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# New classes of three-dimensional ideal-MHD equilibria

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Six ansatzes are investigated for their potential to allow three-dimensional (3D) ideal magnetohydrodynamic (MHD) equilibria. The ansatzes are based on a Clebsch representation for the magnetic field, $\mathbf{B} = \nabla H \times \nabla k$, and a 'generalized Clebsch representation', $\mathbf{B} = \nabla \times (\nabla K \times \nabla k)$, with $\nabla k$ being one of the coordinate directions of a cylindrical coordinate system. Three classes of equilibria, all with a straight magnetic axis, are obtained. Equilibria of the first class have a purely poloidal magnetic field of the Clebsch type with $k = z$ and include the 3D equilibria already known. Equilibria of the other two classes have a purely toroidal (i.e. here longitudinal) magnetic field and pressure surfaces that can be chosen such that poloidal sections are closed. The second class is based on a Clebsch representation with $k = \theta$. Solutions contain a free function of $\theta$ that determines the poloidal sections of the pressure surfaces at, say, $z = 0$. The behaviour in the toroidal direction is then fixed but not periodic. For the third class, the generalized Clebsch representation with $k = z$ is used. The equilibria are similar to those of the second class, with two important differences. They contain no free function and field lines are not planar. Finally, 3D vacuum fields, which exhibit 3D magnetic surfaces, are presented. They have the same geometry as the equilibria of the third class, and in fact can be obtained as a certain limit from these equilibria. Possible applications of the equilibria found are mentioned.

## 1. Introduction

The governing equations of ideal MHD equilibria are

$$\begin{aligned} \mathbf{j} \times \mathbf{B} &= \nabla P \,, \\ \mathbf{j} &= \nabla \times \mathbf{B} \,, \end{aligned} \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \,, \tag{2}$$

where $\mathbf{B}$, $\mathbf{j}$ and $P$ denote the magnetic field, current density and pressure respectively. Solutions of (1) and (2) have a broad range of applications in thermonuclear fusion research, geophysics, solar physics and astrophysics. In most cases symmetric equilibria furnish a correct model or at least a satisfactory approximation of the true situation; in many cases, however, a 3D model is desirable, and in some

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cases, e.g. for stellarator-type configurations in fusion research, 3D equilibria are absolutely necessary.

However, the construction of 3D equilibria is notoriously difficult. This is because (1) and (2) represent a system of partial differential equations of mixed elliptic–hyperbolic type, for which no general theory on the existence and uniqueness of solutions is available. It is not even clear what a well-posed initial–boundary-value problem in general looks like. Only if additional assumptions are made can the problem be rendered tractable. If equilibria are required to exhibit plane, axial or helical symmetry, (1) and (2) can be reduced to a single quasilinear elliptic equation, for which an elaborate existence theory as well as many explicit solutions are known (Morozov and Solovev 1966; Solovev and Shafranov 1970; Edenstrasser 1980*a*). These symmetries, furthermore, are known to be the only continuous symmetries that allow such a reduction (Edenstrasser 1980*b*).

On the other hand, non-symmetric magnetic fields in a bounded volume (e.g. a sphere or a torus) generically exhibit field line chaos connected with island formation and ergodic regions (Reiman and Boozer 1984; Cary and Kotschenreuter 1985; Zheligovsky 1993). As a consequence of the equilibrium equations (1), magnetic lines lie entirely in surfaces of constant pressure. Chaotic field lines therefore do not seem to be compatible with a smooth pressure function, and consequently, 3D fields are unlikely candidates for well-defined equilibria.

The subject has been more closely investigated for configurations of fusion interest. These have closed toroidal pressure surfaces nested around a single magnetic axis. The axisymmetric case is governed by the well-known Lüst–Schlüter–Grad–Shafranov equation (Lüst and Schlüter 1957; Grad and Rubin 1958; Shafranov 1957), which satisfactorily settles the existence problem. If axisymmetry is lost, however, only partial results are known. For example, if the magnetic axis lies in a plane, configurations with purely poloidal magnetic field (symmetric with respect to the plane) are virtually ruled out (Salat 1995). Other non-existence results refer to quite special configurations such as the 'isodynamic' (Palumbo 1984) or the 'quasi-helical' (Garren and Boozer 1991) stellarator. In the general 3D case with finite twist of field lines all we have is Grad's (1967) conjecture that well-defined equilibria do not exist. From a practical point of view, non-axisymmetric fusion devices like stellarators obviously tolerate a certain amount of fine-scaled field-line chaos. It is nevertheless of general interest whether Grad's non-existence conjecture holds true or not.

