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Exponential valuations on lattice polygons

Károly Böröczky, Mátyás Domokos, Ansgar Freyer, Christoph Haberl, Gergely Harcos, and Jin Li

Abstract. We classify translatively exponential and $\mathrm{GL}(2,\mathbb{Z})$ covariant valuations on lattice polygons with values in the space of real (complex) measurable functions. A typical example of such valuations is induced by the Laplace transform, but as it turns out there are many more. The argument uses the ergodicity of the linear action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{R}^2$, and some elementary properties of the Fibonacci numbers.

1 Exponential valuations

By a *polytope* $P \subset \mathbb{R}^n$, we mean the convex hull of finitely many points of $\mathbb{R}^n$. The polytope P is called a *segment* if $\dim P = 1$, and P is called a *polygon* if $\dim P = 2$. Let $\mathcal{F}$ be a family of compact convex sets in $\mathbb{R}^n$ such that if $P \cup Q$ is convex for $P, Q \in \mathcal{F}$, then $P \cup Q \in \mathcal{F}$ and $P \cap Q \in \mathcal{F}$. Examples of such families are the family $\mathcal{K}^n$ of all convex compact sets in $\mathbb{R}^n$, the family $\mathcal{P}^n$ of all polytopes in $\mathbb{R}^n$ and the family $\mathcal{P}(\mathbb{Z}^n)$ of all lattice polytopes; namely, convex hulls of finitely many points of $\mathbb{Z}^n$ (see McMullen [16]). If $\mathcal{A}$ is a cancellative monoid (cancellative semigroup with identity element $0_{\mathcal{A}}$), then a function $Z : \mathcal{F} \to \mathcal{A}$ is called a *valuation* if the following holds: if $P \cup Q$ is convex for $P, Q \in \mathcal{F}$, then

$$Z(P \cup Q) + Z(P \cap Q) = Z(P) + Z(Q). \tag{1}$$

We say that the valuation $Z : \mathcal{F} \to \mathcal{A}$ is *simple* if $Z(P) = 0_{\mathcal{A}}$ for any $P \in \mathcal{F}$ with $\dim P \leq n-1$. Typically, one would consider valuations intertwining with some natural group actions, as we will shortly see.

While the idea of valuations on convex polytopes played a crucial role in Dehn's solution of Hilbert's third problem already around 1900, after sporadic results, the systematic study of valuations only started with Hadwiger's celebrated characterization of the intrinsic volumes as the basis of the space of continuous isometry invariant valutions from 1957. For the breathtaking developments of the last seven

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decades, see, for example, the monograph Alesker [2], and the survey papers Alesker [1], Ludwig [13] and Ludwig and Mussnig [14]. The theory of valuations on lattice polytopes has been flourishing since the classical paper by Betke and Kneser [3] in 1985 characterizing unimodular invariant valuations on lattice polytopes (see, e.g., Böröczky and Ludwig [4, 5], Jochemko and Sanyal [9, 10] and Ludwig and Silverstein [15]).

To state the result stimulating our research, let $L_c^1(\mathbb{R}^n)$ denote the family of Lebesgue integrable real functions with compact support on $\mathbb{R}^n$. In addition, for measurable $\Omega \subset \mathbb{R}^n$, let $\mathcal{M}(\Omega)$ denote the family of real-valued measurable functions on Ω where measurable means Lebesgue measurable without stating it in this article. Actually, all the arguments and statements in this article apply to the case if we fix a finite dimensional real vector space $\mathcal{V}$, and $\mathcal{M}(\Omega)$ denotes the family of measurable functions $\Omega \to \mathcal{V}$, including the case of complex-valued measurable functions. However, for simplicity, we just discuss real-valued functions. For $f \in L_c^1(\mathbb{R}^n)$, its Laplace transform is

$$\mathcal{L}f(x) = \int_{\mathbb{R}^n} e^{-\langle x,y\rangle} f(y)\, dy.$$

Li andMa [12] extended the definition of the Laplace transform to a convex compact set K by applying $\mathcal{L}$ to the characteristic function; namely,

$$\mathcal{L}K = \mathcal{L}\mathbf{1}_K = \int_K e^{-\langle x,y\rangle}\, dy.$$

Inspired by the properties of the Laplace transform applied to the characteristic functions of compact convex sets, Li and Ma [12] considered valuations that are translatively exponential and $\mathrm{GL}(n,\mathbb{R})$ covariant. For a subgroup $G \subset \mathrm{GL}(n,\mathbb{R})$, we say that a valuation $Z : \mathcal{K}^n \to \mathcal{M}(\mathbb{R}^n)$ is translatively exponential and G covariant if for any $x \in \mathbb{R}^n$, we have

$$Z(\Phi K)(x) = |\det \Phi| \cdot Z(K)(\Phi^T x) \text{ for any } \Phi \in G; \tag{2}$$

$$Z(K+z)(x) = e^{-\langle z,x\rangle} Z(K)(x) \text{ for any } z \in \mathbb{R}^n, \tag{3}$$

respectively. The Hausdorff distance $\delta_H(C,K)$ between compact convex sets K, $C \subset \mathbb{R}^n$ is the minimal $r \geq 0$ such that $K \subset C + rB^n$ and $C \subset K + rB^n$ where B^n is the unit ball in $\mathbb{R}^n$ centered at the origin. We say that a valuation $Z : \mathcal{K}^n \to \mathcal{M}(\mathbb{R}^n)$ is continuous if for any convergent sequence $K_i \to K$ of compact convex sets with respect to the Hausdorff distance, $Z(K_i)$ tends pointwise to $Z(K)$. In addition, $C(\mathbb{R}^n)$ denotes the space of real continuous functions on $\mathbb{R}^n$.

Theorem 1 (Li and Ma [12]) *Any continuous translatively exponential and* $\mathrm{GL}(n,\mathbb{R})$ *covariant valuation* $Z : \mathcal{K}^n \to C(\mathbb{R}^n)$ *is of the form* $Z = c\mathcal{L}$ *for a constant* $c \in \mathbb{R}$.

Even if – as we will shortly see – there is an abundance of translatively exponential and $\mathrm{GL}(2,\mathbb{Z})$ covariant valuations $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^n)$ on lattice polytopes, we still believe that it is not so for translatively exponential and $\mathrm{GL}(n,\mathbb{R})$ covariant valuations $Z : \mathcal{P}^n \to \mathcal{M}(\mathbb{R}^n)$ on polytopes in $\mathbb{R}^n$.

Conjecture 2 *Any translatively exponential and* $\mathrm{GL}(n,\mathbb{R})$ *covariant valuation* $Z : \mathcal{P}^n \to \mathcal{M}(\mathbb{R}^n)$ *is of the form* $Z = c\mathcal{L}$ *for a constant* $c \in \mathbb{R}$ *(Figure 1).*

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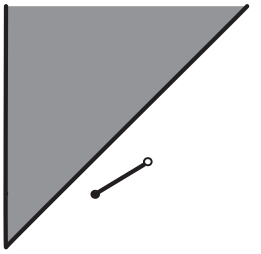


Figure 1: The domain $\widetilde{\Omega}_2$.

In this article, we consider some valuation Z on lattice polygons (of $\mathbb{Z}^2$) with values in the space of measurable functions that satisfy (4) and (5): If $x \in \mathbb{R}^2$, then

$$Z(P+z)(x) = e^{\langle z,x\rangle} Z(P)(x) \quad \text{for any } z \in \mathbb{Z}^2; \tag{4}$$

$$Z(\Phi P)(x) = Z(P)(\Phi^T x) \quad \text{for any } \Phi \in \mathrm{GL}(2,\mathbb{Z}) \tag{5}$$

where (5) is a restriction of (2) to $\mathrm{GL}(2,\mathbb{Z})$ as $|\det \phi| = 1$ for $\Phi \in \mathrm{GL}(2,\mathbb{Z})$. One example is the "positive Laplace transform"

$$\mathcal{L}_+(P)(x) = \int_{\mathbb{R}^n} e^{\langle x,y\rangle} \mathbf{1}_P(y)\, dy = \int_P e^{\langle x,y\rangle}\, dy.$$

We observe that $x \mapsto Z(P)(x)$ satisfies (4) if and only if $x \mapsto Z(P)(-x)$ satisfies (3). We note that Freyeret al. [8] characterized the translatively exponential and $\mathrm{GL}(2,\mathbb{Z})$ covariant valuations with values in the space of formal power series in two variables.

