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Electrostatic fluctuations in plasmas containing suprathermal particles

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The spectral density of electrostatic fluctuations is investigated in a plasma containing an excess of suprathermal particles. As a model for the statistical distribution of the suprathermal particles, the kappa velocity distribution is used. It is found that the Debye length λ_D for such a plasma depends strongly on the spectral index κ of the velocity distribution, and can be much smaller than is commonly found for Maxwellian plasmas. Consequently the plasma parameter $g = 1/n\lambda_D^3$ can be larger, and plasma effects that depend on particle discreteness more important, than for a Maxwellian plasma. Consequently, levels of fluctuation are higher, and their extent in wavenumber and frequency greater, in plasmas containing suprathermals. Results for a number of different 'concentrations' of suprathermal particles are discussed and interpreted in terms of the normal- as well as transient-mode behaviour of the plasma. Both the cases $T_e \gg T_i$ and $T_e = T_i$, as well as $\kappa_e \ll \kappa_i$ and $\kappa_e \gg \kappa_i$, are investigated.

1. Introduction

In the language of statistical mechanics, the state of a plasma at time t is represented by a single point

$$\mathbf{X} = \{\mathbf{x}_1, \mathbf{v}_1; \mathbf{x}_2, \mathbf{v}_2; \ldots; \mathbf{x}_N, \mathbf{v}_N\}$$

in phase or Γ space. In the above N is the total number of plasma particles, comprising any number of ion and electron species. At time t the probability of finding the system in the volume element $d^N\mathbf{x}\, d^N\mathbf{v}$ about $\mathbf{X}$ is given by

$$F(\mathbf{x}_1, \mathbf{v}_1; \mathbf{x}_2, \mathbf{v}_2; \ldots; \mathbf{x}_N, \mathbf{v}_N; t)\, d^N\mathbf{x}\, d^N\mathbf{v},$$

where F satisfies the familiar Liouville equation.

The function $F(\mathbf{X}, t)$ determines the expectation value, or ensemble average, of the *macroscopic* plasma observables such as density, total charge or induced electric and magnetic fields. For example, if $\mathbf{E}_M(\mathbf{x}, t)$ is the macroscopic electric field induced in the plasma then

$$\mathbf{E}_M(\mathbf{x}, t) = \langle \mathbf{E}(\mathbf{x}, t; \mathbf{X}') \rangle = \int d^N\mathbf{x}' d^N\mathbf{v}' F(\mathbf{x}'_1, \mathbf{v}'_1; \ldots; \mathbf{x}'_N, \mathbf{v}'_N; t) \mathbf{E}(\mathbf{x}, t; \mathbf{x}'_1, \mathbf{v}'_1; \ldots; \mathbf{x}'_N, \mathbf{v}'_N),$$

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where $\mathbf{E}(\mathbf{x}, t; \mathbf{x}'_1, \mathbf{v}'_1; \ldots; \mathbf{x}'_N, \mathbf{v}'_N)$ is the induced *microscopic* plasma electric field, which depends, in general, on the positions and velocities of *all* the plasma particles. The above relation is quite general, and one can substitute $O(\mathbf{x}, t; \mathbf{x}'_1, \mathbf{v}'_1; \ldots; \mathbf{x}'_N, \mathbf{v}'_N)$ in place of $\mathbf{E}$, where O is any macroscopic observable.

In a plasma in thermal equilibrium, or in a stable stationary or near-stationary state, it is often the case that various macroscopic observables vanish, i.e. $\langle O(\mathbf{x}, t; \mathbf{X}') \rangle = 0$. However, spontaneous fluctuations occur, so that, for example,

$$\langle \mathbf{E}^2(\mathbf{x}, t; \mathbf{X}') \rangle = \int d^N\mathbf{x}' d^N\mathbf{v}' F(\mathbf{x}'_1, \mathbf{v}'_1; \ldots; \mathbf{x}'_N, \mathbf{v}'_N; t) \mathbf{E}^2(\mathbf{x}, t; \mathbf{x}'_1, \mathbf{v}'_1; \ldots; \mathbf{x}'_N, \mathbf{v}'_N) \neq 0.$$

These fluctuations are caused by the random, or thermal, motions of the *individual* plasma particles and the influence of such motions on the positions and velocities of other nearby plasma particles, i.e. correlations. Indeed, it is the very discreteness of the plasma particles that gives rise to such fluctuations.

Effects due to a plasma particle's discreteness emerge at order $g = 1/n\lambda_D^3$ in the BBGKY hierarchy of kinetic equations. As is well known, these equations arise from moments of the Liouville equation (see e.g. Liboff 1969, pp. 90–98), and are the basic equations of plasma kinetic theory. The commonly used Vlasov theory of plasmas is the lowest-order moment of the Liouville equation, and represents a truncation of the BBGKY hierarchy at order g^0. It does not predict any fluctuations, because it is essentially a fluid theory in phase space: it becomes a precise description of the plasma in the limit that

$$q \to 0, \quad m \to 0, \quad n \to \infty$$

such that the *fluid* parameters q/m and nq remain constant (see e.g. Rostoker and Rosenbluth 1960). In this limit the individual plasma particles are smeared out into a continuous fluid in phase space, coupling of individual plasma particles to the radiation field is lost, and consequently there can be no fluctuations of the electric or magnetic fields.

A rigorous theory of fluctuations can be derived from the BBGKY hierarchy; however, a physically appealing and much more tractable approach is to use the concept of 'dressed' test particles (Rostoker and Rosenbluth 1960; Rostoker 1964*a*,*b*). The theory of test particles can be used to predict many effects, such as fluctuations of macroscopic quantities, which occur at order g. The lack of generality of the test-particle theory is compensated for by a substantial gain in tractability.

Space and astrophysical plasmas are frequently observed to possess particle populations that are not in thermal equilibrium, i.e. their single-particle velocity distribution functions are not Maxwellian. Moreover, these deviations from thermal equilibrium often take the form of a high energy tail – with an approximate power-law dependence on velocity – on one or more of the population's particle distribution functions, i.e. the plasma contains a superabundance of particles with velocities in excess of the thermal speed. These particles are known variously in the space and astrophysical literature as suprathermal or superthermal particles. We shall adopt the former terminology in this paper.

The theory of normal wave modes in plasmas containing suprathermals has recently been discussed by a number of authors. Many plasma kinetic models (Summers and Thorne 1991; Thorne and Summers 1991; Meng *et al.* 1992; Xue *et al.* 1993, 1996; Mace and Hellberg 1995; Mace 1996*a*,*b*) of these suprathermal tail particles have assumed that the single-particle velocity distribution on the nonthermal

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component(s) obey the so-called kappa distribution

$$f_\alpha(v) = (\pi\kappa_\alpha\theta_\alpha^2)^{-3/2}\frac{\Gamma(\kappa_\alpha+1)}{\Gamma(\kappa_\alpha-\frac{1}{2})}\left(1+\frac{v^2}{\kappa_\alpha\theta_\alpha^2}\right)^{-(\kappa_\alpha+1)}, \tag{1}$$

where $\theta_\alpha = [(2\kappa_\alpha-3)/\kappa_\alpha]^{1/2}(T_\alpha/m_\alpha)^{1/2}$ is a modified thermal speed,

$$T_\alpha \equiv \int d^3v\, f_\alpha(v)(\tfrac{1}{2}m_\alpha v^2) = 2\pi m_\alpha \int dv\, v^4 f_\alpha(v),$$

$\Gamma(z)$ is the gamma function and κ_α is chosen so as to best fit the observed power-law tail spectral index. This often means that κ_α must be chosen to be a *real-valued* quantity (see e.g. Christon *et al.* 1988).

The study of normal wave modes and instabilities in plasmas with suprathermals has been facilitated by the introduction by Summers and Thorne (1991) of a plasma dispersion function denoted by Z_κ^*, based on the kappa distribution (1), just as the Fried and Conte Z function, is based on the Maxwellian velocity distribution. This function, which is valid only for integer values of the spectral index κ, was generalized with the introduction of the function Z_κ, to include arbitrary *real* values of κ, by Mace and Hellberg (1995). Moreover, Mace and Hellberg (1995) showed the close relation of Z_κ to the Gauss hypergeometric function ${}_2F_1[1, 2\kappa+2; \kappa+2; z]$, establishing a useful connection that enables manipulation of dispersion relations using the well-established theory of the hypergeometric function. Grabbe and Venturino (1996) have performed an asymptotic expansion of Z_κ in real values of κ. Their asymptotic expansions involve the Fried and Conte Z function, and are useful for the kinetic description of plasmas that have intermediate values of κ. Summers *et al.* (1996) have employed an analysis similar to that of Mace and Hellberg (1995) to obtain closed-form expressions for $Z_{n+1/2}(\zeta)$ where $n = 1, 2, \ldots$, in terms of rational polynomials in ζ and log functions.

Summers *et al.* (1994) showed that when a constant magnetic field $\mathbf{B}_0$ permeates the plasma, the dielectric-tensor elements, valid for arbitrary wavevector direction, can be expressed as doubly infinite sums of integrals over products of Bessel functions and Z_κ^*. Xue *et al.* (1996) have recently investigated numerically the growth/damping of oblique electromagnetic ion-cyclotron waves using these expressions. Summers and Thorne (1995) have also generalized these results to the case when the plasma particles can be adequately modelled by a loss-cone-like generalized Lorentzian. Recently Mace (1996*a*) has derived an alternative form for the dielectric tensor of an isotropic generalized Lorentzian plasma. In this form the dielectric tensor has a formal structure similar to Trubnikov's (1959) dielectric tensor for a relativistic plasma in thermal equilibrium. The result is expressed in terms of an integral over the modified Bessel functions $K_\nu(z)$, and, like the tensor of Summers *et al.* (1994), reduces to the known case, involving Z_κ (or Z_κ^*), for wavevectors $\mathbf{k}$ parallel to $\mathbf{B}_0$. These results have been compared with the Maxwellian equivalent and extended to loss-cone-like Lorentzian distributions by Mace (1996*b*).

