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Invariant random subgroups on certain orbits

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Abstract. Let G be a connected Lie group and Sub_G be the space of closed subgroups of G equipped with the Chabauty topology. In this article, we investigate the existence of invariant random subgroups of G supported on various orbits of the conjugation action of G on Sub_G.

Key words: connected Lie groups, algebraic groups, spaces of closed subgroups, Chabauty topology, invariant random subgroups
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1. *Introduction*

Let G be a connected (real) Lie group and Sub_G be the space of (closed) subgroups of G equipped with the Chabauty topology introduced by Chabauty in [6]. The Chabauty topology on Sub_G is generated by the subsets $\{H \in \mathrm{Sub}_G \mid H \cap K = \emptyset\}$, $\{H \in \mathrm{Sub}_G \mid H \cap U \neq \emptyset\}$, where $K \subset G$ is compact and $U \subset G$ is open. With respect to the Chabauty topology, Sub_G is a compact metrizable space (see [4, Lemma E.1.1]). Moreover, a sequence $H_n \in \mathrm{Sub}_G$ converges to $H \in \mathrm{Sub}_G$ if and only if for any $h \in H$, there is a sequence $h_n \in H_n$ such that h_n converges to h, and for any sequence $x_{k_n} \in H_{k_n}$, with $k_{n+1} > k_n$ and $x_{k_n} \to x$, we have $x \in H$ (see [4, Lemma E.1.2]).

The group G acts on Sub_G by conjugation: $(g, H) \mapsto gHg^{-1}$. An invariant random subgroup (IRS) is a probability measure μ on Sub_G that is invariant under the conjugation action of G on Sub_G. Every Dirac measure supported on a closed normal subgroup is an IRS. Also, if Γ is a lattice in G, that is, Γ is a discrete subgroup of G such that G/Γ carries a G-invariant probability measure μ, then the push forward of μ under the map $g\Gamma \mapsto g\Gamma g^{-1}$ defines an IRS on G. Thus, invariant random subgroups can be thought of

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as a generalization of both normal subgroups and lattices. The study of invariant random subgroups and ergodic theory is deeply intertwined. Any IRS on Sub_G can be realized as the stabilizer of a measure-preserving action of G on a probability space (cf. [1, 3] for more details).

We say that a Borel subset $\mathcal{O}$ of Sub_G supports an IRS μ if $\mu(\mathcal{O}) = 1$: a Borel subset $\mathcal{O}$ supporting an IRS μ need not be the support of μ, but $\overline{\mathcal{O}}$ contains the support of μ.

In recent times, IRS have attracted many studies (cf. [1–3, 5, 9] and the references therein). In this article, we are interested in the existence of invariant random subgroups supported on certain orbits of the conjugation action of G on Sub_G. We first consider the orbit of semisimple Levi subgroups. A maximal connected semisimple Lie subgroup of a Lie group G is called a Levi subgroup (see [13, p. 245] and [14, p. 19]). It is known that every connected Lie group has a Levi subgroup [13, Theorem 3.18.3]. Let L_G be the set of all semisimple Levi subgroups of G. Then, $L_G = \{xLx^{-1} \mid x \in N\}$, where N is the nilradical of G [13, Theorem 3.18.3]. In general, Levi subgroups need not be closed [14, Example on p. 19], but Levi subgroups are closed for linear groups (see [13, Exercises 40 and 41, Ch. 3] and [14, Proposition 6.1, §6 of Ch. 1]).

Let G be a connected Lie group with a semisimple Levi subgroup that is closed. An IRS μ on G is called a Levi IRS if $\mu(L_G) = 1$.

THEOREM 1.1. *Let G be a connected Lie group with nilradical N and L be a closed semisimple Levi subgroup of G. Then, the following are equivalent:*

(1) *there is an IRS supported on the set of all semisimple Levi subgroups;*
(2) *there is an N-invariant probability measure supported on semisimple Levi subgroups;*
(3) *the semisimple Levi subgroup L is normal in G.*

In particular, if G has a Levi IRS μ, then L is a normal subgroup and μ is the Dirac measure supported on the Levi subgroup L of G.

The following is a consequence of the above theorem.

COROLLARY 1.2. *The set of semisimple Levi subgroups L_G is closed in Sub_G if and only if L is normal.*

Thus, any connected Lie group G admitting a Levi IRS is of the form $G = L \times S$, where L is semisimple and S is solvable. In particular, for $n \geq 2$, $\mathrm{SL}(n, \mathbb{R}) \times \mathbb{R}^n$ admits a Levi IRS, whereas $\mathrm{SL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$ does not admit a Levi IRS.