In spite of the negative expectations just mentioned, there are two classes of 3D equilibria without continuous symmetries. The first is a toroidal configuration with purely toroidal magnetic field that is mirror-symmetric with respect to a poloidal plane. The mirror symmetry prevents ergodization of field lines. The existence of such equilibria has been proved for low plasma pressure (Lortz 1970*a*). Note, however, that the pertinent pressure surfaces are in general not poloidally closed. Whether such equilibria having, additionally, closed toroidal pressure surfaces exist at all is not quite clear; only for configurations far from axisymmetry ('bumpy' configurations) is existence plausible (Lortz 1970*b*). The other class comprises equilibria with straight magnetic axis and field lines lying in planes orthogonal to the axis. Poloidal sections of the pressure surfaces exhibit mirror symmetry with respect to two, mutually orthogonal, planes and are either closed (Woolley 1977) or open (Shivamoggi 1986). In the closed case the ellipticity is an arbitrary function along the axis. In particular, a periodic function can be chosen, a case often re-

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ferred to as the 'topological torus'. Recently, the closed case was generalized so that mirror symmetry is lost (Salat and Kaiser 1995). To the best of our knowledge, this is the first example of a 3D equilibrium with the geometry of a (topological) torus without continuous or mirror symmetry.

As explained above, there is no general procedure available yet for constructing 3D equilibria, and in order to make progress, one has to fall back on certain ansatzes for the magnetic field. The aim of this paper is to explore such an ansatz thoroughly and with some degree of completeness. As a result, we recover the known straight-axis equilibria and also find two new classes of 3D equilibria. The ansatz is based on the poloidal/toroidal decomposition valid for any solenoidal vector field $\mathbf{B}$ (see e.g. Moffatt 1978):

$$\mathbf{B} = \nabla H \times \nabla k + \nabla \times (\nabla K \times \nabla k). \tag{3}$$

Here $H$ and $K$ are arbitrary scalar functions and $\nabla k$ is a particular (generalized) coordinate direction to be specified, which can be completed to a non-singular (generalized) coordinate system in the domain considered (Gerlich 1972). In most applications the decomposition (3) is used in Cartesian or spherical coordinates (Moffatt 1978); we have used it in cylindrical coordinates $(r, \theta, z)$ instead. More precisely, we investigate the six ansatzes

$$\mathbf{B} = \nabla H \times \nabla k, \qquad k = r, \theta, z, \tag{4}$$

and

$$\mathbf{B} = \nabla \times (\nabla K \times \nabla k), \qquad k = r, \theta, z. \tag{5}$$

Note that the terms 'poloidal' and 'toroidal' have different meanings in the decomposition (3) (there (4) is called a toroidal field and (5) a poloidal one) and in a torus (or cylinder) geometry. In order to avoid confusion, in the following (4) is called a Clebsch representation and (5) a generalized Clebsch representation.

It turned out that in four cases, namely for all Clebsch representations and for the generalized Clebsch representation with $k = z$, the problem can be reduced to a 2D elliptic equation, perpendicular to the particular direction specified, and certain nonlinear compatibility conditions. In three out of these four cases non-trivial solutions of the reduced problem, i.e. 3D equilibria, could be found. The first class is based on the Clebsch ansatz with $k = z$, and comprises the known equilibria (Woolley 1977; Shivamoggi 1986; Salat and Kaiser 1995). In this context, the extension of the open equilibria of Shivamoggi (1986) to the non-mirror-symmetric case is also discussed. The second class is based on the Clebsch ansatz with $k = \theta$. The magnetic field is purely toroidal and field lines lie on surfaces $\theta = \text{const}$. The solution contains a free function of $\theta$ that determines the poloidal sections of the pressure surfaces at, say, $z = 0$. In particular, poloidal sections can be chosen to be closed; the behaviour in the toroidal direction is then fixed but not periodic. The third class uses the generalized Clebsch ansatz with $k = z$. The magnetic field is again purely toroidal, but field lines are no longer planar. The solution does not contain a free function. The pressure surfaces again exhibit closed poloidal sections and are not periodic in the toroidal direction. Two points should be noted in connection with the third class of equilibria. First, it is an example of a 3D equilibrium with non-planar field lines. So, planar field lines are no prerequisite of 3D equilibria, as could be assumed from the examples so far known. Secondly, the equilibrium has a well-defined vacuum limit; the vacuum field is also 3D and has 3D magnetic surfaces.