The main goal of this article is to characterize translatively exponential and $\mathrm{GL}(2,\mathbb{Z})$ covariant valuations $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$. We write e_1, e_2 to denote the orthonormal basis of $\mathbb{R}^2$ also generating $\mathbb{Z}^2$, and set $T = [e_1, e_2, o]$ where $o = (0,0)$ stands for the origin, and $[x_1, \dots, x_k]$ stands for the convex hull of $x_1, \dots, x_k \in \mathbb{R}^2$. In the formulas below

$$\frac{e^t - 1}{t} \text{ is identified with } \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!};$$

namely, when we write $\varphi(t) = \frac{e^t-1}{t}$, we mean the positive analytic function $\varphi(t) = \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!}$ on $\mathbb{R}$ satisfying $\varphi(0) = 1$. For the golden ratio $\tau = \frac{\sqrt{5}+1}{2}$, we set (see Figure 1)

$$\widetilde{\Omega}_2 = \{(\tau s, s) : 1 \le s < \tau\} \cup \{(x_1, x_2) : 0 \le x_1 \le x_2\}.$$

First, we characterize simple translatively exponential and $\mathrm{GL}(2,\mathbb{Z})$ covariant valuations where a valuation $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$ is *simple* if $Z(P) = 0$ for any $P \in \mathcal{P}(\mathbb{Z}^2)$ with $\dim P \le 1$ (Figure 2).

Theorem 3 *Simple translatively exponential and* $\mathrm{GL}(2,\mathbb{Z})$ *covariant valuations* $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$ *are parameterized uniquely by functions in* $\mathcal{M}(\widetilde{\Omega}_2)$ *in the following way.*

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(i) *For any measurable function* $\tilde{\rho} : \widetilde{\Omega}_2 \to \mathbb{R}$, *there exists a unique measurable extension* $\rho : [0,\infty)^2 \to \mathbb{R}$ *satisfying* $\tilde{\rho} = \rho|_{\widetilde{\Omega}_2}$ *and*

$$(2x+y)\rho(x,y) = (x+y)\rho(x,x+y) + x\rho(x+y,x) \tag{6}$$

for $x, y \geq 0$, *and a unique translatively exponential and* $\mathrm{GL}(2,\mathbb{Z})$ *covariant simple valuation* $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$ *such that* $Z(T)(0,0) = \tilde{\rho}(0,0)$, *and* $f_2 = Z(T)$ *satisfies*

$$f_2(x,y) = \frac{e^x}{y} \cdot \frac{e^{y-x}-1}{y-x} \cdot \rho(y-x,x) - \frac{1}{y} \cdot \frac{e^x-1}{x} \cdot \rho(x,y-x) \tag{7}$$

for $0 \leq x \leq y$ *and* $y > 0$.

(ii) *For any simple translatively exponential and* $\mathrm{GL}(2,\mathbb{Z})$ *covariant valuation* $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$, *there exists some measurable* $\rho : [0,\infty)^2 \to \mathbb{R}$ *satisfying (6) and (7).*

Remark. The "positive Laplace transform" $\mathcal{L}_+$ corresponds to the constant one function $\rho \equiv 1$. On the other hand, Freyeret al. [8] determined all translatively exponential and $\mathrm{GL}(2,\mathbb{Z})$ covariant valuations with values in the space of formal power series in two variables. It follows that there exists an abundance of corresponding simple valuations with values in the space of analytic functions that are different from the positive Laplace transform, and we exhibit one in Example 15.

We write $|X|$ to denote the Lebesgue measure of a measurable set $X \subset \mathbb{R}^2$. We need the following consequence of the fact that the linear action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{R}^2$ is ergodic (see Section 5). Proposition 4 about Lebesgue measurable $\mathrm{SL}(2,\mathbb{Z})$ invariant functions is used in the statement of Theorem 5 characterizing translatively exponential and $\mathrm{GL}(2,\mathbb{Z})$ covariant valuations.

Proposition 4 *If* $f \in \mathcal{M}(\mathbb{R}^2)$ *is invariant under* $\mathrm{SL}(2,\mathbb{Z})$, *then there exists a constant* $c \in \mathbb{R}$ *such that* $f(x) = c$ *for almost everywhere* $x \in \mathbb{R}^2$.

Remark. In particular, any function $f \in \mathcal{M}(\mathbb{R}^2)$ invariant under $\mathrm{GL}(2,\mathbb{Z})$ can be constructed in the following way from a $c \in \mathbb{R}$ and an $X \subset \mathbb{R}^2$ with $|X| = 0$. Writing X_0 to denote the image of X under the action of $\mathrm{GL}(2,\mathbb{Z})$, we have $|X_0| = 0$, and associating an arbitrary $c_{\mathcal{O}} \in \mathbb{R}$ to any orbit $\mathcal{O}$ of $\mathrm{GL}(2,\mathbb{Z})$ intersecting X, we define $f(x) = c$ for $x \in \mathbb{R}^2 \backslash X_0$, and $f(x) = c_{\mathcal{O}}$ if $x \in \mathcal{O}$ for an orbit $\mathcal{O}$ of $\mathrm{GL}(2,\mathbb{Z})$ intersecting X.

In Theorem 5, we use (see Figure 2)

$$\widetilde{\Omega}_1 = \{(0,x_2) : x_2 \geq 0\} \cup \left\{(x_1,x_2) : 0 \leq x_2 \leq \frac{1}{2}x_1\right\}.$$

The space of $\mathrm{GL}(2,\mathbb{Z})$ invariant measurable functions in $\mathbb{R}^2$ – that are characterized in Proposition 4 –, is denoted by $\mathcal{M}(\mathbb{R}^2)^{\mathrm{GL}(2,\mathbb{Z})}$.

Theorem 5 *Translatively exponential and* $\mathrm{GL}(2,\mathbb{Z})$ *covariant valuations* $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$ *are parameterized uniquely by* $\mathcal{M}(\mathbb{R}^2)^{\mathrm{GL}(2,\mathbb{Z})}$, $\mathcal{M}(\widetilde{\Omega}_1)$ *and* $\mathcal{M}(\widetilde{\Omega}_2)$ *as follows:*

(i) *For any* $\mathrm{GL}(2,\mathbb{Z})$ *invariant measurable function* $f_0 : \mathbb{R}^2 \to \mathbb{R}$ *(cf. Proposition 4), any measurable functions* $\tilde{f}_1 : \widetilde{\Omega}_1 \to \mathbb{R}$, *and* $\tilde{\rho} : \widetilde{\Omega}_2 \to \mathbb{R}$, *there exists a unique*

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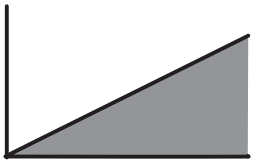


Figure 2: The domain $\widetilde{\Omega}_1$.

translatively exponential and $\mathrm{GL}(2,\mathbb{Z})$ *covariant valuation* $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$ *such that* $Z(\{o\}) = f_0$, $f_1|_{\widetilde{\Omega}_1} = \tilde{f}_1$ *for* $f_1 = Z([o, e_1])$, *and*

$$Z(T)(x,y) = f_2(x,y) + \frac{1}{2} f_1(x,y) + \frac{1}{2} f_1(y,-x) + \frac{e^x}{2} f_1(-x+y,-x) \tag{8}$$

holds for the f_2 *defined by (7) where* $f_2 = Z_2(T)$ *for a simple translatively exponential* $\mathrm{GL}(2,\mathbb{Z})$ *covariant valuation* $Z_2 : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$ *and* ρ *is constructed from* $\tilde{\rho}$ *as in Theorem 3 (i).*

(ii) *For a translatively exponential and* $\mathrm{GL}(2,\mathbb{Z})$ *covariant valuation* $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$, $f_0 = Z(\{o\})$ *is a measurable* $\mathrm{GL}(2,\mathbb{Z})$ *invariant function and*

$$Z(\{p\})(z) = e^{\langle p,z\rangle} f_0(z) \text{ for } p \in \mathbb{Z}^2 \text{ and } x \in \mathbb{R}^2, \tag{9}$$

where there exists constant $c \in \mathbb{R}$ *such that* $f_0 = c$ *almost everywhere.*

In addition, there exists a simple translatively exponential and $\mathrm{GL}(2,\mathbb{Z})$ *covariant valuation* $Z_2 : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$ *such that* $f_1 = Z([o, e_1])$ *and* $f_2 = Z_2(T)$ *satisfy (8).*

Remark. In particular, any translatively exponential and $\mathrm{GL}(2,\mathbb{Z})$ covariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$ can be represented as $Z = Z_1 + Z_2$ where Z_2 is a simple valuation and Z_1 is the valuation constructed in Proposition 11 (see Section 3) satisfying $Z_1(\{o\}) = Z(\{o\})$ and $Z_1([o, e_1]) = Z([o, e_1])$.

Concerning the structure of the article, properties of related group actions are discussed in Section 2, and Section 3 presents some fundamental examples. For a translatively exponential and $\mathrm{GL}(2,\mathbb{Z})$ covariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$, the algebraic properties characterizing $Z(\{o\}) = f_0$, $Z([o, e_1]) = f_1$ and $Z_2(T) = f_2$ for the corresponding translatively exponential and $\mathrm{GL}(2,\mathbb{Z})$ covariant simple valuation $Z_2 : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$ are described in Lemma 9 and in Section 4.

The restrictions of translatively exponential and $\mathrm{GL}(2,\mathbb{Z})$ covariant valuations to points and lattice segments are characterized in Sections 5 and 6, and Theorem 5 is actually proved in Section 6. The study of geometric properties of simple translatively exponential and $\mathrm{GL}(2,\mathbb{Z})$ covariant valuations starts in Section 7, and finally, Section 8 verifies Theorem 3 using the Fibonacci sequence.