As in the case of Levi subgroups, the set of all maximal compact subgroups of a connected Lie group is also the orbit of a maximal compact subgroup under the conjugation action. We next provide a necessary and sufficient condition for the set of maximal compact subgroups to support an IRS.

THEOREM 1.3. *Let G be a connected Lie group and K be a maximal compact subgroup of G. Also, let K_G be the set of all maximal compact subgroups of G. Then, there is an IRS supported on K_G if and only if K is normal in G.*

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As an application of Theorems 1.1 and 1.3, we obtain the following on the cocompactness of certain finite covolume subgroups.

COROLLARY 1.4. *Let G be a connected Lie group and H be a closed subgroup of G such that G/H carries a G-invariant probability measure. Suppose H is either a Levi subgroup or a maximal compact subgroup. Then, G/H is a compact group.*

We now assume that G is a real algebraic group, that is, G is the group of $\mathbb{R}$-points of an algebraic group defined over $\mathbb{R}$. Let B be a Borel subgroup of G, that is, B is a maximal connected solvable subgroup of G. Also, let B_G be the set of all Borel subgroups of G. Then, B_G is the orbit of B under the conjugation action. We now study the IRS supported on B_G, which involves the amenability of the group G equipped with its analytic topology.

We say that a locally compact group H is amenable if there is a left-invariant mean on $L^\infty(G)$. Abelian groups, compact groups, and solvable Lie groups are amenable; see [**12**] for more details regarding amenable groups. The next result concerns the existence of an IRS supported on B_G.

THEOREM 1.5. *Let G be a connected real algebraic group and B_G be the set of all Borel subgroups of G. Then, B_G is closed in* Sub_G*, and there is an IRS supported on B_G if and only if G, equipped with its analytic topology, is amenable.*

Example 1.6. Let $G = SO(3,\mathbb{R}) \ltimes \mathbb{R}^3$. We identify $O(2,\mathbb{R})$ as a subgroup of $SO(3,\mathbb{R})$ by the map $\alpha \mapsto \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \det(\alpha) \end{smallmatrix}\right)$. Then, $B = SO(2,\mathbb{R}) \ltimes \mathbb{R}^3$ is a Borel subgroup of G and the normalizer of B is $O(2,\mathbb{R}) \ltimes \mathbb{R}^3$. Thus, the $SO(3,\mathbb{R})$-invariant probability measure on the quotient space $SO(3,\mathbb{R})/O(2,\mathbb{R})$ induces an IRS on the set of Borel subgroups B_G of G.

Next, we consider the orbit of a maximal diagonalizable subgroup D. Here, by a diagonalizable subgroup, we mean diagonalizable over $\mathbb{R}$, that is, an $\mathbb{R}$-split torus. The set of all maximal diagonalizable subgroups then consists of conjugates of D; we denote this set by D_G.

THEOREM 1.7. *Let G be a connected real algebraic group and D_G be the set of all maximal diagonalizable subgroups of G. Then, the following are equivalent:*

(1) *there is an IRS supported on D_G;*
(2) *D is central in G;*
(3) *D is normal in G.*

Moreover, if statement (1) holds, then any Levi subgroup of G is compact and the solvable radical is a compact extension of its nilradical.

2. *Preliminary*

In this section, we now collect some results that connect the dynamics of G on Sub_G with projective linear actions on Grassmannians. Given a finite-dimensional vector space V, let $\mathrm{Gr}_k(V)$ be the Grassmannian manifold of k-dimensional subspaces of V. The following lemma compares the Grassmannian topology and the Chabauty topology of $\mathrm{Gr}_k(V)$ seen

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as a subset of Sub_V. Recall that the metric on $\mathrm{Gr}_k(V)$ is given by

$$d(W, W') = \|P_W - P_{W'}\|$$

for $W, W' \in \mathrm{Gr}_k(V)$, with P_W and P'_W being orthogonal projections onto the subspaces W and W', respectively. One may look at [9] for the measurability of the inclusion map $\mathfrak{i}\colon \mathrm{Gr}_k(V) \to \mathrm{Sub}_V$.