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The first two classes of equilibria, based on the Clebsch ansatz, are discussed in Sec. 2. The third class, based on the generalized Clebsch ansatz, is discussed in Sec. 3. In Sec. 4 some conclusions are drawn, and possible applications of the new equilibria are mentioned. For the sake of completeness, the remaining ansatzes, which did not lead to 3D equilibria, are briefly dealt with in two appendices.

## 2. Clebsch representation

### *2.1. The ansatz* $\mathbf{B} = \nabla H \times \nabla z$

With the ansatz

$$\mathbf{B} = \nabla H \times \nabla z \tag{6}$$

for the magnetic field $\mathbf{B}$, the divergence constraint (2) is automatically satisfied and the equilibrium equations (1) reduce to

$$\left.\begin{aligned} (\Delta_{xy} H)\, \partial_x H + \partial_x P &= 0 \,, \\ (\Delta_{xy} H)\, \partial_y H + \partial_y P &= 0 \,, \end{aligned}\right\} \tag{7}$$

$$\partial_z [\, (\partial_x H)^2 + (\partial_y H)^2 + 2P \,] = 0 \,. \tag{8}$$

Here we work for convenience with Cartesian coordinates $(x, y, z)$, i.e. $H = H(x, y, z)$, $P = P(x, y, z)$, and $\Delta_{xy}$ denotes the 2D Laplacian in the $(x, y)$ plane,

$$\Delta_{xy} = \partial_{xx}^2 + \partial_{yy}^2 \,. \tag{9}$$

The integrability condition for $P$ (see (7)),

$$\partial_x \partial_y P - \partial_y \partial_x P = (\partial_x H)\, \partial_y \Delta_{xy} H - (\partial_y H)\, \partial_x \Delta_{xy} H = 0 \,, \tag{10}$$

requires that $\Delta_{xy} H$ satisfy locally the relation

$$\Delta_{xy} H = f(H, z) \tag{11}$$

with arbitrary profile function $f$. Equation (8) is solved by

$$P(x, y, z) = Q(x, y) - \tfrac{1}{2} [\, (\partial_x H)^2 + (\partial_y H)^2 \,] \,, \tag{12}$$

where $Q(x, y)$ is an arbitrary function. Equations (7) yield

$$\left.\begin{aligned} \partial_x Q &= (\partial_y H)\, \partial_{xy}^2 H - (\partial_x H)\, \partial_{yy}^2 H \,, \\ \partial_y Q &= (\partial_x H)\, \partial_{xy}^2 H - (\partial_y H)\, \partial_{xx}^2 H \,. \end{aligned}\right\} \tag{13}$$

In order to ensure that $Q$ does not depend on $z$, $H$ has to satisfy in addition the relations following from (13):

$$\left.\begin{aligned} \partial_z [\, (\partial_x H)\, \partial_{xy}^2 H - (\partial_y H)\, \partial_{xx}^2 H \,] &= 0 \,, \\ \partial_z [\, (\partial_y H)\, \partial_{xy}^2 H - (\partial_x H)\, \partial_{yy}^2 H \,] &= 0 \,. \end{aligned}\right\} \tag{14}$$

So, instead of solving (7) and (8) for $H$ and $P$ directly, one can first solve the 2D elliptic problem (11) with arbitrarily given function $f$ and then try to satisfy the compatibility conditions (14) for the $z$ dependence of $H$.

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Those few MHD equilibria known so far that depend on three space coordinates independently (Woolley 1977; Shivamoggi 1986; Salat and Kaiser 1995) not only fit into this scheme, with a linear ansatz for $f$,

$$f \;=\; a_0(z) \;+\; a_1(z)\, H \;, \tag{15}$$

with $a_0(z)$ and $a_1(z)$ being arbitrary, but follow naturally from it. In the next section further 3D solutions are obtained from a different ansatz but with the same scheme. Since (11) with $f$ given by (15) is linear, the general solution is in the form of a sum of an inhomogeneous part $H_i$ and a homogeneous part $H_h$. The homogeneous part has to satisfy

$$\partial^2_{xx} H_h \;+\; \partial^2_{yy} H_h \;=\; a_1\, H_h \;. \tag{16}$$

A product ansatz with respect to $x$ and $y$ is made, while the parametric dependence on $z$ is kept without restrictions:

$$H_h \;=\; \alpha(x,z)\,\beta(y,z) \;. \tag{17}$$

Equation (16) then yields the conditions

$$\left.\begin{aligned} \partial^2_{xx}\alpha \;=\; -\kappa^2(z)\,\alpha \;, \qquad \partial^2_{yy}\beta \;=\; -\lambda^2(z)\,\beta \;, \\ \kappa^2 \;+\; \lambda^2 \;=\; -a_1 \;, \end{aligned}\right\} \tag{18}$$

to which we return below. The general homogeneous solution is, eventually, an arbitrary superposition of solutions of the type (17).

In dealing with an inhomogeneous solution of (11) and (15), a distinction has to be made with regard to $a_1$. This leads to the consideration of two classes of solutions of the full system of equations (11), (14) and (15):

*2.1.1. Case I:* $a_1(z) = 0$. In this case an inhomogeneous solution of (11) and (15) is

$$H_i \;=\; a(z)\,x^2 \;+\; 2b(z)\,x\,y \;+\; c(z)\,y^2 \;, \tag{19}$$

where $a(z)$, $b(z)$ and $c(z)$ are arbitrary functions. (The relation $a + c = \frac{1}{2}a_0$, which follows from (11), does not restrict $a$ or $c$, since $a_0$ is arbitrary.) Inserting the sum of this $H_i$ and the most general solution $H_h$ from (17) and (18) into the compatibility conditions (14) produces rather unwieldy expressions. The simplest way out is to take $H_h = 0$, i.e.

$$H \;=\; H_i \;. \tag{20}$$

In this case (14) merely yield the condition

$$[a(z)c(z) - b^2(z)]' \;=\; 0 \;, \tag{21}$$

where the prime denotes differentiation with respect to the single argument involved. Of the three coefficients $a(z)$, $b(z)$ and $c(z)$ this leaves two of them free. If (12) for the pressure is combined with (13), (19) and (20), one obtains

$$P \;=\; P_0 \;-\; 2\,[a(z) \;+\; c(z)]\,[a(z)\,x^2 \;+\; 2b(z)\,x\,y \;+\; c(z)\,y^2] \;, \tag{22}$$

where $P_0$ = const is the (arbitrary) pressure on the axis $x = y = 0$. Provided that

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$ac-b^2 > 0$, the surfaces of constant pressure have concentric elliptical cross-section. At fixed $P$ the half-axis ratio of the ellipse and the orientation of its major axis with respect to the $x$ axis, for example, can be chosen as arbitrary functions along the axis. In the case $ac - b^2 < 0$ the cross-section consists of a pair of hyperbolas, thus leading to open pressure surfaces.

In cylindrical coordinates $(r, \theta)$, where $x = r\cos\theta,\ y = r\sin\theta$, the pressure assumes a particularly simple form, provided that $a(z),\ b(z)$ and $c(z)$ are written in terms of two arbitrary functions $u(z),\ v(z)$ and an arbitrary constant $c_1$, in the form

$$\left.\begin{aligned} a(z) &= \frac{-c_1}{(1-u^2)^{1/2}}\,[\,1 - u(z)\cos v(z)\,]\ , \\ b(z) &= \frac{c_1}{(1-u^2)^{1/2}}\ u(z)\sin v(z)\ , \\ c(z) &= \frac{-c_1}{(1-u^2)^{1/2}}\,[\,1 + u(z)\cos v(z)\,]\ , \end{aligned}\right\} \tag{23}$$

namely

$$P \;=\; P_0 \;-\; 4c_1^2\,\frac{r^2}{1-u^2(z)}\,\{\,1 \;-\; u(z)\,\cos[\,2\theta - v(z)\,]\,\}\ . \tag{24}$$

This form shows that $u(z)$ and $v(z)$ correspond to the deviation of the cross-section from a circle ($u = 0$) and to a helical twist respectively. For $u^2 < 1$, the cross-sections are ellipses. An example is shown in Fig. 1. The functions $u(z)$ and $v(z)$ were taken arbitrarily as $u(z) = 0.6+0.1\sin 2z$ and $v(z) = 0.2(z-3.5)$. The $z$ axis extends from $-1$ to 3.5. Two nested pressure surfaces with $\hat{P} \equiv (P-P_0)/4c_1^2 = -1.44$ (inner surface) and $-4$ (outer surface) are presented. (In the figure captions, for the sake of simplicity, hats are omitted.) The grid lines that are closed around the axis coincide with magnetic field lines. They are plane ellipses of constant ellipticity at fixed value of $z$.

