\right| \\
&+ 2o^3(1) \cdot \max_{1\le i\le n}\left|\sum_{i=1}^{n}\int_{A_i} E_m(t_i, s)\mathrm{d}s\mu_i\right| + 2o^3(1) \\
\le&o^2(1) \cdot \left[5o^2(1) + \sigma_\eta^2 + \sigma_\mu^2 + 2o(1)E|\eta_i| + 2o(1)E|\mu_i| + 2o(1)\right] \xrightarrow{a.s.} 0.
\end{aligned} \tag{43}$$

From Lemma A.1 and Lemma A.2, we get directly that

$$|L_2| \le \left[O\left(n^{-2\gamma}\right) + O\left(\tau_m^2\right)\right] \cdot o(1) \xrightarrow{a.s.} 0. \tag{44}$$

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Next, we will prove that

$$L_3 \xrightarrow{a.s.} \sigma^2 + \sigma_\mu^2 \beta^2. \tag{45}$$

It is easy to check that

$$\begin{aligned}
&\left|L_3 - \sigma^2 - \sigma_\mu^2 \beta^2\right| \\
=&\left|\frac{1}{n}\sum_{i=1}^{n} \hat{\xi}_i^2 - \sigma^2 - \sigma_\mu^2 \beta^2\right| \\
\le&\left|\frac{1}{n}\sum_{i=1}^{n}\left[\left(\varepsilon_i - \sum_{j=1}^{n} S_{ij}\varepsilon_j\right)^2 - \sigma^2\right]\right| + 2\left|\frac{1}{n}\sum_{i=1}^{n}\left(\varepsilon_i - \sum_{j=1}^{n} S_{ij}\varepsilon_j\right)\left(\mu_i - \sum_{j=1}^{n} S_{ij}\mu_j\right)\beta\right| \\
&+\left|\frac{1}{n}\sum_{i=1}^{n}\left(\mu_i - \sum_{j=1}^{n} S_{ij}\mu_j\right)^2 \beta^2 - \sigma_\mu^2 \beta^2\right| \\
:=&L_{31} + 2L_{32} + L_{33}.
\end{aligned} \tag{46}$$

Let

$$\varsigma_i := \varepsilon_i^2 - \sigma^2 = \varepsilon_i^2 - \mathrm{E}\varepsilon_i^2 = \left(\varepsilon_i^+\right)^2 - \left(\mathrm{E}\varepsilon_i^+\right)^2 - \left(\left(\varepsilon_i^-\right)^2 - \left(\mathrm{E}\varepsilon_i^-\right)^2\right) := \varsigma_i^+ - \varsigma_i^-.$$

We can see from Lemma A.3 that $\left\{\varsigma_i^+, i \ge 1\right\}$ and $\left\{\varsigma_i^-, i \ge 1\right\}$ are both ANA sequences and their mixing coefficients are not greater than $\rho^-(N_\varepsilon)$. It is easy to verify that $E\varsigma_i^\pm = 0$, $\mathrm{Var}\left(\varsigma_i^\pm\right) < \infty$. Furthermore, it follows from the Kolmogorov's strong law of large numbers, Lemma A.6, Lemma A.7, and (30) that

$$\begin{aligned}
|L_{31}| \le& \left|\frac{1}{n}\sum_{i=1}^{n}\left(\varepsilon_i^2 - \sigma^2\right)\right| + 2\left|\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\left(\sum_{j=1}^{n} S_{ij}\varepsilon_j\right)\right| + \left|\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{j=1}^{n} S_{ij}\varepsilon_j\right)^2\right| \\
\le& o(1) + 2\max_{1\le i\le n}\left|\frac{1}{n}\sum_{j=1}^{n} S_{ij}\varepsilon_j\right| \cdot \left|\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\right| + \max_{1\le i\le n}\left|\sum_{j=1}^{n} S_{ij}\varepsilon_j\right|^2 \\
=& o(1) + o(1)\cdot o(1) + o^2(1) \xrightarrow{a.s.} 0.
\end{aligned} \tag{47}$$