LEMMA 2.1. *Let V be a finite-dimensional vector space and $\mathrm{Gr}_k(V)$ be the Grassmannian manifold of subspaces of V of dimension k. Let $\mathfrak{i}\colon \mathrm{Gr}_k(V) \to \mathrm{Sub}_V$ be the canonical inclusion. Then, $\mathrm{Gr}_k(V)$ is homeomorphic to its image. In particular, the set of all subspaces of dimension k is a closed set in the Chaubauty topology.*

Proof. Let $\mathfrak{i}\colon \mathrm{Gr}_k(V) \to \mathrm{Sub}_V$ be the canonical inclusion and $Y = \mathfrak{i}(\mathrm{Gr}_k(V))$. Consider the canonical actions of the locally compact group $GL(V)$ on $\mathrm{Gr}_k(V)$ and on Sub_V. Fix $W \in \mathrm{Gr}_k(V)$. Then, the stabilizer of W and the stabilizer of $i(W)$ are the same, say H. Since $\mathrm{Gr}_k(V)$ is compact, and the canonical action of $GL(V)$ on $\mathrm{Gr}_k(V)$ is continuous and transitive, $Gr_k(V) \simeq GL(V)/H$. Therefore, $GL(V)/H$ is compact. Since the map $gH \mapsto g\mathfrak{i}(W)$ is a continuous surjective map on Y, we see that Y is compact. This implies that $GL(V)/H \simeq Y$. This shows that $\mathrm{Gr}_k(V)$ is homeomorphic to its image under $\mathfrak{i}$. □

It is known that if a sequence of closed subgroups H_n of a Lie group G converges to a closed subgroup H, then $\dim(H) \geq \limsup \dim(H_n)$ (cf. [15, Theorem 1]). However, in the case of closed subgroups of the same dimension, we have the following proposition.

PROPOSITION 2.2. *Let G be a connected Lie group and $H_n \to H$ in Sub_G. Let $\mathfrak{G}$, $\mathfrak{H}_n$, and $\mathfrak{H}$ be the Lie algebras of G, H_n, and H, respectively. If $\dim(H_n) = \dim(H)$ for all n, then $\mathfrak{H}_n \to \mathfrak{H}$ in $Sub_{\mathfrak{G}}$.*

Remark 2.3. Let Φ be the map sending a closed subgroup to its Lie algebra. Proposition 2.2 proves the continuity of Φ when restricted to certain subsets of Sub_G. The map Φ is known to be a measurable map (cf. [16]) which is sufficient for our purpose.

Proof. Let V be an accumulation point of $\mathfrak{H}_n$ in $\mathrm{Sub}_{\mathfrak{G}}$. By passing to a subsequence, we may assume that $\mathfrak{H}_n \to V$ in $\mathrm{Sub}_{\mathfrak{G}}$. Then, V is a vector subspace. The existence of structural constants of the Lie bracket $[\cdot, \cdot]$ of $\mathfrak{G}$ implies that the Lie bracket is continuous in both variables together (see [13, §2.2] for structural constants of a Lie algebra). This implies that V is a Lie subalgebra of $\mathfrak{G}$. For $v \in V$, there are $v_n \in \mathfrak{H}_n$ such that $v_n \to v$, and hence, $tv_n \to tv$ for all $t \in \mathbb{R}$. Let $\exp\colon \mathfrak{G} \to G$ be the exponential map. Then, $\exp(tv_n) \to \exp(tv)$ for all $t \in \mathbb{R}$. Since H_n converges to H, $\exp(tv) \in H$ for all $t \in \mathbb{R}$. This implies that $v \in \mathfrak{H}$. Thus, $V \subset \mathfrak{H}$. By assumption, $\dim(\mathfrak{H}_n) = \dim(\mathfrak{H})$ for all n, and hence by Lemma 2.1, $\dim(V) = \dim(\mathfrak{H}_{\mathfrak{n}})$. Since $V \subset \mathfrak{H}$, we get that $V = \mathfrak{H}$. Thus, $\mathfrak{H}_n \to \mathfrak{H}$ in $\mathrm{Sub}_{\mathfrak{G}}$. □

3. *On the support of random subgroups invariant under unipotent transformations*

In this section, we consider invariant measures on Grassmannian for projective linear actions and first prove a result on the support of the invariant measure. The result of the

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following kind is known as Furstenberg's Lemma (cf. [9, §5]) and it has been proved for minimally almost periodic groups in [8, Lemma 3] and when Γ (as in Lemma 3.1) is a certain semisimple analytic group in [9, Proposition 5.1]. Here, we prove the following using a result of [7].