2 Action of affine lattice automorphisms

We write $\mathcal{G}(\mathbb{Z}^2)$ to denote the group of all affine transforms $\Xi x = \Phi x + a$ for $\Phi \in \mathrm{GL}(2,\mathbb{Z})$ and $a \in \mathbb{Z}^2$, and hence $\mathcal{G}(\mathbb{Z}^2)$ is the group of all affine automorphisms of $\mathbb{Z}^2$.

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Let us define a $\mathcal{G}(\mathbb{Z}^2)$ action on $\mathcal{M}(\mathbb{R}^2)$. For $f \in \mathcal{M}(\mathbb{R}^2)$ and an affine $\Xi x = \Phi x + a$ with $\Phi \in \mathrm{GL}(2,\mathbb{Z})$ and $a \in \mathbb{Z}^2$, we consider $\Xi \cdot f \in \mathcal{M}(\mathbb{R}^2)$ where

$$(\Xi \cdot f)(x) = e^{\langle a,x\rangle} f(\Phi^T x). \tag{10}$$

Formula (10) defines an action of $\mathcal{G}(\mathbb{Z}^2)$ because the identity map leaves any $f \in \mathcal{M}(\mathbb{R}^2)$ invariant, and for any $\aleph \in \mathcal{G}(\mathbb{Z}^2)$, we have

$$(\aleph \circ \Xi) \cdot f = \aleph \cdot (\Xi \cdot f). \tag{11}$$

To prove (11), let $\aleph x = \Psi x + b$ for $\Psi \in \mathrm{GL}(2,\mathbb{Z})$ and $b \in \mathbb{Z}^2$. Hence, $(\aleph \circ \Xi)(x) = \Psi\Phi x + \Psi a + b$ and

$$\Big((\aleph \circ \Xi) \cdot f\Big)(x) = e^{\langle \Psi a+b,x\rangle} f(\Phi^T\Psi^T x) = e^{\langle b,x\rangle} e^{\langle a,\Psi^T x\rangle} f(\Phi^T\Psi^T x)$$
$$= e^{\langle b,x\rangle} (\Xi \cdot f)(\Psi^T x) = \Big(\aleph \cdot (\Xi \cdot f)\Big)(x).$$

We observe that this action of a $\Xi \in \mathcal{G}(\mathbb{Z}^2)$ satisfies that for a valuation $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$, we have $\Xi \cdot Z(P) = Z(\Xi P)$ for $P \in \mathcal{P}(\mathbb{Z}^2)$.

Formula (11) readily yields the following property.

Lemma 6 *Let $P \in \mathcal{P}(\mathbb{Z}^2)$, and let $H \subset \mathcal{G}(\mathbb{Z}^2)$ be the subgroup of all $\Xi \in \mathcal{G}(\mathbb{Z}^2)$ such that $\Xi P = P$.*

For a set $\{\Xi_i\}_{i\in I}$ of group generators of H, if $\xi(P) \in \mathcal{M}(\mathbb{R}^2)$ satisfies $\Xi_i \cdot \xi(P) = \xi(P)$ for $i \in I$, then setting $\xi(\aleph P) \in \mathcal{M}(\mathbb{R}^2)$ via $\xi(\aleph P) = \aleph \cdot \xi(P)$ for any $\aleph \in \mathcal{G}(\mathbb{Z}^2)$ is well-defined, and satisfies

$$\xi\Big(\Gamma(\aleph P)\Big) = \Gamma \cdot \xi(\aleph P) \ \text{ for any } \Gamma \in \mathcal{G}(\mathbb{Z}^2).$$

For a $P \in \mathcal{P}(\mathbb{Z}^2)$, Lemma 7 is used to determine the subgroup $\mathcal{G}(\mathbb{Z}^2)$ fixing P. For a $\Xi \in \mathcal{G}(\mathbb{Z}^2)$ with $\Xi x = \Phi x + a$, $\Phi \in \mathrm{GL}(2,\mathbb{Z})$, and $a \in \mathbb{Z}^2$, we set $\det \Xi = \det \Phi$. Since $\det : \mathcal{G}(\mathbb{Z}^2) \to \{-1,1\}$ is a group homomorphism, and a vertex of P is mapped to a vertex of the image by an affine transform (where endpoints of segments are understood as vertices), we deduce the following. We call a $\Xi \in \mathcal{G}(\mathbb{Z}^2)$ orientation preserving if $\det \Xi = 1$.

Lemma 7 *Let $P \in \mathcal{P}(\mathbb{Z}^2)$ such that o is a vertex, let $H \subset \mathcal{G}(\mathbb{Z}^2)$ be the subgroup of all $\Xi \in \mathcal{G}(\mathbb{Z}^2)$ such that $\Xi P = P$, and let $H_+ \subset H$ be the subgroup of orientation preserving transformations in H.*

(i) *$\Xi(v)$ is a vertex of P for any vertex v of P and $\Xi \in H$;*
(ii) *if there exists $\widetilde{\Xi} \in \mathcal{G}(\mathbb{Z}^2)$ with $\widetilde{\Xi} P = P$ and $\det \widetilde{\Xi} = -1$, then H_+ and $\widetilde{\Xi}$ generate H;*
(iii) *if P is a polygon and $\Xi(o) = \Xi'(o)$ for $\Xi, \Xi' \in H_+$, then $\Xi = \Xi'$.*

For a subgroup $G \subset \mathcal{G}(\mathbb{Z}^2)$, we say that a valuation $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$ is G covariant if

$$Z(\Xi P)(x) = \Big(\Xi \cdot Z(P)\Big)(x) \ \text{ for any } \Xi \in G, P \in \mathcal{P}(\mathbb{Z}^2) \text{ and } x \in \mathbb{R}^2. \tag{12}$$

In particular, Z is $\mathcal{G}(\mathbb{Z}^2)$ covariant if and only if Z is translatively exponential (*cf.* (4)) and $\mathrm{GL}(2,\mathbb{Z})$ covariant (*cf.* (5)).

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Next, we show that a $\mathcal{G}(\mathbb{Z}^2)$ covariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$ is determined by its value at $\{o\}$, $[o, e_1]$, and T. We note that McMullen [16] proved the Inclusion–Exclusion principle for any valuation $Z : \mathcal{P}(\mathbb{Z}^m) \to \mathcal{A}$ where $\mathcal{A}$ is a cancellative semigroup; namely, if P is a d-dimensional lattice polytope, $1 \le d \le m$, and $P = Q_1 \cup \ldots \cup Q_k$ for d-dimensional lattice polytopes $Q_1, \ldots, Q_k$, $k \ge 3$, such that the non-empty intersection of any subfamily of $Q_1, \ldots, Q_k$ is a lattice polytope, then

$$Z(P) = \sum_{i=1}^{k} \sum_{1 \le j_1 < \cdots < j_i \le k} (-1)^{i-1} Z(Q_{j_1} \cap \cdots Q_{j_i}) \tag{13}$$

where we set $Z(\varnothing) = 0$.

We say that a lattice segment $[p, q]$ for $p, q \in \mathbb{Z}^2$ is a *primitive lattice segment* if the only lattice points it contains are its endpoints, and a lattice triangle is an *empty triangle* if the only lattice points it contains are its vertices.

Proposition 8 *For any $\mathcal{G}(\mathbb{Z}^2)$ covariant valuations $Z, Z' : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$,*

(i) *if $Z(\{o\}) = Z'(\{o\})$ and $Z([o, e_1]) = Z'([o, e_1])$, then $Z - Z'$ is a simple $\mathcal{G}(\mathbb{Z}^2)$ covariant valuation;*

(ii) *if $Z(\{o\}) = Z'(\{o\})$, $Z([o, e_1]) = Z'([o, e_1])$ and $Z(T) = Z'(T)$, then $Z = Z'$.*

Proof Let $Z(\{o\}) = Z'(\{o\})$ and $Z([o, e_1]) = Z'([o, e_1])$. For any $p \in \mathbb{Z}^2$, (4) yields that

$$Z(\{p\})(x) = e^{\langle x, p \rangle} Z(\{o\})(x) = e^{\langle x, p \rangle} Z'(\{o\})(x) = Z'(\{p\})(x).$$

For a primitive lattice segment $[p, q]$, $p \ne q \in \mathbb{Z}^2$, there exists $\Phi \in \mathrm{SL}(2, \mathbb{Z})$ such that $[p, q] = p + \Phi[o, e_1]$; therefore, $Z([p, q]) = Z'([p, q])$ by (4) and (5). Now a lattice segment s containing $k \ge 3$ lattice points can be subdivided into $k - 1$ primitive lattice segments, and hence the Inclusion–Exclusion principle (13), together with the just established property that Z and Z' agree on lattice points and primitive segments, yields that $Z(s) = Z'(s)$. In turn, we deduce (i).