Similar to the proof of (29), we can get $\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\mu_i = o(1)$. Furthermore, from the Kolmogorov's strong law of large numbers, Lemma A.1, Lemma A.7, (14), (29), and (30), we know that

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$$
\begin{aligned}
|L_{32}| \leq & \beta \left| \frac{1}{n} \sum_{i=1}^{n} \left[\varepsilon_i \mu_i - \left(\sum_{j=1}^{n} S_{ij} \varepsilon_j \right) \mu_i - \left(\sum_{j=1}^{n} S_{ij} \mu_j \right) \varepsilon_i \right] \right| + \beta \left| \frac{1}{n} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} S_{ij} \varepsilon_j \right) \left(\sum_{j=1}^{n} S_{ij} \mu_j \right) \right| \\
\leq & \beta \left| \frac{1}{n} \sum_{i=1}^{n} \mu_i \left(\sum_{j=1}^{n} S_{ij} \varepsilon_j \right) \right| + \beta \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \left(\sum_{j=1}^{n} S_{ij} \mu_j \right) \right| + \beta \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \mu_i \right| \\
& + \beta \left| \frac{1}{n} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} S_{ij} \varepsilon_j \right) \left(\sum_{j=1}^{n} S_{ij} \mu_j \right) \right| \\
\leq & \beta \max_{1 \leq i \leq n} \left| \sum_{j=1}^{n} \int_{A_j} E_m(t_i, s) \, \mathrm{d}s \varepsilon_j \right| \cdot \left(\frac{1}{n} \sum_{i=1}^{n} |\mu_i| \right) + \beta \max_{1 \leq i \leq n} \left| \sum_{j=1}^{n} \int_{A_j} E_m(t_i, s) \, \mathrm{d}s \mu_j \right| \cdot \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \right| \\
& + \beta \cdot \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \mu_i \right| + \max_{1 \leq i \leq n} \left| \sum_{k=1}^{n} \int_{A_k} E_m(t_i, s) \, \mathrm{d}s \varepsilon_k \right| \cdot \max_{1 \leq i \leq n} \left| \sum_{j=1}^{n} \int_{A_j} E_m(t_i, s) \, \mathrm{d}s \mu_j \right| \\
= & \beta \left[o(1) \cdot E|u_i| + 2o^2(1) + o(1) \right] \xrightarrow{a.s.} 0.
\end{aligned} \tag{48}
$$

According to the Kolmogorov's strong law of large numbers and (15), we have that

$$
\begin{aligned}
|L_{33}| = & \beta^2 \left| \frac{1}{n} \sum_{i=1}^{n} \mu_i^2 - \sigma_\mu^2 - 2 \frac{1}{n} \sum_{i=1}^{n} \mu_i \left(\sum_{j=1}^{n} S_{ij} \mu_j \right) + \frac{1}{n} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} S_{ij} \mu_j \right)^2 \right| \\
\leq & \beta^2 \left[\left| \frac{1}{n} \sum_{i=1}^{n} \mu_i^2 - \sigma_\mu^2 \right| + 2 \left| \frac{1}{n} \sum_{i=1}^{n} \mu_i \left(\sum_{j=1}^{n} S_{ij} \mu_j \right) \right| + \left| \frac{1}{n} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} S_{ij} \mu_j \right)^2 \right| \right] \\
\leq & \beta^2 \left| \frac{1}{n} \sum_{i=1}^{n} \mu_i^2 - \sigma_\mu^2 \right| + 2\beta^2 \max_{1 \leq i \leq n} \left| \sum_{j=1}^{n} \int_{A_j} E_m(t_i, s) \, \mathrm{d}s \mu_j \right| \cdot \left| \frac{1}{n} \sum_{i=1}^{n} \mu_i \right| \\
& + \beta^2 \max_{1 \leq i \leq n} \left| \sum_{j=1}^{n} \int_{A_j} E_m(t_i, s) \, \mathrm{d}s \mu_j \right|^2 \\
= & \beta^2 \left[o(1) + 2 \cdot o(1) \cdot E|\mu_i| + o^2(1) \right] \xrightarrow{a.s.} 0.
\end{aligned} \tag{49}
$$

Formula (45) is established by (46)–(49).