LEMMA 3.1. *Let V be a finite-dimensional vector space and Γ be a subgroup of $GL(V)$. Also, let U be a subgroup of Γ consisting of unipotent transformations. Let $\mathrm{Gr}_k(V)$ be the Grassmannian manifold of k-dimensional subspaces of V and λ be a probability measure on $\mathrm{Gr}_k(V)$, which is invariant under the induced action of Γ on $\mathrm{Gr}_k(V)$. Also, let*

$$I_\lambda = \{\tau \in GL(V) \mid \tau \text{ is trivial on } S\},$$

where S is the support of λ. Then, Γ is contained in an algebraic group $\tilde{\Gamma}$ such that I_λ is a normal subgroup of $\tilde{\Gamma}$, $\tilde{\Gamma}/I_\lambda$ is a compact group, and S consists of U-invariant subspaces.

We first recall a result of [7], which is central to the proof of the lemma.

THEOREM. (See [7, Corollary 2.6]) *Let G and H be almost algebraic subgroups of $GL(n, R)$ for some $n \geq 1$, and suppose that H is contained in G. Also, let μ be a finite measure on G/H. If*

$$G_\mu = \{g \in G \mid \text{the } g\text{-action on } G/H \text{ preserves } \mu\}$$

and

$$J_\mu = \{g \in G \mid gx = x \text{ for all } x \in \operatorname{supp} \mu\}$$

(where the action refers to the action by translation on the left), then G_μ is an $\mathbb{R}$-algebraic subgroup of G containing J_μ as a normal subgroup and G_μ/J_μ is compact. In particular, G_μ is an amenable group if and only if J_μ is amenable.

Proof of Lemma 3.1. Let λ be the Γ-invariant probability measure on $\mathrm{Gr}_k(V)$. Let $G_\lambda = \{\tau \in GL(V) \mid \tau(\lambda) = \lambda\}$ and $I_\lambda = \{\tau \in GL(V) \mid \tau \text{ is trivial on } S\}$, where S is the support of λ. Since $\mathrm{Gr}_k(V) \simeq GL(V)/H$, where H is the stabilizer of a k-dimensional subspace, by [7, Corollary 2.6], we get that G_λ is a real algebraic group and I_λ is a normal algebraic subgroup of G_λ such that G_λ/I_λ is compact. Since τI_λ is unipotent in G_λ/I_λ for any unipotent $\tau \in G_\lambda$ and compact groups have no unipotent elements, we get that any unipotent transformation in G_λ is in I_λ. This proves the result. □

The following proposition is crucial to the proof of our results. It describes the action of the unipotent group on orbits that support invariant measures: for a connected Lie group G, $\mathrm{Aut}(G)$ has a natural action on the Lie algebra of G by identifying each $\tau \in \mathrm{Aut}(G)$ with its differential $d\tau$.

PROPOSITION 3.2. *Let G be a connected Lie group with the Lie algebra $\mathfrak{G}$ and S be a closed connected subgroup of G with the Lie algebra $\mathfrak{S}$. Let Γ be a σ-compact subgroup of $\mathrm{Aut}(G)$ and U be a subgroup of Γ consisting of unipotent automorphisms. If either:*

(1) *there is a Γ-invariant probability measure λ on $\mathrm{Sub}_{\mathfrak{G}}$ with $\lambda(\{d\tau(\mathfrak{S}) \mid \tau \in \Gamma\}) = 1$; or*

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(2) *there is a* Γ*-invariant probability measure* μ *on* Sub_G *with* $\mu(\{\tau(S) \mid \tau \in \Gamma\}) = 1$*; then* S *is invariant under the normal closure of* U *in* Γ*, that is, the smallest normal subgroup of* Γ *containing* U*.*

Proof. Let $X = \mathrm{Gr}_k(\mathfrak{G})$ be the Grassmannian manifold of k-dimensional subspaces of $\mathfrak{G}$, where k is the dimension of $\mathfrak{S}$. Consider the canonical action of Γ on X given by

$$(\tau, V) \mapsto d\tau(V) \quad \text{for } \tau \in \Gamma \text{ and } V \in X.$$

If condition (1) holds, then in view of Lemma 2.1, we get that λ is a Γ-invariant probability measure on X. By Lemma 3.1, λ is supported on U-invariant subspaces. Since λ is Γ-invariant, the support of λ is Γ-invariant. Now, for any W in support of λ, $\alpha \in U$, and $\tau \in \Gamma$, we have $d\alpha(d\tau^{-1}(W)) = d\tau^{-1}(W)$. Therefore, $d(\tau \circ \alpha \circ \tau^{-1})(W) = W$. This shows that $\mathfrak{S}$ is invariant under the smallest normal subgroup of Γ that contains U. Since S is a connected Lie subgroup, S is invariant under the smallest normal subgroup of Γ containing U.