Let us assume that in addition, $Z(T) = Z'(T)$. For an empty lattice triangle $T = [v_1, v_2, v_3]$, the vectors $v_2 - v_1$ and $v_3 - v_1$ form a basis of $\mathbb{Z}^2$; therefore, there exists a $\Phi \in \mathrm{SL}(2, \mathbb{Z})$ such that $T = v_1 + \Phi T$, which in turn yields that $Z(T) = Z'(T)$ by (4) and (5). Finally, a two-dimensional lattice polygon P can be written as $P = T_1 \cup \cdots \cup T_k$ for empty triangles $T_1, \ldots, T_k$ such that $T_i \cap T_j$ is either empty, or a common vertex, or a common side. Thus the Inclusion–Exclusion principle (13) together with the property that Z and Z' agree on lattice points, lattice segments, and empty triangles yields that $Z(P) = Z'(P)$. ∎

It follows from Proposition 8 that in order to characterize a translatively exponential and $\mathrm{GL}(2, \mathbb{Z})$ covariant (or equivalently, $\mathcal{G}(\mathbb{Z}^2)$ covariant) valuation Z on lattice polygons, all we need to characterize are

$$Z(\{o\}) = f_0 \text{ and } Z([o, e_1]) = f_1 \text{ and } Z(T) = f_2. \tag{14}$$

Let us establish some algebraic properties of f_0 and f_1 in (14). According to Trott [17], $\mathrm{GL}(2, \mathbb{Z})$ as a group is generated by $\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$, but we do not use this result directly.

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Lemma 9 *Let $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$ be a $\mathcal{G}(\mathbb{Z}^2)$ covariant valuation. Then $Z(\{o\}) = f_0$ and $Z([o, e_1]) = f_1$ satisfy the following properties:*

$$f_0(\Phi x) = f_0(x) \text{ for any } \Phi \in \mathrm{GL}(2,\mathbb{Z}) \text{ and } x \in \mathbb{R}^2; \tag{15}$$

$$f_1(-x_1, -x_2) = e^{-x_1} f_1(x_1, x_2) \text{ for } (x_1, x_2) \in \mathbb{R}^2; \tag{16}$$

$$f_1(x_1, x_2) = f_1(x_1, x_1 + x_2) \text{ for} (x_1, x_2) \in \mathbb{R}^2; \tag{17}$$

$$f_1(x_1, x_2) = f_1(x_1, -x_2) \text{ for} (x_1, x_2) \in \mathbb{R}^2. \tag{18}$$

Proof Formula (15) follows from (5). The property (16) follows from (4) and the relation $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} [o, e_1] = [o, e_1] - e_1$. Next (17) says that $Z([o, e_1])$ is covariant under $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and (18) is the consequence of the fact that $Z([o, e_1])$ also is covariant under $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. ∎

3 Some examples of $\mathcal{G}(\mathbb{Z}^2)$ covariant valuations on lattice polygons

In this section, we construct two fundamental examples of $\mathcal{G}(\mathbb{Z}^2)$ covariant valuations on $\mathcal{P}(\mathbb{Z}^2)$. The first is the "positive Laplace transform" that is the basic example of simple valuations satisfying (4) and (5), and the second in Propositions 11 extends a "would be" valuation defined on lattice points and lattice segments.

Example 10 (Positive Laplace transform) For any lattice polygon P, we define

$$\mathcal{L}_+(P)(x) = \int_P e^{\langle x,y\rangle}\, dy, \tag{19}$$

thus $\mathcal{L}_+$ is a $\mathcal{G}(\mathbb{Z}^2)$ covariant simple valuation.

We note that (19) yields that

$$\mathcal{L}_+(T)(x_1, x_2) = \frac{x_1 e^{x_2} - x_2 e^{x_1} + x_2 - x_1}{x_1 x_2 (x_2 - x_1)}; \tag{20}$$

or in other words, we have

$$\mathcal{L}_+(T)(x_1, x_1 + x_2) = \frac{1}{x_1 + x_2} \left(e^{x_1} \cdot \frac{e^{x_2} - 1}{x_2} - \frac{e^{x_1} - 1}{x_1} \right). \tag{21}$$

Proposition 11 *For $f_0, f_1 \in \mathcal{M}(\mathbb{R}^2)$ satisfying (15)–(18), there exists a $\mathcal{G}(\mathbb{Z}^2)$ covariant valuation $Z_1 : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$ satisfying $Z_1(\{o\}) = f_0$ and $Z_1([o, e_1]) = f_1$ and*

$$Z_1(T)(x_1, x_2) = \frac{1}{2} f_1(x_1, x_2) + \frac{1}{2} f_1(x_2, -x_1) + \frac{e^{x_1}}{2} f_1(-x_1 + x_2, -x_1).$$

Proof For points, we define

$$Z_1(\{z\})(x) = e^{\langle x,z\rangle} f_0(x) \text{ for } z \in \mathbb{Z}^2 \text{ and } x \in \mathbb{R}^2. \tag{22}$$

As $\mathrm{GL}(2,\mathbb{Z}) \subset \mathcal{G}(\mathbb{Z}^2)$ is the stabilizer subgroup of $\{o\}$, (15) and Lemma 6 yield that Z_1 as defined in (22) is well-defined and is $\mathcal{G}(\mathbb{Z}^2)$ covariant on lattice points.

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Turning to segments, we write H to denote the stabilizer subgroup of $[o, e_1]$ in $\mathcal{G}(\mathbb{Z}^2)$, $H_+ \subset H$ to denote the orientation preserving stabilizers, and $H_0 \subset \mathrm{SL}(2,\mathbb{Z})$ to denote the subgroup of all $\Phi \in \mathrm{SL}(2,\mathbb{Z})$ with $\Phi e_1 = e_1$. Now H_0 consists of the matrices of the form $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ for an $a \in \mathbb{Z}$, and hence it is generated by $\Phi_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Since $\Xi_1[o, e_1] = [o, e_1]$ for $\Xi_1 x = -x + e_1 \in H_+ \backslash H_0$, and $H_0 \subset H_+$ is a subgroup of index 2 by Lemma 7 (i), we deduce that H_+ is generated by Φ_0 and Ξ_1. It follows from Lemma 7 (ii) that H is generated by Φ_0, Ξ_1 and $\Phi_- = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

For any primitive lattice segment $[p, q]$ there exists $\Xi \in \mathcal{G}(\mathbb{Z}^2)$ such that $\Xi[o, e_1] = [p, q]$; namely, take $\Xi x = \Phi x + p$ where $q - p = \Phi e_1$ for $\Phi \in \mathrm{SL}(2,\mathbb{Z})$. Since H is generated by Φ_0, Ξ_1, and Φ_- where the condition (18) on f_1 corresponds to the Φ_- invariance of $[o, e_1]$, it follows from Lemma 6 that if $[p, q]$ is a primitive lattice segment, then we may define

$$Z_1([p, q]) = \Xi \cdot f_1 \tag{23}$$

for any $\Xi \in \mathcal{G}(\mathbb{Z}^2)$ with $\Xi[o, e_1] = [p, q]$. Lemma 6 also yields that Z_1 is $\mathcal{G}(\mathbb{Z}^2)$ covariant on lattice segments.

Now a general lattice segment $[a, b]$ for $a \neq b$, $a, b \in \mathbb{Z}^2$ is divided into primitive lattice segments by the lattice points contained in $[a, b]$ different from a, b, and hence, using the abbreviation "pls" for primitive lattice segments, we define

$$Z_1([a, b]) = \sum_{\substack{s \subset [a,b] \\ s \text{ pls}}} Z_1(s) - \sum_{\substack{z \in [a,b] \cap \mathbb{Z}^2 \\ z \neq a,b}} Z_1(\{z\}). \tag{24}$$

Since Z_1 is translatively exponential and $\mathrm{GL}(\mathbb{Z}^2)$ covariant on points and primitive lattice segments, Z_1 is $\mathcal{G}(\mathbb{Z}^2)$ covariant on lattice segments and lattice points.

In addition, if P is a lattice polygon, then we define

$$Z_1(P) = \sum_{\substack{s \subset \partial P \\ s \text{ pls}}} \frac{1}{2} Z_1(s) + \sum_{z \in (\mathrm{int} P) \cap \mathbb{Z}^2} Z_1(\{z\}) \tag{25}$$

where the second sum is zero if $(\mathrm{int} P) \cap \mathbb{Z}^2 = \varnothing$. Z_1 is readily $\mathcal{G}(\mathbb{Z}^2)$ covariant also on polygons. In addition, the (primitive) edges of T are $[o, e_1]$, $\Phi_1[o, e_1]$, and $e_1 + \Phi_2[o, e_1]$ for $\Phi_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ and $\Phi_2 = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, thus (23) and (25) yield

$$Z_1(T)(x, y) = \frac{1}{2} f_1(x, y) + \frac{1}{2} f_1(y, -x) + \frac{e^x}{2} f_1(-x + y, -x).$$

Therefore, all we have to prove is that the function Z_1 on $\mathcal{P}(\mathbb{Z}^2)$ defined by (22)–(25) is a valuation; namely, if $P \cup Q$ is convex for $P, Q \in \mathcal{P}(\mathbb{Z}^2)$, then

$$Z_1(P) + Z_1(Q) = Z_1(P \cap Q) + Z_1(P \cup Q). \tag{26}$$

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If $P \subset Q$ or $Q \subset P$, then (26) readily holds. Therefore, besides that $P \cup Q$ are convex, we assume that either P and Q are lattice segments not containing each other, or P and Q are lattice polygons not containing each other.