By Cauchy's inequality, (43) and (44), we can prove

$$
L_4^2 \leq \left[\frac{1}{n} \sum_{i=1}^{n} \tilde{X}_i^2 \left(\beta - \widehat{\beta}_n \right)^2 \right] \cdot \left[\frac{1}{n} \sum_{i=1}^{n} \tilde{g}^2(t_i) \right] = L_1 \cdot L_2 \xrightarrow{a.s.} 0. \tag{50}
$$

By Cauchy's inequality, (43) and (45), we have that

$$
L_5^2 \leq \left[\frac{1}{n} \sum_{i=1}^{n} \tilde{X}_i^2 \left(\beta - \widehat{\beta}_n \right)^2 \right] \cdot \left[\frac{1}{n} \sum_{i=1}^{n} \tilde{\xi}_i^2 \right] = L_1 \cdot L_3 \xrightarrow{a.s.} 0. \tag{\{4.42\}}
$$

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By Cauchy's inequality, (44) and (45), we get directly that

$$L_6^2 \leq \frac{1}{n}\sum_{i=1}^{n} \tilde{g}^2(t_i) \cdot \frac{1}{n}\sum_{i=1}^{n} \tilde{\xi}_i^2 = L_2 \cdot L_3 \xrightarrow{a.s.} 0. \qquad (\{4.43\})$$

Therefore, by combining (42)–(45) and (50)–(4.43), the theorem is proved. □

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Appendix A.

To prove the main results of the paper, we need the following useful lemmas. The first one is the basic property of wavelet estimation, which comes from [1].

Lemma A.1. *Suppose that Assumptions* (**H**1)–(**H**3) *hold. For each* $t \in [0,1]$, *we can conclude that:*
(i) $|E_0(t,s)| \le \frac{C_k}{(1+|t-s|)^k}$, $|E_m(t,s)| \le \frac{2^m C_k}{(1+2^m|t-s|)^k}$, *where* $k \in \mathbb{N}$, $\mathrm{C_k}$ *is a constant depending on* k *only;*
(ii) $\sup_{0\le s\le 1} |E_m(t,s)| = O(2^m)$;
(iii) $\sup_{t\in[0,1]} \int_0^1 |E_m(t,s)|\,\mathrm{d}s \le C$;
(iv) $\int_{A_i} |E_m(t,s)|\,\mathrm{d}s = O(2^m/n)$, *where* $A_i = [s_{i-1}, s_i)$ *represents the arbitrarily divided interval of* $[0,1]$.

The second lemma is about the important properties of wavelet estimators, which is given by [10].

Lemma A.2. *Suppose that Assumptions* (**H**1)–(**H**3) *hold. Then*

$$\sup_{t\in[0,1]} \left| f(t) - \sum_{i=1}^{n} \left(\int_{A_i} E_m(t,s)\mathrm{d}s \right) f(t_i) \right| = O(n^{-\gamma}) + O(\tau_m)$$

and

$$\sup_{t\in[0,1]} \left| g(t) - \sum_{i=1}^{n} \left(\int_{A_i} E_m(t,s)\mathrm{d}s \right) g(t_i) \right| = O(n^{-\gamma}) + O(\tau_m),$$

where

$$\tau_m = \begin{cases} 2^{-m(\alpha-1/2)}, & 1/2 < \alpha < 3/2; \\ \sqrt{m}\cdot 2^{-m}, & \alpha = 3/2; \\ 2^{-m}, & \alpha > 3/2. \end{cases}$$

The next one is the basic property of ANA random variables, which comes from [26].