From the point of view of fluctuations and correlations, perhaps one of the more interesting characteristics of a plasma containing suprathermal particles and modelled by the kappa distribution is the dependence of the Debye shielding length on the power-law index κ. It will be shown later (see also Chateau and Meyer-Vernet (1991), who deduced this length from the wave dispersion relation, and, more recently, Bryant (1996)) that the Debye shielding length in a plasma comprising

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particle populations with kappa distributions is given by

$$\lambda_D = \left(\sum_\alpha \frac{n_\alpha q_\alpha^2}{\left(\frac{\kappa_\alpha - \frac{3}{2}}{\kappa_\alpha - \frac{1}{2}} \right) \epsilon_0 T_\alpha} \right)^{-1/2} = \left(\sum_\alpha \frac{1}{\lambda_{\kappa\alpha}^2} \right)^{-1/2}, \tag{2}$$

with the Debye length of particle population α given by

$$\lambda_{\kappa\alpha} = \left[\left(\frac{\kappa_\alpha - \frac{3}{2}}{\kappa_\alpha - \frac{1}{2}} \right) \frac{\epsilon_0 T_\alpha}{n_\alpha q_\alpha^2} \right]^{1/2}. \tag{3}$$

One observes that the smaller the value of κ_α, the smaller is the Debye length of that species and the smaller the total Debye length. For example, for space plasmas where the value of $\kappa_\alpha \approx 2$ is not uncommon, the Debye length is only

$$\lambda_{D\alpha} \approx \frac{1}{\sqrt{3}} \lambda_{D\alpha,M} \approx 0.58 \lambda_{D\alpha,M}$$

where $\lambda_{D\alpha,M} = (\epsilon_0 T_\alpha / n_\alpha q_\alpha^2)^{1/2}$ is the familiar definition of the Debye length for a Maxwellian plasma; and hence the interaction volume $\sim \lambda_{D\alpha}^3$ is reduced by a factor of five. For values of κ_α approaching the theoretical lower limit (see Sec. 2) of $\kappa_\alpha = \frac{3}{2}$, the Debye length approaches zero, while in the opposite limit, i.e. as $\kappa_\alpha \to \infty$, the familiar shielding length for a Maxwellian plasma is obtained.

A consequence of these potentially small values of λ_D is that the plasma parameter

$$g \equiv \frac{1}{n \lambda_D^3}, \tag{4}$$

whose *smallness* may be interpreted as a measure of the degree to which plasma collective effects dominate over single-particle behaviour, is usually larger than for an ordinary Maxwellian plasma. With the possibility of finite but small, as opposed to infinitesimal, g, we expect higher-order (in g, i.e. effects at orders greater that $O(g^0)$) correlations and their physical effects, and single-particle behaviour to play an enhanced role in space plasmas containing suprathermal particles.

It is these effects, which depend on the discreteness of the plasma particles, that will be the topic of this paper. In particular, we shall be interested in fluctuations of the electric field in a plasma containing suprathermal particles. Such an investigation is important in two respects. First, it provides us with a theoretical baseline fluctuation level for a stable, *nonthermal* plasma. This permits one to recognize immediately, in measurements of fluctuations, whether the source is a plasma instability. This can be important when the plasma contains suprathermals, since the level of fluctuations produced by particle thermal motion is higher than it is for a Maxwellian, which might lead an observer to deduce incorrectly – on the basis of trends for a Maxwellian plasma – that some kind of instability mechanism is operative. Secondly, the spectroscopy of thermal noise near the plasma frequency is currently used for measuring electron plasma parameters in space plasmas (see Meyer-Vernet and Perche 1989, and references therein) and a knowledge of how suprathermals affect this region is imperative for a better understanding and more precise measurements.

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Chateau and Meyer-Vernet (1991) briefly investigated fluctuations in a plasma with electrons whose single-particle distribution function was given by the kappa distribution (1). However, they assumed stationary ions, considered only integer values of κ, and were primarily concerned with antennae effects. In this paper we assume that both ion and electron velocity distributions are given by the kappa distribution; we allow each component α to possess its own *real-valued* spectral index κ_α; and we investigate in detail the effects of spectral index and temperature on the spectral density of the electric field fluctuations. We re-analyse the normal modes in such a plasma in terms of the correct Debye length for a plasma comprising kappa distributions, and, wherever possible, we use these results to interpret the numerical results obtained from the fluctuation theory.

The organization of the paper is as follows. In Sec. 2 we introduce the basic equations and some notation that we shall use in our calculation of fluctuations and related phenomena. Discussion of collective plasma wave effects, i.e. the plasma normal wave modes follows in Sec. 3. In Sec. 4 we outline the calculation of the spectral density of the fluctuations, and in Sec. 5 we numerically evaluate the spectral density of fluctuations of the electric field and discuss these results in the light of conclusions drawn from Sec. 2 and 3. Finally we summarize some of our results and conclusions in Sec. 6.

2. Basic equations

In the test-particle theories that follow, we consider only the Coulomb fields produced by the particles, i.e. we consider a purely electrostatic situation. This is a reasonable assumption provided the plasma is not relativistically hot. The basic equations are therefore the Vlasov equation for each particle species α,

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} - \frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{x}}\phi(\mathbf{x},t)\cdot\frac{\partial}{\partial \mathbf{v}}\right] f_\alpha(\mathbf{x},\mathbf{v},t) = 0, \tag{5}$$

and Poisson's equation

$$\frac{\partial^2}{\partial \mathbf{x}^2}\phi(\mathbf{x},t) = -\frac{\rho_{\text{plas}}(\mathbf{x},t)}{\epsilon_0} - \frac{\rho_{\text{test}}(\mathbf{x},t)}{\epsilon_0}, \tag{6}$$

where the charge density of the plasma is given by

$$\rho_{\text{plas}}(\mathbf{x},t) = \sum_\alpha n_\alpha q_\alpha \int d^3\mathbf{v}\, f_\alpha(\mathbf{x},\mathbf{v},t) \tag{7}$$

and the charge density due to a moving test particle is given by

$$\rho_{\text{test}}(\mathbf{x},t) = q_T\delta(\mathbf{x} - \mathbf{x}_0' - \mathbf{v}'t). \tag{8}$$

We assume that the test charge produces a small perturbation of the plasma equilibrium state; this permits the following linearization scheme (see e.g. Krall and Trivelpiece 1973, p. 369):

$$\phi(\mathbf{x},t) = \phi^{(0)} + \varepsilon\phi^{(1)}(\mathbf{x},t), \tag{9a}$$

$$f_\alpha(\mathbf{x},\mathbf{v},t) = f_\alpha^{(0)}(\mathbf{v}) + \varepsilon f_\alpha^{(1)}(\mathbf{x},\mathbf{v},t). \tag{9b}$$

Equations (5)–(9*b*) will form the basic set of equations for this paper. Before continuing, we shall briefly motivate our choice of Debye length for a plasma containing suprathermals and introduce some other parameters.

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We seek the perturbation to the potential generated by a stationary test charge, i.e. we set $\mathbf{v}' = 0$ in (8). Since the charge is stationary, we seek a time-stationary solution to (5) and (6), after linearization. This is readily accomplished through the use of the Fourier transform (Krall and Trivelpiece 1973, p. 558); the final inversion for $\phi^{(1)}(\mathbf{x})$ can be carried out using residue theory. This yields the result (Krall and Trivelpiece 1973, p. 558)

$$\phi^{(1)}(\mathbf{x}) = \frac{q_T}{4\pi\epsilon_0 \mid \mathbf{x} - \mathbf{x}_0' \mid} \exp\left[- \mid \mathbf{x} - \mathbf{x}_0' \mid \left(-2 \sum_\alpha \omega_{p\alpha}^2 \int d^3\mathbf{v} \, \frac{\partial}{\partial v^2} f_\alpha^{(0)}(v^2) \right)^{1/2} \right], \tag{10}$$

where the plasma frequency $\omega_{p\alpha}$ of particle species α is defined by

$$\omega_{p\alpha}^2 = \frac{n_\alpha q_\alpha^2}{\epsilon_0 m_\alpha}.$$

This solution is of the form

$$\phi^{(1)}(\mathbf{x}) = \frac{q_T}{4\pi\epsilon_0 \mid \mathbf{x} - \mathbf{x}_0' \mid} \exp\left(- \frac{\mid \mathbf{x} - \mathbf{x}_0' \mid}{\lambda_D} \right),$$

which permits us to identify the Debye length λ_D with the quantity

$$\lambda_D \equiv \left(-2 \sum_\alpha \omega_{p\alpha}^2 \int d^3\mathbf{v} \, \frac{\partial}{\partial v^2} f_\alpha(v) \right)^{-1/2}. \tag{11}$$

Substituting the kappa distribution for the $f_\alpha(v)$ yields (2), which permits us to define the Debye length per particle population by (3), where we introduce the notation $\lambda_{\kappa\alpha}$ to avoid confusion with the traditional notation for the Debye length $\lambda_{D\alpha}$, and to emphasize the dependence of this parameter on κ. Bryant (1996) has recently derived a similar expression for the Debye length in a plasma comprising kappa distributions, and our notation conforms to his.

It is clear that the definition of the Debye length given by (3) has associated with it the usual physical interpretation of the distance travelled by a typical thermal particle ($v \sim \theta_\alpha$, where $\theta_\alpha = [(2\kappa_\alpha - 3)/\kappa_\alpha]^{1/2}(T_\alpha/m_\alpha)^{1/2}$ and $T_\alpha = \langle \frac{1}{2} m_\alpha v^2 \rangle_v$) in a typical α-plasma oscillation time $\sim 1/\omega_{p\alpha}$.

It is interesting to investigate (3) for the limiting values of κ, i.e. $\kappa \to \frac{3}{2}$ and $\kappa \to \infty$. The lower κ limit is the constraint that arises from the physical requirement that the average particle kinetic energy not diverge. Clearly, as $\kappa_\alpha \to \frac{3}{2}$ (from above), the Debye length $\lambda_{\kappa\alpha} \to 0$ independently of plasma temperature. Thus the interaction volume $\sim \lambda_{\kappa\alpha}^3$ for the Coulomb electric field vanishes! That is, the Coulomb field of an oppositely charged, stationary particle is completely screened by the particles of species α; only the exponentially small 'residual' field, which contributes to the mean field, remains. As a consequence of this complete screening, particles of species α will be *very strongly* correlated with plasma particles of opposite charge. As alluded to in Sec. 1, this means that fluctuations, correlations and other still higher-order effects, such as Bremsstrahlung, can be quite important in plasmas containing suprathermals.

For $\kappa_\alpha \to \infty$ the Debye length reduces to the familiar Debye shielding length for a Maxwellian plasma. This is not unexpected, since in this limit the kappa distribution reduces to the Maxwellian.

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The Debye length or electrostatic shielding distance in a plasma modelled with generalized Lorentzian distributions is thus always smaller than the equivalent length for a Maxwellian plasma $\lambda_{D,M}$, and therefore discrete-particle effects are relatively more important than they are for Maxwellian plasmas. In terms of the fluctuations that are investigated later, we shall see that this fact is manifested by a large increase in the level of intensity.