Case 1 $\dim(P \cap Q) < \dim P = \dim Q$ In other words, either P and Q are collinear lattice segments whose intersection is a common endpoint, or P and Q are lattice polygons whose intersection is a common side, and in addition, $P \cup Q$ is convex. In the first case (24), and in the second case (24) and (25) directly yield (26).

Case 2 $\dim(P \cap Q) = \dim P = \dim Q \geq 1$ Let us introduce a notion of multiplicity of a primitive lattice segment s and a $z \in \mathbb{Z}^2$ with respect to a general lattice segment $[a, b]$ for $a \neq b$, $a, b \in \mathbb{Z}^2$ and a lattice polygon N where we set $(a, b) = [a, b] \setminus \{a, b\}$:

$$m_{[a,b]}(s) = 1 \text{ if } s \subset [a, b], \text{ and } m_{[a,b]}(s) = 0 \text{ otherwise;}$$
$$m_{[a,b]}(z) = 1 \text{ if } z \in (a, b), \text{ and } m_{[a,b]}(z) = 0 \text{ otherwise;}$$
$$m_N(s) = 1 \text{ if } s \subset \partial N, \text{ and } m_N(s) = 0 \text{ otherwise;}$$
$$m_N(z) = 1 \text{ if } z \in \operatorname{int} N, \text{ and } m_N(z) = 0 \text{ otherwise.}$$

In particular,

$$Z_1([a, b]) = \sum_{s \text{ pls}} m_{[a,b]}(s) \cdot Z_1(s) - \sum_{z \in \mathbb{Z}^2} m_{[a,b]}(z) \cdot Z_1(\{z\}) \tag{27}$$

$$Z_1(N) = \sum_{s \text{ pls}} m_N(s) \cdot \frac{1}{2} Z_1(s) + \sum_{z \in \mathbb{Z}^2} m_N(z) \cdot Z_1(\{z\}). \tag{28}$$

If $P, Q \in \mathcal{P}(\mathbb{Z}^2)$ satisfy that $P \cup Q$ is convex and $\dim(P \cap Q) = \dim P = \dim Q \geq 1$, and if s is a primitive lattice segment and $z \in \mathbb{Z}^2$, then we claim that

$$m_P(s) + m_Q(s) = m_{P \cap Q}(s) + m_{P \cup Q}(s) \tag{29}$$

$$m_P(z) + m_Q(z) = m_{P \cap Q}(z) + m_{P \cup Q}(z). \tag{30}$$

If $\dim(P \cap Q) = \dim P = \dim Q = 1$, then $(\operatorname{relint} P) \cup (\operatorname{relint} Q) = \operatorname{relint}(P \cup Q)$ (where $\operatorname{relint} P$ is the open segment determined by P), and hence $z \in (\operatorname{relint} P) \cap (\operatorname{relint} Q)$ if and only if $z \in \operatorname{relint} P$ and $z \in \operatorname{relint} Q$. Since $s \subset P \cap Q$ for a primitive lattice segment s if and only if $s \subset P$ and $s \subset Q$, we conclude (29) and (30).

Let $\dim(P \cap Q) = \dim P = \dim Q = 2$, and hence

$$\text{no line separates } P \text{ and } Q. \tag{31}$$

For (30), if $x \in \operatorname{int}(P \cup Q)$ and $x \in \partial P$, then there exists a supporting line ℓ to P at x. Now any small semicircle centered at x and separated from P by ℓ is contained in Q, and (31) yields that $x \in \operatorname{int} Q$. The similar argument if $x \in \operatorname{int}(P \cup Q)$ and $x \in \partial Q$ shows that $(\operatorname{int} P) \cup (\operatorname{int} Q) = \operatorname{int}(P \cup Q)$, which in turn implies (30). For (29), we deduce from the convexity of $P \cup Q$ that each vertex of $P \cap Q$ is a vertex of P or Q (see, e.g., McMullen [16]), and hence the primitive lattice segment s is contained in $\partial(P \cup Q)$ or $\partial(P \cap Q)$ if and only if s is contained in ∂P or ∂Q. We deduce from (31) that if $s \subset (\partial P) \cap (\partial Q)$, then $s \subset \partial(P \cup Q)$ and $s \subset \partial(P \cap Q)$; therefore, (29) holds, as well.

Combining (27)–(30) yields (26), and in turn Proposition 11. ∎

Propositions 8 (i) and 11 yield the following.

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Corollary 12 *For any $\mathcal{G}(\mathbb{Z}^2)$ covariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$, let Z_1 be the $\mathcal{G}(\mathbb{Z}^2)$ covariant valuation satisfying $Z_1(\{o\}) = Z(\{o\})$ and $Z_1([o, e_1]) = Z([o, e_1])$ constructed in Proposition 11. Then $Z_2 = Z - Z_1$ is a $\mathcal{G}(\mathbb{Z}^2)$ covariant simple valuation.*

4 Algebraic properties of simple valuations

Having Lemma 9 and Corollary 12 at hand, an algebraic characterization of simple $\mathcal{G}(\mathbb{Z}^2)$ covariant valuations leads to an algebraic characterization of all $\mathcal{G}(\mathbb{Z}^2)$ covariant valuations.

Lemma 13 *For any simple $\mathcal{G}(\mathbb{Z}^2)$ covariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$, $Z(T) = f_2$ satisfies the following properties: For $x, y \in \mathbb{R}$, we have*

$$f_2(-x + y, -x) = e^{-x} f_2(x, y); \tag{32}$$

$$f_2(x, y) + e^{x+y} f_2(-x, -y) = f_2(x, x + y) + f_2(x + y, y); \tag{33}$$

$$f_2(x, y) = f_2(y, x). \tag{34}$$

Proof The $\mathcal{G}(\mathbb{Z}^2)$ covariance yields (32) because $T - e_1 = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} T$.

For (33), we observe that each diagonal of the square $[0,1]^2$ cuts the square into two empty triangles. In particular, $[0,1]^2$ can be written as $T \cup (e_1 + e_2 - T)$ on the one hand, and the union of the triangles $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} T$ and $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} T$ on the other hand. We conclude (33) from evaluating $Z([0,1]^2)$ in the two ways; and using the simpleness of Z and the $\mathcal{G}(\mathbb{Z}^2)$ covariance of Z.

Finally, (34) follows from $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} T$ and the $\mathcal{G}(\mathbb{Z}^2)$ covariance of Z. ∎

Proposition 14 *For any $f_2 \in \mathcal{M}(\mathbb{R}^2)$ satisfying the properties (32)–(34) in Lemma 13, there exists a unique simple $\mathcal{G}(\mathbb{Z}^2)$ covariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$ such that $Z(T) = f_2$.*

Proof We write H to denote the subgroup of stabilizers of T of $\mathcal{G}(\mathbb{Z}^2)$, and let $H_+ \subset H$ be the subgroup of orientation preserving transformations.

In accordance with Lemma 7, H_+ has three elements corresponding to the vertices of T; namely, the identity, $\Xi_+ x = \Phi_+ x + e_1$ for $\Phi_+ = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$ where $\Xi_+ T = T$, and $\Xi_+^{-1} x = \Phi_+^{-1} x + e_2$ where $\Phi_+^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$. It followsfrom Lemma 7 (ii) that H is generated by Ξ_+ and $\Phi_- = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Since the property (32) of f_2 corresponds to the Ξ_+ invariance of T, and the property (34) of f_2 corresponds to the Φ_- invariance of T, it follows from Lemma 6 that we may define $Z_2(N)$ for any empty triangle N by choosing a $\Xi \in \mathcal{G}(\mathbb{Z}^2)$ with $N = \Xi T$ and setting

$$Z_2(N) = \Xi \cdot f_2. \tag{35}$$

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In addition, the Z_2 defined as in (35) is $\mathcal{G}(\mathbb{Z}^2)$ covariant on empty lattice triangles according to Lemma 6.

Next we call a parallelogram $P \in \mathcal{P}(\mathbb{Z}^2)$ an empty parallelogram if contains no other lattice points then its vertices. In this case, $P = \Xi[0,1]^2$ for a $\Xi \in \mathcal{G}(\mathbb{Z}^2)$.

