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Weak specification properties and large deviations for non-additive potentials

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Abstract. We obtain large deviation bounds for the measure of deviation sets associated with asymptotically additive and sub-additive potentials under some weak specification properties. In particular, a large deviation principle is obtained in the case of uniformly hyperbolic dynamical systems. Some applications to the study of the convergence of Lyapunov exponents are given.

1. *Introduction*

The purpose of the theory of large deviations is to study the rates of convergence of sequences of random variables to some limit distribution. Some applications of these ideas into the realm of dynamical systems have been particularly useful to estimate the velocity at which time averages of typical points of ergodic invariant measures converge to the space average as guaranteed by Birkhoff's ergodic theorem. More precisely, given a continuous transformation f on a compact metric space M and a reference measure ν, one interesting question is to obtain sharp estimates for the ν-measure of the deviation sets $\{x \in M : (1/n) \sum_{j=0}^{n-1} g(f^j(x)) > c\}$ for all continuous functions $g : M \to \mathbb{R}$ and real numbers c. We refer the reader to [**1, 11, 16–18, 24–27, 31, 33, 35, 42, 43, 45**] and the references therein for an account of recent large deviations results.

Since many relevant quantities in dynamical systems arise from non-additive sequences, e.g. the largest Lyapunov exponent for higher-dimensional dynamical systems, Kingman's ergodic theorem becomes in many situations crucial in the study of deviation sets $\{x \in M : \varphi_n(x) > cn\}$ with respect to some not necessarily additive sequence $\Phi = \{\varphi_n\}_n$ of continuous functions. Inspired by the pioneering work of Young [**43**] our purpose in this direction is to provide sharp large deviations estimates for a wide class of non-additive sequences of continuous potentials. Our approach uses ideas from the non-additive

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thermodynamical formalism and we estimate the measure of deviation sets in the case that the reference measure satisfies a weak Gibbs property. Some recent results considering the thermodynamical formalism of almost additive or sub-additive sequences of potentials include [**4, 19, 20, 22, 29**]: in all cases the authors proved that there exists a unique equilibrium state μ_Φ and it is absolutely continuous with respect to a Gibbs measure ν_Φ with density bounded away from zero and infinity. Building on [**3**], Méson and Vericat [**28**] obtained bounds for large deviations processes for a family of non-additive potentials $\Phi = \{\varphi_n\}$, namely those such that $\varphi_n - \varphi_{n-1} \circ f$ converge uniformly. Our purpose here is to extend the theory beyond the uniformly hyperbolic context and to consider a broad class of sub-additive, almost additive and asymptotically additive sequences of potentials. For simplicity, we will refer to the previous classes of potentials as non-additive sequences of potentials.

Let us mention that since we consider some non-uniformly hyperbolic dynamical systems or dynamical systems that admit mistakes, these in general do not satisfy the usual specification property. In fact, if on the one hand the specification property holds robustly in the uniformly hyperbolic setting [**38**], on the other hand dynamical systems with the specification property are rare even among partially hyperbolic dynamical systems in dimension three [**39**]. For that reason we assume that the dynamical system satisfies some weaker specification property. In fact, as a physical process evolves it is natural for the evolving process to change or produce some errors in the evaluation of orbits. However, a self-adaptable system should decrease errors over time. This is a motivation for our study of the large deviations for non-additive potentials when the systems admit mistakes. In fact, not only do the notions of specification and topologically mixing coincide for every one-dimensional *continuous* mapping (see [**6**]), but also weaker specification properties hold in the presence of non-uniform hyperbolicity.

Roughly, one proves that the measure of the set of points whose sequences of values remain far from the space average with respect to the equilibrium measure decreases exponentially fast. In particular, we obtain a large deviations principle for non-additive sequences in the uniformly expanding setting. As important applications we estimate the rate of convergence of the maximal Lyapunov exponent for some open families of linear cocycles and non-conformal expanding maps. We refer the reader to §4 for precise statements and details.

The remainder of this paper is organized as follows. In §2, we present some definitions and fundamental notions necessary to state our results. In §3 we present statements of the main results in this paper. Some examples are given in §4 while the proofs of the main results are given in §5. Finally, in the Appendix A we estimate the measure of mistake dynamical balls in the uniformly expanding setting.

2. *Preliminaries*

Throughout this paper, (M, f) denotes a continuous dynamical systems in the sense that $f : M \to M$ is a (piecewise) continuous transformation on the compact metric space M with a metric d. Invariant Borel probability measures are associated with (M, f). Let $\mathcal{M}_f$ and $\mathcal{E}_f$ denote the space of f-invariant Borel probability measures and the set of f-invariant ergodic Borel probability measures, respectively.

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2.1. *Specification properties.* Specification properties are very useful to obtain existence of equilibrium states as well as large deviation principles. Here we introduce and discuss some different notions.

Definition 2.1. We say that a map f satisfies the *specification property* if for any $\varepsilon > 0$ there exists an integer $N = N(\varepsilon) \geq 1$ such that the following holds: for every $k \geq 1$, any points $x_1, \ldots, x_k$, and any sequence of positive integers $n_1, \ldots, n_k$ and $p_1, \ldots, p_k$ with $p_i \geq N(\varepsilon)$, there exists a point x in M such that

$$d(f^j(x),\ f^j(x_1)) \leq \varepsilon \quad \text{for all } 0 \leq j \leq n_1$$

and

$$d(f^{j+n_1+p_1+\cdots+n_{i-1}+p_{i-1}}(x),\ f^j(x_i)) \leq \varepsilon$$

for every $2 \leq i \leq k$ and $0 \leq j \leq n_i$.

The previous notion is slightly weaker than that introduced by Bowen [**7**], which requires that any finite sequence of pieces of orbit is well approximated by periodic orbits. Although the robust specification property for diffeomorphisms is satisfied only by uniformly hyperbolic dynamical systems (see Sakai *et al* [**38**]) we know by Blokh [**6**] that the notions of specification and topologically mixing coincide for every one-dimensional *continuous* mapping. This is no longer true if the one-dimensional map fails to be continuous (see, e.g., [**10**]).

To define another weak form of specification we first recall the definitions of the mistake function and mistake dynamical balls which are due to Thompson [**40**] and Pfister and Sullivan [**32**]. Given $\varepsilon_0 > 0$, the function $g : \mathbb{N} \times (0, \varepsilon_0] \to \mathbb{N}$ is called a *mistake function* if, for all $\varepsilon \in (0, \varepsilon_0]$ and all $n \in \mathbb{N}$, $g(n, \varepsilon) \leq g(n+1, \varepsilon)$ and $\lim_n g(n, \varepsilon)/n = 0$. By a slight abuse of notation, we set $g(n, \varepsilon) = g(n, \varepsilon_0)$ for every $\varepsilon > \varepsilon_0$. Moreover, for any subset of integers $\Lambda \subset [0, N]$, we will use the family of distances in the metric space M given by $d_\Lambda(x, y) = \max\{d(f^i x,\ f^i y) : i \in \Lambda\}$ and consider the balls $B_\Lambda(x, \varepsilon) = \{y \in M : d_\Lambda(x, y) < \varepsilon\}$. Hence, we can now consider mistake dynamical balls. Given a mistake function g, $\varepsilon > 0$ and $n \geq 1$, the (n, ε)-*mistake dynamical ball* $B_n(g; x, \varepsilon)$ of radius ε and length n associated with g is defined by

$$\begin{aligned} B_n(g; x, \varepsilon) &= \{y \in M \mid y \in B_\Lambda(x, \varepsilon) \text{ for some } \Lambda \in I(g; n, \varepsilon)\} \\ &= \bigcup_{\Lambda \in I(g;n,\varepsilon)} B_\Lambda(x, \varepsilon), \end{aligned}$$

where $I(g; n, \varepsilon) = \{\Lambda \subset [0, n-1] \cap \mathbb{N} \mid \#\Lambda \geq n - g(n, \varepsilon)\}$. A set $F \subset Z$ is $(g; n, \varepsilon)$-separated for Z if every $x, y \in F$ with $x \neq y$ implies that $d_\Lambda(x, y) > \varepsilon$ for all $\Lambda \in I(g; n, \varepsilon)$. The dual definition is as follows. A set $E \subset Z$ is $(g; n, \varepsilon)$-spanning for Z, if for all $z \in Z$, there exists $x \in E$ and $\Lambda \in I(g; n, \varepsilon)$ such that $d_\Lambda(x, z) \leq \varepsilon$.

Definition 2.2. Let g be a mistake function. We say that f satisfies the *g-almost specification property* if there exists $\varepsilon > 0$ and a positive integer $N(g, \varepsilon)$ such that the following property holds: for every $k \geq 1$, any points $x_1, \ldots, x_k$, and any positive integers $n_1, \ldots, n_k$ with $n_i \geq N(g, \varepsilon)$,

$$\bigcap_{i=1}^{k} f^{-\sum_{j=0}^{i-1} n_j}(B_{n_i}(g; x_i, \varepsilon)) \neq \emptyset,$$

where $n_0 = 0$.

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The latter property holds for all β-transformations (see [**32, 40**]). In fact, Thompson [**40**] introduced a more general property where the value ε is replaced by several values $\varepsilon_1, \ldots, \varepsilon_k$. However, this weaker requirement is suitable for our purposes.

2.2. *Non-additive potentials.* Let $C(M)$ denote the space of continuous functions from M to $\mathbb{R}$. A sequence $\Phi = \{\varphi_n\} \subset C(M)$ is a *sub-additive* (respectively *super-additive*) sequence of potentials if $\varphi_{m+n} \le \varphi_m + \varphi_n \circ f^m$ (respectively $\varphi_{m+n} \ge \varphi_m + \varphi_n \circ f^m$) for every $m, n \ge 1$.

We say that the sequence $\Phi = \{\varphi_n\} \subset C(M)$ is an *almost additive* sequence of potentials if there exists a uniform constant $C > 0$ such that $\varphi_m + \varphi_n \circ f^m - C \le \varphi_{m+n} \le \varphi_m + \varphi_n \circ f^m + C$ for every $m, n \ge 1$. Finally, we say that $\Phi = \{\varphi_n\} \subset C(M)$ is an *asymptotically additive* sequence of potentials if for any $\xi > 0$ there exists a continuous function φ_ξ such that

$$\limsup_{n\to\infty} \frac{1}{n} \|\varphi_n - S_n\varphi_\xi\| < \xi, \tag{2.1}$$

where $S_n\varphi_\xi = \sum_{j=0}^{n-1} \varphi_\xi \circ f^j$ denotes the usual Birkhoff sum, and $\|\cdot\|$ is the sup norm on the Banach space $C(M)$. Let $\mathcal{A}$ denote the set of asymptotically additive potentials. The following result establishes the relation between these notions.

PROPOSITION 2.1. *The following properties hold:*

(1) *if $\Phi = \{\varphi_n\}$ is almost additive there exists $C > 0$ such that the sequence $\Phi_C = \{\varphi_n + C\}$ is sub-additive and $\Phi_{-C} = \{\varphi_n - C\}$ is super-additive;*

(2) *if $\Phi = \{\varphi_n\}$ is almost additive then it is asymptotically additive, and for any $\xi > 0$ there exists $k = k(\xi) \ge 1$ so that $\limsup_{n\to\infty}(1/n)\|\varphi_n - S_n((1/k)\varphi_k)\| < \xi$.*

Proof. Part (1) is obvious from the definitions. Part (2) is contained in [**21**, Proposition A.5] or [**46**, Proposition 2.1]. □

Example 4.6 provides examples of sub-additive and super-additive potentials that are almost additive, while in Example 4.7 we exhibit some sub-additive and super-additive potentials that are asymptotically additive. By Kingman's sub-additive ergodic theorem it follows that, for every sub-additive potential $\Phi = \{\varphi_n\}$ and every f-invariant ergodic probability measure μ,

$$\lim_{n\to\infty} \frac{1}{n}\varphi_n(x) = \inf_{n\ge1} \frac{1}{n} \int \varphi_n \, d\mu =: \mathcal{F}_*(\mu, \Phi) \quad \text{for } \mu\text{-almost every } x. \tag{2.2}$$

In fact, Feng and Huang [**21**] proved that the same property holds for asymptotically additive potentials and, consequently, for all almost additive ones. In that paper, Feng and Huang also proved that the map $\mu \mapsto \mathcal{F}_*(\mu, \Phi)$ is continuous (respectively upper semi-continuous) if Φ is asymptotically additive (respectively sub-additive).