3. Normal modes

In the electrostatic approximation, the dielectric function for a multi-species plasma can be written as (see e.g. Krall and Trivelpiece 1973, p. 373)

$$\epsilon(\mathbf{k},\omega) = 1 + \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2} \int d^3\mathbf{v}\, \frac{\mathbf{k}\cdot\partial f_{\alpha}^{(0)}(\mathbf{v})/\partial\mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v}}, \qquad \mathrm{Im}\,\omega > 0, \tag{12}$$

where, for convenience in inverting the Laplace transform, this result can be analytically continued to $\mathrm{Im}\,\omega \leqslant 0$ via the Landau contour. Employing the kappa distribution (1) for the particle distributions in (12), Summers and Thorne (1991) and later Mace and Hellberg (1995) derived the following dielectric function:

$$\epsilon(\mathbf{k},\omega) = 1 + 2\sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2\theta_{\alpha}^2}\left[\frac{\kappa_{\alpha} - \frac{1}{2}}{\kappa_{\alpha}} + \frac{\omega}{k\theta_{\alpha}} Z_{\kappa_{\alpha}}\left(\frac{\omega}{k\theta_{\alpha}}\right)\right], \tag{13}$$

where $Z_{\kappa}(\zeta)$ is the dispersion function of Mace and Hellberg (1995), which is valid for arbitrary real, positive κ.

As is well known, the normal modes of the plasma are defined as the zeros of $\epsilon(\mathbf{k},\omega)$; these zeros define the poles whose residues give, in the time-asymptotic limit, the largest contribution to the inverse Laplace transform of the electrostatic plasma potential. Indeed, it is only in this time-asymptotic sense that we may talk about a 'dispersion relation'. For short times after the initial charge disturbance, and as we shall see later for the higher-order (in g) fluctuations, it is not merely the zeros of the dielectric function that influence the plasma behaviour. Instead, the dielectric function as a whole must be considered. Nevertheless, knowledge of the normal modes is useful in interpreting the fluctuation results, since these weakly damped wave modes are the ones that undergo the largest thermal excitation. These normal modes have been investigated previously by Thorne and Summers (1991) and Mace and Hellberg (1995), and we provide only a brief review here. For the ion-acoustic wave we re-interpret the results in terms of the relevant Debye length $\lambda_{\kappa\alpha}$.

3.1. Low-frequency ion-acoustic wave

The dispersion relation for ion-acoustic waves in a plasma modelled by kappa distributions has been obtained before (Thorne and Summers 1991; Meng *et al.* 1992). These papers assumed that the ions and electrons shared the same spectral index κ. Here we allow ions and electrons each to have their own spectral index κ_{α}, and, moreover, we write the dispersion relation and damping in terms of the *natural* Debye length for the plasma $\lambda_{\kappa\alpha}$, which yields substantial simplification.

The dispersion relation for ion-acoustic waves can then be written as (cf. Thorne and Summers 1991; Meng *et al.* 1992)

$$\omega_r^2 = \frac{\omega_{pi}^2}{1 + 1/k^2\lambda_{\kappa e}^2} = \frac{k^2\lambda_{\kappa e}^2\omega_{pi}^2}{1 + k^2\lambda_{\kappa e}^2}, \tag{14a}$$

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and for the damping rate we obtain

$$\gamma = -\frac{\pi^{1/2}}{2^{3/2}} \frac{\Gamma(\kappa_e - 1)}{(\kappa_e - \frac{1}{2})^{1/2}\Gamma(\kappa_e + \frac{1}{2})} \frac{\omega_{pi}}{\omega_{pe}} \frac{k\lambda_{\kappa e}\omega_{pi}}{(1 + k^2\lambda_{\kappa e}^2)^2}$$
$$\times \left[1 + \frac{1}{2\kappa_e - 1}\left(\frac{\omega_{pi}^2/\omega_{pe}^2}{1 + k^2\lambda_{\kappa e}^2}\right)\right]^{-(\kappa_e+1)}$$
$$-\frac{\pi^{1/2}}{2^{3/2}} \frac{\Gamma(\kappa_i + 1)}{(\kappa_e - \frac{1}{2})^{3/2}\Gamma(\kappa_i - \frac{1}{2})} \left(\frac{\kappa_e - \frac{3}{2}}{\kappa_i - \frac{3}{2}}\right)^{3/2} \left(\frac{T_e}{T_i}\right)^{3/2} \frac{k\lambda_{\kappa e}\omega_{pi}}{(1 + k^2\lambda_{\kappa e}^2)^2}$$
$$\times \left[1 + \frac{1}{2\kappa_e - 1}\left(\frac{\kappa_e - \frac{3}{2}}{\kappa_i - \frac{3}{2}}\right)\left(\frac{T_e}{T_i}\right)\left(\frac{1}{1 + k^2\lambda_{\kappa e}^2}\right)\right]^{-(\kappa_i+1)}. \tag{14b}$$

The first term in (14*b*) corresponds to the Landau damping by electrons and is important at small wavenumbers, $k\lambda_{\kappa e} \ll 1$, and the second term correponds to Landau damping by the ion component, which dominates at large wavenumbers. The term involving $[1 + \ldots]^{-(\kappa_e+1)}$ is of the form $[1 + O(m_e/m_i)]^{-(\kappa_e+1)}$, which in most instances can be approximated by unity.

The well-known physics of the ion-acoustic wave is evident from the first equality in (14*a*). The ion-acoustic wave is a collective mode associated with the ions, which, in the absence of mobile electrons, would oscillate collectively with a frequency $\omega_r = \omega_{pi}$. This frequency is so low that when electrons are present, they are able, through their greater mobility, to follow the motion of the ions. The electrons respond adiabatically to the ion motion, thereby acting to screen, on length scales larger than $\lambda_{\kappa e}$, their electrostatic fields. The result of this screening is that an ion 'sees' only the sum of the residual fields, i.e. that attenuated part beyond distances of $\lambda_{\kappa e}$, of all the other ions, and the frequency of ion oscillations is modified by the 'screening factor' $1/(1 + 1/k^2\lambda_{\kappa e}^2)^{1/2}$.

It is important to note that it is the Debye length $\lambda_{\kappa e}$ that plays a role here, and not the 'traditional' Debye length – which is valid only for Maxwellian plasmas. When the Debye length $\lambda_{\kappa e}$ is used, the physics of the ion-acoustic wave in a plasma containing suprathermal particles is immediately apparent and is essentially the same as that for a Maxwellian plasma. It can be seen that the screening factor approaches unity for wavelengths less than $\lambda_{\kappa e}$. This is because for distances less than $\lambda_{\kappa e}$ the bare ion electric microfield is exposed, i.e. at these wavelengths there is no Debye shielding by the suprathermal electrons and the wave reduces to an ion oscillation.

We have plotted in Fig. 1 the frequency and damping rate of the ion-acoustic wave for a plasma where $T_e/T_i = 100$. The curves were obtained through numerical solution for the zeros of $\epsilon(\mathbf{k}, \omega)$ given by (13), without approximation. Normalization of the wavenumber is with respect to $1/\lambda_{De}$, rather than $1/\lambda_{\kappa e}$. This was done to emphasize the dependence of the wave dispersion on κ_e, when viewed in terms of 'conventional' parameters. One can clearly observe the dependence of the ion sound speed v_{si}^{κ} on κ_e, which is predicted by (14*a*):

$$v_{si}^{\kappa} = \lambda_{\kappa e}\omega_{pi} = \left[\left(\frac{\kappa_e - \frac{3}{2}}{\kappa_e - \frac{1}{2}}\right)\frac{T_e}{m_i}\right]^{1/2}. \tag{15}$$

From the first equality in the above, one observes that the ion-acoustic speed in

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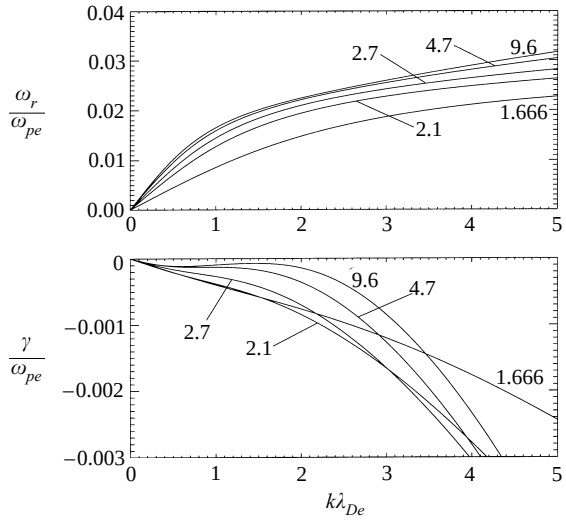


Figure 1. Real (*a*) and imaginary (*b*) parts of the complex frequency for the ion-acoustic wave, obtained through numerical solution of (13). In both Figs 1 and 2, strongly damped waves (such that $1/|\gamma| < 2\pi/\omega_r$, i.e. the damping time is less than the wave period) are indicated by dashes. The parameter $\kappa_e = \kappa_i$ labels the curves. Other parameters are $T_e/T_i = 100$ and $m_i/m_e = 1836$.

a plasma comprising kappa distributions is the Debye shielding length divided by a typical ion-oscillation time $\tau_{pi} = 1/\omega_{pi}$, i.e. it has the same physical meaning as it does in a Maxwellian plasma. Note the monotonic dependence of the dispersion curves on the parameter $\kappa_e = \kappa_i$. Furthermore, to within graphical accuracy, the dispersion curves for the cases κ_e, $\kappa_i > 5$ are essentially equivalent, but this observation does not hold for the damping rate.

For the Landau damping of the ion-acoustic wave, the general trend is that at small wavenumbers it increases as $\kappa_e = \kappa_i$ is decreased (see Fig. 1), i.e. as the number of suprathermal particles is increased. Indeed, when there is an abundance of suprathermal particles, i.e. $\kappa_i = \kappa_e < 3$ say, there is no window in $k\lambda_{De}$ space where one could claim that the ion-acoustic wave is very-weakly damped. This contrasts strongly with familiar theories of ion-acoustic wave damping in Maxwellian plasmas, where the wave is weakly damped at intermediate wavenumbers $k\lambda_{De} \approx 1$ (see the curve labelled $\kappa_e = \kappa_i = 9.6$ for a 'quasi-Maxwellian' example).

