*Ergod. Th. & Dynam. Sys.* (2000), **20**, 1391–1403
*Printed in the United Kingdom* © *2000 Cambridge University Press*

# Non-accessible critical points of Cremer polynomials

JAN KIWI

*Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Casilla 306, Correo 22, Santiago, Chile*
*(e-mail: jkiwi@riemann.mat.puc.cl)*

(*Received* 31 *July* 1998 *and accepted in revised form* 26 *January* 1999)

*Abstract.* It is shown that a polynomial with a Cremer periodic orbit has a non-accessible critical point in its Julia set provided that the Cremer periodic orbit is approximated by small cycles. Also, this paper contains a new proof of the Douady–Shishikura inequality for the number of non-repelling cycles of a complex polynomial.

## 1. *Introduction*

Cremer fixed points occur as fixed points of polynomial maps from the complex plane onto itself. Following Douady, Sullivan [**S1**] and Lyubich [**L**], the presence of Cremer fixed points forces the Julia set to be not locally connected. When local connectivity fails it is of interest to determine which points in the Julia set are accessible from the complement of the Julia set and which are not (if any).

According to Douady, Hubbard, Sullivan and Yoccoz, repelling periodic points are always accessible (e.g. see [**H, M1**]; for disconnected Julia sets see [**EL**]) and it 'often' happens that Cremer fixed points are 'surrounded' by repelling cycles. Let us be more precise, if every neighborhood of a Cremer fixed point contains infinitely many repelling cycles, we say that the Cremer fixed point is approximated by small cycles. For quadratic polynomials, Yoccoz has shown that this is always the case [**Y**]. For polynomials of higher degree it is an open problem. Some results and a more detailed discussion about this problem can be found in [**M1, PM1**].

In this paper we show how the accessibility of the small cycles implies the non-accessibility of a critical point.

THEOREM 1.1. *Let $P$ be a polynomial with a Cremer fixed point $\hat{z}$ that is approximated by small cycles. Then there exists a critical point which is not accessible from the complement of the Julia set.*

We conclude that the critical point of a quadratic polynomial with a Cremer periodic point is not accessible, thus answering a question posed by Milnor in [**BL**].

https://doi.org/10.1017/S0143385700000754 Published online by Cambridge University Press

COROLLARY 1.2. *The critical point of a quadratic polynomial with a Cremer periodic point is not accessible from the complement of the Julia set.*

Under the assumption that the Julia set is connected, each point that is accessible from the complement of the filled Julia set is the landing point of some external ray. In this case we prove a stronger statement by showing the existence of a non-accessible critical value with the following property. An external ray (if there is one) that accumulates at this critical value also accumulates at the Cremer fixed point.

THEOREM 1.3. *Let $P$ be a polynomial with connected Julia set $J_P$ and with a Cremer fixed point $\hat{z}$ that is approximated by small cycles. Then there exists a critical value $v$ which is not accessible from $\mathbb{C} \setminus J_P$ such that: if $v$ belongs to the closure $\overline{R}$ of some external ray $R$, then $\hat{z}$ also belongs to $\overline{R}$.*

For some quadratic polynomials our result can be deduced from a stronger one of Douady and Sørensen (compare [**Sø**]). More precisely, for a generic quadratic polynomial with a Cremer fixed point, they showed that there is an external ray that does not land; furthermore, this external ray accumulates in a set containing the Cremer fixed point and its preimage. It follows that for such quadratic polynomials every external ray that accumulates at the critical value also accumulates at the Cremer fixed point (see Corollary 3.6, §3).

Pérez-Marco [**PM2**] has shown that rational functions with Cremer periodic points always have some non-accessible points in their Julia set. For a 'large' family of infinitely renormalizable quadratic polynomials, he proves the non-accessibility of the critical point; and for a generic quadratic polynomial with a Cremer fixed point, Pérez-Marco shows that the critical point is not accessible. We also refer the reader to [**PM3, PM4**] for more results about the semi-local dynamics around Cremer fixed points.

This paper is organized as follows. In §2, we briefly summarize some definitions and results from polynomial dynamics. For general background in complex dynamics see [**M1, L**].

In §3, we prove some preliminary results. Also, we show how one of them leads to a new proof (for polynomials with connected Julia set) of a well-known result by Douady [**Bl**] and Shishikura [**Sh**]. That is, *for a polynomial of degree $d$, the total number of Cremer cycles and cycles of bounded Fatou components is strictly less than $d$* (Corollary 3.5). Later, at the end of §4, we deduce the above inequality for polynomials which may have disconnected Julia sets. In §4, we prove our main results. The proof of Theorem 1.3 is by contradiction and the idea is a refinement of the following line of argument. Under the assumption that certain critical values are accessible, we construct an open region around the Cremer fixed point containing at most finitely many cycles. This region is obtained by cutting the complex plane along an appropriate collection of external rays. The correct choice of these external rays is based on the results from §3. In order to prove Theorem 1.1 in the disconnected case we use some results on polynomial-like mappings and then we apply Theorem 1.3 to find a non-accessible critical point.

https://doi.org/10.1017/S0143385700000754 Published online by Cambridge University Press

2. *Cremer points and accessible points*

Consider a polynomial $P : \mathbb{C} \to \mathbb{C}$ of degree $d \geq 2$ with a fixed point $\hat{z}$ of *multiplier* $\lambda = P'(\hat{z})$. The fixed point is called *repelling* if $|\lambda| > 1$, *attracting* if $|\lambda| < 1$ and *parabolic* if $\lambda$ is a root of unity.

For $\lambda = e^{2\pi i\theta}$ where $\theta$ is irrational, we have to distinguish between two possibilities. If the fixed point $\hat{z}$ belongs to the Julia set $J_P$ we say that $\hat{z}$ is a *Cremer fixed point*. Otherwise, if $\hat{z}$ lies in the Fatou set $F_P$, call $\hat{z}$ a *Siegel fixed point*.