An open pressure surface, requiring $c_1^2 < 0$, is shown in Fig. 2. It has $u(z) = -6 - 0.3\sin z,\ v = 0$ and $\hat{P} = 9$. The $z$ axis in the middle extends from $-14$ to $+14$.

The MHD equilibria with magnetic field $\mathbf{B}$ and pressure $P$, given by (6), (19), (20) and by (22) or (24) respectively, were obtained in Salat and Kaiser (1995), while the subcase $b(z) = 0$ (or $v(z) = 0$) was already discovered by Woolley (1977). Finally, a non-trivial harmonic polynomial for $H_h$ either does not give a physically distinct solution or does not solve (14), as discussed by Salat and Kaiser (1995).

*2.1.2. Case II:* $a_1(z) \neq 0$. In this case an inhomogeneous solution of (11) is

$$H_i \;=\; -\,a_0(z)/a_1(z)\ . \tag{25}$$

According to (6)–(8), neither the magnetic field nor the pressure is changed if an arbitrary function of $z$ is added to $H$. Nothing is lost, therefore, by putting $a_0 = H_i = 0$. Hence it follows from (17) that

$$H \;=\; \alpha(x,z)\,\beta(y,z)\ , \tag{26}$$

with $\alpha$ and $\beta$ subject to (18). A further restriction follows from the compatibility conditions (14), namely

$$(\partial_x\alpha)^2 \;=\; -\kappa^2\,\alpha^2\ , \qquad \left(\partial_y\beta\right)^2 \;=\; -\lambda^2\,\beta^2\ . \tag{27}$$

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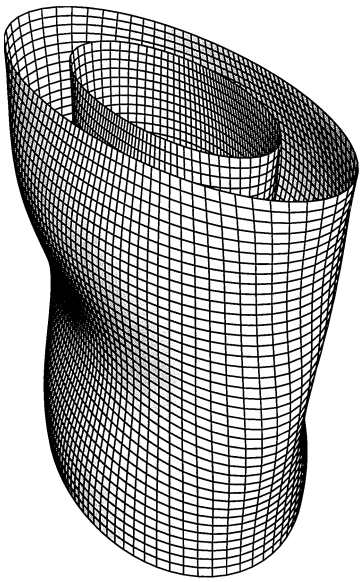


**Figure 1.** Nested pressure surfaces $P(r,\theta,z) = -r^2\{1 - u(z)\cos[2\theta - v(z)]\}/[1 - u^2(z)]$. Closed grid lines are magnetic field lines. Here $u(z) = 0.6 + 0.1\sin 2z$, $v(z) = 0.2(z - 3.5)$, $P = -1.44$ (inner surface) and $P = -4$ (outer surface).

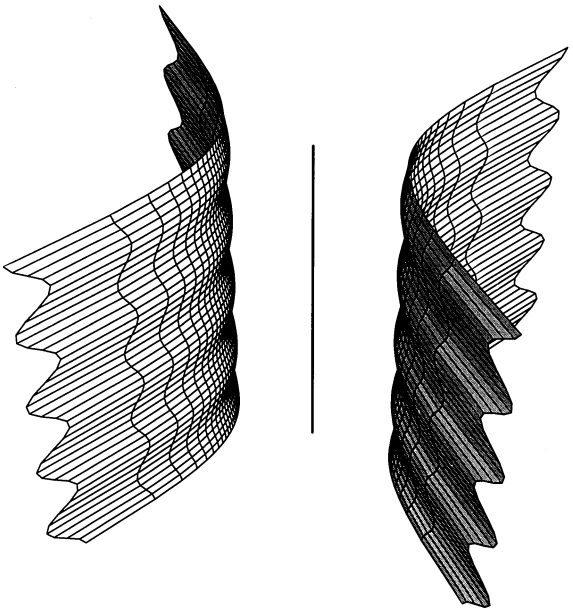


**Figure 2.** Two branches of the open pressure surface $P = 9$, with axis in between. Here $u(z) = -6 - 0.3\sin z$, $v(z) = 0$. $P$ is defined as in Fig. 1.