Suppose condition (2) holds. Let $\mathcal{O} = \{\tau(S) \mid \tau \in \mathrm{Aut}(G)\}$ and $\phi\colon \mathcal{O} \to X$ be defined by $\phi(\tau(S)) = d\tau(\mathfrak{S})$. Since the Lie algebra of $\tau(S)$ is $d\tau(\mathfrak{S})$, ϕ is well defined. Since $d(\alpha \circ \beta) = d\alpha \circ d\beta$, ϕ is equivariant. By Proposition 2.2 and Lemma 2.1, we get that ϕ is continuous. Let $\lambda = \phi(\mu)$. Then, λ is a Γ-invariant probability measure and $\lambda(\{\tau(\mathcal{S}) \mid \tau \in \Gamma\}) = 1$. Now, the result follows from the previous case. □

4. *IRS on orbits*

In this section, we show the existence of invariant random subgroups supported on various orbits of the conjugation action of a connected Lie group G on Sub_G. In other words, we discuss the proofs of Theorems 1.1, 1.3, 1.5, and 1.7. We also discuss certain structural results related to these theorems.

For a group G and $x \in G$, let $i_x\colon G \to G$ be the inner automorphism of G defined by $i_x(g) = xgx^{-1}$ for all $g \in G$. For any $H \subset G$, let $\mathrm{Inn}(H) = \{i_x \mid x \in H\}$.

Proof of Theorem 1.1. It is sufficient to prove that (2) implies (3). Let L be a semisimple Levi subgroup of G and N be the nilpotent radical of G as in the statement of the theorem. Then, L is a connected Lie subgroup of G. Suppose there is an N-invariant probability measure μ supported on semisimple Levi subgroups. Since any two semisimple Levi subgroups conjugate in G by an element of N, $\mu(\{xLx^{-1} \mid x \in N\}) = 1$. Let $\mathfrak{G}$ be the Lie algebra of G. Since N is a nilpotent normal subgroup of G, $\mathrm{Ad}(x)$ is a unipotent transformation on $\mathfrak{G}$ for all $x \in N$. Therefore, applying Proposition 3.2 by taking $S = L$ and $\Gamma = U = \mathrm{Inn}(N)$, we get that L is normalized by N. Since any two semisimple Levi subgroups are conjugate by an element of N, we get that L is normal in G. □

Proof of Corollary 1.2. Suppose the set of all semisimple Levi subgroups is closed in Sub_G. Then, L_G is compact. Since N is amenable, L_G admits an N-invariant probability measure. Therefore, Theorem 1.1 implies that L is normal in G. □

Proof of Theorem 1.3. Suppose that there is an IRS μ such that $\mu(K_G) = 1$, where K_G is the set of all maximal compact subgroups of G with K being a maximal compact subgroup of G. We show that K is normal in G. We first show that K is normalized by any

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Ad-unipotent subgroup of G. Since K is a maximal compact subgroup of G, K is connected and $K_G = \{xKx^{-1} \mid x \in G\}$ [11, Theorem 3.1, Ch. XV]. Applying Proposition 3.2 by taking $S = K$, $\Gamma = \mathrm{Inn}(G)$, and $U = \mathrm{Inn}(N)$ for any Ad-unipotent subgroup N, we get that K is normalized by any Ad-unipotent subgroup of G.

Let L be a semisimple Levi subgroup of G and K_l be a maximal compact subgroup of L. Then, $K_l \subset gKg^{-1}$ for some $g \in G$. Replacing K by a conjugate, we may assume that K contains K_l. This, in particular, implies that K is normalized by compact simple factors of L. Let L_1 be a non-compact simple factor of L. Then, L_1 is generated by the Ad-unipotent elements of L_1. Therefore, L_1 normalizes K. Thus, the semisimple Levi subgroup L normalizes K.