3.2. High-frequency Langmuir wave

Thorne and Summers (1991) and later Mace and Hellberg (1995) derived, through an expansion of (13) in asymptotic series, the following dispersion relation and damping rate respectively for the Langmuir wave:

$$\omega_r^2 = \omega_{pe}^2 + 3k^2 v_e^2, \tag{16a}$$

$$\gamma = -\pi^{1/2} \frac{\Gamma(\kappa_e + 1)}{\Gamma(\kappa_e - \frac{1}{2})} \omega_{pe} (2\kappa_e - 3)^{\kappa_e - 1/2} (k^2 \lambda_{De}^2)^{\kappa_e - 1/2}. \tag{16b}$$

The Langmuir wave is a high-phase-velocity wave, and Debye shielding does not play an important role in its properties; hence $\lambda_{\kappa e}$ does not appear in (16*a*).

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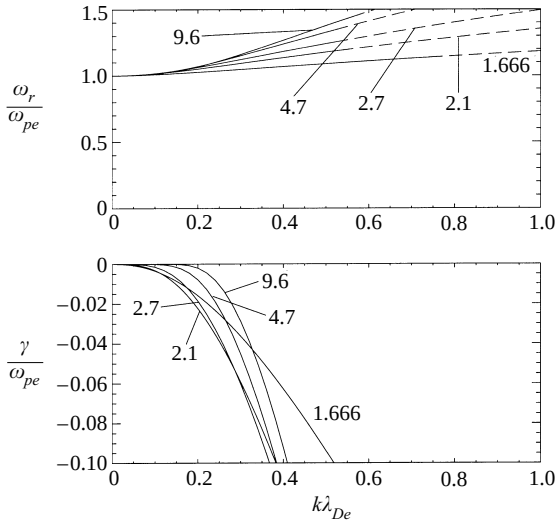


Figure 2. As for Fig. 1, but for the Langmuir wave.

Equations (16) are valid only for very small wavenumbers (Thorne and Summers 1991; Mace and Hellberg 1995). We have plotted in Fig. 2 the dispersion relation and damping rate of the Langmuir wave obtained through numerical solution for the zeros of (13) for a plasma in which $T_e/T_i = 100$. At larger wavenumbers, i.e. for $k\lambda_{De} > 0.2$ say, the dispersion relation (16*a*) clearly fails, since the wave dispersion becomes strongly dependent on the spectral index κ_e (for a more detailed discussion see Mace and Hellberg 1995).

4. Debye shielding of a moving test charge and fluctuations

The calculation of the spectral densities of various fluctuating plasma quantites can be done in a physically and mathematically rigorous manner by averaging these quantities at two consecutive times in phase space (Rostoker 1961). That is, one averages over the joint probability function $D_2(\mathbf{X}_\alpha;\ldots;\mathbf{X}_\gamma;t;\mathbf{X}'_\alpha;\ldots;\mathbf{X}'_\gamma;t')$, which expresses the probability density that the system will be in the volume $d\mathbf{X}_\alpha\cdots d\mathbf{X}_\gamma$ about the point $\mathbf{X} = \{\mathbf{X}_\alpha;\ldots;\mathbf{X}_\gamma\}$ at time t given that it was in the volume $d\mathbf{X}'_\alpha\cdots d\mathbf{X}'_\gamma$ about the point $\mathbf{X}' = \{\mathbf{X}'_\alpha;\ldots;\mathbf{X}'_\gamma\}$ at time t' (Rostoker 1961). The subscripts α, β, ... denote the different plasma particle species.

A perhaps more physically appealing method – and one that is mathematically simpler and yields the same result – is the method of superposition of dressed test particles (Rostoker 1961, 1964*a*,*b*). In this method we single out one plasma particle and treat it as if it were immersed in a dielectric medium characterized by the dielectric function $\epsilon(\mathbf{k},\omega)$, which is the same dielectric function that one can derive from the linearized Vlasov–Poisson system of equations (see (12)). The singled-out particle polarizes the plasma, i.e. it acquires a shielding cloud – the so-called 'dressing' – consisting of an increased density of oppositely charged particles and a deficiency of those with like charge. Treating each plasma particle in turn as a shielded test particle and summing the fields induced by these particles yields the total plasma microelectric field. Performing the ensemble average of the square of this quantity using the Liouville function $F(\mathbf{X}_\alpha;\ldots;\mathbf{X}_\gamma;t) \equiv D_1(\mathbf{X}_\alpha;\ldots;\mathbf{X}_\gamma;t)$

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permits one to identify the spectral density. The theory is not new; however, we outline it here for completeness and to make clear the assumptions that go into the calculation of the spectral density.

To lowest order in the strength of interaction between a moving test charge and the background plasma, the test charge moves in a straight line:

$$\mathbf{x}'(t) = \mathbf{x}'_0 + \mathbf{v}'t. \tag{17}$$

The charge density associated with such a test particle is given by

$$\rho_T(\mathbf{x}, t; \mathbf{x}', \mathbf{v}') = q_T \delta(\mathbf{x} - \mathbf{x}'(t)) = q_T \delta(\mathbf{x} - \mathbf{x}'_0 - \mathbf{v}'t). \tag{18}$$

We linearize (5) and (6) in the usual manner and take first the Laplace transform in time and then the Fourier transform in each spatial component. The resultant system of algebraic equations

$$(s + i\mathbf{k}\cdot\mathbf{v}) f_\alpha^{(1)}(\mathbf{k}, \mathbf{v}, s) - i\frac{q_\alpha}{m_\alpha}\phi^{(1)}(\mathbf{k}, s)\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{v}} f_\alpha^{(0)}(\mathbf{v}) = f_\alpha^{(1)}(\mathbf{k}, \mathbf{v}, 0), \tag{19}$$

$$-\mathbf{k}^2\phi^{(1)}(\mathbf{k}, s) = -\frac{1}{\epsilon_0}\sum_\alpha n_\alpha q_\alpha \int d^3\mathbf{v}\, f_\alpha^{(1)}(\mathbf{k}, \mathbf{v}, s) - \frac{q_T}{\epsilon_0}\frac{\exp(-i\mathbf{k}\cdot\mathbf{x}'_0)}{s + i\mathbf{k}\cdot\mathbf{v}'} \tag{20}$$

are readily solved for $\phi^{(1)}(\mathbf{k}, s)$. We shall assume that initially, i.e. at $t = 0$, the test particle does not possess a shielding cloud, and hence we set $f_\alpha^{(1)}(\mathbf{k}, \mathbf{v}, 0) = 0$ for all particle species α.

In the presence of the plasma the test particle aquires a shielding cloud. If the time required by the test particle to acquire this shielding cloud is much less than the time between close-encounter Coulomb interactions ('collisions') – which disrupt the cloud – then the electric field produced by the test particle, or quasiparticle, is given by its time-asymptotic form. Thus in inverting the Laplace transform we consider only the contribution from the pole at $s = -i\mathbf{k}\cdot\mathbf{v}'$; the poles due to the zeros $s_j(\mathbf{k})$ of $\epsilon(\mathbf{k}, s)$ represent waves that Landau-damp in time for a stable plasma, and produce exponentially vanishing contributions in the time-asymptotic limit. We thus obtain the contribution to the (microscopic) electric field due to the presence of a test particle with $\mathbf{x}'$, $\mathbf{v}'$:

$$\mathbf{E}_T^{(1)}(\mathbf{x}; \mathbf{x}', \mathbf{v}') = -\frac{q_T}{\epsilon_0}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{i\mathbf{k}}{k^2}\frac{\exp[i\mathbf{k}\cdot(\mathbf{x} - \mathbf{x}')]}{\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}')}, \tag{21}$$

where $\epsilon(\mathbf{k}, \omega)$ is given by (12). It should be noted that there is implicit time dependence in (21) due to the time dependence of the position of the quasiparticle $\mathbf{x}' = \mathbf{x}'(t)$.

To calculate the mean square of $\mathbf{E} = \mathbf{E}^{(1)}$, and hence the spectral density, we assume that the plasma consists of N_α dressed particles, or quasiparticles of each species α in volume V. The total electric microfield $\mathbf{E}(\mathbf{x}, t; \mathbf{X}'_\alpha; \ldots; \mathbf{X}'_\gamma)$ is obtained by summing the electric microfields produced by each quasiparticle, i.e. we treat each particle in the plasma, in turn, as a Debye-shielded quasiparticle or test particle. Thus we see that (21) is of the nature of a Green function. In this case the macroscopic mean square electric field is given by

$$\langle \mathbf{E}^2(\mathbf{x}, t; \mathbf{X}'_\alpha; \ldots; \mathbf{X}'_\gamma)\rangle = \int d^{N_\alpha}\mathbf{X}'_\alpha \;\cdots\; d^{N_\gamma}\mathbf{X}'_\gamma \; F(\mathbf{X}'_\alpha; \ldots; \mathbf{X}'_\gamma; t) \times \sum_{i=1}^{N_\alpha+\ldots+N_\gamma}\;\sum_{j=1}^{N_\alpha+\cdots+N_\gamma} \mathbf{E}(\mathbf{x}; \mathbf{x}'_i, \mathbf{v}'_i)\cdot\mathbf{E}(\mathbf{x}; \mathbf{x}'_j, \mathbf{v}'_j), \tag{22}$$

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where $F(\mathbf{X}'_\alpha; \ldots; \mathbf{X}'_\gamma; t)$ is the density of quasiparticles in phase or Γ space, and $\mathbf{X}'_\alpha = \{\mathbf{x}'_{\alpha 1}, \mathbf{v}'_{\alpha 1}; \ldots; \mathbf{x}'_{\alpha N_\alpha}, \mathbf{v}'_{\alpha N_\alpha}\}$. We shall neglect correlations among the *quasiparticles*. This means that the only terms that survive in the above double summation are those for which $i = j$ (Rostoker 1964*a*). We are then able to carry out the integrations of all but one particle, yielding

$$\langle \mathbf{E}^2(\mathbf{x}, t; \mathbf{X}'_\alpha; \ldots; \mathbf{X}'_\gamma) \rangle = \sum_\alpha n_\alpha \int d^3\mathbf{x}' \int d^3\mathbf{v}' \, f_\alpha(\mathbf{v}') \mathbf{E}^2(\mathbf{x}; \mathbf{x}', \mathbf{v}'), \tag{23}$$

where, as is well known, the single-particle distribution function is defined by

$$f_\alpha(\mathbf{x}_{\alpha 1}, \mathbf{v}_{\alpha 1}, t) = V \int \prod_{\beta \neq \alpha} d^{N_\beta}\mathbf{X}_\beta \, d^3\mathbf{x}_{\alpha 2} \, d^3\mathbf{v}_{\alpha 2} \cdots d^3\mathbf{x}_{\alpha N_\alpha} d^3\mathbf{v}_{\alpha N_\alpha} \, F(\mathbf{X}_\alpha; \ldots; \mathbf{X}_\gamma; t). \tag{24}$$