For a generic choice of $\theta$, Cremer [**Cr**] showed that there exists a sequence of periodic points converging to $\hat{z}$, in this case $\hat{z}$ belongs to the Julia set. In contrast, for a full Lebesgue measure set of angles $\theta$, Siegel [**Si**] proved that $\hat{z}$ lies in the Fatou set. We refer the reader to [**M1, PM1**] for a more detailed exposition about this dichotomy.

A modification of Cremer's proof (compare [**M1**]) shows that for a generic choice of $\theta$, every neighborhood of a Cremer fixed point $\hat{z}$ contains infinitely many cycles. We say that $\hat{z}$ *is approximated by small cycles*. According to Yoccoz [**Y**] quadratic Cremer fixed points are always approximated by small cycles. It is not known whether this is true for polynomials of arbitrary degree. For some interesting results about this problem see [**PM1**].

A point $z$ in the Julia set $J_P$ is called *accessible* from a Fatou component $V$ if there exists a path $\gamma$ contained in $V$ that ends at $z$, i.e.

$$\gamma : [0, 1) \to V$$

and $\gamma(r) \to z$ as $r \to 1$. Accessibility is a property completely invariant under the dynamics, $z$ is accessible from $V$ if and only if $P(z)$ is accessible from $P(V)$. When $z$ is not accessible from any Fatou component sometimes we just say that $z$ is a *non-accessible point*.

Let us now assume that $K_P$ is connected. For simplicity we also assume that $P$ is a monic polynomial. The complement of the closed unit disk $\overline{\Delta}$ is conformally isomorphic to the complement of $K_P$. Moreover, there exists a unique conformal isomorphism:

$$\phi : \mathbb{C} \setminus \overline{\Delta} \to \mathbb{C} \setminus K_P$$

such that

$$P \circ \phi(z) = \phi(z^d),$$

and that is asymptotic to identity near infinity (see [**M1**]).

The question of which points in $J_P$ are accessible from $\mathbb{C} \setminus K_P$ is closely related to the boundary behavior of the map $\phi$. We say that

$$R_t = \phi(\{re^{2\pi it} : r \in (1, \infty)\})$$

is the *external ray* with angle $t \in \mathbb{R}/\mathbb{Z}$. An external ray $R_t$ *lands at* $z$ if $\phi(re^{2\pi it}) \to z$ as $r \to 1$. Therefore, the landing points of external rays are accessible from $\mathbb{C} \setminus K_P$. A result of Lindelöf assures that the converse is also true, every point accessible from $\mathbb{C} \setminus K_P$ is the landing point of some external ray (compare [**Ru**]).

The external ray $R_t$ is called a *periodic ray* if $P^{\circ n}(R_t) = R_t$ for some positive integer $n$. In particular, if $P(R_t) = R_t$ we say that the external ray $R_t$ is a *fixed ray*. Periodic

https://doi.org/10.1017/S0143385700000754 Published online by Cambridge University Press

rays always land, moreover their landing points are parabolic or repelling periodic points. Conversely, parabolic and repelling periodic points are the landing points of at least one periodic ray (see [**H, M1**]).

If an external ray $R$ lands, then the closure $\overline{R}$ is the union of the external ray and its landing point. Otherwise, if $R$ fails to land, $\overline{R}$ is the union of the external ray and a non-trivial connected subset of the Julia set. For short, the closure $\overline{R}$ of an external ray $R$ is called a *closed ray*.

## 3. *Separation lemmas*

For polynomials with connected Julia sets, Poirier showed that the union of all closed fixed rays 'separates' invariant Fatou components and Cremer fixed points. This is stated as a corollary of a result by Goldberg and Milnor (compare [**GM**, Theorem 3.3 and Corollary 3.5]). We go one step further and show that after cutting the complex plane along an appropriate collection of external rays, each Cremer periodic point and periodic Fatou component is 'separated' from the others.

LEMMA 3.1. *Let $P$ be a polynomial with connected Julia set. Then there exists a finite union $\mathcal{C}$ of closed periodic rays such that*

$$P(\mathcal{C}) = \mathcal{C}.$$

*Each component of $\mathbb{C} \setminus \mathcal{C}$ contains at most one Cremer periodic point or periodic Fatou component.*

Recall that periodic rays always land, hence the corresponding closed ray is the union of the periodic ray and its landing point. In this section, the closed rays taken into consideration are always the closure of external rays that land.

Lemma 3.1 is an immediate consequence of the following (compare with [**GM**, Theorem 3.3]):

THEOREM 3.2. (Goldberg and Milnor) *Consider a polynomial $P$ with connected Julia set. Denote by $\mathcal{C}$ the union of all closed fixed rays:*

$$\mathcal{C} = \overline{R}_0 \cup \overline{R}_{1/d-1} \cup \cdots \cup \overline{R}_{d-2/d-1}.$$

*If $U$ is a connected component of $\mathbb{C} \setminus \mathcal{C}$, then one and only one of the following holds:*

- *$U$ contains exactly one invariant Fatou component;*
- *$U$ contains exactly one fixed point which belongs to the Julia set.*

*Proof of Lemma 3.1.* If $P$ has no Cremer periodic points and no bounded periodic Fatou components there is nothing to prove. Otherwise, let $m$ be the minimum common multiple of the following list of integers: periods of periodic Fatou components, and periods of Cremer periodic points. The number of non-repelling cycles is finite, hence this list is also finite and $m$ is a well-defined integer.