2.3. *Non-additive topological pressure and equilibrium states.* We recall the notions of non-additive topological pressure. For any $\varepsilon > 0$ and $n \in \mathbb{N}$ we consider the (n, ε)-dynamical balls $B_n(x, \varepsilon) := \{y \in M : d_n(x, y) < \varepsilon\}$, where $d_n(x, y) = \max_{0\le i<n} d(f^i x, f^i y)$. We say that a set $E \subset M$ is (n, ε)-*separated* if all distinct $x, y \in E$ satisfy $y \notin B_n(x, \varepsilon)$. If $\Phi = \{\varphi_n\}$ is a non-additive (namely, sub-additive, almost additive

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or asymptotically additive) family of potentials, the *topological pressure of f with respect to* Φ is defined by

$$P(f, \Phi) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup_{E_n} \{Z_n(\Phi, E_n, \varepsilon)\},$$

where $Z_n(\Phi, E_n, \varepsilon) = \sum_{y \in E_n} \exp(\varphi_n(y))$ and the supremum is taken over all (n, ε)-separated sets. The following variational principle relates the non-additive pressure with the natural modifications of the measure-theoretic free energy. Recall first a very useful formula to compute the metric entropy due to Katok [**23**, Theorem I.I]: if η is an f-invariant ergodic probability measure, then

$$h_\eta(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon, \delta) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon, \delta), \tag{2.3}$$

where $N(n, \varepsilon, \delta)$ is the minimum number of (n, ε)-dynamical balls necessary to cover a set of η-measure larger than δ. More generally, it was proven in [**40**] that for any mistake function g under the previous assumptions it also holds that

$$h_\eta(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(g; n, \varepsilon, \delta) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log N(g; n, \varepsilon, \delta), \tag{2.4}$$

where the term $N(g; n, \varepsilon, \delta)$ stands for the minimum number of (n, ε)-mistake dynamical balls necessary to cover a set of η-measure larger than δ. A generalization of formulas (2.3) and (2.4) to measure theoretic pressure was given in [**13**] and [**14**].

THEOREM 2.1. *Let $f : M \to M$ be a continuous map on the compact metric space M, and $\Phi = \{\varphi_n\}$ a non-additive potential (namely, sub-additive, almost additive or asymptotically additive). Then*

$$P(f, \Phi) = \sup\{h_\mu(f) + \mathcal{F}_*(\mu, \Phi) : \mu \in \mathcal{M}_f, \mathcal{F}_*(\mu, \Phi) \neq -\infty\}.$$

We refer the reader to [**4, 12, 21, 29**] for the proof of this variational principle and further details on topological pressure of non-additive potentials. An f-invariant probability measure μ that attains the supremum is called *equilibrium state* for f with respect to Φ. In many situations these arise as invariant measures absolutely continuous with respect to weak Gibbs measures.

Definition 2.3. Given a sequence of functions $\Phi = \{\varphi_n\}$, we say that a probability measure ν is a *weak Gibbs measure with respect to* Φ *on* $\Lambda \subset M$ if the set Λ has full ν-measure and there exists $\varepsilon_0 > 0$ such that for every $x \in \Lambda$ and $0 < \varepsilon < \varepsilon_0$ there exists a sequence of positive constants $(K_n)_{n \geq 1}$ (depending only on ε) satisfying $\lim_{n \to \infty} (1/n) \log K_n = 0$ and, for every $n \geq 1$,

$$K_n^{-1} \leq \frac{\nu(B_n(x, \varepsilon))}{e^{-nP(f,\Phi)+\varphi_n(x)}} \leq K_n.$$

We say that ν is a *Gibbs measure with respect to* Φ if there exists $K > 0$ such that the same property holds with $K_n = K$ independent of n.

The previous notion of Gibbs measure is a generalization of the usual one obtained in [**4, 29**] in the uniformly hyperbolic setting. In the case of additive potentials, these weak

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Gibbs measures appear in dynamics with some non-uniform hyperbolicity as, e.g., [**41, 44**]. Let us focus on some results concerning the existence of equilibrium states in the uniformly hyperbolic setting. Given a basic set Ω for an Axiom A diffeomorphism f, it is known that every almost additive potential $\Phi = \{\varphi_n\}$ satisfying the bounded variation statement

$$\text{there exists } A, \delta > 0 \text{ such that } \sup_{n\in\mathbb{N}} \gamma_n(\Phi, \delta) \le A,$$

where $\gamma_n(\Phi, \delta) = \sup\{|\varphi_n(y) - \varphi_n(z)| : y, z \in B_n(x, \delta)\}$, admits a unique equilibrium state μ_Φ which coincides with the Gibbs measure with respect to Φ (see [**4, 29**] for the proof). This condition above was introduced by Bowen [**8**] to obtain uniqueness of equilibrium states for expansive maps with the specification property. We say that a sequence of continuous functions $\Psi = \{\psi_n\}$ satisfy the *weak Bowen condition* if there exist $\delta > 0$ and a sequence of positive real numbers $\{a_n\}_n$ such that $\limsup_{n\to\infty} a_n/n = 0$ and

$$\gamma_n(\Phi, \delta) \le a_n \quad \text{for all } n \ge 1. \tag{2.5}$$

3. *Statement of the results*

Here we state our main results of this paper. The first one is a modified Brin–Katok local entropy formula for dynamical systems when some errors are admissible. We prove that the exponential decreasing rate of the measure of the mistake dynamical ball is equal to the measure-theoretic entropy.

PROPOSITION A. *Given an f-invariant ergodic measure μ and a mistake function g, the limits*

$$\underline{h}_\mu(g;\, f, x) = \lim_{\varepsilon\to 0} \liminf_{n\to\infty} -\frac{1}{n} \log \mu(B_n(g;\, x, \varepsilon))$$

and

$$\overline{h}_\mu(g;\, f, x) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log \mu(B_n(g;\, x, \varepsilon))$$

exist for μ-almost every x and coincide with the measure theoretic entropy $h_\mu(f)$.

Let us mention that, although the statement of the previous proposition was expected, the proof of the later formulas does not follow the original strategy of Brin and Katok. Note that the mistake dynamical balls $B_n(g;\, x, \varepsilon)$ take into account not only the size n, but also the amount of allowed mistakes $g(n, \varepsilon)$. In particular, the mistake dynamical balls $B_n(g;\, x, \varepsilon)$ may not even satisfy the inclusion $B_{n+1}(g;\, x, \varepsilon) \subset B_n(g;\, x, \varepsilon)$, e.g. if $g(n, \varepsilon)$ is much larger than $g(n-1, \varepsilon)$. Since this fact is not standard, an estimation of the measure of mistake dynamical balls for uniformly expanding maps is given in Appendix A. Furthermore, the ergodicity assumption in Proposition A is not crucial. Given $\mu \in \mathcal{M}_f$, by the ergodic decomposition theorem we know that μ can be decomposed as a convex combination of ergodic measures, $\mu = \int \mu_x \, d\mu(x)$. Applying Proposition A to each ergodic component μ_x and using $h_\mu(f) = \int h_{\mu_x}(f) \, d\mu(x)$ we obtain the following corollary.

COROLLARY 3.1. *Given any $\mu \in \mathcal{M}_f$, the limits $\underline{h}_\mu(g;\, f, x)$ and $\overline{h}_\mu(g;\, f, x)$ do exist for μ-almost every x and the measure theoretic entropy $h_\mu(f)$ satisfies*

$$h_\mu(f) = \int \underline{h}_\mu(g;\, f, x) \, d\mu(x) = \int \overline{h}_\mu(g;\, f, x) \, d\mu(x).$$

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Large deviation bounds for asymptotically additive observables. We are also interested in studying the rate of convergence in Kingman's sub-additive theorem. More precisely, given a Borel probability measure m on the space M, we study the rate at which the m-measure of the sets

$$B(n) = \left\{ x \in M : \left| \frac{1}{n}\varphi_n(x) - \mathcal{F}_*(\mu, \Phi) \right| > c \right\}$$

goes to zero as n tends to infinity, with respect to our reference and not necessarily invariant probability measure m. Given a mistake function g and a Borel probability ν, we define

$$h_m(g;\, f, x) = \lim_{\varepsilon \to 0} \limsup_{n\to\infty} -\frac{1}{n} \log m(B_n(g;\, x, \varepsilon))$$

and $h_m(g;\, f, \nu) = \nu - \operatorname{ess\,sup} h_m(g;\, f, x)$. It follows from Proposition A that we have $h_\nu(g;\, f, x) = h_\nu(f)$ for ν-almost every x and every $\nu \in \mathcal{E}_f$. To provide more precise bounds for $h_m(g;\, f, x)$, we introduce two sets of functions as follows.

Given a constant K and a mistake function g, define $\mathcal{V}_K^+(g)$ as the set of sequences $\Phi \in \mathcal{A}$ for which there exist $\varepsilon_0 > 0$ and a set Υ of full m-measure such that the following property holds: for all $0 < \varepsilon < \varepsilon_0$, there are constants C_n (depending only on ε) so that $\lim_{n\to\infty}(1/n) \log C_n = 0$ and

$$m(B_n(g;\, x, \varepsilon)) \le C_n \exp(-nK + \varphi_n(x)) \quad \text{for all } x \in \Upsilon \text{ and } n \ge 1.$$

We also consider the set $\mathcal{V}_K^-(g)$ as the set of sequences $\Phi \in \mathcal{A}$ for which there exists $\varepsilon_0 > 0$ and a set Υ of full m-measure such that the following property holds: for all $0 < \varepsilon < \varepsilon_0$, there are constants C_n so that $\lim_{n\to\infty}(1/n) \log C_n = 0$ and

$$m(B_n(g;\, x, \varepsilon)) \ge C_n \exp(-nK + \varphi_n(x)) \quad \text{for all } x \in \Upsilon \text{ and } n \ge 1.$$

Observe that both classes of asymptotic potentials $\mathcal{V}_K^+(g)$, $\mathcal{V}_K^-(g)$ depend on the mistake function g and the constant K used in the expressions above. For the mistake function $g \equiv 0$, we will simply write $\mathcal{V}_K^+$ and $\mathcal{V}_K^-$, respectively. We shall refer to the constants C_n above as tempered constants. Finally, for any not necessarily additive sequence of observables $\Phi = \{\varphi_n\}$ and $E \subset \mathbb{R}$ define

$$\overline{R}_m(\Phi, E) = \limsup_{n\to\infty} \frac{1}{n} \log m\left(\left\{ x \in M : \frac{1}{n}\varphi_n(x) \in E \right\} \right)$$

and

$$\underline{R}_m(\Phi, E) = \liminf_{n\to\infty} \frac{1}{n} \log m\left(\left\{ x \in M : \frac{1}{n}\varphi_n(x) \in E \right\} \right).$$

The following abstract results generalize [**43**, Theorem A] to the case of asymptotically additive potentials under some mistake dynamical systems.