Lemma A.3. *Let* $\{X_n,\ n \ge 1\}$ *be a sequence of ANA random variables with mixing coefficients* $\rho^-(s)$. *If* $\{g_n(\cdot),\ n \ge 1\}$ *are nondecreasing or nonincreasing functions, then* $\{g_n(X_n),\ n \ge 1\}$ *is still a sequence of ANA random variables with mixing coefficients not greater than* $\rho^-(s)$.

The fourth lemma is an extension of the fundamental property of ANA random variables, which has been proved by [8].

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Lemma A.4. *Let $\{X_i, 1 \le i \le n\}$ and $\{Y_i, 1 \le i \le n\}$ be sequences of ANA nonnegative random variables with mixing coefficients $\rho^-(s_1)$ and $\rho^-(s_2)$, respectively. Assume that $\{X_i, 1 \le i \le n\}$ and $\{Y_i, 1 \le i \le n\}$ are independent. Then $\{X_iY_i, 1 \le i \le n\}$ is a sequence of ANA random variables with mixing coefficients not greater than $\rho^-(s_1) + \rho^-(s_2)$.*

The fifth lemma is the Rosenthal-type inequality for ANA random variables, which has been established by [18].

Lemma A.5. *Let $\{X_i,\ i \ge 1\}$ be an ANA sequence of random variables with $EX_i = 0$ and $E|X_i|^p < \infty$ for some $p \ge 4$. Assume that $\rho^-(N) \le r$ for $N \ge 1$ and $0 \le r < (\frac{1}{6p})^{p/2}$. Then there exists a positive constant $D = D(p, N, r)$ such that*

$$E \max_{1\le i\le n} \left|\sum_{j=1}^{i} X_j\right|^p \le D\left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} E|X_i|^2\right)^{p/2}\right\}.$$

Note that the complete convergence implies the almost sure convergence in view of the Borel–Cantelli lemma. So the sixth one can be derived from Theorem 4.2 in [23].

Lemma A.6. *Let $p \ge 4$ and let integer $N \ge 1$. Suppose that $\{X_i,\ i \ge 1\}$ be an ANA sequence of random variables with mixing coefficients $\rho^-(N) \le r$ for $0 \le r < (\frac{1}{6p})^{p/2}$. If*

$$\sup_{n\ge 1} \frac{1}{n} \sum_{i=1}^{n} E|X_i|^p < \infty,$$

then for any $\delta > 1/2$,

$$n^{-\delta} \max_{1\le i\le n} \left|\sum_{i=1}^{n} X_i - E\left(\sum_{i=1}^{n} X_i\right)\right| \xrightarrow{a.s.} 0.$$

The last lemma plays an important role in the proof of the theorem.

Lemma A.7. *Let $\{X_i,\ i \ge 1\}$ be an ANA sequence of random variables with $EX_i = 0$ and $\sup_{j\ge 1} E|X_j|^p < \infty$ for some $\mathrm{p} > 2$. Assume that $\rho^-(N) \le r$ for $N \ge 1$ and $0 \le r < (\frac{1}{6p})^{p/2}$. If $\{w_{nj}(t_i),\ 1 \le i, j \le n, n \ge 1\}$ is an array of real numbers satisfying $\max_{1\le i,j\le n} |w_{nj}(t_i)| = O(n^{-1/2})$ and $\max_{1\le i\le n} \sum_{j=1}^{n} |w_{nj}(t_i)|^2 = O(n^{-s}\log^{-1} n)$ for some $\frac{1}{2} \le s \le \frac{1}{p} + 1$. Then*

$$\max_{1\le i\le n} \left|\sum_{j=1}^{n} w_{nj}(t_i)X_j\right| \xrightarrow{a.s.} 0, \quad \text{as} \quad n \to \infty.$$