Periodic Fatou components and Cremer periodic points of $P$ and $P^{\circ m}$ are the same, only their periods differ. In fact, we have chosen $m$ in such a way that the periodic Fatou components are invariant under $P^{\circ m}$ and that the Cremer periodic points are fixed points in its Julia set.

https://doi.org/10.1017/S0143385700000754 Published online by Cambridge University Press

Now consider the union $\mathcal{C}$ of closed rays fixed by $P^{\circ m}$:

$$\mathcal{C} = \overline{R_0} \cup \cdots \cup \overline{R}_{d^m-2/d^m-1}.$$

Since fixed rays land at parabolic or repelling points, this set $\mathcal{C}$ is disjoint from bounded Fatou components and Cremer points. Applying the previous theorem to $P^{\circ m}$, it follows that the set $\mathcal{C}$ has the desired properties. □

Let us introduce some notation. If $\mathcal{C}$ is the union of finitely many closed external rays and $z$ is a point that lies in the complement of $\mathcal{C}$, denote by $U_{\mathcal{C}}(z)$ the connected component of $\mathbb{C} \setminus \mathcal{C}$ that contains $z$. In a similar way, if $V$ is a connected set contained in the complement of $\mathcal{C}$, denote by $U_{\mathcal{C}}(V)$ the connected component of $\mathbb{C} \setminus \mathcal{C}$ that contains $V$.

A refinement of the previous lemma shows that cutting along a larger collection of external rays, critical points and Cremer points that are left 'together', stay 'together' under forward iterations. The corresponding statement for periodic Fatou components also holds.

LEMMA 3.3. *Let $P$ be a polynomial with connected Julia set. Then there exists a finite union $\mathcal{C}$ of closed rays such that:*

(1) $P(\mathcal{C}) \subset \mathcal{C}$;
(2) *each component of $\mathbb{C} \setminus \mathcal{C}$ contains at most one Cremer periodic point or periodic Fatou component;*
(3) *if $\hat{z}$ is a Cremer periodic point and $c$ is a critical point such that $U_{\mathcal{C}}(c) = U_{\mathcal{C}}(\hat{z})$, then $U_{\mathcal{C}}(P^{\circ n}(c)) = U_{\mathcal{C}}(P^{\circ n}(\hat{z}))$ for all $n \geq 1$. Moreover, $c \in J_P$ and $c$ does not belong to the closure of any bounded Fatou component;*
(4) *if $V$ is a bounded periodic Fatou component and $c$ is a critical point such that $U_{\mathcal{C}}(c) = U_{\mathcal{C}}(V)$, then $U_{\mathcal{C}}(P^{\circ n}(c)) = U_{\mathcal{C}}(P^{\circ n}(V))$ for all $n \geq 1$.*

*Remark.* The proof of Corollary 3.4 will show that statements (3) and (4) of the previous lemma are not vacuous in the following sense. Given a Cremer periodic orbit $z_0 = P(z_{n-1}), \ldots, z_{n-1} = P(z_{n-2})$, then for some $z_i$ there exists a critical point $c$ such that $U_{\mathcal{C}}(c) = U_{\mathcal{C}}(z_i)$. The corresponding assertion is also true for bounded periodic Fatou components.

*Proof.* Let $\mathcal{C}_0$ be the union of closed rays from Lemma 3.1. For $k \geq 1$ define inductively $\mathcal{C}_k = P^{-1}(\mathcal{C}_{k-1})$.

We are going to show that conditions (1)–(4) hold for $\mathcal{C}_k$ when $k$ is large enough. Beforehand, let us list some properties of the sets $\mathcal{C}_k$ and their complementary regions:

(a) $\mathcal{C}_k \subset \mathcal{C}_{k+1}$ for $k \geq 0$,
(b) $P(\mathcal{C}_k) \subset \mathcal{C}_k$ for $k \geq 0$,
(c) $U_{\mathcal{C}_{k+1}}(z) \subset U_{\mathcal{C}_k}(z)$ for $k \geq 0$ and $z \in \mathbb{C} \setminus \mathcal{C}_{k+1}$,
(d) $P^{\circ n}(U_{\mathcal{C}_{k+n}}(z)) = U_{\mathcal{C}_k}(P^{\circ n}(z))$ for $k \geq 0$, $n \geq 1$ and $z \in \mathbb{C} \setminus \mathcal{C}_{k+n}$.

All of the above can be deduced from the forward invariance of $\mathcal{C}_0$ ($P(\mathcal{C}_0) = \mathcal{C}_0$).

From Lemma 3.1 and properties (b) and (c) it follows that conditions (1) and (2) hold for all $\mathcal{C}_k$.

https://doi.org/10.1017/S0143385700000754 Published online by Cambridge University Press

CLAIM 1. *Given a critical point $c$ and a Cremer periodic point $\hat{z}$ there exists $k(c,\hat{z}) \geq 0$ such that:*
$(*)$ *if $U_{\mathcal{C}_l}(c) = U_{\mathcal{C}_l}(\hat{z})$, then $U_{\mathcal{C}_l}(P^{\circ n}(c)) = U_{\mathcal{C}_l}(P^{\circ n}(\hat{z}))$ for all $n \geq 1$;*
*holds for all $l \geq k(c,\hat{z})$.*

*Proof of Claim 1.* In the case that $(*)$ holds for all $l \geq 0$, of course we take $k(c,\hat{z}) = 0$. Otherwise, for some $l_0 \geq 0$ we have that $U_{\mathcal{C}_{l_0}}(c) = U_{\mathcal{C}_{l_0}}(\hat{z})$ and $U_{\mathcal{C}_{l_0}}(P^{\circ n}(c)) \neq U_{\mathcal{C}_{l_0}}(P^{\circ n}(\hat{z}))$ for some $n \geq 1$. In this case let $k(c,\hat{z}) = l_0 + n$, it follows from (c) and (d) that $U_{\mathcal{C}_l}(c) \neq U_{\mathcal{C}_l}(\hat{z})$ for all $l \geq k(c,\hat{z})$.

We omit the identical proof of the corresponding claim for periodic Fatou components that follows.