For an empty parallelogram $P \in \mathcal{P}(\mathbb{Z}^2)$, one of the diagonals of P cuts P into the two empty triangles, let them be T_1 and T_2, and let T_3 and T_4 be the two empty triangles in P determined by the other diagonal of P. We claim that

$$Z_2(T_1) + Z_2(T_2) = Z_2(T_3) + Z_2(T_4). \tag{36}$$

Let v be the vertex of T_1 that is not a vertex of T_2, and let $\Xi \in \mathcal{G}(\mathbb{Z}^2)$ such that $P = \Xi[0,1]^2$ and $\Xi o = v$. For $\widetilde{\Xi}_1 = I_2$ and $\widetilde{\Xi}_2 x = -x + e_1 + e_2$, the diagonal $[e_1, e_2]$ of $[0,1]^2$ divides the square into $\widetilde{\Xi}_1 T$ and $\widetilde{\Xi}_2 T$, and $T_i = \Xi\,\widetilde{\Xi}_i T$ for $i = 1, 2$. Possibly after interchanging T_3 and T_4, we may assume that $T_i = \Xi\,\widetilde{\Xi}_i T$ for $i = 3, 4$ where $\widetilde{\Xi}_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\widetilde{\Xi}_4 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, and the diagonal $[o, e_1 + e_2]$ of $[0,1]^2$ divides the square into $\widetilde{\Xi}_3 T$ and $\widetilde{\Xi}_4 T$. Since (33) states that

$$\widetilde{\Xi}_1 \cdot f_2 + \widetilde{\Xi}_2 \cdot f_2 = \widetilde{\Xi}_3 \cdot f_2 + \widetilde{\Xi}_4 \cdot f_2,$$

we conclude (36) from the definition and $\mathcal{G}(\mathbb{Z}^2)$ covariance of Z_2 for empty triangles.

To define $Z_2(Q)$ for a lattice polygon $Q \in \mathcal{P}(\mathbb{Z}^2)$, to any triangulation $\mathcal{T} = \{E_1, \dots, E_k\}$ into empty triangles, we assign

$$\xi(\mathcal{T}) = \sum_{i=1}^{k} Z_2(E_i).$$

We claim that if $\mathcal{T}' = \{E_1', \dots, E_k'\}$ is another triangulation of Q into empty triangles, then

$$\xi(\mathcal{T}) = \xi(\mathcal{T}'). \tag{37}$$

According to Lawson [11] (see de Loera et al. [7, Section 3.4.1],), the two triangulations $\mathcal{T}$ and $\mathcal{T}'$ can be connected by a series of triangulations $\mathcal{T}_0, \dots, \mathcal{T}_m$ of Q into empty triangles such that $\mathcal{T} = \mathcal{T}_0$, $\mathcal{T}' = \mathcal{T}_m$, and $\mathcal{T}_{i+1}$ is obtained by a diagonal flip from $\mathcal{T}_i$. Here a diagonal flip means that we find two empty triangles T_1 and T_2 in $\mathcal{T}_i$ whose union is an empty parallelogram P, and hence $T_1 \cap T_2$ is a diagonal of P, and we replace T_1 and T_2 by the two empty triangles T_3 and T_4 obtained by cutting P into two by the other diagonal of P. We deduce from (36) that $\xi(\mathcal{T}_{i+1}) = \xi(\mathcal{T}_i)$, yielding (37).

It follows from (37) that for any lattice polygon $Q \in \mathcal{P}(\mathbb{Z}^2)$, we can define $Z_2(Q)$ by subdividing Q into empty lattice triangles $E_1, \dots, E_k$, and setting

$$Z_2(Q) = \sum_{i=1}^{k} Z_2(E_i). \tag{38}$$

If $P \in \mathcal{P}(\mathbb{Z}^2)$ is a point or a segment, then we define $Z_2(P) = 0$. Since Z_2 has been known to be $\mathcal{G}(\mathbb{Z}^2)$ contravariant on empty triangles, Z_2 is $\mathcal{G}(\mathbb{Z}^2)$ contravariant on $\mathcal{P}(\mathbb{Z}^2)$.

Therefore, all left to prove is that Z_2 is a simple valuation. To verify this, we only need to check the case when $P, Q \in \mathcal{P}(\mathbb{Z}^2)$ are lattice polygons such that $P \cup$

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Q is convex and $P \cap Q$ is also a lattice polygon. First, we subdivide $P \cap Q$ into empty triangles, which triangulation we extend into a triangulation $\mathcal{T}_P$ of P and a triangulation $\mathcal{T}_Q$ of Q into empty triangles. It follows that $\mathcal{T}_P \cap \mathcal{T}_Q$ is the original triangulation of $P \cap Q$, and $\mathcal{T}_P \cup \mathcal{T}_Q$ is a triangulation of $P \cup Q$. Therefore, (38) yields that $Z_2(P \cup Q) + Z_2(P \cap Q) = Z_2(P) + Z_2(Q)$. ∎

We note that for any $a, b \in \mathbb{N}$, the function

$$(x, y) \mapsto \frac{1}{x - y}\left(\frac{e^y - 1}{y} \cdot x^a y^b - \frac{e^x - 1}{x} \cdot x^b y^a\right) \tag{39}$$

is analytic.

Example 15 Since the analytic function (*cf.* (39))

$$f_2(x, y) = \frac{1}{x - y} \cdot \left(\frac{e^y - 1}{y} \cdot (x^4 - 4x^3 y + x^2 y^2 + 6xy^3 - 3y^4) - \frac{e^x - 1}{x} \cdot (y^4 - 4xy^3 + x^2 y^2 + 6x^3 y - 3x^4)\right)$$

satisfies (32)–(34), it follows from Proposition 14 that there exists a $\mathcal{G}(\mathbb{Z}^2)$ covariant simple valuation $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$ satisfying $Z(T) = f_2$ that is different from the Laplace transform (see (20)), and $Z(P)$ is analytic for any lattice polytope P. In the language of Freyer et al. [8], $Z = \sum_{i>0} L_i^{i+2}$ where L_i^r is the i-homogeneous Ehrhart tensor on lattice polygons with rank r.

5 $Z(\{o\}) = f_0$ is essentially constant

The action of a group G on a space S equipped with a measure μ is called *ergodic* if for any G invariant measurable $X \subset S$, either $\mu(X) = 0$ or $\mu(S \backslash X) = 0$. The following well-known statement is Example 2.2.9 in Zimmer [19].

Proposition 16 *The action of* $\mathrm{SL}(2, \mathbb{Z})$ *on* $\mathbb{R}^2$ *is ergodic.*

Proposition 16 directly yields Proposition 4, that here we recall as Corollary 17, stating that any $\mathrm{SL}(2, \mathbb{Z})$ invariant measurable function is essentially constant (see Proposition 2.1.11 in Zimmer [19]).

Corollary 17 *If* $f \in \mathcal{M}(\mathbb{R}^2)$ *is invariant under* $\mathrm{SL}(2, \mathbb{Z})$*, then there exists a constant* $c \in \mathbb{R}$ *such that* $f(x) = c$ *for almost everywhere* $x \in \mathbb{R}^2$*.*

Proof Let $M = \sup\{t : |\{f \geq t\}| > 0\} \in (-\infty, \infty]$ and $m = \inf\{t : |\{f \leq t\}| > 0\} \in [-\infty, \infty)$, and hence $m \leq M$. If $m = M$, then $c = m = M$ works as $|\{f > c\}| = 0$ and $|\{f < c\}| = 0$ (here we did not even use the $\mathrm{SL}(2, \mathbb{Z})$ invariance of f).

If $m < M$, then choose $c \in (m, M)$. Since exactly one of the pairwise disjoint $\mathrm{SL}(2, \mathbb{Z})$ invariant measurable sets $\{f > c\}$, $\{f = c\}$ and $\{f < c\}$ has nonzero measure, and the union of the first two and the union of the last two have positive measure, we have $|\{f > c\}| = |\{f < c\}| = 0$. ∎

Combining Lemma 9 and Corollary 17 yields the characterization of $f_0 = Z(\{o\})$.

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Proposition 18 *For any translatively exponential and* $\mathrm{GL}(2,\mathbb{Z})$ *covariant valuations* $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$, $f_0 = Z(\{o\})$ *is* $\mathrm{GL}(2,\mathbb{Z})$ *invariant, and hence almost everywhere constant.*

6 Characterization of $Z([o,e_1]) = f_1$, and the proof of Theorem 5

According to Lemma 9 and Proposition 11, for an $f_1 \in \mathcal{M}(\mathbb{R}^2)$, there exists a $\mathcal{G}(\mathbb{Z}^2)$ covariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$ satisfying $Z([o,e_1]) = f_1$ if and only if for any $(x,y) \in \mathbb{R}^2$, we have

$$f_1(x,y) = e^x f_1(-x,-y); \tag{40}$$

$$f_1(x,y) = f_1(x,x+y), \tag{41}$$

$$f_1(x,y) = f_1(x,-y). \tag{42}$$

Proposition 19 *For* $\Omega_1 = \{(0,y) : y \geq 0\} \cup \{(x,y) : 0 \leq y \leq \frac{1}{2}x\}$, *and for any measurable* $\tilde{f}_1 : \Omega_1 \to \mathbb{R}$, *there exists a unique* $f_1 \in \mathcal{M}(\mathbb{R}^2)$ *satisfying (40)–(42) and* $f_1|_{\Omega_1} = \tilde{f}_1$.