The integral over $\mathbf{x}'$ is readily carried out, noting that

$$\int d^3\mathbf{x}' \, \exp\{i[(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{x}']\} = (2\pi)^3 \delta(\mathbf{k}_1 - \mathbf{k}_2),$$

where $\delta(\mathbf{x}) = \delta(x)\delta(y)\delta(z)$ is the three-dimensional Dirac delta function, yielding the following formula for the average electric field energy density level:

$$\langle \mathbf{E}^2(\mathbf{x}, t; \mathbf{X}'_\alpha; \ldots; \mathbf{X}'_\gamma) \rangle = \sum_\alpha \frac{n_\alpha q_\alpha^2}{\epsilon_0^2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int d^3\mathbf{v}' \frac{f_\alpha^{(0)}(\mathbf{v}')}{k^2 \mid \epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}') \mid^2}. \tag{25}$$

Now, for each given $\mathbf{k}$ the dielectric function is a function only of that component of $\mathbf{v}'$ directed along $\mathbf{k}$. We can thus integrate over the components of $\mathbf{v}'$ that are perpendicular to $\mathbf{k}$. Following Krall and Trivelpiece (1973, p. 373), we define the one-dimensional distribution function

$$F^{(0)}(u) = \int d^3\mathbf{v}' \, \delta\left(u - \frac{\mathbf{k} \cdot \mathbf{v}'}{k}\right) f_\alpha^{(0)}(\mathbf{v}'), \tag{26}$$

and the angular frequency $\omega = ku$, which is *real-valued*. With these definitions, we are able to write (25) in the form

$$\tfrac{1}{2}\epsilon_0 \langle \mathbf{E}^2(\mathbf{x}, t; \mathbf{X}'_\alpha; \ldots; \mathbf{X}'_\gamma) \rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_\alpha \frac{\pi n_\alpha q_\alpha^2}{\epsilon_0} \frac{F_\alpha^{(0)}(\omega/k)}{k^3 \mid \epsilon(\mathbf{k}, \omega) \mid^2}. \tag{27}$$

Equation (27) is of the form of an inverse Fourier transform in 'time' as well as in each spatial dimension. This permits us to identify the quantity

$$S(\mathbf{k}, \omega) = \sum_\alpha \frac{\pi n_\alpha q_\alpha^2}{\epsilon_0} \frac{F_\alpha^{(0)}(\omega/k)}{k^3 \mid \epsilon(\mathbf{k}, \omega) \mid^2}, \tag{28}$$

as the spectral density of the electric field fluctuations.

We should like to stress that the quantity ω appearing in the above equations is *not* the complex wave frequency. It is the quantity $\mathbf{k} \cdot \mathbf{v}'$, which is a purely real-valued quantity. This quantity is the frequency for Čerenkov resonance between a test particle and a wave. The real-valuedness of ω means that the expression appearing in (28) is a well-behaved function, since we never encounter a zero of $\epsilon(\mathbf{k}, \omega)$. The spectral density $S(\mathbf{k}, \omega)$ thus remains finite for all real values of $\mathbf{k}$ and ω as one would expect on physical grounds.

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For plasma particle components whose single-particle statistical distribution in velocity space is modelled by the kappa distribution, the dielectric function is given by (13). This equation is valid for all complex ω except for a partial cut along the negative imaginary axis (Mace and Hellberg 1995), and thus can be used for the case for which ω is real, i.e. with $\mathrm{Im}\,\omega = 0$.

The functions $F_\alpha^{(0)}(u)$ are readily calculated for the kappa distribution through use of the standard integral (Prudnikov *et al.* 1986, p. 295, equation(5))

$$\int_0^\infty dv\,(b^2+v^2)^{-(\kappa+1)} = \tfrac{1}{2}b^{-2(\kappa+1/2)}B(\tfrac{1}{2},\kappa+\tfrac{1}{2}) = \tfrac{1}{2}b^{-2(\kappa+1/2)}\frac{\Gamma(\frac{1}{2})\Gamma(\kappa+\frac{1}{2})}{\Gamma(\kappa+1)},$$

where $B(z,w)$ is the beta function and $\Gamma(z)$ is the gamma function. This yields for the one-dimensional distribution functions (cf. Summers and Thorne 1991, Table I)

$$F_\alpha^{(0)}(u) = (\pi\kappa_\alpha\theta_\alpha^2)^{-1/2}\frac{\Gamma(\kappa_\alpha)}{\Gamma(\kappa_\alpha-\frac{1}{2})}\left(1+\frac{u^2}{\kappa_\alpha\theta_\alpha^2}\right)^{-\kappa_\alpha}. \tag{29}$$

5. Numerical results

In this section we present results for a plasma comprising electrons and protons, each of whose single-particle distribution functions in velocity space is given by the kappa distribution. Our task is to evaluate numerically the spectral density (28) using the one-dimensional distribution functions defined by (29) and the dielectric function (13). In each of our figures we plot the natural logarithm of the quantity $S(\mathbf{k},\omega)/(T_e/\omega_{pe}) = \frac{1}{2}\epsilon_0\langle E^2(\mathbf{k},\omega)\rangle/(T_e/\omega_{pe})$, where the magnitude of the logarithm is depicted in terms of the intensity of the shade ('greyness').

As is to be expected, the most intense fluctuation levels appear in the vicinity of the plasma normal modes, i.e. where the values of (real-valued) ω and $\mathbf{k}$ lie in the vicinity of one of the zeros of $\epsilon(\mathbf{k},\omega)$. For the purely electrostatic scenario under consideration here there are only two normal modes: the low-frequency ion-acoustic wave whose dispersion relation is given approximately by (14*a*), and the higher frequency Langmuir wave whose dispersion is given approximately by (16*a*) for small wavenumbers, but becomes strongly κ-dependent at larger wavenumbers (Mace and Hellberg 1995).

These features are apparent in the diagrams. Figures 3–8 illustrate the level of fluctuations on a logarithmic scale, as a function of both ω/ω_{pe} and $k\lambda_{De}$, for increasing values of κ_e, which is set equal to κ_i. The figures represent a sequence in which the number of suprathermal particles, i.e. the suprathermal tail components of each species, are gradually decreased until in the final plot $\kappa_e = \kappa_i = \infty$, which represents a tail-less Maxwellian for each population. We have used the ordinary Maxwellian electron Debye length in the normalization for ease of comparison with previous work (e.g. Lund *et al.* 1996) and to emphasize the κ dependence. The hydrogen plasma ($m_i/m_e = 1836$) has an electron-to-proton temperature ratio of $T_e/T_i = 100$, so that both the Langmuir and ion-acoustic waves appear as normal modes.

The most obvious feature of the figures is that the larger the component of suprathermal particles (small κ_e, κ_i), the more intense is the level of electrostatic fluctuations, and the more widespread in (ω, k) space they are. Comparing the extremes denoted by Fig. 3 (which represents a plasma with a well-developed

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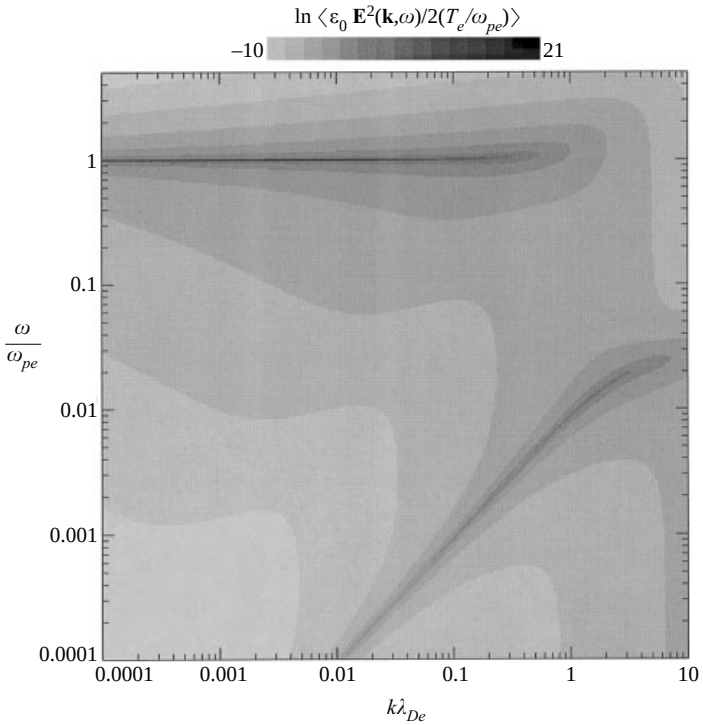


Figure 3. Greyscale plot of $\ln\langle\frac{1}{2}\epsilon_0 E^2(k,\omega)\rangle/(T_e/\omega_{pe})$ as a function of the logarithms of ω/ω_{pe} and $k\lambda_{De}$. Darker shades of grey indicate higher fluctuation levels. The value of $\kappa_e = \kappa_i = 1.666$. Other parameters are $T_e/T_i = 100$ and $m_i/m_e = 1836$.

suprathermal tail) with Fig. 8 (where the tail is completely absent) makes this point abundantly clear.

At this juncture a comment on our numerical technique is in order. In the numerical generation of our plots of $S(\omega, \mathbf{k})$ we have sampled the function on a rectangular matrix of *discrete* points $(\mathbf{\Omega}_n, K_n)$, where n is a positive integer or zero, and $\mathbf{\Omega}_n$ and K_n are respectively the normalized frequency and wavenumber. A consequence of this is that our graphical resolution can be defined by $\Delta\mathbf{\Omega} = \mathbf{\Omega}_{n+1} - \mathbf{\Omega}_n$ in the direction of the $\mathbf{\Omega}$ axis and $\Delta K = K_{n+1} - K_n$ in the direction of the K axis. Thus any feature of $S(\omega, \mathbf{k})$ whose typical scale length L is smaller than either of $\Delta\mathbf{\Omega}$ or ΔK will not be reproduced in the figures owing to our coarse numerical sampling. This has the unfortunate consequence that for Maxwellian plasmas, or plasmas with large κ values, the delta-function-like behaviour of $S(\omega, \mathbf{k})$ (Krall and Trivelpiece 1973, p. 566; Sentman 1982; Lund *et al.* 1994) in the vicinity of $k\lambda_{De} \ll 1$, $\omega \sim \omega_{pe}$ will not be reproduced. As our main aim is the investigation of low to moderate values of κ where the peak of $S(\omega, \mathbf{k})$ in the vicinity of $\omega \approx \omega_{pe}$ for $k\lambda_{De} \ll 1$ is broadened well beyond our sampling resolution, we do not see this as a problem. When such delta-function-like behaviour is known to exist and is not reproduced in our figures, we shall be at pains to point it out. With this caveat, we continue.