CLAIM 2. *Given a critical point $c$ and a bounded periodic Fatou component $V$ there exists $k(c,V) \geq 0$ such that:*
$(*)$ *if $U_{\mathcal{C}_l}(c) = U_{\mathcal{C}_l}(V)$, then $U_{\mathcal{C}_l}(P^{\circ n}(c)) = U_{\mathcal{C}_l}(P^{\circ n}(V))$ for all $n \geq 1$;*
*holds for all $l \geq k(c,V)$.*

Let $N$ be the maximum of the numbers $k(c,\hat{z})$ and $k(c,V)$, where all possible pairs $(c,\hat{z})$ and $(c,V)$ are taken into consideration. Since the number of critical points and non-repelling cycles is finite, the total number of all such pairs is also finite and $N$ is a well-defined integer.

As mentioned before Claim 1, $\mathcal{C}_N$ satisfies conditions (1) and (2). From Claim 2, it follows that (4) holds for $\mathcal{C}_N$. It remains to complete the proof that (3) also holds for $\mathcal{C}_N$.

If $c$ is a critical point and $\hat{z}$ is a Cremer point such that $U_{\mathcal{C}_N}(c) = U_{\mathcal{C}_N}(\hat{z})$, then

$$U_{\mathcal{C}_N}(P^{\circ n}(c)) = U_{\mathcal{C}_N}(P^{\circ n}(\hat{z})) \neq U_{\mathcal{C}_N}(V)$$

for all $n \geq 1$ and for all bounded periodic Fatou components $V$. According to Sullivan [**S2**], every point in the closure of a bounded Fatou component is mapped after a finite number of iterates to the closure of a bounded periodic Fatou component. It follows that $c$ belongs to $J_P$ and $c$ does not belong to the closure of any bounded Fatou component. □

After Fatou, each irrationally neutral cycle has an 'associated' critical point. In fact, each Cremer cycle is contained in the omega limit set of a critical point. Similarly, each boundary point of a Siegel disk belongs to the omega limit set of a critical point. Lemma 3.3 allows us to show that an irrationally neutral cycle has 'associated' a critical point which is not 'shared' with other irrationally neutral cycles.

COROLLARY 3.4. *Let $P$ be a polynomial with connected Julia set $J_P$.*

(a) *Given a Cremer cycle $\{z_1,\ldots,z_p\}$ there exist a critical point $c \in J_P$ such that the omega limit set $\omega(c)$ contains the Cremer cycle $\{z_1,\ldots,z_p\}$ and $\omega(c)$ contains neither any other Cremer points nor any non-preperiodic boundary points of a Siegel disk.*

(b) *Given a cycle of Siegel disks $\{V_1,\ldots,V_p\}$ and a point $z \in \partial V_1 \cup \cdots \cup \partial V_p$ there exist a critical point $c \in J_P$ such that $z$ belongs to the omega limit set $\omega(c)$ and $\omega(c)$ contains neither any Cremer periodic points nor any non-preperiodic boundary points of other Siegel disks.*

https://doi.org/10.1017/S0143385700000754 Published online by Cambridge University Press

The proof is almost identical to the one which appears in [**M1**] for Fatou's results mentioned above.

*Proof.* Consider a finite union $\mathcal{C}$ of closed rays as in Lemma 3.3. Recall that we denote by $U_{\mathcal{C}}(z)$ the connected component of $\mathbb{C} \setminus \mathcal{C}$ which contains $z$.

For part (a), first notice that Lemma 3.3 guarantees that a critical point in $U = U_{\mathcal{C}}(z_1) \cup \cdots \cup U_{\mathcal{C}}(z_p)$ also has its forward orbit contained in $U$. Now suppose that $z_1$ is not in the omega limit set of a critical point of $P$ which lies in $U$. Pick a neighborhood $W \subset U_{\mathcal{C}}(z_1)$ of $z_1$ such that $W$ is disjoint from the forward orbit of the critical points that belong to $U$. It follows that, for all $k > 0$, the inverse branch $P^{-kp} : W \to U_{\mathcal{C}}(z_1)$ which fixes $z_1$ is a well-defined univalent function in $W$. Hence, these maps form a normal family and a subsequence $P^{-k_i p}$ converges locally uniformly to a map $f : W \to U_{\mathcal{C}}(z_1)$ which has derivative of norm 1 at $z_1$. Since $f$ is invertible around $z_1$, we have that $P^{\circ k_i p}$ converges to $f^{-1}$ in a neighborhood $W'$ of $z_1$. However, this is impossible because $J_P \subset P^{\circ k_i p}(W')$ for $k_i$ large enough. Thus, the cycle $\{z_1, \ldots, z_p\}$ is in the omega limit set of at least one critical point $c \in U$ such that $\omega(c) \subset \overline{U}$. Hence $\omega(c)$ cannot contain non-preperiodic boundary points of a Siegel disk or other Cremer cycle because they lie in the complement of $\overline{U}$.

For part (b), denote by $U_{\mathcal{C}}(V_1), \ldots, U_{\mathcal{C}}(V_p)$ the connected components of $\mathbb{C} \setminus \mathcal{C}$ which contain $V_1, \ldots, V_p$, respectively. Pick a point $z$ which is not preperiodic in $\partial V_1$. It is enough to show that $z$ is in the omega limit set of a critical point $c$ with forward orbit contained in

$$U = U_{\mathcal{C}}(V_1) \cup \cdots \cup U_{\mathcal{C}}(V_p).$$

As above we suppose that $z$ is not in the omega limit of a critical point of $P$ which belongs to $U$. Choose a neighborhood $W \subset U_{\mathcal{C}}(V_1)$ of $z$ such that the inverse branches $P^{-kp} : W \to U_{\mathcal{C}}(V_1)$ which take points of $V_1$ onto points of $V_1$ are well defined. Since $P^{\circ p}$ is an irrational rotation in $V_1$ and $P^{-kp}$ is a normal family we have that a subsequence of $P^{-kp}$ converges to the identity in $W$. Just as in the proof of part (a) this leads to a contradiction. □

According to Douady [**Bl**] and Shishikura [**Sh**], the total number of Cremer cycles and cycles of bounded Fatou components of $P$ is strictly less than $d = \deg P$. For polynomials with connected Julia sets, the previous corollary leads to a new proof of this result.