THEOREM A. *Assume that $h_{\mathrm{top}}(f) < \infty$. Then, for each $\Phi \in \mathcal{A}$, $c \in \mathbb{R}$ and mistake function g, the following hold:*

(1) $\underline{R}_m(\Phi, (c, \infty)) \ge \sup\{h_\nu(f) - h_m(g;\, f, \nu) : \nu \in \mathcal{E}_f,\ \mathcal{F}_*(\nu, \Phi) > c\}$;

(2) *for each $\Psi \in \mathcal{V}_K^+(g)$, we have*

$$\overline{R}_m(\Phi, [c, \infty)) \le \sup\{-K + h_\nu(f) + \mathcal{F}_*(\nu, \Psi)\},$$

where the supremum is taken over all $\nu \in \mathcal{M}_f$ satisfying $\mathcal{F}_*(\nu, \Phi) \ge c$;

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(3) *for every* $\Psi \in \mathcal{V}_K^-(g)$ *as above, we have*

$$\underline{R}_m(\Phi, (c, \infty)) \geq \sup\{-K + h_\nu(f) + \mathcal{F}_*(\nu, \Psi)\},$$

where the supremum is taken over all $\nu \in \mathcal{E}_f$ *satisfying* $\mathcal{F}_*(\nu, \Phi) > c$ *and* $\nu(\Upsilon) = 1$*;*

(4) *assume* f *satisfies the* g*-almost specification property. Then, given* $\Psi \in \mathcal{V}_K^-$*,*

$$\underline{R}_m(\Phi, (c, \infty)) \geq \sup\{-K + h_\nu(f) + \mathcal{F}_*(\nu, \Psi)\},$$

where the supremum is taken over all $\nu \in \mathcal{M}_f$ *satisfying* $\mathcal{F}_*(\nu, \Phi) > c$ *and* $\nu(\Upsilon) = 1$.

The previous result has particularly interesting applications to weak Gibbs measures obtained in thermodynamical formalism as we now describe.

THEOREM B. *Assume that* $h_{\text{top}}(f) < \infty$. *Let* $\Phi = \{\varphi_n\}$ *be an almost additive family of potentials with* $P(f, \Phi) > -\infty$ *and let* ν *be a weak Gibbs measure for* f *with respect to* Φ *on* $\Lambda \subset M$. *Assume that either:*

(a) $\Psi = \{\psi_n\}$ *is an asymptotically additive family of potentials; or*

(b) $\Psi = \{\psi_n\}$ *is a sub-additive family of potentials such that:*

  (i) $\Psi = \{\psi_n\}$ *satisfies the weak Bowen condition;*

  (ii) $\inf_{n\geq 1}(\psi_n(x)/n) > -\infty$ *for all* $x \in M$*; and*

  (iii) *the sequence* $\{\psi_n/n\}$ *is equicontinuous.*

Given $c \in \mathbb{R}$*,*

$$\overline{R}_\nu(\Psi, [c, \infty)) \leq \sup\{-P(f, \Phi) + h_\eta(f) + \mathcal{F}_*(\eta, \Phi)\} \tag{UB}$$

holds, where the supremum is over all $\eta \in \mathcal{M}_f$ *such that* $\mathcal{F}(\eta, \Psi) \geq c$. *Moreover,*

$$\underline{R}_\nu(\Psi, (c, \infty)) \geq \sup\{-P(f, \Phi) + h_\eta(f) + \mathcal{F}_*(\eta, \Phi)\},$$

where the supremum is taken over all ergodic *measures* η *satisfying* $\mathcal{F}_*(\eta, \Psi) > c$ *and* $\eta(\Lambda) = 1$. *If, in addition,* f *satisfies the specification property, then*

$$\underline{R}_\nu(\Psi, (c, \infty)) \geq \sup\{-P(f, \Phi) + h_\eta(f) + \mathcal{F}_*(\eta, \Phi)\}, \tag{LB}$$

where the supremum is taken over all $\eta \in \mathcal{M}_f$ *satisfying* $\mathcal{F}_*(\eta, \Psi) > c$ *and* $\eta(\Lambda) = 1$.

Some comments on our assumptions are in order. We use distinct strategies to deal with the two different classes of potentials Ψ. On the one hand, given an asymptotically additive family of potentials $\Psi = \{\psi_n\}$, the potentials ψ_n can be approximated by Birkhoff sums of continuous potentials and, in particular, the functional $\mu \to \mathcal{F}_*(\mu, \Psi)$ is continuous. On the other hand, for the sub-additive setting, conditions (ii) and (iii) will imply the continuity of the previous functional and condition (i) is a bounded distortion property as explained before. In Example 4.1 we explain how to deduce conditions (i) and (iii) in the expanding setting if the family Ψ satisfies some Hölder continuous regularity. Let us also mention that for the lower bound estimate above it is enough for the measure μ to satisfy the non-uniform specification property defined in [**37, 42**], but we shall not use or prove this fact here. Under the uniform hyperbolic assumption, we can also extend a large deviations principle to this non-additive setting as follows.

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COROLLARY A. *Assume that Ω is a basic set for a uniform hyperbolic map f, that $\Phi = \{\varphi_n\}$ is an almost additive sequence of functions such that there exists ν_Φ, a Gibbs measure for Φ, and $\mu_\Phi \ll \nu_\Phi$, a unique equilibrium state for f with respect to Φ. If $\Psi = \{\psi_n\}$ is a family of potentials as in Theorem B, then it satisfies the following large deviations principle: given $c \in \mathbb{R}$ it holds that*

$$\overline{R}_{\nu_\Phi}(\Psi, [c, \infty)) \leq -\inf_{\eta \in \mathcal{M}_f} \{P(f, \Phi) - h_\eta(f) - \mathcal{F}_*(\eta, \Phi) : |\mathcal{F}_*(\eta, \Psi) - \mathcal{F}_*(\mu_\Phi, \Psi)| \geq c\}$$

and also

$$\underline{R}_{\nu_\Phi}(\Psi, (c, \infty)) \geq -\inf_{\eta \in \mathcal{M}_f} \{P(f, \Phi) - h_\eta(f) - \mathcal{F}_*(\eta, \Phi) : |\mathcal{F}_*(\eta, \Psi) - \mathcal{F}_*(\mu_\Phi, \Psi)| > c\}.$$

Let us finish this section with some comments. First note that the uniqueness of the equilibrium state implies that the latter convergence to the average in Kingman's sub-additive ergodic theorem is indeed exponential. Second, since almost additive potentials are indeed asymptotically additive then our results apply for a wide class of non-additive sequences of observables.

4. *Examples and applications*

In this section we discuss the specification properties and large deviation results on a broad class of examples. In particular, we obtain applications to the study of the velocity of convergence for the Lyapunov exponents in open classes of linear cocycles and non-conformal expanding repellers. Our first example concerns sub-additive families of Hölder continuous potentials.

Example 4.1. Let X be a compact metric space and assume that $f : X \to X$ expands distances, that is, there are $\lambda > 1$ and $\varepsilon > 0$ such that $d(f(x), f(y)) \geq \lambda\, d(x, y)$ for all $y \in B(x, \varepsilon)$ and, consequently, $f^n : B_n(x, \varepsilon) \to B(f^n(x), \varepsilon)$ is a bijection. Let $\Psi = \{\psi_n\}_n$ be any sub-additive family of γ-Hölder continuous potentials such that the Hölder constants have at most linear growth, meaning that there exists $K > 0$ such that $\text{Höl}_\gamma(\psi_n) \leq Kn$. We claim that Ψ satisfies assumptions (i) and (iii) in Theorem B. On the one hand, $|\psi_n(x) - \psi_n(y)| \leq Kn\, d(x, y)^\gamma \leq Kn\lambda^{-\gamma n} \operatorname{diam}(X)^\gamma$ for every $y \in B_n(x, \varepsilon)$ and $n \geq 1$. In consequence, $\gamma_n(\Phi, \varepsilon) \leq a_n = Kn\lambda^{-\gamma n} \operatorname{diam}(X)^\gamma$, where $\lim_{n\to\infty} a_n/n = 0$. This proves that Ψ satisfies the weak Bowen condition. On the other hand, the sequence $\{\psi_n/n\}$ is Hölder continuous with uniform constant K. Thus, this sequence is equicontinuous.

Now we provide an illustrative example of a transformation that does satisfy the almost specification property but does not satisfy the strong specification property.

Example 4.2. Consider the piecewise expanding maps of the interval $[0, 1)$ given by $T_\beta(x) = \beta x \pmod 1$, where $\beta > 1$. This family is known as the family of *β-transformations* and it was introduced by Rényi in [**34**]. It was proved by Buzzi [**10**] that for all but countably many values of β the transformation T_β does not satisfy the

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specification property. However, it follows from [**32, 40**] that every β-map satisfies the almost specification property for every unbounded mistake function g. It is well known that for all $\beta > 1$ the topological entropy of T_β is $\log \beta$ and it admits a unique maximal entropy measure μ_β. Moreover, Pfister and Sullivan [**32**] proved a level-2 large deviations principle for the β-transformations with respect to μ_β. In fact, a key point is that the maximal entropy measure μ_β satisfies a weak Gibbs property (see, e.g., [**30**, equation (5.18)]), that is, there are positive constants K_n so that $\lim_{n\to\infty}(1/n)\log K_n = 0$ and, for all $n \geq 1$,

$$K_n^{-1}\beta^{-n} \leq \mu_\beta(B_n(x, \varepsilon)) \leq K_n\beta^{-n}.$$

Since the discontinuities have zero entropy, our results apply also in this piecewise expanding setting. Thus, if $\Phi = \{0\}$ and $\Psi = \{\psi_n\}$ is any family of asymptotically additive potentials (e.g. $\psi_n = \sum_{j=0}^{n-1} \psi \circ T_\beta^j + a_n$, where ψ is a continuous function and $(a_n)_n$ is any sequence of real numbers so that a_n/n is convergent to zero) then Theorem B and Proposition B yield that, for any $c \in \mathbb{R}$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_\beta\left(\left\{x \in M : \frac{1}{n}\psi_n(x) \geq c\right\}\right) \leq \sup_\eta\{-\log\beta + h_\eta(T_\beta)\},$$

where the supremum is taken over all $\eta \in \mathcal{M}_{T_\beta}$ satisfying $\mathcal{F}_*(\eta, \Psi) \geq c$, and also

$$\liminf_{n\to\infty} \frac{1}{n} \log \mu_\beta\left(\left\{x \in M : \frac{1}{n}\psi_n(x) \geq c\right\}\right) \geq \sup_\eta\{-\log\beta + h_\eta(T_\beta)\},$$

where the supremum is taken over all $\eta \in \mathcal{M}_{T_\beta}$ satisfying $\mathcal{F}_*(\eta, \Psi) > c$. Finally, one should mention that Climenhaga *et al* [**15**] have recently obtained a level-2 large deviations principle for symbolic systems equipped with a large class of reference measures, which includes the β-transformations.

In the next example we provide an application of our results in the uniformly expanding context.

Example 4.3. Let $f : M \to M$ be a C^1 map, and let $J \subset M$ be a compact f-invariant set. If J is a maximal topological mixing repeller, Barreira [**4**, p. 289] proved that each almost additive potential $\Phi = \{\varphi_n\}$ with weak Bowen property has a weak Gibbs measure ν_Φ. Thus, Theorem B applies to ν_Φ and any asymptotically additive family of potentials Ψ.

We also point out that every almost additive potential indeed satisfies the weak Bowen condition, see [**46**, Lemma 2.1] for a proof. Thus, any almost additive potential has a weak Gibbs measure. Therefore, if J is a maximal topological mixing repeller, Theorem B applies to any almost additive family of potentials Φ and any asymptotically additive family of potentials Ψ without any additional conditions.

In our next class of examples we estimate the rate of convergence of the maximal Lyapunov exponent for an important open class of linear cocycles.