In Fig. 8, for a Maxwellian plasma, there is in the left-hand portion, $k\lambda_{De} \leqslant 2 \times 10^{-4}$, a virtual absence of thermal fluctuations at the levels we are considering, with this fluctuation-deficient region increasing in size as one proceeds upwards in frequency. As we have previously mentioned, there is, in fact, a very narrow peak

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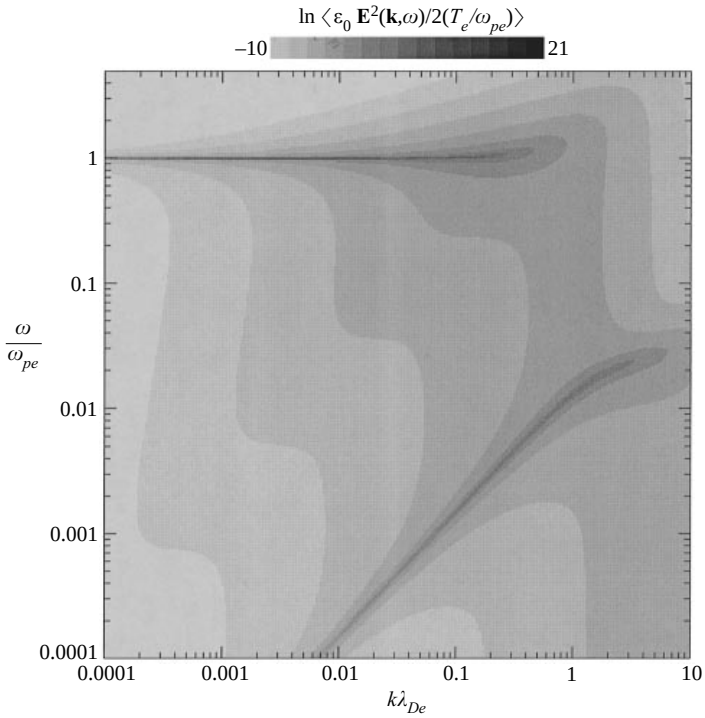


Figure 4. As Fig. 3, but for $\kappa_e = \kappa_i = 2.1$.

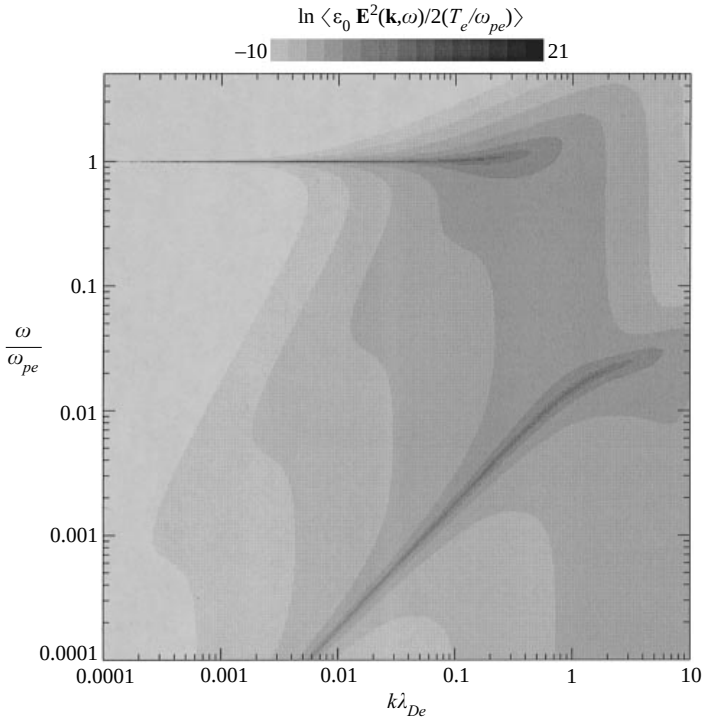


Figure 5. As Fig. 3, but for $\kappa_e = \kappa_i = 2.7$.

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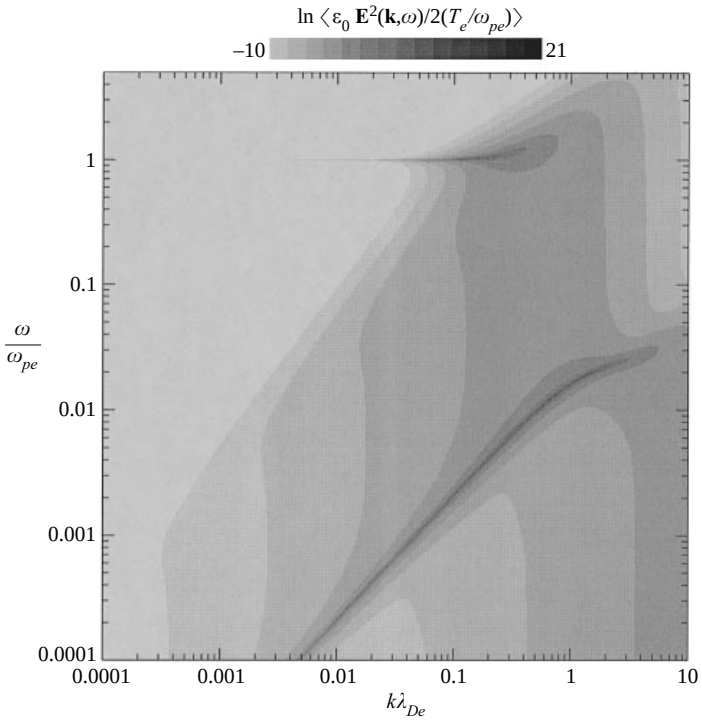


Figure 6. As Fig. 3, but for $\kappa_e = \kappa_i = 4.7$.

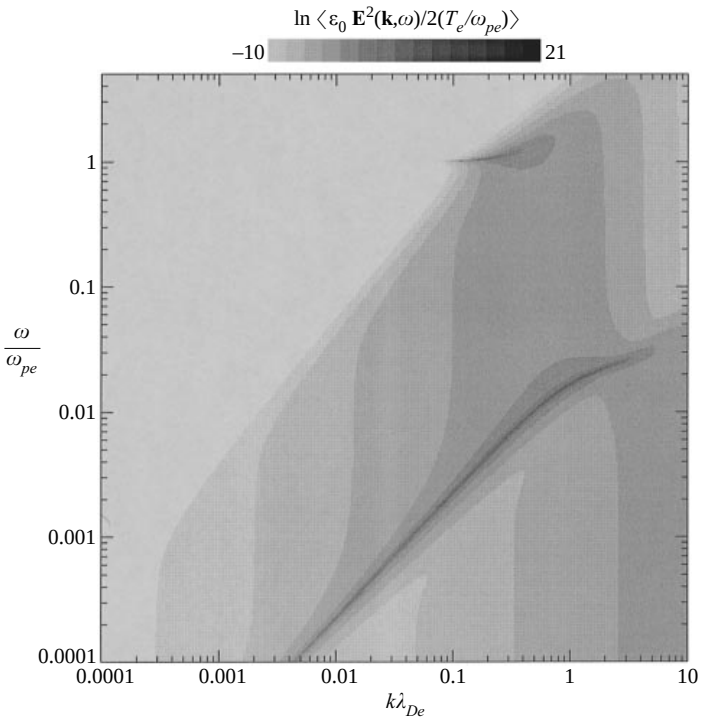


Figure 7. As Fig. 3, but for $\kappa_e = \kappa_i = 9.6$.

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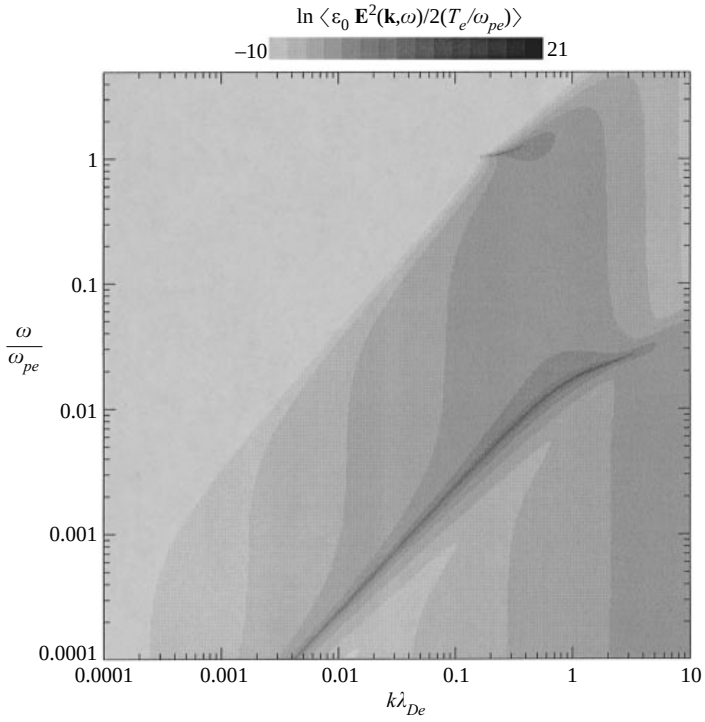


Figure 8. As Fig. 3, but for $\kappa_e = \kappa_i = \infty$, i.e. for Maxwellian distributions.

of fluctuations in a horizontal line at $\omega \approx \omega_{pe}$ extending in the k direction from $k\lambda_{De} \approx 0.0001$ to 0.15.

One also observes the rather rapid changeover to the approximate Maxwellian state as κ_e and κ_i are increased to only moderate values. For example, even at $\kappa_e = \kappa_i = 4.7 \ll \infty$ (Fig. 6), the fluctuation levels and their position in (ω, k) space are beginning to approach a Maxwellian 'configuration'. This is even more evident in Fig. 7, for $\kappa_e = \kappa_i = 9.6$.