COROLLARY 3.5. *Let $P$ be a polynomial with connected Julia set. Then the total number of Cremer cycles and cycles of bounded Fatou components is less than or equal to the number of distinct critical points.*

*Proof.* By the previous Corollary distinct irrationally neutral cycles have 'associated' distinct critical points which lie in the Julia set $J_P$. Also, each cycle of parabolic or attracting basins contains a critical point. □

The next result shows how the existence of an external ray that accumulates at a Cremer fixed point and one of its preimages implies the existence of a non-accessible critical value. Moreover, the non-accessible critical value has the property that if an external ray accumulates at it, this external ray also accumulates at the Cremer fixed point.

https://doi.org/10.1017/S0143385700000754 Published online by Cambridge University Press

COROLLARY 3.6. *Let $P$ be a polynomial with connected $K_P$ and with a Cremer fixed point $\hat{z}$. If there exists an external ray $R_0$ such that $\{\hat{z}, z\} \subset \overline{R_0}$ for some $z \in P^{-1}(\hat{z}) \setminus \{\hat{z}\}$, then there exists a non-accessible critical value $v$ with the following property: every external ray $R$ that accumulates at $v$ also accumulates at $\hat{z}$.*

*Outline of the proof.* Apply Lemma 3.3 to obtain a finite union $\mathcal{C}$ of closed rays. Let $\mathcal{C}_1 = P^{-1}(\mathcal{C})$ and notice that $R_0 \subset U_{\mathcal{C}_1}(\hat{z})$. Call $c_1, \ldots, c_k$ the critical points that are contained in $U_{\mathcal{C}_1}(\hat{z})$. It follows that their critical values $v_1, \ldots, v_k$ are contained in $U_{\mathcal{C}}(\hat{z})$, and the map

$$P : U_{\mathcal{C}_1}(\hat{z}) \setminus P^{-1}(\{v_1, \ldots, v_k\}) \to U_{\mathcal{C}}(\hat{z}) \setminus \{v_1, \ldots, v_k\}$$

is a regular covering.

Take a small disk $D$ centered at $\hat{z}$. Call $D_0$ the preimage of $D$ that contains $\hat{z}$ and denote by $D_1, \ldots, D_l$ the other preimages of $D$ contained in $U_{\mathcal{C}_1}(\hat{z})$. It follows that there exists a closed segment $\alpha$ of $R_0$ contained in $U_{\mathcal{C}_1}(\hat{z}) \setminus D_0 \cup \cdots \cup D_l$ such that $\alpha$ joins the boundary $\partial D_0$ of $D_0$ with the boundary $\partial D_i$ of $D_i$ for some $i \neq 0$. Also, we can assume that $\alpha$ intersects $\partial D_0 \cup \partial D_i$ in exactly two points, which are the endpoints of $\alpha$.

Observe that $P(\alpha)$ is contained in $U_{\mathcal{C}}(\hat{z}) \setminus D$ and intersects $\partial D$ in exactly two points. Recall that $P$ maps the external ray $R_0$ homeomorphically onto its image. In particular, $P$ maps the arc $\alpha$ homeomorphically onto $P(\alpha)$. Hence, the set $\partial D \cup P(\alpha)$ separates the plane into three components. That is, an unbounded component, the disk $D$, and a bounded component $\tilde{D}$. Call $\beta$ the boundary of the Jordan domain $\tilde{D} \subset U_{\mathcal{C}}(\hat{z})$. The curve $\beta$ is the union of a segment of $\partial D$ with $P(\alpha)$. Since a lift of $\beta$ is not a closed curve, $\tilde{D}$ contains a critical value $v$. By Lemma 3.3(3) the critical value $v$ does not belong to the closure of any bounded Fatou component. If an external ray $R$ accumulates at $v$, then $R$ intersects $\partial D$. Taking $D$ arbitrarily small, it follows that $v$ has the desired properties. □

## 4. *Non-accessible points*

In this section we first prove the following.

THEOREM 1.3. *Let $P$ be a polynomial with connected Julia set $J_P$ and with a Cremer fixed point $\hat{z}$ that is approximated by small cycles. Then there exists a critical value $v$ which is not accessible from $\mathbb{C} \setminus J_P$ such that: if $v \in \overline{R}$ for some external ray $R$, then $\hat{z} \in \overline{R}$.*

*Proof of Theorem 1.3.* Start applying Lemma 3.3 to obtain a finite union $\mathcal{C}$ of closed rays. Notice that all the critical points $c_1, \ldots, c_k$ contained in $U_{\mathcal{C}}(\hat{z})$ belong to the Julia set $J_P$. Thus, their critical values $v_1 = P(c_1), \ldots, v_k = P(c_k)$ are also in $J_P$. These critical values are not in the closure of any bounded Fatou component. Therefore, it is enough to show that one of them is not accessible from $\mathbb{C} \setminus K_P$ and that every external ray which accumulates at this critical value also accumulates at the Cremer fixed point.

Let us first consider the case in which the Cremer fixed point is not a critical value. The proof is by contradiction. Assume that there exists external rays $R_{t_1}, \ldots, R_{t_k}$ such that $v_i \in \overline{R_{t_i}}$ and $\hat{z} \notin \overline{R_{t_i}}$. Consider the sets formed by the union of finitely many closed rays:

$$\mathcal{C}_0 = \mathcal{C} \cup \overline{R_{t_1}} \cup \cdots \cup \overline{R_{t_k}}, \quad \mathcal{C}_1 = P^{-1}(\mathcal{C}_0);$$

https://doi.org/10.1017/S0143385700000754 Published online by Cambridge University Press

and the corresponding domains containing the Cremer fixed point $\hat{z}$:

$$U_0 = U_{\mathcal{C}_0}(\hat{z}), \quad U_1 = U_{\mathcal{C}_1}(\hat{z}).$$

The rest of the proof consists in showing that $U_1$ contains at most finitely many cycles. For each repelling cycle contained in $U_1$ there is at least one cycle of external rays contained in $U_1$. Then, it is enough to show that $U_1$ contains at most finitely many cycles of external rays.