Let N denote the nilpotent radical of G. Then, $\mathrm{Ad}(x)$ is unipotent for any $x \in N$ and, hence, N normalizes K. Thus, NK is a closed connected subgroup of G. Since K is a maximal compact subgroup of G, K is a maximal compact subgroup of NK. Since N normalizes K, K is normal in NK. Thus, K is a maximal compact normal subgroup of the connected Lie group NK. Since any two maximal compact subgroups of a connected Lie group are conjugate, K contains any compact subgroup of NK. Let $T = N \cap K$. Since K contains any compact subgroup of NK, K contains any compact subgroup of N. This implies that T is the largest compact subgroup of N and T is normal in N. Therefore, N/T is a simply connected nilpotent Lie group [11, Exercise 1 of Ch. XV]. Since N is normal in G and T is the largest compact subgroup of N, T is normal in G. Since $T \subset K$, replacing G by G/T, we may assume that N is simply connected.

Let $\tilde{G} = G/N$ and $\tilde{K}$ be a maximal compact subgroup of $\tilde{G}$ containing KN/N. Then, $\tilde{K}$ is connected [11, Theorem 3.1, Ch. XV] and there is a closed subgroup H of G containing N such that $H/N = \tilde{K}$. Since N and $\tilde{K}$ are connected, H is also connected [10, Theorem 7.4]. As N is simply connected, H contains a compact subgroup M such that $H = MN$ and $\tilde{K} = H/N \simeq M$ [11, Theorem 3.2, Ch. XII]. Also, $K \subset H$ as $H/N = \tilde{K}$ contains KN/N. Since M is a compact subgroup of H and K is a maximal compact subgroup, $M \subset xKx^{-1}$ for some $x \in H = MN$. However, N normalizes K and, hence, $M \subset K$. This shows that $\tilde{K} = KN/N$.

Now, let R be the solvable radical of G. Then, $N \subset R$ and by [13, Corollary 3.16.4], R/N is the center of $\tilde{G}$ and, hence, $\tilde{K}$ is normalized by R/N. This shows that KN is normalized by R. Since the semisimple Levi subgroup L normalizes K, L normalizes KN as well. It follows that KN is a normal subgroup of G. Since K contains any compact subgroup of NK, K is a normal subgroup of G. □

The existence of an IRS supported on the set of maximal compact subgroups has certain implications on the structure of the connected Lie group G under consideration as described in the following proposition.

PROPOSITION 4.1. *Let G be a connected Lie group with solvable radical R and M be a maximal compact subgroup of G. Suppose M is normal in G. Then:*

(1) *the solvable radical R admits a compact central subgroup T so that R/T is simply connected;*
(2) *any closed simple Lie subgroup of G is compact or has no non-trivial compact connected subgroup.*

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Remark 4.2. The universal cover of $SL(2,\mathbb{R})$ is a simple Lie group without any compact subgroup of positive dimension. Thus, the second possibility in item (2) of Proposition 4.1 may occur.

Proof. Suppose that the maximal compact subgroup M of G is normal in G. Since any two maximal compact subgroups of a connected Lie group are conjugate, M is the largest compact subgroup of G. If R is the solvable radical of G, then $R \cap M$ is a maximal compact subgroup of R. Since R is a connected Lie group, $R \cap M$ is connected. Therefore, the connected compact solvable group $R \cap M$ is abelian. Since M is normal in G, $R \cap M$ is normal in R. Since R is connected, $R \cap M$ is central in R. If $T = R \cap M$, then T is a central subgroup of R and R/T has no compact subgroup. Thus, R/T is simply connected [**11**, Exercise 3, Ch. XII]).

Let S be any simple Lie subgroup of G. Suppose that there exists a non-trivial compact connected subgroup of S. Then, $M \cap S$ is a compact normal subgroup of positive dimension in S. Since S is simple, $S = S \cap M$. □

We now apply the above results to show that a finite covolume subgroup is cocompact if the subgroup is a Levi subgroup or a maximal compact subgroup.