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For real-valued $H$, (18) and (27) are only compatible provided that $\kappa^2 < 0$, $\lambda^2 < 0$ and $\alpha = \alpha_0(z) \exp(\pm|\kappa(z)|x)$, with an analogous $\beta$ (disregarding solutions with $\kappa = 0$ or $\lambda = 0$, which are independent of $x$ or $y$ respectively). It turns out, however, that the pressure depends on $x$ and $y$ in the combination $|\kappa|\, x \pm |\lambda|\, y$, so that the cross-sections of the surfaces of constant pressure are not closed curves. Physically more relevant solutions are obtained if the ansatz (26) is replaced by an ansatz containing the sum of two less general product terms instead of one more general product term:

$$H = a(z)g(x) + c(z)h(y) \,. \tag{28}$$

In this case one obtains from (18) and (14)

$$g''(x) = -\kappa^2 g \,, \qquad h''(y) = -\kappa^2 h \,, \qquad [a(z)c(z)]' = 0 \,, \tag{29}$$

where $\kappa^2 = -a_1$ is now an arbitrary constant. For positive as well as negative $\kappa^2$, there exist solutions of (29) with closed poloidal contours of $P$ and with one free function of $z$, say $a(z)$; these solutions correspond to the second class in Woolley (1977). For $\kappa^2 > 0$, for example, pressure surfaces that are closed around the axis at $x = y = 0$ are obtained for $c(z) = 1/a(z)$ and

$$g(x) = \frac{c_1}{\kappa} \cos \kappa x \,, \qquad h(y) = \frac{c_1}{\kappa} \cos \kappa y \,, \tag{30}$$

where $c_1$ is an arbitrary constant. In this case

$$P = P_0 - \tfrac{1}{2} c_1^2 \left[a^2(z) \sin^2 \kappa x + 2(1 - \cos \kappa x \cos \kappa y) + c^2(z) \sin^2 \kappa y\right] . \tag{31}$$

An example is shown in Fig. 3, where $a(z) = 0.6 + 0.1 \sin 2z$ and $\kappa = 1$ is chosen. $\hat{P} \equiv 2(P - P_0)/c_1^2$ is $-1.96$ (inner surface) and $-3.8025$ (outer surface). The poloidally closed grid lines again coincide with magnetic field lines. It is evident that farther away from the axis they deviate from an elliptical shape. In fact, since the pressure surfaces repeat periodically in $x$ and $y$, the field lines tend towards a separatrix that has a curved roughly diamond-shaped form. The pressure $\hat{P}$ tends toward $\hat{P}_s = -(1 + a^2)^2/a^2$ there.

In the case $\kappa^2 < 0$ results analogous to those of (30) and (31) are obtained. The trigonometric functions are replaced by hyperbolic functions (Woolley, 1977). There again exist pressure surfaces that close around the axis.

### 2.2. *Ansatz* $\mathbf{B} = \nabla H \times \nabla \theta$

Here the following ansatz is made for the magnetic field:

$$\mathbf{B} = \nabla H \times \nabla \theta \,. \tag{32}$$

It is practical to switch from polar coordinates $(r, \theta, z)$ to modified polar coordinates $(\rho, \theta, z)$ where $\rho = \frac{1}{2} r^2$. We thus consider $H = H(\rho, \theta, z)$ and, similarly for the pressure, $P = P(\rho, \theta, z)$. The MHD equations take the form

$$\left.\begin{aligned} \frac{1}{2\rho} (\Delta_{\rho z} H)\, \partial_\rho H + \partial_\rho P &= 0 \,, \\ \frac{1}{2\rho} (\Delta_{\rho z} H)\, \partial_z H + \partial_z P &= 0 \,, \end{aligned}\right\} \tag{33}$$

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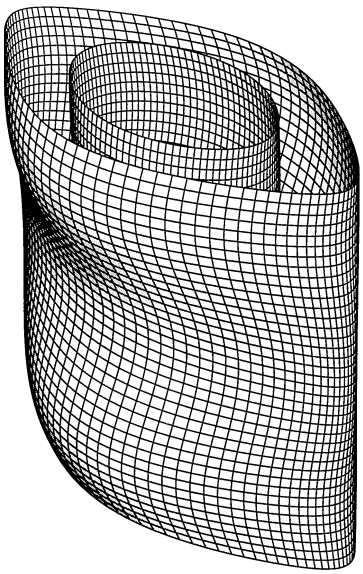