First observe that $U_0$ and $U_1$ are simply connected domains. In fact, each connected component of the complement of a closed and connected subset of the Riemann sphere is simply connected (see [**N**, VI.4.1]). In our situation, the set $\mathcal{C}_0 \cup \{\infty\}$ (respectively $\mathcal{C}_1 \cup \{\infty\}$) is a closed and connected subset of the Riemann sphere and $U_0$ (respectively $U_1$) is a connected component of the complement of $\mathcal{C}_0 \cup \{\infty\}$ (respectively $\mathcal{C}_1 \cup \{\infty\}$).

Now observe that the proper holomorphic map

$$P|_{U_1} : U_1 \to U_0$$

has no critical points. Regular coverings of simply connected domains are homeomorphisms. Therefore, $P|_{U_1}$ is a conformal isomorphism.

The polynomial $P$ acts in the angles of the external rays as the multiplication by $d = \deg P$ map,

$$\begin{aligned} m_d : \mathbb{R}/\mathbb{Z} &\to \mathbb{R}/\mathbb{Z} \\ t &\mapsto d \cdot t. \end{aligned}$$

We are interested in the external rays contained in $U_1$ and their image, the external rays contained in $U_0$. Denote by $B_1$ and $B_0$ the corresponding sets formed by the angles of such rays:

$$B_0 = \{t : R_t \subset U_0\} \subset \mathbb{R}/\mathbb{Z}, \quad B_1 = \{t : R_t \subset U_1\} \subset \mathbb{R}/\mathbb{Z}.$$

As mentioned before, the map $P|_{U_1} : U_1 \to U_0$ is a conformal isomorphism, therefore

$$m_d|_{B_1} : B_1 \to B_0$$

is a cyclic order preserving homeomorphism. Hence, $m_d|_{B_1}$ admits a continuous extension $g : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ which is a non-decreasing circle map (i.e. the lift is non-decreasing) of degree one.

A non-decreasing circle map $g$ of degree one either is free of periodic orbits or all periodic orbits have the same period. If $m_d$ has cycles contained in $B_1$, then all of them have the same period. The map $m_d : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ has finitely many cycles of the same period. We conclude that $m_d$ has at most finitely many cycles contained in $B_1$. An equivalent statement is that there are at most finitely many cycles of external rays contained in $U_1$, which contradicts the assumption that the Cremer fixed point $\hat{z}$ is approximated by small cycles.

Under the assumption that $\hat{z}$ is not a critical value, we have proved that there exists a critical value $v \neq \hat{z}$ such that every ray that accumulates at $v$ also accumulates at $\hat{z}$. It follows that $v$ is not accessible from $\mathbb{C} \setminus K_P$.

https://doi.org/10.1017/S0143385700000754 Published online by Cambridge University Press

In the case that $\hat{z}$ is a critical value we outline how to modify the arguments used before. Call $c_1, \ldots, c_k$ the critical points of $P$ contained in $U_{\mathcal{C}}(\hat{z})$. List without repetition their critical values $v_0 = \hat{z}, v_1, \ldots, v_l$. The proof also proceeds by contradiction. This time, we assume that $R_{t_0}$ lands at $v_0$ and that $v_i \in \overline{R_{t_i}}$ and $\hat{z} \notin \overline{R_{t_i}}$.

As before, consider the sets formed by the union of closed rays:

$$\mathcal{C}_0 = \mathcal{C} \cup \overline{R_{t_1}} \cup \cdots \cup \overline{R_{t_k}}, \quad \mathcal{C}_1 = P^{-1}(\mathcal{C}_0).$$

Now let $U_0$ (respectively $U_1$) be the unique component of the complement of $\mathcal{C}_0$ (respectively $\mathcal{C}_1$) such that $\hat{z}$ lies in the boundary of $U_0$ (respectively $U_1$). The same arguments used before show that $U_0$ contains at most finitely many cycles. Since the union of $U_0$ and a preimage of $\overline{R_{t_0}}$ is an open neighborhood of $\hat{z}$, it follows that $\hat{z}$ cannot be approximated by small cycles. □

THEOREM 1.1. *Let $P$ be a polynomial with a Cremer fixed point $\hat{z}$ that is approximated by small cycles. Then there exists a critical point which is not accessible from $\mathbb{C} \setminus J_P$.*

For the case in which $J_P$ is connected, Theorem 1.1 is an immediate consequence of Theorem 1.3. In the case that $J_P$ is disconnected the idea is to extract from $P$ a *polynomial-like map* which after *straightening* becomes a Cremer polynomial with connected Julia set. Then we can apply Theorem 1.3 to obtain a non-accessible critical point. Before going further we need some definitions and results about polynomial-like mappings (compare [**DH**]).

*Definition.* Let $D \subset \mathbb{C}$ and $D' \subset \mathbb{C}$ be bounded simply connected domains with smooth boundaries such that $\overline{D} \subset D'$. The triple $(f; D, D')$ is called a *polynomial-like map* of degree $d$, if the map $f : \overline{D} \to \overline{D'}$ is a $d$-fold branched covering ($d \geq 2$) which is holomorphic in $D$.

The *filled Julia set* $K_{(f;D,D')}$ of $(f; D, D')$ is the set of points in $D$ for which the forward iterates of $f$ are well defined:

$$K_{(f;D,D')} = \bigcap f^{-n}(\overline{D}).$$

The *Julia set* $J_{(f;D,D')}$ is the boundary of $K_{(f;D,D')}$.

Polynomial-like mappings can be extended to the complex plane in such a way that they are quasiconformally conjugate to a polynomial of the same degree.