Example 4.4. Here we consider cocycles over subshifts of finite type considered by Feng and Lau [**19**] and later by Feng and Käenmäki [**20**]. Let $\sigma : \Sigma \to \Sigma$ be the shift map on the space $\Sigma = \{1, \ldots, \ell\}^{\mathbb{N}}$ endowed with the distance $d(x, y) = 2^{-n}$, where $x = (x_j)_j$, $y = (y_j)_j$ and $n = \min\{j \geq 0 : x_j \neq y_j\}$. Consider the matrices $M_1, \ldots, M_\ell \in \mathcal{M}_{d\times d}(\mathbb{C})$

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such that for every $n \geq 1$ there exist $i_1, \ldots, i_n \in \{1, \ldots, \ell\}$ so that the product matrix $M_{i_1} \cdots M_{i_n} \neq 0$. Then, the topological pressure function is well defined as

$$P(q) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\iota\in\Sigma_n} \|M_\iota\|^q,$$

where $\Sigma_n = \{1, \ldots, d\}^n$ and for any $\iota = (i_1, \ldots, i_n) \in \Sigma_n$ one considers the matrix $M_\iota = M_{i_n} \cdots M_{i_2} M_{i_1}$. Moreover, for any σ-invariant probability measure μ define also the maximal Lyapunov exponent of μ by

$$M_*(\mu) = \lim_{n\to\infty} \frac{1}{n} \sum_{\iota\in\Sigma_n} \mu([\iota]) \log \|M_\iota\|$$

and it holds that $P(q) = \sup\{h_\mu(\sigma) + q\, M_*(\mu) : \mu \in \mathcal{M}_\sigma\}$. Note that this is the variational principle for the potentials $\Psi = \{\psi_n\}$, where $\psi_n(x) = q \log \|M_{\iota_n(x)}\|$, and for any $x \in \Sigma$ we set $\iota_n(x) \in \Sigma_n$ as the only symbol such that x belongs to the cylinder $[\iota_n(x)]$. Assume that the set of matrices $\{M_1, \ldots, M_d\}$ is irreducible over $\mathbb{C}^d$, that is, there is no non-trivial subspace $V \subset \mathbb{C}^d$ such that $M_i(V) \subset V$ for all $i = 1, \ldots, \ell$. Then it follows from [**20**, Proposition 1.2] that there exists a unique equilibrium state μ_q for σ with respect to Ψ and it is a Gibbs measure: there exists $C > 0$ such that

$$\frac{1}{C} \leq \frac{\mu_q([\iota_n])}{e^{-nP(q)}\|M_{\iota_n}\|^q} \leq C$$

for all $\iota_n \in \Sigma_n$ and $n \geq 1$.

Moreover, it is not hard to check that for any $\varepsilon > 0$ we get $y \in B_n(x, \varepsilon)$ if and only if the sequences $x = (x_j)$ and $y = (y_j)_j$ verify $x_j = y_j$ for all $0 \leq j \leq n + [(-\log \varepsilon/\log 2)]$. In consequence, $B_n(x, \varepsilon) \subset [\iota_n(x)]$, where the potential $\log \|M_{\iota_n(x)}\|$ is constant. Therefore, the sub-additive family of potentials $\{\log \|M_{\iota_n(x)}\|\}$ clearly satisfies the weak Bowen condition, and Corollary A yields that, for any $\delta > 0$,

$$\mu_q\left(x \in \Sigma : \left|\frac{1}{n} \log \|M_{\iota_n(x)}\| - M_*(\mu)\right| > \delta\right)$$

decreases exponentially fast.

The next example combines the theory for both additive and non-additive families of potentials in a non-uniformly expanding context.

Example 4.5. Let f be the Manneville–Pomeau map on the interval $[0, 1]$ given by $f(x) = x + x^{1+\alpha} \pmod 1$, for $\alpha \in (0, 1)$. This transformation satisfies the specification property since it is topologically conjugated to the doubling map. Moreover, it is well known that there exists an equilibrium state $\mu \ll \text{Leb}$ for f with respect to the potential $\phi = -\log|f'|$ and there exists a sequence K_n so that $\limsup_{n\to\infty}(1/n) \log K_n = 0$ and such that the measure has the weak Gibbs property:

$$\frac{1}{K_n} \leq \frac{\mu(\mathcal{P}^{(n)}(x))}{|(f^n)'(x)|} \leq K_n$$

for all $x \in [0, 1]$ and $n \geq 1$, where $\mathcal{P}$ is the Markov partition for f, $\mathcal{P}^{(n)} := \bigvee_{j=0}^{n-1} f^{-j}\mathcal{P}$ and $\mathcal{P}^{(n)}(x)$ is the element of the partition $\mathcal{P}^{(n)}$ that contains x. So, our results apply for

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any family of asymptotically additive potentials or sub-additive potentials with the weak Gibbs property $\Psi = \{\psi_n\}$.

Let us mention that Yuri [**44**] proved that typical piecewise C^1-smooth maps f with indifferent periodic points admit invariant ergodic weak Gibbs measures for $-\log|\det Df|$. In particular, our large deviation upper bound results also hold in this context.

The following class of local diffeomorphisms was introduced by Barreira and Gelfert [**5**] in the study of multifractal analysis for Lyapunov exponents associated to non-conformal repellers.

Example 4.6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 local diffeomorphism, and let $J \subset \mathbb{R}^2$ be a compact f-invariant set. Following [**5**], we say that f satisfies the following *cone condition* on J if there exists a number $b \le 1$ and for each $x \in J$ there is a one-dimensional subspace $E(x) \subset T_x\mathbb{R}^2$ varying continuously with x such that

$$Df(x)C_b(x) \subset \{0\} \cup \operatorname{int} C_b(fx),$$

where $C_b(x) = \{(u, v) \in E(x) \bigoplus E(x)^{\perp} : \|v\| \le b\|u\|\}$. It follows from [**5**, Proposition 4] that the latter condition implies that both families of potentials given by $\Phi_1 = \{\log \sigma_1(Df^n(x))\}$ and $\Phi_2 = \{\log \sigma_2(Df^n(x))\}$ are almost additive, where $\sigma_1(L) \ge \sigma_2(L)$ stands for the singular values of the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$, i.e. the eigenvalues of $(L^*L)^{1/2}$ with L^* denoting the transpose of L. Assume that J is a locally maximal topological mixing repeller of f such that:

(i) f satisfies the cone condition on J; and

(ii) f has bounded distortion on J, i.e. there exists some $\delta > 0$ such that

$$\sup_{n\ge 1} \frac{1}{n} \log \sup\{\|Df^n(y)(Df^n(z))^{-1}\| : x \in J \text{ and } y, z \in B_n(x, \delta)\} < \infty.$$

It then follows from [**5**, Theorem 2] that there exists a weak Gibbs measure ν_{σ_i} with respect to the family of potentials Φ_i, for $i = 1, 2$. Moreover, [**4**, Theorem 9] yields that there exists a unique equilibrium state μ_i for (f, Φ_i). So, for any $i, j \in \{1, 2\}$, it follows from Theorem B applied to the weak Gibbs measure ν_{σ_i} and the family of potentials $\Psi = \Phi_j = \{\log \sigma_j(Df^n(x))\}$ that

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \log \nu_{\sigma_i}\left(\left\{x \in M : \frac{1}{n} \log \sigma_j(Df^n(x)) \ge c\right\}\right)\\ \le \sup\left\{-P(f, \Phi_i) + h_\eta(f) + \lim_{n\to\infty} \frac{1}{n} \int \log \sigma_i(Df^n(x))\, d\eta\right\},\end{aligned}$$

where the supremum is over all $\eta \in \mathcal{M}_f$ such that $\inf_{n\ge 1}(1/n) \int \log \sigma_j(Df^n(x))\, d\eta \ge c$, and also

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n} \log \nu_{\sigma_i}\left(\left\{x \in M : \frac{1}{n} \log \sigma_j(Df^n(x)) > c\right\}\right)\\ \ge \sup\left\{-P(f, \Phi_i) + h_\eta(f) + \lim_{n\to\infty} \frac{1}{n} \int \log \sigma_i(Df^n(x))\, d\eta\right\},\end{aligned}$$

where the supremum is taken over all $\eta \in \mathcal{M}_f$ so that $\inf_{n\ge 1}(1/n) \int \log \sigma_j(Df^n(x)) d\eta > c$. As a simple application of the previous formulas, we get that for any $\varepsilon > 0$ the tail of

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the convergence to the largest or smallest Lyapunov exponent (corresponding respectively to $j = 1$ or $j = 2$)

$$\nu_{\sigma_i}\left(\left\{x \in M : \left|\frac{1}{n}\log \sigma_j(Df^n(x)) - \lim_{n\to\infty}\frac{1}{n}\int \log \sigma_j(Df^n(x))\, d\mu_i\right| > \varepsilon\right\}\right)$$

decays exponentially fast as $n \to \infty$.

Finally, in our last example we deal with an example where both families of potentials responsible for computing the largest and smallest Lyapunov exponents are asymptotically additive.

Example 4.7. Let M be a d-dimensional smooth manifold and J a compact expanding invariant set for a C^1 map f. We say that J is an average conformal repeller if all Lyapunov exponents of each ergodic measure are equal and positive. In particular, it follows from [**2**, Theorem 4.2] that

$$\lim_{n\to\infty}\frac{1}{n}(\log \|Df^n(x)\| - \log \|Df^n(x)^{-1}\|^{-1}) = \lim_{n\to\infty}\frac{1}{n}\log\frac{\|Df^n(x)\|}{\|Df^n(x)^{-1}\|^{-1}} = 0$$

uniformly on J. It is easy to see that the family of continuous potentials $\Psi_1 = \{\log \|Df^n(x)\|\}_n$ is sub-additive while $\Psi_2 = \{\log \|Df^n(x)^{-1}\|^{-1}\}_n$ is super-additive. Furthermore, in this setting it is not hard to check that these two families of potentials are asymptotically additive since they can be uniformly approximated by the additive potentials $\{(1/d)\log|\det(Df^n(x))|\}_n$. Let Φ be any family of continuous potentials such that f has a unique equilibrium state μ_Φ for f with respect to Φ, and μ_Φ satisfies the Gibbs property. Then it follows from Corollary A that

$$\mu_\Phi\left(x \in J : \left|\frac{1}{n}\log \|Df^n(x)\| - \lambda(\mu_\Phi)\right| > \delta \text{ or } \left|\frac{1}{n}\log \|Df^n(x)^{-1}\|^{-1} - \lambda(\mu_\Phi)\right| > \delta\right)$$

decrease exponentially fast, where in this average conformal setting we consider $\lambda(\mu_\Phi) = \inf_n(1/n)\int \log \|Df^n(x)\|\, d\mu_\Phi = \sup_n(1/n)\int -\log \|Df^n(x)^{-1}\|\, d\mu_\Phi$ to be the average Lyapunov exponent for μ_Φ. A final remark is that f admits equilibrium states with respect to the potentials Ψ_1 and Ψ_2 by upper semi-continuity of the metric entropy.

5. *Proof of the main results*

5.1. *Proof of Proposition A.*

We first recall a covering lemma for mistake dynamical balls of points with slow recurrence to the boundary of a partition.

LEMMA 5.1. (Rousseau and Varandas [**36**, Lemma 3.2]) *Let $\mathcal{Q}$ be a finite partition of M and consider $\varepsilon > 0$ arbitrary small. Let V_ε denote the ε-neighborhood of the boundary $\partial\mathcal{Q}$. For any $\alpha > 0$, there exists $\gamma > 0$ (depending only on α), such that for every $x \in M$ satisfying $\sum_{j=0}^{n-1}\chi_{V_\varepsilon}(f^j x) < \gamma n$, the mistake dynamical ball $B_n(g; x, \varepsilon)$ can be covered by $e^{\alpha n}$ cylinders of $\mathcal{Q}^{(n)}$ for sufficiently large n.*

We are now in a position to prove our generalized Brin–Katok local entropy formula whose proof exploits the ergodicity of the measure.