Proof of Lemma A.7. Without loss of generality, we assume that $w_{nj}(t_i) \ge 0,\ 1 \le i, j \le n, n \ge 1$ (otherwise, we can note that $w_{nj}(t_i) = (w_{nj}(t_i))^+ - (w_{nj}(t_i))^-$. For any $0 < q < \frac{1}{2} - \frac{1}{2(p-1)}$ and $1 \le i, j \le n$, denote

$$Y_{n,i,j} = -n^{-q} I(w_{nj}(t_i)X_j < -n^{-q}) + w_{nj}(t_i)X_j I(|w_{nj}(t_i)X_j| < n^{-q}) + n^{-q} I(w_{nj}(t_i)X_j > n^{-q}),$$

$$Z_{n,i,j} = [w_{nj}(t_i)X_j + n^{-q}] I(w_{nj}(t_i)X_j < -n^{-q}) + [w_{nj}(t_i)X_j - n^{-q}] I(w_{nj}(t_i)X_j > n^{-q}).$$

Since $EX_j = 0$, for $1 \le j \le n$ and $n \ge 1$, it can be argued that

$$\begin{aligned}
\max_{1\le i\le n} \left|\sum_{j=1}^{n} EY_{n,i,j}\right| &\le \max_{1\le i\le n} \sum_{j=1}^{n} E\left|w_{nj}(t_i)X_j\right| I(|w_{nj}(t_i)X_j| > n^{-q}) \\
&\le Cn^{(p-1)q}\left[\max_{1\le i\le n} \sum_{j=1}^{n} w_{nj}^2(t_i)\right]\left[\max_{1\le i,j\le n} w_{nj}(t_i)\right]^{p-2} \sup_{j\ge 1} E\left|X_j\right|^p \\
&\le Cn^{(p-1)q-s-(p-2)/2}\log^{-1} n \le Cn^{(p-1)q-q-(p-2)/2}\log^{-1} n \\
&\le Cn^{(p-2)(q-1/2)} \to 0, \quad \text{as} \quad n \to \infty. \qquad (51)
\end{aligned}$$

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Hence, we know by Lemma A.3 that $\{Y_{n,i,j} - EY_{n,i,j}, 1 \le i,j \le n\}$ are also ANA random variables with zero mean and mixing coefficients $\rho^-(N) < r$ for $0 \le r < (\frac{1}{6p})^{p/2}$. Take $\tau > \max\{p, 1+2/q\}$. From (51), Markov's inequality and Lemma A.5, we have that for all $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\max_{1\le i\le n}\left|\sum_{j=1}^{n} Y_{n,i,j}\right| > \varepsilon\right)\\
\le\ & C\sum_{n=1}^{\infty}\sum_{i=1}^{n} P\left(\left|\sum_{j=1}^{n} Y_{n,i,j} - EY_{n,i,j}\right| > \varepsilon/2\right)\\
\le\ & C\sum_{n=1}^{\infty}\sum_{i=1}^{n} E\left(\left|\sum_{j=1}^{n}(Y_{n,i,j} - EY_{n,i,j})\right|^{\tau}\right)\\
\le\ & C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} E\left|Y_{n,i,j} - EY_{n,i,j}\right|^{\tau} + C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\left(\sum_{j=1}^{n} E|Y_{n,i,j} - EY_{n,i,j}|^2\right)^{\tau/2}\\
:=\ & M_{n1} + M_{n2}.
\end{aligned} \tag{52}$$

For M_{n1}, we can obtain by C_r-inequality, Jensen's inequality and Markov's inequality that