THEOREM 4.1. (Douady and Hubbard) *If $(f; D, D')$ is a polynomial-like map of degree $d$, then there exists a quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$ and a polynomial $Q$ of degree $d$ such that $\phi \circ f = Q \circ \phi$ on $\overline{D}$. Moreover, $\phi(K_{(f;D,D')}) = K_Q$.*

Now we have all the ingredients to prove Theorem 1.1 except for the following.

LEMMA 4.2. *Let $P$ be a polynomial and $C$ a connected component of $K_P$ such that $P(C) = C$. If $C$ is not a repelling fixed point then there exists a polynomial-like map $(P; D, D')$ such that $C = K_{(P;D,D')}$.*

Let us defer the proof of this lemma.

https://doi.org/10.1017/S0143385700000754 Published online by Cambridge University Press

*Proof of Theorem 1.1.* Let $C$ be the connected component of $K_P$ that contains the Cremer fixed point $\hat{z}$. It follows that $P(C) = C$. Extract a polynomial-like map $(P; D, D')$ as in Lemma 4.2. After 'straightening' $(P; D, D')$ we obtain a polynomial $Q$ and a homeomorphism $\phi$ such that $\phi \circ f = Q \circ \phi$ on $\overline{D}$ and $K_Q = \phi(C)$ (Theorem 4.1). Notice that $K_Q$ is connected. Hence, $J_Q = \phi(\partial C)$ is connected.

Every neighborhood $U \subset D$ of $\hat{z}$ contains infinitely many cycles of $P$, hence every neighborhood of $\phi(\hat{z})$ contains infinitely many cycles of $Q$. Therefore, $\phi(\hat{z})$ is a Cremer fixed point of $Q$ that is approximated by small cycles. Now we are under the assumptions of Theorem 1.3 for connected Julia sets, let $c \in J_Q$ be a critical point of $Q$ that is not accessible from $\mathbb{C} - J_Q$.

It is not difficult to check that $\phi^{-1}(c)$ is critical point of $P$ that belongs to $\partial C \subset J_P$. Also, paths in $\mathbb{C} \setminus J_Q$ correspond under $\phi^{-1}$ to paths in $\mathbb{C} \setminus \partial C$. Thus, the critical point $\phi^{-1}(c)$ of $P$ is not accessible from $\mathbb{C} \setminus \partial C \supset \mathbb{C} \setminus J_P$. □

In order to prove Lemma 4.2 we need to work with the *Green function* $G$ of $K_P$:

$$G : \mathbb{C} \to \mathbb{R}_{\geq 0}$$
$$z \mapsto \lim_{n\to\infty} \frac{\log_+ |P^{\circ n}(z)|}{d^n}$$

where $\log_+ s = \max\{0, \log s\}$.

It can be shown that $G$ is continuous, vanishes in $K_P$ and $G \circ P = d \cdot G$. The Green function is harmonic in $\mathbb{C} \setminus K_P$, in particular differentiable. If $r > 0$ is a regular value of $G$, then each component of $\{z : G(z) \leq r\}$ is a closed topological disk with smooth boundary. A point $z \in \mathbb{C} \setminus K_P$ is a critical point of $G$ if and only if some forward iterate of $z$ is a critical point of $P$. Thus, $G$ has regular values arbitrarily close to zero (see [**M1**, §18]).

*Proof of Lemma 4.2.* Let us denote by $V(r)$ the connected component of $\{z : G(z) \leq r\}$ containing $C$. It is not difficult to check that

$$C = \bigcap V(r)$$

where the intersection is taken over all the regular values $r$ of $G$.

In particular, if a critical point $c$ of $P$ is such that $c \in V(r)$ for all regular values $r$ of $G$, then $c \in C$. It follows that there exists a regular value $r_0$ of $G$ such that all the critical points of $P$ contained in $V(r_0)$ belong to $C$.

Consider the topological disks with smooth boundary $D' = \operatorname{int} V(r_0)$ and $D = \operatorname{int} V(r_0/d)$. The proper holomorphic map $P : D \to D'$ is a $d$-fold branched covering. Under the assumption that $C$ is not a repelling fixed point, $P|_D$ is not a conformal isomorphism. Hence, $d \geq 2$ and $(P; D, D')$ is a polynomial-like map.

The critical points of the polynomial-like map $(P; D, D')$ are contained in $C$, therefore they never escape from $D$. It follows that the filled Julia set $K_{(P;D,D')}$ is connected and $C \subset K_{(P;D,D')} \subset K_P$; but $C$ is a connected component of $K_P$, so in fact $C = K_{(P;D,D')}$. □

COROLLARY 1.2. *The critical point of a quadratic polynomial $P$ with a Cremer periodic point $\hat{z}$ is not accessible from the complement of the Julia set.*

https://doi.org/10.1017/S0143385700000754 Published online by Cambridge University Press

The proof of Corollary 1.2 consists in showing that Cremer periodic points of quadratic polynomials are approximated by small cycles. This is obtained by putting together results of Douady, Hubbard and Yoccoz. According to Douady and Hubbard a quadratic polynomial $P$ with a Cremer periodic point (not fixed) is renormalizable. After an appropriate renormalization of $P$ we obtain a quadratic polynomial with a Cremer fixed point which, according to Yoccoz, is approximated by small cycles.

*Proof.* Let $n$ be the period of $\hat{z}$. It follows from the work of Douady and Hubbard [**DH**] that there exists a quadratic-like map $(P^{\circ n}, D, D')$ which after straightening becomes a quadratic polynomial $Q$ with a Cremer fixed point. More precisely, call $\phi$ the quasiconformal conjugacy between $(P^{\circ n}, D, D')$ and $Q$ (Theorem 4.1), then $\phi(\hat{z})$ is a Cremer fixed point of $Q$. According to Yoccoz [**Y**], $\phi(\hat{z})$ is approximated by small cycles of $Q$. Therefore, $\hat{z}$ is approximated by small cycles of $P^{\circ n}$. Now apply Theorem 1.1 to $P^{\circ n}$ and obtain a non-accessible critical point of $P^{\circ n}$. Since the critical points of $P^{\circ n}$ are preimages of the critical point of $P$ and accessibility is a completely invariant property, it follows that the critical point of $P$ is not accessible from the complement of the Julia set. □

Now we outline how to deduce from Corollary 3.4 the Douady–Shishikura inequality for a polynomial with disconnected Julia set.