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Proof of Proposition A. First we note that the limits in the statement of Proposition A are indeed well defined almost everywhere. Given $n \geq 1$, $\varepsilon > 0$ and $x \in M$, it is clear that $B_n(x, \varepsilon) \subset B_n(g; x, \varepsilon)$. Thus, the Brin–Katok formula (see [**9**]) immediately yields

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log \mu(B_n(g; x, \varepsilon)) \leq \lim_{\varepsilon\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log \mu(B_n(x, \varepsilon)) = h_\mu(f)$$

for almost every x. Hence, to complete the proof of the proposition it is enough to show that, for μ-almost every x, one has

$$\underline{h}_\mu(g; f, x) = \lim_{\varepsilon\to 0} \liminf_{n\to\infty} -\frac{1}{n} \log \mu(B_n(g; x, \varepsilon)) \geq h_\mu(f).$$

Fix $\alpha > 0$ arbitrary and let γ be given by Lemma 5.1. Consider a finite partition $\mathcal{Q}$ of M such that $\mu(\partial\mathcal{Q}) = 0$ and $h_\mu(f) \leq h_\mu(f, \mathcal{Q}) + \gamma$. If $\varepsilon > 0$ is small enough the ε-neighborhood V_ε of ∂Q satisfies $\mu(V_\varepsilon) < \gamma/2$. For each positive integer N, set

$$\Gamma_N = \left\{ x \in M : \frac{1}{n} \sum_{j=0}^{n-1} \chi_{V_\varepsilon}(f^j(x)) < \gamma \text{ and } \mu(\mathcal{Q}^{(n)}(x)) \leq e^{-n(h_\mu(f,\mathcal{Q})-\gamma)},\ \forall n \geq N \right\}.$$

Using the ergodicity of μ it follows that $\Gamma_N \subset \Gamma_{N+1}$ and $\mu(\bigcup_N \Gamma_N) = 1$. By Lemma 5.1, one has that for any $x \in \Gamma_N$ the mistake dynamical ball $B_n(g; x, \varepsilon)$ can be covered by $e^{\alpha n}$ cylinders of $\mathcal{Q}^{(n)}$. Therefore, we obtain that

$$\begin{aligned}\mu(B_n(g; x, \varepsilon) \cap \Gamma_N) &\leq \mu\left(\bigcup \{Q \in \mathcal{Q}^{(n)} : B_n(g; x, \varepsilon) \cap Q \neq \emptyset \text{ and } Q \cap \Gamma_N \neq \emptyset\}\right) \\ &\leq e^{\alpha n} e^{-nh_\mu(f,\mathcal{Q})+\gamma n} \leq e^{\alpha n} e^{-nh_\mu(f)+2\gamma n}\end{aligned}$$

for all $x \in \Gamma_N$ and $n \geq N$. For each $\varepsilon > 0$ and positive integer n fixed it follows that $\mu(B_n(g; x, \varepsilon) \cap \Gamma_N) \to \mu(B_n(g; x, \varepsilon))$ as N tends to infinity. Therefore, for any arbitrary constant $\xi > 0$ it follows that $\mu(B_n(g; x, \varepsilon)) \leq e^\xi \mu(B_n(g; x, \varepsilon) \cap \Gamma_N)$ provided that N is large and $x \in \Gamma_N$. In particular, if N is fixed as above, then

$$\liminf_{n\to\infty} -\frac{1}{n} \log \mu(B_n(g; x, \varepsilon)) \geq h_\mu(f) - 2\gamma - \alpha - \xi.$$

Considering ε small enough, the constants involved above converge to zero proving that $\lim_{\varepsilon\to 0} \liminf_{n\to\infty} -(1/n) \log \mu(B_n(g; x, \varepsilon)) \geq h_\mu(f)$ as claimed. □

5.2. *Proof of Theorem A.* This subsection is devoted to the proof of Theorem A.

Proof of Theorem A(1). Pick $\nu \in \mathcal{E}_f$ with $\mathcal{F}_*(\nu, \Phi) > c$ and let $U_n = \{x \in M : (1/n)\varphi_n(x) > c\}$. It suffices to show that

$$\liminf_{n\to\infty} \frac{1}{n} \log m(U_n) \geq h_\nu(f) - h_m(g; f, \nu).$$

Without loss of generality, assume that $h_m(g; f, \nu) < \infty$, otherwise there is nothing to prove since $h_\nu(f) \leq h_{\text{top}}(f) < \infty$. The strategy for the proof is to approximate the family of asymptotically additive functions by appropriate Birkhoff sums associated with some continuous observable. Indeed, fix a sufficiently small $\xi > 0$ so that $\mathcal{F}_*(\nu, \Phi) > c + 2\xi$,

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and φ_ξ is given by (2.1) approximating Φ. We have

$$\widetilde{U}_n = \left\{ x \in M : \frac{1}{n} S_n \varphi_\xi(x) > c + \xi \right\} \subset U_n$$

for all sufficiently large n. Moreover, if $\delta > 0$ is small enough we can assume that $\mathcal{F}_*(\nu, \Phi) > c + 2\xi + \delta$. By uniform continuity of φ_ξ, there exists $\varepsilon_\delta > 0$ such that $|\varphi_\xi(x) - \varphi_\xi(y)| < \delta/2$ whenever $d(x, y) \le \varepsilon_\delta$. We claim that this guarantees that if $(1/n)S_n\varphi_\xi(x) > c + \xi + \delta$, then $B_n(g; x, \varepsilon) \subset \widetilde{U}_n \subset U_n$ for all sufficiently large n. Indeed, for each $y \in B_n(g; x, \varepsilon)$ there exists $\Lambda \subset I(g; n, \varepsilon_0)$ so that $y \in B_\Lambda(x, \varepsilon)$, and so

$$\sum_{i=0}^{n-1} \varphi_\xi(f^i x) \le \sum_{i \in \Lambda} \left[\varphi_\xi(f^i y) + \frac{\delta}{2} \right] + \sum_{i \notin \Lambda} \|\varphi_\xi\| \le \sum_{i=0}^{n-1} \left[\varphi_\xi(f^i y) + \frac{\delta}{2} \right] + C g(n, \varepsilon), \tag{5.1}$$

where $C = 2\|\varphi_\xi\| + \delta$. This implies that $S_n\varphi_\xi(y) \ge S_n\varphi_\xi(x) - Cg(n, \varepsilon) - \delta n/2$. By the definition of a mistake function, we have $(1/n)S_n\varphi_\xi(y) > c + \xi$ for all sufficiently large n, which means that $y \in \widetilde{U}_n \subset U_n$ as claimed.

Fix now an arbitrarily small $\gamma > 0$. By the modified Katok entropy formula (2.4) we can choose $0 < \varepsilon_1 < \varepsilon_\delta$ such that $\liminf_{n\to\infty}(1/n) \log N(g; n, 2\varepsilon, \frac{1}{2}) \ge h_\nu(f) - \gamma$ for all $\varepsilon < \varepsilon_1$. Pick also $0 < \varepsilon_2 < \varepsilon_1$ such that

$$\nu\left(\left\{ x \in M : \limsup_{n\to\infty} -\frac{1}{n} \log m(B_n(g; x, \varepsilon_2)) \le h_m(g; f, \nu) + \gamma \right\} \right) > \frac{2}{3}. \tag{5.2}$$

Since $\int \varphi_\xi \, d\nu > \mathcal{F}_*(\nu, \Phi) - \xi > c + \xi + \delta$, one can use (5.2) and the Birkhoff ergodic theorem to choose a measurable set $\mathcal{L} \subset M$ with $\nu(\mathcal{L}) \ge \frac{1}{2}$ and a positive integer N such that for any $n \ge N$ the following two properties hold for each $x \in \mathcal{L}$:

(i) $(1/n)S_n\varphi_\xi(x) > c + \xi + \delta$;

(ii) $m(B_n(g; x, \varepsilon_2)) \ge \exp(-n(h_m(g; f, \nu) + 2\gamma))$.

For each sufficiently large n, let E_n be a maximal set of $(g; n, 2\varepsilon_2)$-separated points contained in $\mathcal{L}$. It is not hard to check that $\mathcal{L} \subset \bigcup_{x \in E_n} B_n(g; x, 2\varepsilon_2)$ and that the mistake dynamical balls $\{B_n(g; x, \varepsilon_2) : x \in E_n\}$ are disjoint. Using the fact that $\sharp E_n \ge N(g; n, 2\varepsilon_2, \frac{1}{2})$, it follows that

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{n} \log m(U_n) &\ge \liminf_{n\to\infty} \frac{1}{n} \log m(\widetilde{U}_n) \\
&\ge \liminf_{n\to\infty} \frac{1}{n} \log \sum_{x \in E_n} m(B_n(g; x, \varepsilon_2)) \\
&\ge \liminf_{n\to\infty} \frac{1}{n} \log \sharp E_n \exp(-n(h_m(g; f, \nu) + 2\gamma)) \\
&\ge h_\nu(f) - h_m(g; f, \nu) - 3\gamma.
\end{aligned}$$

The arbitrariness of γ implies the desired result. □

Proof of Theorem A(2). Let $U_n = \{x \in M : (1/n)\varphi_n(x) \ge c\}$ and $\Psi = \{\psi_n\} \in \mathcal{V}_K^+(g)$. By the definition of $\mathcal{V}_K^+(g)$, there exists a set Υ of full m-measure and constants C_n and ε such

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that $m(B_n(g; x, \varepsilon)) \le C_n \exp(-nK + \psi_n(x))$ for all $x \in \Upsilon$ and $n \ge 1$. We will assume without loss of generality that $U_n \subset \Upsilon$, since otherwise we just consider $U_n \cap \Upsilon$. It suffices to construct a measure $\nu \in \mathcal{M}_f$ with $\mathcal{F}_*(\nu, \Phi) \ge c$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log m(U_n) \le -K + h_\nu(f) + \mathcal{F}_*(\nu, \Psi).$$

Let E_n be a maximal $(g; n, \varepsilon)$-separated set contained in U_n. Note that every $(g; n, \varepsilon)$-separated set is also a (n, ε)-separated set. If E_n is not a maximal (n, ε)-separated set, consider a maximal (n, ε)-separated set E_n' containing E_n. Now, define the probability measures

$$\mu_n = \sum_{x\in E_n'} \frac{e^{\psi_n(x)}}{Z_n} \cdot \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i x},$$

where $Z_n = \sum_{x\in E_n'} e^{\psi_n(x)}$, and let ν be a weak* limit of μ_n. It is not hard to check that $\nu \in \mathcal{M}_f$. Moreover, it follows from the proof of [**12**, Theorem 1.1] that $\limsup_{n\to\infty}(1/n) \log Z_n \le h_\nu(f) + \mathcal{F}_*(\nu, \Psi)$. Now, since $U_n \subset \bigcup_{x\in E_n} B_n(g; x, \varepsilon)$ and the constants C_n are tempered, we get

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \log m(U_n) &\le \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x\in E_n} m(B_n(g; x, \varepsilon)) \\
&\le \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x\in E_n} C_n \exp(-nK + \psi_n(x)) \\
&\le -K + \limsup_{n\to\infty} \frac{1}{n} \log Z_n \\
&\le -K + h_\nu(f) + \mathcal{F}_*(\nu, \Psi).
\end{aligned}$$

It remains to prove that $\mathcal{F}_*(\nu, \Phi) \ge c$. For each small $\xi > 0$, let φ_ξ be given by (2.1) approximating Φ. Since μ_n is a linear combination of measures $(1/n) \sum_{i=0}^{n-1} \delta_{f^i x}$ for $x \in E_n'$ and

$$\int \varphi_\xi \, d\left(\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i x}\right) = \frac{1}{n} S_n \varphi_\xi(x) \ge \frac{1}{n} \varphi_n(x) - \xi \ge c - \xi,$$

we deduce that $\int \varphi_\xi \, d\mu_n \ge c - \xi$. As a consequence, $c - \xi \le \int \varphi_\xi \, d\nu \le \mathcal{F}_*(\nu, \Phi) + \xi$. The arbitrariness of ξ implies that $\mathcal{F}_*(\nu, \Phi) \ge c$. This completes the proof of the second part of the theorem. □