One of the more striking features of the inclusion of large numbers of suprathermal particles is the broadening and the intensification of the region of fluctuations at and around the Langmuir branch. This is very apparent in Figs 3 and 4, where there are well-developed tail components, becoming less so in Figs 6 and 7, where the suprathermal tails are quite diminished by comparison. The narrowing, with increasing values of κ, of the peak of fluctuations around ω_{pe} to an approximately delta-function-like distribution in ω for $k\lambda_{De} \ll 1$ is apparent in the sequence of Figs 3–8. The position of the omitted delta-function-like peak in Figs 6–8 should be readily apparent when the sequence of Figs 3–8 is viewed with the above-mentioned narrowing in mind.

The broadening and intensification of fluctuations around ω_{pe} in the presence of suprathermal particles can have important implications for observers, since the spectral features in this frequency regime are frequently used as diagnostics for electron parameters (see Meyer-Vernet and Perche 1989, and references therein).

When the temperature of the ions is equal to the temperature of the electrons and $\kappa_i = \kappa_e$, the ion-acoustic wave is strongly ion Landau-damped, since the ion-acoustic speed approaches the ion thermal speed (see (15)). In this case the term

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due to the ions, i.e. the second term in (14*b*), dominates the rate at which the wave is temporally damped. Thus the 'wave' is not a normal mode of the plasma at all, since the damping time $\tau_d = 1/\mid\gamma\mid$ is less than the wave period $\tau = 2\pi/\omega$, meaning that any excitation of this branch is significantly damped within a wave period.

As far as fluctuations in the frequency range of the ion-acoustic branch are concerned this means that there is no sharply defined excitation of the ion-acoustic mode. Instead, one finds in this vicinity a broad region in (ω, k) space where there are enhanced fluctuations due to the continuous thermal excitation of wavenumbers and frequencies near this quasimode (sometimes referred to as a higher-order mode – see Gledhill and Hellberg 1986; Mace and Hellberg 1990).

The Langmuir wave, on the other hand, is only very slightly affected by the ion temperature, as long as the masses are sufficiently disparate – as in our hydrogen plasma. In the (ω, k) vicinity of the Langmuir wave the level of fluctuations and their distribution is very much the same as in the $T_i \ll T_e$ case of Figs 3–8. There is, however, a decrease in the *maximum* level of the fluctuations; for example, for $\kappa_e = \kappa_i = 1.666$ the maximum level of $\ln\langle\frac{1}{2}\epsilon_0 E^2(k,\omega)/(T_e/\omega_{pe})\rangle$ decreased from 21 for the case $T_e/T_i = 100$ to 19 for the isothermal case $T_i = T_e$, i.e. by a factor of approximately 7.

We illustrate these features in Figs 9 and 10. We show only the cases for which $\kappa_e = \kappa_i = 1.666$ (Fig. 9) and $\kappa_e = \kappa_i = 9.6$ (Fig. 10) for brevity. The trend is the same as that discussed in the previous paragraph for all those values of κ_e and κ_i depicted in Figs 3–8. For reasons previously discussed, the delta-function-like behaviour of $S(\omega, \mathbf{k})$ is not apparent in Fig. 10, but its shape can be inferred by comparison with the dark line situated at ω_{pe} in Fig. 9.

It is possible, in the ion term of the analytical expression for the damping rate for the ion-acoustic wave (14*b*), to group the ion and electron spectral indices with the temperature ratio as follows:

$$\left[\frac{(\kappa_e - \frac{3}{2})/(\kappa_e - \frac{1}{2})}{\kappa_i - \frac{3}{2}}\right]\left(\frac{T_e}{T_i}\right).$$

This combination of parameters arises from the ratio of the ion sound speed v_{si}^{κ} (see (15)) to the ion thermal speed θ_i, and plays a role analogous to the temperature ratio T_e/T_i in the analytical expression for the ion Landau damping of ion-acoustic waves in a Maxwellian plasma. It is therefore tempting, in a plasma containing suprathermals, to hold T_e/T_i constant and vary the ratio $[(\kappa_e - \frac{3}{2})/(\kappa_e - \frac{1}{2})]/(\kappa_i - \frac{3}{2})$ to observe whether the wave Landau damping can be reduced in a manner similar to that produced by increasing the temperature ratio T_e/T_i in a Maxwellian plasma. Note, however, that the dynamic range of this parameter is severely restricted. The electron contribution to this ratio, $(\kappa_e - \frac{3}{2})/(\kappa_e - \frac{1}{2})$, is always less than unity, and approaches this value asymptotically as κ_e is increased. To ensure that the ratio $[(\kappa_e - \frac{3}{2})/(\kappa_e - \frac{1}{2})]/(\kappa_i - \frac{3}{2})$ is greater than unity requires (i) large κ_e, i.e. $\kappa_e \gg \frac{3}{2}$, and (ii) small κ_i, i.e. $\kappa_i - \frac{3}{2} < 1$. These conditions are fairly restrictive, and mean that the electrons must be quasi-Maxwellian while the ions have been *very* efficiently energized to suprathermal speeds. Consequently, we do not expect that varying the ratio κ_e/κ_i would produce dramatic changes in the Landau damping rate of the ion-acoustic wave.

In Figs 11 and 12 we illustrate the level of fluctuations for plasmas containing electrons and ions with widely disparate numbers of suprathermals. In each of

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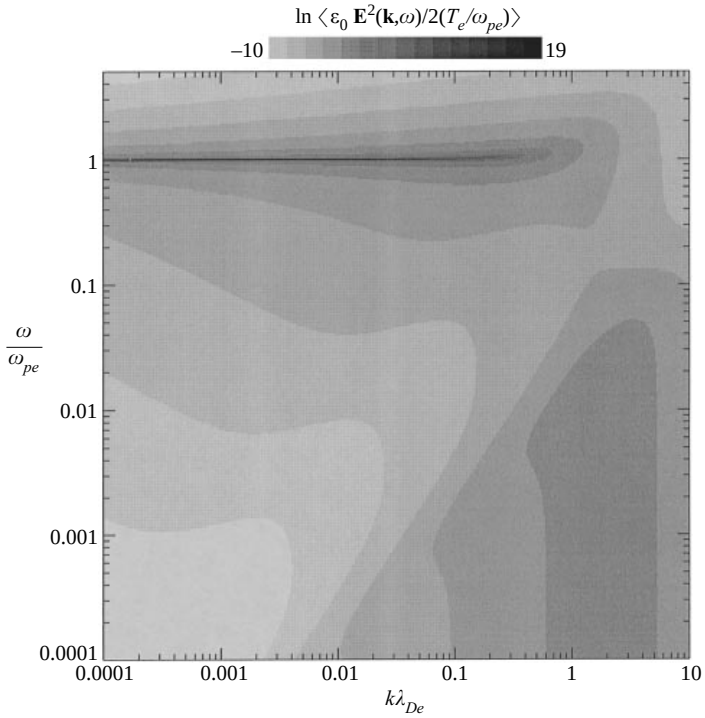


Figure 9. Greyscale plot of $\ln\langle\frac{1}{2}\epsilon_0 E^2(k,\omega)\rangle/(T_e/\omega_{pe})$ as a function of the logarithms of ω/ω_{pe} and $k\lambda_{De}$. Darker shades of grey indicate higher fluctuation levels. The value of $\kappa_e = \kappa_i = 1.666$. Other parameters are $T_e/T_i = 1$ and $m_i/m_e = 1836$.

these figures the plasma is isothermal, i.e. $T_e = T_i$, but the electrons and ions have very different values of κ.

In Fig. 11 the electrons have $\kappa_e = 1.666$ and thus possess a superabundance of high-energy particles, while the ions have $\kappa_i = 10.5$, only a slight excess of high-energy particles, and hence approximate a Maxwellian distribution in their statistical behaviour. One notes that the behaviour of $E^2(k,\omega)$ is very similar to that of Fig. 9, in which both electrons and ions had spectral indices of 1.666. The lack of suprathermal ions in Fig. 11 does not significantly alter the pattern of fluctuations in (ω, k) space. It does, however, reduce by a factor of e^1 the maximum level of fluctuations.

In Fig. 12 we illustrate the reverse scenario. The electrons have $\kappa_e = 10.5$, i.e. the electrons have only a small excess of very fast particles, while the ions have $\kappa_i = 1.666$, and hence an abundance of suprathermal particles. The ion-acoustic wave is strongly Landau-damped, as we should expect for $T_e = T_i$, but of interest in this figure is the enhanced level of fluctuations around the Langmuir branch. To see this, compare Fig. 12 with Fig. 10, where in the latter the ions and electrons have the same temperature and $\kappa_i = \kappa_e = 9.6$, which is close to the electron spectral index, $\kappa_e = 10.5$, in Fig. 12. When the ions have an equally large spectral index and thus as few energetic particles as the electrons (Fig. 10), the level of fluctuations at small k near ω_{pe} is too low to be depicted. However, increasing the number of ion suprathermals relative to the electrons, as in Fig. 12, produces, surprisingly, a significant level of fluctuations near ω_{pe} for $k\lambda_{De} \ll 1$. Thus the addition of

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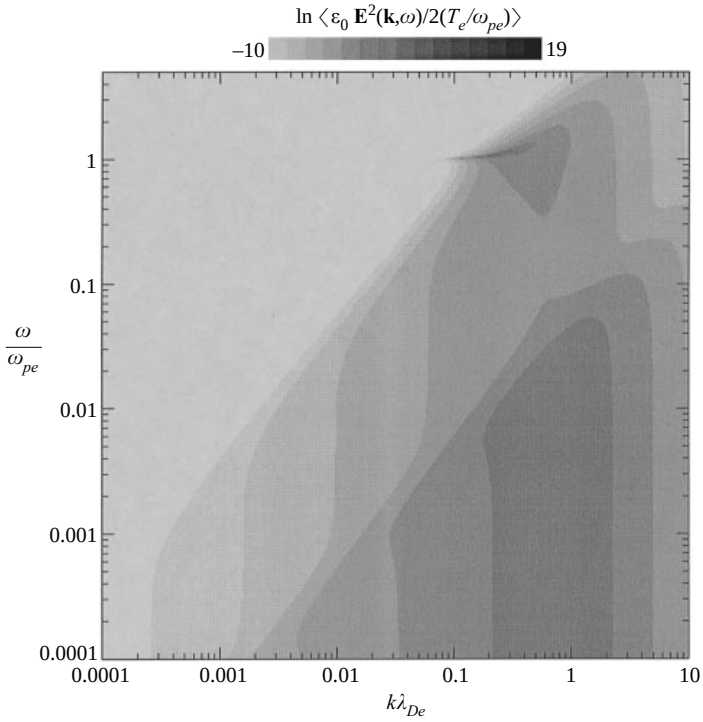


Figure 10. As Fig. 9, but for $\kappa_e = \kappa_i = 9.6$.

suprathermal ions can drastically alter the physics in the vicinity of ω_{pe}. The *excess* of mobile, suprathermal ions are able to thermally excite, through their random motion, even very high-phase-velocity waves such as the Langmuir wave. This result is at the same time surprising and interesting from the point of view that so many kinetic and fluid theoretical analyses of electron waves in the literature ignore ion dynamics altogether, because they are based on a Maxwellian distribution.