THEOREM 4.3. (Douady and Shishikura) *If $P$ is a polynomial of degree $d$ then the total number of Cremer cycles and cycles of bounded Fatou components is less or equal than $d-1$.*

*Proof.* Consider a cycle of connected components $K_1, \ldots, K_p$ of the filled Julia set $K_P$. It is enough to show that the total number $n$ of Cremer cycles and cycles of bounded Fatou components in

$$K_1 \cup \cdots \cup K_p$$

is strictly less than the number of critical points of $P$ in $K_1 \cup \cdots \cup K_p$. Observe that $n$ is also the number of Cremer cycles and cycles of bounded Fatou components of $P^{\circ p}$ in $K_1$. By combining Corollary 3.4, Theorem 4.1 and Lemma 4.2 we conclude that $n$ is less or equal than the number of disjoint forward critical orbits of $P^{\circ p}$ in $K_1$. After observing that the number of disjoint critical orbits of $P^{\circ p}$ in $K_1$ coincides with the number of disjoint critical orbits of $P$ in $K_1 \cup \cdots \cup K_p$ we conclude that $n$ is less than or equal to the number of critical points of $P$ in $K_1 \cup \cdots \cup K_p$. □

*Acknowledgements.* I am grateful to John Milnor for many insightful and inspiring discussions. I would like to thank Misha Lyubich and Alfredo Poirier for many helpful conversations. This work has been available as a preprint since Spring 1995. Since then I have received many comments and suggestion. I am particularly grateful to Adrien Douady and Mitsu Shishikura for observing that our initial form of the proof of their inequality contained the proof of the slightly sharper statement given in Corollary 3.4. This work was partially supported by a *'Beca Presidente de la República' (Chile)*. This work was produced during the author's graduate studies at the Mathematics Department, State University of New York at Stony Brook.

https://doi.org/10.1017/S0143385700000754 Published online by Cambridge University Press

## REFERENCES


**[BL]** B. Bielefeld and M. Lyubich (editors). Problems in holomorphic dynamics. *Linear and Complex Analysis Problem Book 3*, Vol. III. Eds. V. P. Havin and N. K. Nikolskii. Springer 1994. (IMS Preprints 1992/7.)

**[Bl]** P. Blanchard. Complex analytic dynamics on the Riemann sphere. *Bull. Amer. Math. Soc.* **11** (1984), 85–141.

**[Cr]** H. Cremer. Zum Zentrumproblem. *Math. Ann.* **98** (1927), 151–163.

**[DH]** A. Douady and J. H. Hubbard. On the dynamics of polynomial-like mappings. *Ann. Sci. Ec. Norm. Sup. 4$^e$ série* **18** (1985), 287–343.

**[EL]** A. E. Eremenko and G. M. Levin. Periodic points of polynomials. *Ukrainian Math. J.* **1** (1990), 563–633.

**[GM]** L. Goldberg and J. Milnor. Fixed points of polynomial maps, part II. Fixed point portraits. *Ann. Sci. Ec. Norm. Sup. 4$^e$ série* **26** (1993), 51–98.

**[H]** J. H. Hubbard. Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz. *Topological Methods in Modern Mathematics*. Publish or Perish, 1993, pp. 467–511.

**[L]** M. Lyubich. The dynamics of rational transforms: the topological picture. *Russian Math. Surveys* **41**(4) (1986), 43–117.

**[M1]** J. Milnor. *Dynamics in One Complex Variable: Introductory Lectures.* Vieweg, 1999.

**[N]** M. Newman. *Elements of the Topology of Plane Sets*. Cambridge University Press, 1939.

**[PM1]** R. Pérez-Marco. Solution complete au probleme de Siegel de linearisation d'une application holomorphe au voisinage d'un point fixe (d'apres J.-C. Yoccoz). Sém. Bourbaki 753, February 1992. *Astérisque* **206** (1992), 273–310.

**[PM2]** R. Pérez-Marco. Topology of Julia sets and hedgehogs. *Preprint*, Université Paris-Sud, 1994.

**[PM3]** R. Pérez-Marco. Fixed points and circle maps. *Acta Math.* **179**(2) (1997), 243–294.

**[PM4]** R. Pérez-Marco. Sur une question de Dulac et Fatou. *C. R. Acad. Sci. Paris Sér. I Math.* **321**(8) (1995), 1045–1048.

**[Ru]** W. Rudin. *Real and Complex Analysis*. McGraw-Hill, 1987.

**[Si]** C. L. Siegel. Iteration of analytic functions. *Ann. Math.* **43** (1942), 607–612.

**[Sh]** M. Shishikura. On quasiconformal surgery of rational functions. *Ann. Sci. Ec. Norm. Sup. 4$^e$ série* **20** (1987), 1–29.

**[Sø]** D. E. K. Sørensen. Accumulation theorems for quadratic polynomials. *Ergod. Th. & Dynam. Sys.* **16**(3) (1996), 555–590.

**[S1]** D. Sullivan. Conformal dynamical systems. *Geometric Dynamics (Lecture Notes in Mathematics, 1007)*. Ed. J. Palis. Springer, 1983, pp. 725–752.

**[S2]** D. Sullivan. Quasiconformal homeomorphism and dynamics I, solution of the Fatou–Julia problem on wandering domains. *Ann. Math.* **122** (1985), 401–418.

**[Y]** J.-C. Yoccoz. Théorème de Siegel, nombres de Bruno et polynômes quadratiques. *Petits diviseurs en dimension 1*, *Astérisque* **231** (1995), 3–88.


https://doi.org/10.1017/S0143385700000754 Published online by Cambridge University Press