Proof of Theorem A(3). Let $U_n = \{x \in M : (1/n)\varphi_n(x) \ge c\}$ and $\Psi = \{\psi_n\} \in \mathcal{V}_K^-(g)$. Without loss of generality, assume that $U_n \subset \Upsilon$, since otherwise we just consider $U_n \cap \Upsilon$. Pick an arbitrary f-invariant and ergodic measure ν satisfying that $\mathcal{F}_*(\nu, \Phi) > c$ and $\nu(\Upsilon) = 1$. It follows from the definition of $\mathcal{V}_K^-(g)$ and the ergodicity of ν that

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log m(B_n(g; x, \varepsilon)) \le K - \mathcal{F}_*(\nu, \Psi), \quad \nu\text{-almost every } x \in \Upsilon,$$

and, consequently, $h_m(g; f, \nu) \le K - \mathcal{F}_*(\nu, \Psi)$. Therefore, it follows from the first item of this theorem that $\underline{R}_m(\Phi, (c, \infty)) \ge \sup\{-K + h_\nu(f) + \mathcal{F}_*(\nu, \Psi)\}$, where the supremum is taken over all $\nu \in \mathcal{E}_f$ that satisfy the statements $\mathcal{F}_*(\nu, \Phi) > c$ and $\nu(\Upsilon) = 1$. This proves the third assertion of the theorem. □

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Proof of Theorem A(4). Fix an asymptotically additive family of potentials $\Psi \in \mathcal{V}_K^-$. Let $U_n = \{x \in M : (1/n)\varphi_n(x) > c\}$ and assume, without loss of generality, that $U_n \subset \Upsilon$. Pick an invariant measure $\nu \in \mathcal{M}_f$ with $\mathcal{F}_*(\nu, \Phi) > c$ and $\nu(\Upsilon) = 1$. It suffices to prove that

$$\liminf_{n\to\infty}(1/n)\log m(U_n) \geq -K + h_\nu(f) + \mathcal{F}_*(\nu, \Psi) - 5\gamma$$

for any preassigned $\gamma > 0$. We will divide the proof into several lemmas.

LEMMA 5.2. *For every $\delta > 0$ and $\gamma > 0$, there exists $\mu \in \mathcal{M}_f$ such that $\mu = \sum_{i=1}^k a_i\mu_i$, where $\sum_{i=1}^k a_i = 1$ and $\mu_i \in \mathcal{E}_f$, which satisfies the statements:*

(i) $\mu_i(\Upsilon) = 1$, $i = 1, \ldots, k$;

(ii) $|\mathcal{F}_*(\nu, \Phi) - \mathcal{F}_*(\mu, \Phi)| < \delta$;

(iii) $h_\mu(f) + \mathcal{F}_*(\mu, \Psi) \geq h_\nu(f) + \mathcal{F}_*(\nu, \Psi) - 2\gamma$.

Proof. Although the argument follows by a small modification of the standard one in [**43**] we will prove it here for completeness. Let dist* denote the metric on the space of all Borel probability measures on M. It follows by uniform continuity of the functional $\eta \mapsto \mathcal{F}_*(\eta, \Phi)$ (see [**21**]) that, for any given $\delta > 0$ and $\gamma > 0$, there exists $\varepsilon_0 > 0$ such that

$$\text{dist}^*(\tau_1, \tau_2) < \varepsilon_0 \Rightarrow |\mathcal{F}_*(\tau_1, \Phi) - \mathcal{F}_*(\tau_2, \Phi)| < \delta \text{ and } |\mathcal{F}_*(\tau_1, \Psi) - \mathcal{F}_*(\tau_2, \Psi)| < \gamma.$$

Let $\mathcal{P} = \{P_1, \ldots, P_2\}$ be a partition of $\mathcal{M}_f$ with $\text{diam}\mathcal{P} = \max_{1\leq i\leq k}|P_i| < \varepsilon_0$, where $|\cdot|$ denotes the diameter of a set. Using the ergodic decomposition theorem, there is a Borel probability measure π on $\mathcal{M}_f$ such that $\pi(\mathcal{E}_f) = 1$ and $\mathcal{F}_*(\nu, \mathcal{G}) = \int \mathcal{F}_*(\tau, \mathcal{G})\, d\pi(\tau)$ for any asymptotically additive potential $\mathcal{G}$. We refer the reader to [**21**] for a proof of this fact. Moreover, let $\mathcal{E}_f' = \{\tau \in \mathcal{E}_f : \tau(\Upsilon) = 1\}$, then $\pi(\mathcal{E}_f') = 1$. Let $a_i = \pi(P_i)$ and pick $\mu_i \in P_i \cap \mathcal{E}_f'$ satisfying $h_{\mu_i}(f) > h_\tau(f) - \gamma$ for all $\tau \in P_i \cap \mathcal{E}_f'$. Then we have that

$$a_i h_{\mu_i}(f) \geq \int_{P_i\cap\mathcal{E}_f'} [h_\tau(f) - \gamma]\, d\pi(\tau) = \int_{P_i} h_\tau(f)\, d\pi(\tau) - \gamma a_i.$$

Summing over i, it follows that the probability measure $\mu = \sum_{i=1}^k a_i\mu_i$ satisfies $h_\mu(f) \geq h_\nu(f) - \gamma$. Moreover,

$$\mathcal{F}_*(\mu, \Psi) = \sum_{i=1}^k a_i\mathcal{F}_*(\mu_i, \Psi) \geq \sum_{i=1}^k \int_{P_i} (\mathcal{F}_*(\tau, \Psi) - \gamma)\, d\pi(\tau) = \mathcal{F}_*(\nu, \Psi) - \gamma.$$

Hence, we have that $h_\mu(f) + \mathcal{F}_*(\mu, \Psi) \geq h_\nu(f) + \mathcal{F}_*(\nu, \Psi) - 2\gamma$. A similar argument shows that $|\mathcal{F}_*(\nu, \Phi) - \mathcal{F}_*(\mu, \Phi)| < \delta$. This completes the proof of the lemma. □

Fix a small $\xi > 0$, let ψ_ξ and φ_ξ be the continuous functions given as in (2.1) approximating the sequences Ψ and Φ, respectively. Fix $\delta = (\mathcal{F}_*(\nu, \Phi) - c)/4 > 0$ and a small $\gamma > 0$. Using the uniform continuity of ψ_ξ and φ_ξ, the Birkhoff ergodic theorem and Proposition A we can choose $\varepsilon > 0$ sufficiently small and N sufficiently large so that the following properties hold:

(i) $d(x, y) < \varepsilon \Rightarrow |\psi_\xi(x) - \psi_\xi(y)| < \gamma$ and $|\varphi_\xi(x) - \varphi_\xi(y)| < \delta$;

(ii) for each $n \geq N$ and $1 \leq i \leq k$, there exist at least $e^{[a_i n](h_{\mu_i}(f)-\gamma)}$ points $x_1^{(i)}, \ldots, x_{n_i}^{(i)}$ that are $(2g; [a_i n], 4\varepsilon)$-separated with the property that, for each $1 \leq j \leq n_i$, we have:

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(a) $S_{[a_i n]}\psi_\xi(x_j^{(i)}) \le [a_i n](\int \psi_\xi \, d\mu_i + \gamma)$;

(b) $S_{[a_i n]}\varphi_\xi(x_j^{(i)}) \ge [a_i n](\int \varphi_\xi \, d\mu_i - 2\delta)$.

Note that $n_i \ge e^{[a_i n](h_{\mu_i}(f)-\gamma)}$. Since f has the g-almost specification property, for each k-tuple $(j_1, \dots, j_k)$ with $1 \le j_i \le n_i$ we can choose a point

$$y = y_{j_1 \cdots j_k} \in \bigcap_{i=1}^{k} f^{-\sum_{j=0}^{i-1}[a_i n]}(B_{[a_i n]}(g; x_{j_i}, \varepsilon)).$$

Let $F = \{y_{j_1 \cdots j_k} : (j_1, \dots, j_k),\ 1 \le j_i \le n_i\}$ and $\hat{n} = \sum_i [a_i n]$. Now we prove the following lemma.

LEMMA 5.3. *The set F is $(\hat{n}, 2\varepsilon)$-separated, i.e. $B_{\hat{n}}(y, \varepsilon) \cap B_{\hat{n}}(y', \varepsilon) = \emptyset$ for any distinct $y, y' \in F$. In particular, $\#F \ge \exp(\hat{n}(h_\mu(f) - \gamma))$.*

Proof. Given $y \ne y' \in F$, take distinct k-tuples $(j_1, \dots, j_k) \ne (j_1', \dots, j_k')$ such that $y = y_{j_1 \cdots j_k}$ and $y' = y_{j_1' \cdots j_k'}$. We assume without loss of generality that $j_1 \ne j_1'$. Consequently, $y \in B_{[a_1 n]}(g; x_{j_1}, \varepsilon)$ and $y' \in B_{[a_1 n]}(g; x_{j_1'}, \varepsilon)$. Therefore, there exist $\Lambda_i \in I(g; [a_1 n], \varepsilon)$ for $i = 1, 2$ such that $d_{\Lambda_1}(x_{j_1}, y) < \varepsilon$ and $d_{\Lambda_2}(x_{j_1'}, y') < \varepsilon$. If $\Lambda = \Lambda_1 \cap \Lambda_2$ then $[a_1 n] \ge \#\Lambda \ge [a_1 n] - 2g([a_1 n], \varepsilon])$ and so

$$\begin{aligned} d_{\hat{n}}(y, y') &\ge d_\Lambda(y, y') \ge d_\Lambda(x_{j_1}, x_{j_1'}) - d_\Lambda(y, x_{j_1}) - d_\Lambda(y', x_{j_1'}') \\ &> 4\varepsilon - \varepsilon - \varepsilon = 2\varepsilon. \end{aligned}$$

Hence, $B_{\hat{n}}(y, \varepsilon) \cap B_{\hat{n}}(y', \varepsilon) = \emptyset$ as claimed. By the previous reasoning all $y = y_{j_1 \cdots j_k}$ are distinct and so the cardinality of F is bounded from below by $n_1 \cdots n_k \ge \exp(\hat{n}(h_\mu(f) - \gamma))$. This finishes the proof of the lemma. □

We proceed with the following auxiliary lemma.

LEMMA 5.4. *If $\xi, \delta > 0$ are small enough, then for every large n and $y \in F$:*

(i) $B_{\hat{n}}(y, \varepsilon) \subset U_{\hat{n}}$;

(ii) $(1/\hat{n})\psi_{\hat{n}}(y) \ge \mathcal{F}_*(\mu, \Psi) - 2\gamma$.