**Figure 3.** Nested pressure surfaces $P(x,y,z) = -[a^2(z)\sin^2 x + 2(1 - \cos x \cos y) + a^{-2}(z)\sin^2 y]$. Closed grid lines are magnetic field lines. Here $a(z) = 0.6 + 0.1\sin 2z$, $P = -1.96$ (inner surface) and $P = -3.8025$ (outer surface).

$$\partial_\theta \left[ (\partial_\rho H)^2 + \frac{1}{2\rho}(\partial_z H)^2 + 2P \right] = 0 , \tag{34}$$

where $\Delta_{\rho z}$ denotes the 2D Stokes operator in the $(\rho,z)$ plane,

$$\Delta_{\rho z} = 2\rho\,\partial^2_{\rho\rho} + \partial^2_{zz} . \tag{35}$$

The analysis of (33) and (34) proceeds analogously to that in the previous section. The integrability condition $\partial_\rho\partial_z P - \partial_z\partial_\rho P = 0$ for $P$ yields

$$\Delta_{\rho z} H = 2\rho\, f(H,\theta), \tag{36}$$

with arbitrary profile function $f$. Equation (34) is solved by

$$P(\rho,\theta,z) = Q(\rho,z) - \frac{1}{2}\left[ (\partial_\rho H)^2 + \frac{1}{2\rho}(\partial_z H)^2 \right] , \tag{37}$$

where $Q(\rho,z)$ is an arbitrary function. Equations (33) yield

$$\left.\begin{aligned} \partial_\rho Q &= \frac{1}{2\rho}\left[ (\partial_z H)\,\partial^2_{\rho z} H - (\partial_\rho H)\,\partial^2_{zz} H - \frac{1}{2\rho}(\partial_z H)^2 \right] , \\ \partial_z Q &= (\partial_\rho H)\,\partial^2_{\rho z} H - (\partial_z H)\,\partial^2_{\rho\rho} H . \end{aligned}\right\} \tag{38}$$

In order to ensure that $Q$ does not depend on $\theta$, $H$ has to satisfy the relations

$$\left.\begin{aligned} \partial_\theta \left[ (\partial_z H)\,\partial^2_{\rho z} H - (\partial_\rho H)\,\partial^2_{zz} H - \frac{1}{2\rho}(\partial_z H)^2 \right] &= 0 , \\ \partial_\theta [(\partial_\rho H)\,\partial^2_{\rho z} H - (\partial_z H)\,\partial^2_{\rho\rho} H] &= 0 . \end{aligned}\right\} \tag{39}$$

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By analogy with (15), the following linear ansatz for $f$ is made:

$$f \;=\; a_0(\theta) \;+\; a_1(\theta)\, H \;, \tag{40}$$

with arbitrary functions $a_0(\theta)$ and $a_1(\theta)$. Again, we seek solutions of (36), (40) as the sum of an inhomogeneous solution $H_i$ and homogeneous solutions $H_h$. The latter have to satisfy

$$2\rho\, \partial^2_{\rho\rho} H_h \;+\; \partial^2_{zz} H_h \;=\; 2\rho\, a_1\, H_h \;. \tag{41}$$

The product ansatz

$$H_h \;=\; \alpha(\rho,\theta)\, \beta(z,\theta) \tag{42}$$

yields the conditions

$$\partial^2_{\rho\rho}\alpha \;=\; -\kappa^2(\rho,\theta)\, \alpha \;, \qquad \partial^2_{zz}\beta \;=\; -\lambda^2(\theta)\, \beta \;. \tag{43}$$

While $\lambda(\theta)$ is arbitrary, the dependence of $\kappa$ on $\rho$ is fixed and is given by

$$2\rho\, \kappa^2 \;+\; \lambda^2 \;=\; -2\rho\, a_1 \;. \tag{44}$$

Postponing the treatment of (43), we again consider the two possible cases with respect to $a_1$.