$$\begin{aligned}
M_{n1} \le\ & C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} E\left|Y_{n,i,j}\right|^{\tau}\\
\le\ & C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[E|w_{nj}(t_i)X_j|^{\tau} I(|w_{nj}(t_i)X_j| \le n^{-q}) + n^{-\tau q} P(|w_{nj}(t_i)X_j| > n^{-q})\right]\\
\le\ & C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} E|w_{nj}(t_i)X_j|^{p}\left[n^{-(\tau-p)q} I(|w_{nj}(t_i)X_j| \le n^{-q}) + n^{-(\tau-p)q}\right]\\
\le\ & C\sum_{n=1}^{\infty} n^{-(\tau-p)q}\sum_{i=1}^{n}\left[\max_{1\le i\le n}\sum_{j=1}^{n} w_{nj}^2(t_i)\right]\left[\max_{1\le i,j\le n} w_{nj}(t_i)\right]^{p-2}\sup_{j\ge 1} E\left|X_j\right|^{p}\\
\le\ & C\sum_{n=1}^{\infty} n^{-(\tau-p)q+1-s-(p-2)/2}\log^{-1} n \le C\sum_{n=1}^{\infty} n^{-(\tau-1)q+1}\log^{-1} n < \infty.
\end{aligned} \tag{53}$$

By C_r-inequality and Jensen's inequality, we can get that

$$\begin{aligned}
M_{n2} \le\ & C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\left(\sum_{j=1}^{n} E|Y_{n,i,j} - EY_{n,i,j}|^2\right)^{\tau/2} \le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\left(\sum_{j=1}^{n} E|Y_{n,i,j})|^2\right)^{\tau/2}\\
\le\ & C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\left[\sum_{j=1}^{n} w_{nj}^2(t_i) E|X_j|^2 I(|w_{nj}(t_i)X_j| \le n^{-q}) + n^{2q} P(|w_{nj}(t_i)X_j| > n^{-q})\right]^{\tau/2}\\
\le\ & C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\left[\max_{1\le i\le n}\sum_{j=1}^{n} w_{nj}^2(t_i)\sup_{j\ge 1} E\left|X_j\right|^{p}\right]^{\tau/2}\\
\le\ & C\sum_{n=1}^{\infty} n^{-\tau s/2+1}\log^{-1} n < C\sum_{n=1}^{\infty} n^{-\tau q/2+1}\log^{-1} n < \infty.
\end{aligned} \tag{54}$$

According to (52)–(54), combined with Borel–Cantelli lemma, it can be inferred that

$$\max_{1\le i\le n}\left|\sum_{j=1}^{n} Y_{n,i,j}\right| \xrightarrow{a.s.} 0.$$

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Next, we will prove that $\max_{1\le i\le n}\left|\sum_{j=1}^{n} Z_{n,i,j}\right| \xrightarrow{a.s.} 0$. Since $0 < q < \frac{1}{2} - \frac{1}{2(p-1)}$, we can get

$$\sum_{j=1}^{\infty} j^{-1/2} E|X_j| I(|X_j| > Cj^{1/2-q}) \le \sum_{j=1}^{\infty} j^{-1/2} j^{(1/2-q)(1-p)} \sup_{j\ge 1} E\left|X_j\right|^p \le \sum_{i=1}^{\infty} j^{(p-1)q-p/2} < \infty,$$

which implies

$$\sum_{j=1}^{\infty} j^{-1/2} |X_j| I(|X_j| > Cj^{1/2-q}) < \infty \ \ a.s.$$

Therefore, we can derive by $|Z_{n,i,j}| \le |w_{nj}(t_i)X_j| I(|w_{nj}(t_i)X_j| > n^{-q})$ and the Kronecker's lemma that

$$\begin{aligned}
\max_{1\le i\le n}\left|\sum_{j=1}^{n} Z_{n,i,j}\right| &\le \max_{1\le i\le n}\sum_{j=1}^{n}\left|w_{nj}(t_i)X_j\right| I(|w_{nj}(t_i)X_j| > n^{-q}) \\
&\le Cn^{-1/2}\sum_{j=1}^{n}\left|X_j\right| I(|X_j| > n^{1/2-q}) \\
&\le Cn^{-1/2}\sum_{j=1}^{n}\left|X_j\right| I(|X_j| > j^{1/2-q}) \\
&\xrightarrow{a.s.} 0.
\end{aligned}$$

This completes the proof of the lemma. □

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