Proof. Fix a small $\xi > 0$ and let φ_ξ be a continuous function approximating the sequence Φ as before. For each $z \in B_{\hat{n}}(y, \varepsilon)$ with $y = y_{j_1 \cdots j_k}$ consider the images $y_i = f^{([a_1 n] + \cdots + [a_{i-1} n])}(y)$ $(1 \le i \le k)$ and note that

$$\begin{aligned} \frac{1}{\hat{n}}\varphi_{\hat{n}}(z) &\ge \frac{1}{\hat{n}} S_{\hat{n}}\varphi_\xi(z) - \xi \ge \frac{1}{\hat{n}} \sum_{i=1}^{k} (S_{[a_i n]}\varphi_\xi(y_i) - [a_i n]\delta) - \xi \\ &\ge \frac{1}{\hat{n}} \sum_{i=1}^{k} [S_{[a_i n]}\varphi_\xi(x_{j_i}) - 2[a_i n]\delta - 2g([a_i n], \varepsilon)\|\varphi_\xi\|] - \xi \\ &\ge \frac{1}{\hat{n}} \sum_{i=1}^{k} \left[[a_i n]\left(\int \varphi_\xi \, d\mu_i - 2\delta \right) - 2g([a_i n], \varepsilon)\|\varphi_\xi\| \right] - 2\delta - \xi \\ &\ge \frac{1}{\hat{n}} \sum_{i=1}^{k} [[a_i n]\mathcal{F}_*(\mu_i, \Phi) - 2g([a_i n], \varepsilon)\|\varphi_\xi\|] - 4\delta - 2\xi \\ &\ge \mathcal{F}_*(\nu, \Phi) - 5\delta - 2\xi \end{aligned}$$

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for all sufficiently large n. Therefore, $(1/\hat{n})\varphi_{\hat{n}}(z) > c$ provided that δ and ξ are sufficiently small. This completes the proof of (i) above. Since the arguments to prove (ii) are analogous we shall omit the proof. □

We are now in a position to finish the proof of Theorem A(4). Indeed, combining the previous lemmas we obtain that

$$\begin{aligned}\limsup_{\hat{n}\to\infty}\frac{1}{\hat{n}}\log m(U_{\hat{n}}) &\geq \limsup_{\hat{n}\to\infty}\frac{1}{\hat{n}}\log\sum_{y\in F} m(B_{\hat{n}}(y,\varepsilon))\\ &\geq \limsup_{\hat{n}\to\infty}\frac{1}{\hat{n}}\log\sum_{y\in F} C_{\hat{n}}\exp(-\hat{n}K+\psi_{\hat{n}}(y))\\ &\geq -K+h_\mu(f)+\mathcal{F}_*(\mu,\Psi)-3\gamma\\ &\geq -K+h_\nu(f)+\mathcal{F}_*(\nu,\Psi)-5\gamma,\end{aligned}$$

which proves the last statement and finishes the proof of the theorem. □

5.3. *Proof of Theorem B.* Here we prove our second main result, which estimates the measure of the deviation sets in terms of some thermodynamical quantities for asymptotically additive and sub-additive families of potentials.

Case I: $\Psi=\{\psi_n\}_n$ asymptotically additive sequence of continuous functions.

Upper bound. Since the upper bound is similar for both asymptotically additive and sub-additive potentials we shall focus on the first case. So, let $\Psi=\{\psi_n\}\in\mathcal{A}$ be an asymptotically additive sequence of continuous functions, ν be a weak Gibbs measure with respect to Φ on $\Lambda\subset M$ and take $c\in\mathbb{R}$. Therefore, there exists $\varepsilon_0>0$ such that for every $0<\varepsilon<\varepsilon_0$ there exists a sequence of positive constants $(K_n)_{n\geq 1}$ (depending only on ε) satisfying $\lim_{n\to\infty}(1/n)\log K_n=0$ and

$$K_n^{-1}\leq\frac{\nu(B_n(x,\varepsilon))}{e^{-nP(f,\Phi)+\varphi_n(x)}}\leq K_n\quad\text{for all } n\geq 1 \text{ and } x\in\Lambda.$$

In consequence, $\Phi\in\mathcal{V}_K^+(g)$ with respect to the reference measure ν, where we consider the mistake function $g\equiv 0$, $C_n=K_n$ and $K=P(f,\Phi)$. In this way, (UB) is a direct consequence of Theorem A(2). This finishes the proof of the upper bound in Theorem B.

Lower bound using ergodic measures. The same reasoning as above yields that $\Phi\in\mathcal{V}_K^-(g)$ with respect to the reference measure ν, where we consider the mistake function $g\equiv 0$, $C_n=K_n^{-1}$ and $K=P(f,\Phi)$. In this way, the lower bound estimate using ergodic measures is a direct consequence of Theorem A(3). However, for completeness we shall prove this fact to collect some constants needed for the proof of (LB). Let $\Psi=\{\psi_n\}$ be an asymptotically additive sequence of continuous functions, $c\in\mathbb{R}$ be fixed and $\beta>0$ be arbitrary. Denote by U_n the set of points $x\in M$ so that $\psi_n(x)>cn$, without loss of generality, we assume that $U_n\subset\Lambda$. We claim that if η is any f-invariant and ergodic probability measure so that $\eta(\Lambda)=1$ and $\mathcal{F}_*(\eta,\Psi)>c$, then

$$\liminf_{n\to\infty}\frac{1}{n}\log\nu(U_n)\geq -P(f,\Phi)+h_\eta(f)+\mathcal{F}_*(\eta,\Phi)-2\beta$$

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for any preassigned $\beta > 0$ with $\mathcal{F}_*(\eta, \Psi) > c + \beta$. By ergodicity, one can pick $N \geq 1$ large and ε_0 small so that $N(n, 2\varepsilon, \frac{1}{2}) \geq e^{[h_\eta(f)-\beta]n}$ for all $0 < \varepsilon < \varepsilon_0$ and $n \geq N$, and also the set F of points $x \in M$ satisfying $\mathcal{F}_*(\eta, \Phi) - \beta < (1/n)\varphi_n(x) < \mathcal{F}_*(\eta, \Phi) + \beta$ and $\mathcal{F}_*(\eta, \Psi) - \beta < (1/n)\psi_n(x) < \mathcal{F}_*(\eta, \Psi) + \beta$ for all $n \geq N$ has η-measure at least $\frac{1}{2}$.

Now, take $\xi = (\mathcal{F}_*(\eta, \Psi) - \beta - c)/4 > 0$ and let ψ_ξ be given by approximation (2.1). Up to smaller ε_0 and larger $N \geq 1$ we may assume that $|\psi_\xi(x) - \psi_\xi(y)| < \xi$ whenever $d(x, y) < \varepsilon_0$ (ψ_ξ is uniformly continuous) and $\|\psi_n - S_n\psi_\xi\| \leq \xi n$ for all $n \geq N$. Using our construction it is clear that $F \subset U_n$. We claim that

$$B_n(x, \varepsilon) \subset U_n \quad \text{for all } x \in F. \tag{5.3}$$

In fact, if $x \in F$ and $y \in B_n(x, \varepsilon)$ using $(1/n)\psi_n(x) \geq \mathcal{F}_*(\eta, \Psi) - \beta = c + 4\xi$, we get

$$\begin{aligned} |\psi_n(x) - \psi_n(y)| &\leq |\psi_n(x) - S_n\psi_\xi(x)| + |\psi_n(y) - S_n\psi_\xi(y)| \\ &\quad + |S_n\psi_\xi(x) - S_n\psi_\xi(y)| < 3\xi n \end{aligned}$$

and consequently $\psi_n(y) > cn$, proving our claim. Therefore, $U_n \supset \bigcup_{x\in F} B_n(x, \varepsilon) \supset F$ for every $0 < \varepsilon < \varepsilon_0$ and $n \geq N$. Moreover, if $F_n \subset F$ is a maximal $(n, 2\varepsilon)$-separated set, one can use the fact that the dynamical balls $B_n(x, \varepsilon)$ centered at points in F_n are pairwise disjoint and contained in U_n. This yields

$$\begin{aligned} \nu(U_n) &\geq \nu\left(\bigcup_{x\in F_n} B_n(x, \varepsilon)\right) = \sum_{x\in F_n} \nu(B_n(x, \varepsilon)) \\ &\geq \sum_{x\in F_n} K_n^{-1} e^{-P(f,\Phi)n+\varphi_n(x)} \\ &\geq K_n^{-1} \exp(-P(f, \Phi) + h_\eta(f) + \mathcal{F}_*(\eta, \Phi) - 2\beta)n \end{aligned}$$

whenever $n \geq N$, which proves our claim since the sequence $\{K_n\}$ is tempered. The arbitrariness of β and the measure implies that

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu\left\{x \in M : \frac{1}{n}\psi_n(x) > c\right\} \geq -P(f, \Phi) + h_\eta(f) + \mathcal{F}_*(\eta, \Psi)$$

for every f-invariant, ergodic probability measure η such that $\mathcal{F}_*(\eta, \Psi) > c$ and $\eta(\Lambda) = 1$. In consequence, we obtain the second assertion in Theorem B.

Lower bound over all invariant measures. Here we deduce the general lower bound over all invariant measures under the assumption that f satisfies some weak specification properties.

Fix $c > 0$ and an asymptotically additive potential $\Psi = \{\psi_n\}$. Let $U_n = \{x \in M : (1/n)\psi_n(x) > c\}$. Without loss of generality, we assume that $U_n \subset \Lambda$. We claim that for any f-invariant probability measure η satisfying $\eta(\Lambda) = 1$ and $\mathcal{F}_*(\eta, \Psi) > c$ it holds that

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu\left[x \in M : \frac{1}{n}\psi_n(x) > c\right] \geq -P(f, \Phi) + h_\eta(f) + \mathcal{F}_*(\eta, \Phi).$$

We will make use of the following approximation result that is a reformulation of Lemma 5.2 and in a position to finish the proof of the theorem for asymptotically additive sequences.

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LEMMA 5.5. *Let $\eta = \int \eta_x \, d\eta(x)$ be the ergodic decomposition of an f-invariant probability measure η such that $\eta(\Lambda) = 1$. Given $\gamma > 0$ and a finite set of asymptotically additive potentials $(\Psi_j)_{1 \le j \le r}$, there are positive real numbers $(a_i)_{1 \le i \le k}$ satisfying $a_i \le 1$ and $\sum a_i = 1$, and finitely many points $x_1, \ldots, x_k$ such that the ergodic measures $\eta_i = \eta_{x_i}$ from the ergodic decomposition satisfy:*

(i) $\eta_i(\Lambda) = 1$;
(ii) $h_{\hat{\eta}}(f) \ge h_\eta(f) - \gamma$; *and*
(iii) $|\mathcal{F}_*(\Psi_j, \hat{\eta}) - \mathcal{F}_*(\Psi_j, \eta)| < \gamma$ *for every* $1 \le j \le r$;
where $\hat{\eta} = \sum_{i=1}^k a_i \eta_i$.

Given a small $\gamma > 0$, using the lemma, there are ergodic probability measures $(\eta_i)_i$ so that a probability measure $\hat{\eta} = \sum_{i=1}^k a_i \eta_i$ satisfies $h_{\hat{\eta}}(f) \ge h_\eta(f) - \gamma$, and such that $|\mathcal{F}_*(\Psi, \hat{\eta}) - \mathcal{F}_*(\Psi, \eta)| < \gamma$ and $|\mathcal{F}_*(\Phi, \hat{\eta}) - \mathcal{F}_*(\Phi, \eta)| < \gamma$. Proceeding as before, there are $N \ge 1$ large and ε_0 small so that, for every $1 \le i \le k$, the set F^i of points $x \in \Lambda$ such that $\mathcal{F}_*(\eta_i, \Phi) - \gamma < (1/n)\varphi_n(x) < \mathcal{F}_*(\eta_i, \Phi) + \gamma$ and $\mathcal{F}_*(\eta_i, \Psi) - \gamma < (1/n)\psi_n(x) < \mathcal{F}_*(\eta_i, \Psi) + \gamma$ for every $n \ge N$ and $0 < \varepsilon < \varepsilon_0$ has η_i-measure at least $\frac{1}{2}$. We may assume also without loss of generality that the minimum number of $(n, 4\varepsilon)$-dynamical balls needed to cover a set of η_i-measure $\frac{1}{2}$ satisfies $N_i(n, 4\varepsilon, \frac{1}{2}) \ge e^{[h_{\eta_i}(f) - \gamma]n}$ for all $0 < \varepsilon < \varepsilon_0$ and $n \ge N$. So, pick a finite set $F_n^i \subset F^i$ so that F_n^i is a maximal $([a_i n], 4\varepsilon)$-separated set in F^i and $\# F_n^i \ge e^{(h_{\eta_i}(f) - \gamma)\,[a_i n]}$. Moreover, by the specification property, for every sequence $(x_1, x_2, \ldots, x_k)$ with $x_i \in F_n^i$ there exists $x \in M$ that ε-shadows each x_i during $[a_i n]$ iterates with a time lag of $N(\varepsilon, n)$ iterates in between. Set $\tilde{n} = \sum_i [a_i n] + kN(\varepsilon, n)$ and consider the set

$$\mathcal{L}_{\tilde{n}} = \bigcup \{B_{\tilde{n}}(x, \varepsilon) : x = x_{i_1, \ldots, i_k}\}.$$

Using similar arguments as in the proofs of Lemmas 5.3 and 5.4, we can show that the dynamical balls in the latter union have the following properties:

(i) the dynamical balls $B_{\tilde{n}}(x, \varepsilon)$ and $B_{\tilde{n}}(y, \varepsilon)$ are disjoint for $x \ne y$;
(ii) the number of the dynamical balls is larger than $\exp \tilde{n}(h_{\hat{\eta}}(f) - \gamma)$;
(iii) each dynamical ball $B_{\tilde{n}}(x, \varepsilon)$ is contained in $U_{\tilde{n}}$;
(iv) $(1/\tilde{n})\varphi_{\tilde{n}}(y) \ge \mathcal{F}_*(\hat{\eta}, \Phi) - 2\gamma$ for every $y \in \mathcal{L}_{\tilde{n}}$.