Proof The key observation is that (41) yields that

$$f_1(x,y) = f_1(x,kx+y) \text{ for } k \in \mathbb{Z} \text{ and } x,y \in \mathbb{R}. \tag{43}$$

Now knowing f_1 on $\{(x,y) : 0 \leq y \leq \frac{1}{2}x\}$ determines $f_1(x,y)$ for $x > 0$ and $|y| \leq \frac{1}{2}x$ via (42). We observe that

$$f_1\left(x,-\frac{1}{2}x\right) = f_1\left(x,\frac{1}{2}x\right) \text{ for } x > 0. \tag{44}$$

Next knowing f_1 on $\{(x,y) : |y| \leq \frac{1}{2}x\}$ determines $f_1(x,y)$ for $x > 0$ and $y \in \mathbb{R}$ via (43) where (44) ensures that f_1 is well-defined. In addition, knowing f_1 on $\{(0,y) : y \geq 0\}$ determines $f_1(0,y)$ for $y \in \mathbb{R}$ as (40) implies $f_1(0,-y) = f_1(0,y)$. Then, knowing f_1 on $\{(x,y) : x > 0\}$ determines $f_1(x,y)$ for $x < 0$ and $y \in \mathbb{R}$ via (40). Finally, the f_1 constructed this way satisfies (40)– (42). ∎

Proof of Theorem 5. Combining Proposition 11, Corollary 12, and Propositions 18 and 19 implies Theorem 5. ∎

7 Toward a geometric characterization of $Z(T) = f_2$ for simple valuations

According to Lemma 13 and Proposition 14 in Section 4, if $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{M}(\mathbb{R}^2)$ is a $\mathcal{G}(\mathbb{Z}^2)$ covariant simple valuation, then $Z(T) = f_2$ is characterized by the following properties (cf. (32)–(34) in Lemma 13):

$$f_2(-x+y,-x) = e^{-x} f_2(x,y); \tag{45}$$

$$f_2(x,y) + e^{x+y} f_2(-x,-y) = f_2(x,x+y) + f_2(x+y,y); \tag{46}$$

$$f_2(x,y) = f_2(y,x). \tag{47}$$

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Next we provide a partially geometric characterization of functions satisfying (45)–(47). As the first step, we concentrate only on the properties (45) and (47). Let

$$\widetilde{\Omega} = \{(x, y) \in \mathbb{R}^2 : \ y \geq x \geq 0\}.$$

We consider the positive hull $\mathrm{pos}\{u, v\} = \{\alpha u + \beta v : \alpha, \beta \geq 0\}$ of $u, v \in \mathbb{R}^2$.

Lemma 20 *Given any measurable function $\tilde{f} : \widetilde{\Omega} \to \mathbb{R}$, there exists a unique measurable $f_2 : \mathbb{R}^2 \to \mathbb{R}$ satisfying (45) and (47) extending $\tilde{f}$.*

Proof We consider the $\Phi \in \mathrm{SL}(n, \mathbb{Z})$ defined by

$$\Phi(x, y) = (-x + y, -x),$$

and hence $\Phi^{-1}(x, y) = (-y, x - y)$, and (45) is equivalent to

$$\begin{aligned} f_2(\Phi(x, y)) &= e^{-x} f_2(x, y) \\ e^{-y} f_2(x, y) &= f_2(\Phi^{-1}(x, y)) = f_2(-y, x - y). \end{aligned} \tag{48}$$

We observe that Φ maps the quadrant $[0, \infty)^2 = \mathrm{pos}\{e_1, e_2\}$ onto $\mathrm{pos}\{e_1, -e_1 - e_2\}$, while $\Phi^{-1} = \Phi^2$ maps $[0, \infty)^2$ onto $\mathrm{pos}\{e_2, -e_1 - e_2\}$.

Given a measurable function $\tilde{f} : \widetilde{\Omega} \to \mathbb{R}$, we extend it to an f on $[0, \infty)^2$ by setting $f(x, y) = \tilde{f}(y, x)$ if $0 \leq y \leq x$. This function satisfies that $f(\Phi(x, y)) = e^{-x} f(x, y)$ if $(x, y), \Phi(x, y) \in [0, \infty)^2$ because $x = 0$ in this case. Using the properties of Φ, we can extend f in a unique way to a function f_2 on $\mathbb{R}^2$ satisfying (48), and hence (45).

Finally, we need to check whether f_2 satisfies (47). It does hold on $[0, \infty)^2$ by definition, and the $\Phi[0, \infty)^2$ and $\Phi^{-1}[0, \infty)^2$ are images of each other by the linear transformation $(x, y) \mapsto (y, x)$. Therefore, all we need to check is that $f_2(x, y) = f_2(y, x)$ for $(x, y) \in \Phi^{-1}[0, \infty)^2$ with $x \neq y$, and hence $x < 0$ and $y > x$. We have $(x, y) = \Phi^{-1}(y - x, -x)$ and $(y, x) = \Phi(-x, y - x)$, and hence (48) implies

$$\begin{aligned} f_2(x, y) &= e^x f(y - x, -x) \\ f_2(y, x) &= e^x f(-x, y - x) \end{aligned}$$

where $f(y - x, -x) = f(-x, y - x)$ by the construction of f. ∎

Based on Lemma 20, we plan to include the condition (46) in the characterization, and hence we rewrite (46) into forms suiting better the domains in Lemma 20.

Lemma 21 *Assuming that $f_2 : \mathbb{R}^2 \to \mathbb{R}$ satisfies (45) and (47), we have f_2 also satisfies (46) if and only if for any $x, y \in \mathbb{R}$, we have*

$$f_2(x, y) + e^x f_2(y - x, y) = f_2(x, x + y) + f_2(y, x + y). \tag{49}$$

Proof We observe that using the substitution $a = -x + y$ and $-x = b$, (45) is equivalent with $e^{-b} f_2(a, b) = f_2(-b, a - b)$, and in turn (45) is equivalent with $e^y f_2(-x, -y) = f_2(y, y - x)$ for any $x, y \in \mathbb{R}$. Therefore, under the condition of symmetry; namely, that f satisfies (47), the condition (45) is equivalent with $e^{x+y} f_2(-x, -y) = e^x f_2(y - x, y)$ for any $x, y \in \mathbb{R}$, and then (46) is equivalent to (49). ∎

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8 The proof of Theorem 3

Concerning (49), we observe that if $(x, y) \in \widetilde{\Omega}$ where

$$\widetilde{\Omega} = \{(x, y) \in \mathbb{R}^2 : y \geq x \geq 0\},$$

then $(y - x, y)$, $(x, x + y)$, and $(y, x + y)$ all lie in $\widetilde{\Omega}$. Therefore, it makes sense to consider (49) as a property of a function on $\widetilde{\Omega}$.

Proposition 22 *Given any measurable function $\tilde{f} : \widetilde{\Omega} \to \mathbb{R}$ satisfying (49), there exists a unique measurable $f_2 : \mathbb{R}^2 \to \mathbb{R}$ satisfying (45)–(47) extending $\tilde{f}$.*

Proof Using Lemma 20, first we extend $\tilde{f}$ to the unique measurable function f_2 on $\mathbb{R}^2$ satisfying (45) and (47), and we use (45) in both forms

$$e^{-a} f_2(a, b) = f_2(-a + b, -a) \text{ and } e^{-b} f_2(a, b) = f_2(-b, a - b). \tag{50}$$

We note that for f_2, the formulas $f_2(a, b) = f_2(b, a)$ and (50) yield

$$\begin{aligned} f_2(x, y) + e^x f_2(y - x, y) &= f_2(y, x) + e^x e^{y-x} f_2(x, x - y) \\ &= f_2(y, x) + e^y f_2(x - y, x). \end{aligned} \tag{51}$$

Now we verify that f_2 satisfies (49) for any $x, y \in \mathbb{R}$.

Case 1 $x, y \geq 0$ If $y \geq x$, then $(x, y) \in \widetilde{\Omega}$, and (49) holds. If $0 \leq y < x$, then combining (49) for $(y, x) \in \widetilde{\Omega}$ and (51) implies (49) again.

Case 2 $x \leq 0$ and $y \geq 0$ Now $(-x, y) \in [0, \infty)^2$; therefore, Case 1 yields that

$$f_2(-x, y) + e^{-x} f_2(y + x, y) = f_2(-x, y - x) + f_2(y, y - x).$$

We multiply through by e^x, and observe that (50) and $f_2(a, b) = f_2(b, a)$ imply

$$e^x f_2(-x, y) + f_2(y + x, y) = f_2(x + y, x) + f_2(y + x, y) = f_2(x, x + y) + f_2(y, y + x),$$

which is the right-hand side of (49). On the other hand, again (50) and $f_2(a, b) = f_2(b, a)$ yield

$$\begin{aligned} e^x f_2(-x, y - x) + e^x f_2(y, y - x) &= f_2(y, x) + e^x f_2(y, y - x) \\ &= f_2(x, y) + e^x f_2(y - x, y), \end{aligned}$$

which is the left-hand side of (49). In turn, we conclude (49).