This phenomenon can be understood in terms of (28), which does not discriminate between particle species in terms of their ability to excite fluctuations (the summation contains terms involving each particle distribution function divided by the dielectric function). That is, from the point of view of thermal excitation, it is not so much the type of particle as the number in the region of the wave phase velocity. The ion dynamics do not play a large role in the wave *dispersion* at high frequency, i.e. the dielectric function is not strongly dependent on ion parameters in this region, and thus for simple normal-mode analyses one is perfectly justified in ignoring the ion dynamics. However, the level of thermal *excitation* does, in this case, depend strongly on the form of the ion distribution function, and one cannot necessarily ignore ion dynamics when thermal excitation of high-frequency waves is concerned.

We should like to caution, however, that we have employed a non-relativistic form for the kappa distribution. Owing to the slow power-law fall-off with velocity of this distribution, a significant fraction of particles can have velocities exceeding the speed of light, c. A relativistically correct formulation of the suprathermal tail particles would necessarily require $f(\mathbf{p}) \to 0$ as $|\mathbf{p}| \to \infty$, where $\mathbf{p}$ is the relativistic

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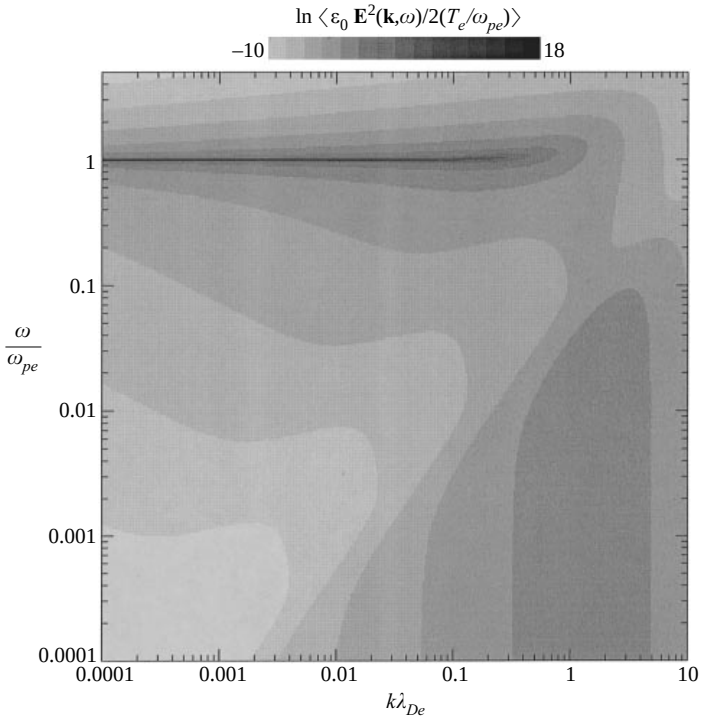


Figure 11. Greyscale plot of $\ln\langle \frac{1}{2}\epsilon_0 E^2(k,\omega)\rangle/(T_e/\omega_{pe})$ as a function of the logarithms of ω/ω_{pe} and $k\lambda_{De}$. Darker shades of grey indicate higher fluctuation levels. The value of $\kappa_e = 1.666$ and $\kappa_i = 10.5$. Other parameters are $T_e/T_i = 1$ and $m_i/m_e = 1836$.

momentum of a particle, and thus prevent the condition $v < c$ from being violated. While the Maxwellian distribution is not relativistically correct either, its faster fall-off with velocity makes it somewhat less prone to $v > c$ artefacts.

6. Summary and conclusions

We have investigated fluctuations in the electric field, and specifically their spectral density, in a plasma containing a superabundance of suprathermal particles. Suprathermal particles with an approximate power-law fall-off in velocity are a frequent feature of space plasmas. They are generated through a number of physical mechanisms, of which Fermi acceleration by shock waves (Drury 1983) and stochastic acceleration by electromagnetic (and electrostatic) waves are two. It is well known – and simple theories (Drury 1983; Blandford and Eichler 1987; Jones and Ellison 1991) predict – that particle acceleration by shock waves produces power-law tails. The power-law index predicted by these simple theories depends, remarkably, only on the shock compression ratio, and the power index itself is in general a real-valued quantity. The smaller the value of the power-law index, the larger the component of suprathermal particles. Part of the attractiveness of shock acceleration theory is that shocks are ubiquitous in space and astrophysics.

One of the surprising features of a plasma containing suprathermals is the strong dependence of the plasma Debye length on the spectral indices of the component power laws. This Debye length, as we have shown (see also Chateau and Meyer-

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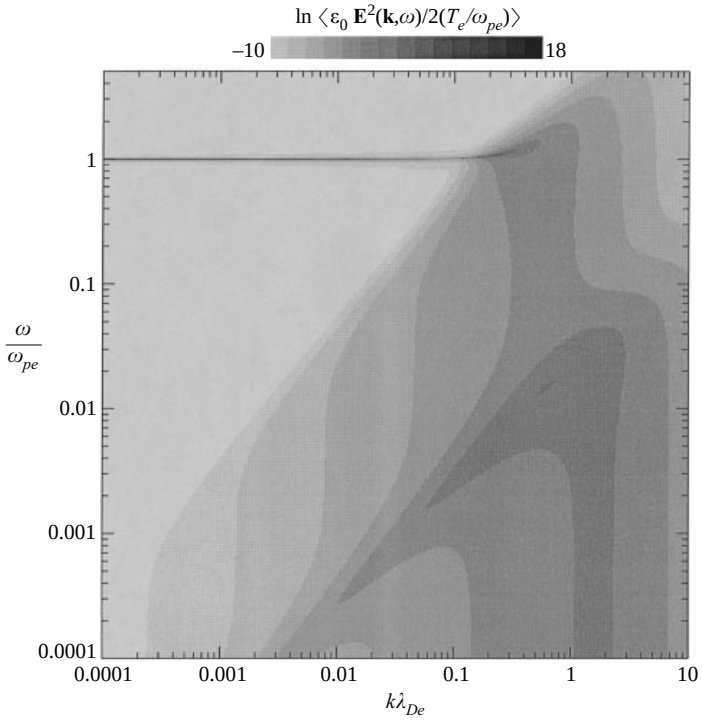


Figure 12. As Fig. 11, but for $\kappa_e = 10.5$ and $\kappa_i = 1.666$.

Vernet 1991; Bryant 1996), is always smaller than it is for a Maxwellian plasma, and consequently particle spatial correlations in plasmas containing suprathermals are relatively more important than they are for Maxwellians. Moreover, we have shown that for similar plasma densities and temperatures the plasma parameter $g = 1/n\lambda_D^3$ is larger than it is for a Maxwellian plasma, and consequently effects due to particle discreteness will be more important. What this means for fluctuations, as we have observed, is that the maximum level of electrostatic fluctuations in a plasma containing an excess of suprathermals can exceed that expected in a similar Maxwellian plasma by several hundred percent. Only when the spectral indices κ_α of the particles exceed about 10, and hence the fraction of suprathermal tail particles of each component is dramatically reduced, does the fluctuation level approach, and its spectral density closely resemble, the Maxwellian case.

The distribution of most intense fluctuations in (ω, k) space was centred on the electrostatic normal modes, i.e. the Langmuir wave and – if the electron temperature is significantly larger than the ion temperature – the ion-acoustic wave. For the case $T_e = T_i$ the level of fluctuations in the region below the ion plasma frequency ω_{pi} drops moderately, and one cannot clearly distinguish, fluctuations in an ion-acoustic mode; however, there are still *significant* levels of fluctuations at and below ω_{pi}.

It has been argued occasionally in the literature (see e.g. Lund *et al.* 1994) that the effects brought about by varying the spectral index κ for a kappa distribution are akin to varying the temperature of the Maxwellian, with the implication that one can crudely extrapolate results pertaining to a Maxwellian distribution to the

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kappa distribution by increasing the temperature. This is not true, in general, for the following reasons.

It is true that decreasing the spectral index produces more particles in the tail of the distribution, as does increasing the temperature of the Maxwellian. However, whereas increasing the temperature of the Maxwellian produces an overall 'flattening' of that distribution, decreasing κ in the kappa distribution produces a sharp steepening at velocities near the thermal speed. This can produce, in the suprathermal case, markedly different Landau damping rates for waves with phase velocities of the order of $v_{\rm th}$. When the wave phase velocity greatly exceeds the thermal speed one is on safer ground. However, care should be exercised in this region also, since the kappa distribution – with its power-law fall-off in velocity – is *always* steeper than the Maxwellian in the tail region, and, moreover, the probability of finding a particle there is much greater.

We have investigated, briefly, such effects in this paper. For the ion-acoustic wave we find that, with an initially isothermal and near-Maxwellian plasma, decreasing the spectral index of the electron distribution, κ_e, i.e. increasing the energetic tail electron population, *does not* produce the same effect as increasing the electron temperature relative to the ion temperature. Physical considerations severely restrict the dynamic range of the relevant 'temperature-ratio-like' parameter $[(\kappa_e - \frac{3}{2})/(\kappa_e - \frac{1}{2})]/(\kappa_i - \frac{3}{2})$ in this case. Consequently the level of fluctuations in the vicinity of the ion-acoustic mode is little affected by varying this ratio, while varying the temperature ratio produces a much larger effect.

An interesting effect emerges when one begins with a near-Maxwellian plasma and preferentially accelerates ions to suprathermal energies. In this case the presence of an overabundance of suprathermal ions produces enhanced levels of electrostatic fluctuations around ω_{pe}. This has important implications for the use of the spectral features around ω_{pe} as diagnostics for electron parameters (see Meyer-Vernet and Perche 1989, and references therein) in space plasmas, since the suprathermal ion population contributes significantly to the level of fluctuations. However, we expect that the inclusion of relativistic effects for the particles would diminish this phenomenon somewhat.

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