Since ν is a weak Gibbs measure for Φ on $\Lambda \subset M$ and $\tilde{n} \ge N$ it follows that $\nu(B_{\tilde{n}}(x, \varepsilon)) \ge K_{\tilde{n}}^{-1} e^{-\tilde{n} P(f, \Phi) + \varphi_{\tilde{n}}(x)}$. Therefore, this yields that

$$\begin{aligned} \nu(U_{\tilde{n}}) &\ge \sum_x \nu(B_{\tilde{n}}(x, \varepsilon)) \ge K_{\tilde{n}}^{-1} \exp[-P(f, \Phi) + h_{\hat{\eta}}(f) + \mathcal{F}_*(\hat{\eta}, \Phi) - 3\gamma]\tilde{n} \\ &\ge K_{\tilde{n}}^{-1} \exp[-P(f, \Phi) + h_\eta(f) + \mathcal{F}_*(\eta, \Phi) - 5\gamma]\tilde{n} \end{aligned}$$

for every large n. Since the sequence $\{K_n\}$ is tempered and γ was chosen arbitrarily, then we get

$$\liminf_{n \to \infty} \frac{1}{n} \log \nu(U_n) \ge -P(f, \Phi) + h_\eta(f) + \mathcal{F}_*(\eta, \Phi),$$

which finishes the proof of Theorem B in the case of asymptotically additive sequences Ψ.

Case II: $\Psi = \{\psi_n\}_n$ sub-additive sequence of continuous potentials. Throughout the remainder of the proof, assume that $\Psi = \{\psi_n\}_n$ is a sub-additive sequence of continuous potentials that satisfy the weak Bowen condition, that $(\psi_n/n)_n$ is equicontinuous and

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that $\inf_{n\geq 1}\psi_n(x)/n > -\infty$ for every $x \in M$ as in Theorem B. Since the proof of upper and lower bounds are similar to the previous ones for asymptotically additive potentials we will only highlight the main differences.

Upper bound. Let $\Psi = \{\psi_n\}$ be as above, ν be a weak Gibbs measure with respect to Φ on $\Lambda \subset M$ and take $c \in \mathbb{R}$. Therefore, there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ there exists a sequence of positive constants $(K_n)_{n\geq 1}$ (depending only on ε) satisfying $\lim_{n\to\infty}(1/n)\log K_n = 0$ and

$$K_n^{-1} \leq \frac{\nu(B_n(x,\varepsilon))}{e^{-nP(f,\Phi)+\varphi_n(x)}} \leq K_n \quad \text{for all } n \geq 1 \text{ and } x \in \Lambda.$$

In consequence, $\Phi \in \mathcal{V}_K^+(g)$ with respect to the reference measure ν, where we consider the mistake function $g \equiv 0$, $C_n = K_n$ and $K = P(f, \Phi)$. On the other hand, using that $\{\psi_n/n\}$ is equicontinuous, $\psi_n(x) \leq n\|\psi_1\|$ and that $\inf_{n\geq 1}\psi_n(x)/n > -\infty$ for every $x \in M$ it follows from the Arzéla–Ascoli theorem that the sequence $\{\psi_n/n\}$ admits a subsequence uniformly convergent to some continuous function g. Therefore, for any $\varepsilon > 0$ small there exists a positive integer $k \geq 1$ such that $\|(\psi_k/k) - g\| < \varepsilon$. For each sufficiently large n, it follows from [**13**, Lemma 2.2] that there exists a constant C depending only on k such that $\psi_n \leq S_n(\psi_k/k) + C$. Consequently, we have that

$$\frac{\psi_n(x)}{n} \leq \frac{1}{n}S_n\left(\frac{\psi_k}{k}\right)(x) + \frac{C}{n} \leq \frac{1}{n}S_n g(x) + 2\varepsilon$$

for all x and sufficiently large n. Thus, for any $\varepsilon > 0$ one has the inclusion

$$\left\{\frac{1}{n}\psi_n \geq c\right\} \subset \left\{\frac{1}{n}S_n g \geq c - 2\varepsilon\right\}$$

provided that n is large. Now we proceed as in the proof of Theorem A(2). Fix $\varepsilon > 0$ to be arbitrary small and take a maximal (n, ε)-separated set E_n contained in $\{S_n g \geq (c - 2\varepsilon)n\}$. Then, the same arguments as before show that any weak* limit μ of the probability measures

$$\mu_n = \sum_{x\in E_n} \frac{e^{\varphi_n(x)}}{Z_n} \cdot \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i x},$$

where $Z_n = \sum_{x\in E_n} e^{\varphi_n(x)}$, satisfies (UB).

It remains to prove that $\mathcal{F}_*(\mu, \Psi) \geq c$. Applying the sub-additive ergodic theorem to the invariant measure μ, there exist an f-invariant function $\tilde{\psi}$ and a subset Υ with $\mu(\Upsilon) = 1$ such that

$$\tilde{\psi}(x) = \lim_{n\to\infty} \frac{\psi_n(x)}{n} \quad \text{for all } x \in \Upsilon.$$

This implies that $\tilde{\psi}(x) = g(x)$ for each point $x \in \Upsilon$, and consequently,

$$\mathcal{F}_*(\mu, \Psi) = \int \tilde{\psi}\, d\mu = \int g\, d\mu.$$

Since the same argument holds for any f-invariant probability measure this also implies that the map $\mu \mapsto \mathcal{F}_*(\mu, \Psi)$ is continuous since the function g is continuous. Proceeding

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as in the proof of Theorem A(2), we have

$$\int g\, d\mu = \lim_{n\to\infty} \int g\, d\mu_n \geq c - 2\varepsilon.$$

The arbitrariness of ε implies that $\mathcal{F}_*(\mu, \Psi) \geq c$. This finishes the proof of the upper bound in this sub-additive setting.

Lower bounds. The proof of this second case goes along the same lines as the first one. For that reason we shall just highlight the differences. Fix $c > 0$ and let $\Psi = \{\psi_n\}$ be a sub-additive family of continuous potentials as above, and assume without loss of generality that $U_n = \{x \in M : \psi_n(x) > cn\} \subset \Lambda$. We claim that

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu\left[x \in M : \frac{1}{n}\psi_n(x) > c\right] \geq -P(f, \Phi) + h_\eta(f) + \mathcal{F}_*(\eta, \Phi)$$

for any f-invariant probability measure η so that $\eta(\Lambda) = 1$ and $\mathcal{F}_*(\eta, \Psi) > c$. In fact, the requirement of the former asymptotically additive assumption used to deduce the claim for ergodic measures was only to control the variation of this sequence on dynamical balls. See (5.3) for the precise statement. To overlap this difficulty in this setting we use the weak Bowen property as follows.

LEMMA 5.6. *Let $\Psi = \{\psi_n\}_n$ be a sub-additive potential with the weak Bowen property and $a \in \mathbb{R}$. If $\psi_n(x) > an$, then for all $\gamma > 0$ there exists $N \geq 1$ large so that all $y \in B_n(x, \varepsilon)$ satisfy $\psi_n(y) > (a - \gamma)n$.*

Proof. Let $a \in \mathbb{R}$ and $\gamma > 0$ be arbitrary. In fact, using the fact that $\mathrm{var}(\psi_n|_{B_n(x,\varepsilon)}) \leq a_n$ for some sequence $a_n/n \to 0$, it is immediate that $|\psi_n(x) - \psi_n(y)| \leq a_n$ and consequently

$$\frac{1}{n}\psi_n(y) \geq \frac{1}{n}\psi_n(x) - \frac{a_n}{n} \geq a - \gamma$$

for every $y \in B_n(x, \varepsilon)$, provided that n is large. This proves the lemma. □

Hence the remainder of the argument to prove the claim in the case that η is ergodic is analogous to that of Case I. In the case that η is not ergodic an extra approximation argument by ergodic measures given by Lemma 5.5 is necessary. The proof of that lemma uses the continuity of the functional $\mu \to \mathcal{F}_*(\Psi, \mu)$. This is not necessarily continuous for general sub-additive sequences but in our context $\mathcal{F}_*(\Psi, \mu) = \int g\, d\mu$ varies continuously with μ. Since the remaining argument in the proof follows that in Case I we shall omit the details. This finishes the proof of Theorem B.

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A. *Appendix. Estimates for measures of mistake dynamical balls*

In this appendix we provide an estimate for the measure of mistake dynamical balls with respect to the Gibbs measures obtained through the additive thermodynamical formalism for uniformly expanding transformations. In this setting there is a unique equilibrium state for f with respect to any Hölder continuous potential ϕ, and it is equivalent to the unique Gibbs measure for f with respect to ϕ.

PROPOSITION B. *Let X be a compact manifold, $f : X \to X$ be a uniformly expanding map, ϕ be a Hölder continuous potential and $\mu = \mu_\phi$ be the unique equilibrium state for f with respect to ϕ. Let g be any mistake function. There exists $C > 0$ and for any $\xi > 0$ there exists a measurable set $X_\xi \subset X$ such that $\mu(X_\xi) \geq 1 - \xi$ and*

$$e^{-Pn+S_n\phi(x)} \leq \mu(B_n(x, \varepsilon)) \leq \mu(B_n(g; x, \varepsilon)) \leq Ce^{2\xi n - Pn + S_n\phi(x)}$$

for all $x \in X_\xi$, $n \geq 1$ and small ε, where $P = P_{top}(f, \phi)$ denotes the topological pressure of f with respect to Φ.

Proof. Let $\xi > 0$ be arbitrary and fixed. Since f is uniformly expanding then there exists a finite Markov partition $\mathcal{Q}$. Moreover, the unique equilibrium state μ for f with respect to ϕ verifies $\mu(\partial\mathcal{Q}) = 0$ and satisfies the Gibbs property: there exists $C > 0$ such that

$$\frac{1}{C} \leq \frac{\mu(\mathcal{Q}^n(x))}{e^{-Pn+S_n\phi(x)}} \leq C$$

for all $x \in X$ and every $n \neq 1$, where P is the topological pressure of f with respect to ϕ and $\mathcal{Q}^n(x)$ is the element of the partition $\mathcal{Q}^n = \bigvee_{j=0}^{n-1} f^{-j}(\mathcal{Q})$ that contains x. Using Lemma 5.1, it follows that there exists a measurable set $X_\xi \subset X$ such that $\mu(X_\xi) \geq 1 - \xi$ and for which $B_n(g; x, \varepsilon) \subset \bigcup\{Q \in \mathcal{Q}^{(n)} : Q \cap B_n(g; x, \varepsilon) \neq \emptyset\}$, and the cardinality of such sequence is bounded by $e^{\xi n}$ provided that ε is small. The result follows immediately. □

Let us mention that the previous estimate also holds in the case of the non-additive thermodynamical formalism using the same approximation argument as above.

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