Case 3 $x \geq 0$ and $y \leq 0$ Now Case 2 yields that

$$f_2(y, x) + e^y f_2(x - y, x) = f_2(y, y + x) + f_2(x, y + x),$$

and hence (51) implies (49).

Case 4 $x, y < 0$ As $-x > 0$ and $y < 0$, Case 3 yields that

$$f_2(-x, y) + e^{-x} f_2(y + x, y) = f_2(-x, y - x) + f_2(y, y - x).$$

Therefore, the argument in Case 2 yields (49). ∎

According to Proposition 22, the task to construct a measurable function $f_2 : \mathbb{R}^2 \to \mathbb{R}$ satisfying (45)–(47) – that is needed for Theorem 3 by Lemma 13 and Proposition

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14 – is equivalent to construct a measurable function $\tilde{f} : \widetilde{\Omega} \to \mathbb{R}$ satisfying

$$\tilde{f}(x,y) + e^x \tilde{f}(y-x,y) = \tilde{f}(x,x+y) + \tilde{f}(y,x+y). \tag{52}$$

When we write $\varphi(t) = \frac{e^t-1}{t}$ in the formulas below, we mean the positive analytic function on $\mathbb{R}$ with $\varphi(0) = 1$; namely, $\varphi(t) = \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!}$, and hence the reciprocal of φ is the series defining the Bernoulli numbers $B_0, B_1, \ldots$ by the formula (cf. Zagier [18])

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} \cdot t^n.$$

The following statement follows by direct calculations.

Lemma 23 *A natural bijection between measurable functions $\tilde{f} : \widetilde{\Omega} \to \mathbb{R}$ satisfying (52) and $\rho : [0,\infty)^2 \to \mathbb{R}$ satisfying (54) is induced by (53) and (55) as follows:*

(i) *If a measurable function $\tilde{f} : \widetilde{\Omega} \to \mathbb{R}$ satisfies (52), then the function $\rho : [0,\infty)^2 \to \mathbb{R}$ defined by*

$$\rho(x,y) = \frac{x}{e^x-1} \cdot \frac{x+y}{e^{x+y}-1} \cdot \left(\tilde{f}(x,x+y) + e^x \cdot \tilde{f}(y,x+y)\right) \tag{53}$$

satisfies

$$(2x+y)\rho(x,y) = (x+y)\rho(x,x+y) + x\rho(x+y,x), \tag{54}$$

$\rho(0,0) = 2\tilde{f}(0,0)$, and for $(x,y) \in \widetilde{\Omega}\backslash(0,0)$, the formula

$$\tilde{f}(x,y) = \frac{e^x}{y} \cdot \frac{e^{y-x}-1}{y-x} \cdot \rho(y-x,x) - \frac{1}{y} \cdot \frac{e^x-1}{x} \cdot \rho(x,y-x). \tag{55}$$

(ii) *If a function $\rho : [0,\infty)^2 \to \mathbb{R}$ satisfies (54), then the function $\tilde{f} : \widetilde{\Omega} \to \mathbb{R}$ defined by (55) satisfies (52).*

Remark. The case when the original simple valuation is the positive Laplace transform is equivalent with the property that ρ is the constant 1 function in (54) (cf. (20)).

Proof Substituting the formula (55) for $\tilde{f}$ into the left side of (52) results in the expression $\frac{(e^x-1)(e^y-1)}{xy}\rho(x,y-x)$, and into the right-hand side of (52) results in the expression

$$\frac{(e^x-1)(e^y-1)}{xy(x+y)}(y\rho(x,y) + x\rho(y,x)).$$

It follows that

$$y\rho(x,y) + x\rho(y,x) = (x+y)\rho(x,y-x),$$

which is equivalent to (54) by replacing y by $y-x$. This proves (ii), and (i) can be verified analogously. ∎

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To provide a characterization of the measurable function $\rho : [0,\infty)^2 \to \mathbb{R}$ in Lemma 23, we consider the golden ratio

$$\tau = \frac{\sqrt{5}+1}{2},$$

which satisfies the properties

$$\tau^2 = 1+\tau \text{ and } \frac{1}{\tau} = \tau - 1 \text{ and } \frac{1+\tau}{\tau} = \tau. \tag{56}$$

In addition, recall that

$$\widetilde{\Omega}_2 = \{(\tau s, s) : 1 \le s < \tau\} \cup \{(x,y) : 0 \le x \le y\}. \tag{57}$$

Lemma 24 *If $\tilde\rho$ is any (or Lebesgue measurable, or Borel measurable) function on $\widetilde{\Omega}_2$ into a real vectorspace, then there exists a unique (Lebesgue measurable, or Borel measurable) function ρ on $[0,\infty)^2$ satisfying*

$$(2x+y)\rho(x,y) = (x+y)\rho(x,x+y) + x\rho(x+y,x), \tag{58}$$

whose restriction to $\widetilde{\Omega}_2$ is $\tilde\rho$.

Proof If $x = 0$ and $y \ge 0$, then both the left-hand side and the right-hand side of (58) is $y\rho(0,y)$, thus (58) provides no restriction on ρ. If $x > 0$ and $y = 0$, then (58) reads as $2x\rho(x,0) = 2x\rho(x,x)$; therefore, $\rho(x,0) = \rho(x,x)$. In particular, we may assume that $x, y > 0$ for $(x,y) \in \widetilde{\Omega}_2$ in (58). Since then $(x+y,x) \notin \widetilde{\Omega}_2$, (58) provides no restriction on $\tilde\rho$ on $\widetilde{\Omega}_2$. However, we still need to define ρ on $\Omega_* = \{(x,y) : 0 < y < x\}$.

We recall the Fibonacci sequence, which we index as

$$F_0 = 0, \text{ and } F_1 = 1, \text{ and } F_{n+2} = F_n + F_{n+1} \text{ for } n \ge 0.$$

For $n \ge 0$, setting $\frac{1}{0} = \infty$, the Fibonacci sequence (cf. Burton [6]) satisfies

$$\begin{aligned} \frac{F_{n+1}}{F_n} > \frac{F_{n+3}}{F_{n+2}} > \tau \text{ if } n \ge 0 \text{ even;} \\ \frac{F_{n+1}}{F_n} < \frac{F_{n+3}}{F_{n+2}} < \tau \text{ if } n \ge 1 \text{ odd;} \\ \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \tau. \end{aligned}$$

In particular, for $n \ge 0$, we can define

$$\Omega_n = \left\{(x,y) : \frac{F_{n+1}}{F_n} > \frac{x}{y} \ge \frac{F_{n+3}}{F_{n+2}}\right\} \text{ if } n \ge 0 \text{ is even;}$$

$$\Omega_n = \left\{(x,y) : \frac{F_{n+1}}{F_n} < \frac{x}{y} \le \frac{F_{n+3}}{F_{n+2}}\right\} \text{ if } n \ge 1 \text{ is odd.}$$

We observe that $\{\Omega_n\}_{n\ge 0}$ is a partition of $\Omega_* \backslash \ell$ for the open half line $\ell = \{(a,b) : a = \tau b \text{ and } b > 0\}$ by the properties above of the Fibonacci numbers.

We note that for the points of Ω_* and for $n \ge 0$, we have

$$\text{if } (x,y) \in \Omega_*, \text{ then } \quad (x, x+y) \in \widetilde{\Omega}_2; \tag{59}$$

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$$(x, y) \in \Omega_n \text{ if and only if } \quad (x + y, x) \in \Omega_{n+1} \tag{60}$$

$$x, y > 0 \text{ and } \frac{x}{y} = \tau \text{ if and only if } \quad y > 0 \text{ and } \frac{x + y}{x} = \tau. \tag{61}$$

We observe that for any x, y, $(x + y, x) \in \Omega_0 = \{(a, b) : a \geq 2b > 0\}$ if and only if $(x, y) \in \{(a, b) : 0 < a \leq b\} \subset \widetilde{\Omega}_2$; therefore, we can define ρ on Ω_0 *via* (58). It can be done in a unique way because if $(x, y) \in \Omega_0$, then $(x + y, x) \in \Omega_1$ by (60). Using (59) and (60), we define ρ on any Ω_n, $n \geq 0$, by induction on $n \geq 0$. This extension of ρ onto $\Omega_* \backslash \ell$ is also unique according to (59) and (60). Therefore, all we need to do is to define ρ on the open half line ℓ.

For $m \in \mathbb{Z}$, let $s_m = \{(a, b) : a = \tau b \text{ and } \tau^m \leq b < \tau^{m+1}\} \subset \ell$, and hence $\{s_m\}_{m \in \mathbb{Z}}$ provides a partition of ℓ, and $\ell \cap \widetilde{\Omega}_2 = s_0$. Since $(x, y) \in s_m$ if and only if $(x + y, x) \in s_{m+1}$ according to (61), we can define ρ on s_n for $n \geq 0$ by induction on n *via* (58), and similarly we can define ρ on s_{-n} for $n \geq 0$ by induction on n again *via* (58), completing the proof of Lemma 24. ∎

Proof of Theorem 3. Combining Lemma 13, Proposition 14, Proposition 22, and Lemmas 23 and 24 yields Theorem 3. ∎

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