Proof of Corollary 1.4. Suppose H is a Levi subgroup of G. Then, the stabilizer, say G_{xH}, at any $xH \in G/H$ is xHx^{-1}. Therefore, using the map $xH \mapsto G_{xH}$, we obtain an IRS supported on the set of semisimple Levi subgroups. By Theorem 1.1, we get that G has a unique semisimple Levi subgroup and is normal in G. This implies that G/H is a locally compact group. Since G/H carries a G-invariant probability measure, G/H is compact. The other case, *viz.* when H is a maximal compact subgroup, can be proved similarly by using Theorem 1.3. □

Now, we assume that G is a real algebraic group. Also, let B be a Borel subgroup of G and B_G be the set of all Borel subgroups of G, that is, $B_G = \{xBx^{-1} \mid x \in G\}$.

Proof of Theorem 1.5. Since B is a Borel subgroup of G, G/B is compact and, hence, there is a compact set C in G such that $G = CB$. Since any two Borel subgroups are conjugate in G, $B_G = \{xBx^{-1} \mid x \in C\}$. It follows that, B_G is a compact subset of Sub_G.

Suppose that there is an IRS μ such that $\mu(B_G) = 1$. To prove that G is amenable, it is sufficient to prove that any semisimple Levi subgroup of G is compact. Let R be the solvable radical of G. Then, $R \subset B$ and, hence, any Borel subgroup of G is a lift of a Borel subgroup of G/R. Thus, replacing G by G/R, we may assume that G is semisimple. Now, let L be a simple factor of G. If L is non-compact, then L is generated by unipotent elements. Applying Proposition 3.2 by taking $S = B$, $\Gamma = \mathrm{Inn}(L)$ and $U = \mathrm{Inn}(N)$, where N is a maximal unipotent subgroup of L, we get that B is normalized by L. Thus, $B \cap L$ is a solvable normal subgroup of L containing a Borel subgroup of L. This is a contradiction to the simplicity of L. Hence, any simple factor of G is compact. This completes the proof. □

Proof of Theorem 1.7. Let D be a maximal diagonalizable subgroup of G and U be the unipotent radical of G. Then, there is a reductive Levi subgroup L of G containing D such that $G = LU$. Let $L' = [L, L]$. Then, L' is a semisimple Levi subgroup of G.

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Let μ be an IRS with $\mu(D_G) = 1$. Then, applying Proposition 3.2 by taking $S = D$, $\Gamma = \mathrm{Inn}(G)$ and $U = \mathrm{Inn}(N)$, where N is a unipotent subgroup of G, we get that any unipotent subgroup of G normalizes D. If S is a non-compact simple algebraic subgroup of G, then the maximal split diagonalizable subgroup D_S of S is contained in a conjugate of D. Replacing S by a conjugate of S, we may assume that $D_S \subset D$. Since S is non-compact, S is generated by unipotent elements and, hence, S normalizes D. Therefore, $D \cap S$ is a normal subgroup of positive dimension, which is a contradiction to the simplicity of S. Thus, the semisimple Levi subgroup $L' = [L, L]$ of G is compact. This implies that $D \subset Z(L)$ because $L = [L, L]Z(L)$.

Since the unipotent subgroups normalize D, D is normalized by U. Since $D \cap U$ is trivial, D is centralized by U also. Again, since $G = LU$, D is central in G. This proves that (1) implies (2); (2) implies (3) is trivial. If D is normal, then δ_D, the Dirac measure at D is the required IRS. This proves that (3) implies (1).

Now, suppose condition (1) holds. Then, the proof of the first implication shows that the Levi subgroup is compact and D is central. Let R be the solvable radical of G. Then, $D_R = R \cap D$ is a central subgroup of R and, hence, $D_R U$ is a nilpotent subgroup of R, where U is the unipotent radical of G. Since $R/D_R U$ is an anisotropic torus (that is, without any split torus subgroup), we get that $R/D_R U$ is compact. □

COROLLARY 4.3. *Let G be a connected real algebraic group and D be a maximal diagonalizable subgroup of G. Suppose G/D has a G-invariant probability measure. Then, G is a direct product of D and a compact group.*

Proof. Let m be the G-invariant probability measure on G/D. Then, the stabilizer, G_{xD}, of $xD \in G/D$ is xDx^{-1}. Therefore, the map $xD \mapsto xDx^{-1}$ gives rise to an IRS supported on maximal diagonalizable subgroups. By Theorem 1.7, we get that D is normal in G and the semisimple Levi subgroup of G is compact. Since G/D has a G-invariant probability measure, G/D is a compact group. Therefore, the unipotent radical of G is trivial. This implies that G is reductive with compact semisimple Levi subgroup and, hence, D is central. It follows that G is a direct product of D and a compact subgroup. □

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