*2.2.1. Case I:* $a_1(\theta) = 0$. In this case an inhomogeneous solution of (36) and (40) is

$$H_i \;=\; \tfrac{1}{2} a_0(\theta)\, \rho^2 \;. \tag{45}$$

In contrast to the previous section, here $H_i$ does not depend on all three coordinates. It is possible, however, to bring in the missing $z$ dependence via a simple solution of the homogeneous part. For $\lambda = 0$, $\kappa$ also vanishes and (43) are trivial to solve. The sum of $H_h$ and $H_i$ is then a polynomial of the form

$$H \;=\; \tfrac{1}{2} a_0(\theta)\, \rho^2 \;+\; a(\theta)\rho \;+\; b(\theta) z \;+\; c(\theta) z\rho \;, \tag{46}$$

where $a$, $b$ and $c$ are arbitrary functions of $\theta$. (An irrelevant additional term that depends on $\theta$ alone was omitted.) From the nonlinear compatibility conditions (39), one obtains

$$\left.\begin{aligned} \partial_\theta\, (\, b\, a_0 \;-\; a\, c \;-\; c^2\, z\,) \;&=\; 0 \;, \\ \partial_\theta\, (\, b^2 \;-\; c^2 \rho^2\,) \;&=\; 0 \;. \end{aligned}\right\} \tag{47}$$

This finally yields $b = \text{const}$, $c = \text{const}$ and

$$b\, a_0'(\theta) \;-\; c\, a'(\theta) \;=\; 0 \;. \tag{48}$$

Thus the solution in general contains one free function $a_0(\theta)$ or $a(\theta)$. In conformity with the switched roles of $z$ and $\theta$ in relation to the last section, the freedom in the present configuration refers to the angular dependence around the axis and not to the dependence along it.

From these results for $H$ and from (37) and (38), one obtains for the pressure

$$P \;=\; P_0 \;-\; \tfrac{1}{2} a^2(\theta) \;-\; a_0(\theta)\, [\, \tfrac{1}{2} a_0(\theta)\, \rho^2 \;+\; a(\theta)\rho \;+\; bz \;+\; cz\rho\,] \;, \tag{49}$$

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where $P_0$ is an arbitrary constant. The terms in square brackets are nothing but $H$, so that, with (32), $\mathbf{B}\cdot\nabla P = 0$ is confirmed. On switching back to the radial variable $r$, the magnetic field components $(B_r, B_\theta, B_z)$ are found to be

$$\left.\begin{aligned} B_r &= -\frac{b}{r} - \tfrac{1}{2}cr\,, \\ B_\theta &= 0\,, \\ B_z &= \tfrac{1}{2}a_0(\theta)\,r^2 + a(\theta) + cz\,. \end{aligned}\right\} \tag{50}$$

In order to make the magnetic field non-singular on the axis, it is necessary to have $b = 0$. According to (48), this implies $a(\theta) = \text{const}$. As a consequence, the pressure on the axis $\rho \equiv \frac{1}{2}r^2 = 0$ is independent of the polar angle $\theta$, as it should be. $a_0(\theta)$ is the remaining free function of the solution. The components of the current density $\mathbf{j}$ are

$$j_r = \tfrac{1}{2}ra_0'(\theta)\,, \qquad j_\theta = -ra_0(\theta)\,, \qquad j_z = 0\,. \tag{51}$$

The surfaces of constant pressure are most easily discussed by solving (49) for $\hat{z} \equiv a + cz$. With $P_{00} = P_0 - \frac{1}{2}a^2$ being the pressure on the $z$ axis, one has

$$\hat{z}(r,\theta) = -\frac{1}{4a_0(\theta)}\left[8\frac{P - P_{00}}{r^2} + a_0^2(\theta)r^2\right]\,. \tag{52}$$

If the pressure decreases away from the axis, i.e. $P < P_{00}$, then $\hat{z}(r)$ at fixed $\theta$ is a monotonic function of $r$, ranging from $-\infty$ to $\infty$. These limits are reached for $r \to 0$ or $r \to \infty$ respectively (depending on the sign of $a_0$). At fixed $z = \text{const}$ the distance $r$ to the axis is single-valued, and for periodic $a_0(\theta)$ the curves $r(\theta)$ are closed. Thus the surfaces $P = \text{const}$ wrap around the axis and are more and more radially compressed in one $z$ direction and blow up in the opposite direction. This feature is rather unphysical globally. With proper boundary conditions at two values of $z$, however, the solution could be physically relevant for the region in between.

An example of this configuration is given in Fig. 4. The pressure $\hat{P} \equiv 8(P - P_{00})$ on the two surfaces shown, nested around the $z$ axis, has the values $-0.4$ (inner surface) and $-0.9$ (outer surface). The free function $a_0(\theta)$ is chosen as $a_0(\theta) = -1 + 0.1\cos\theta + 0.1\cos 2\theta$. $\hat{z}$ extends from $-2.6$ to $0.6$. The longitudinal grid lines correspond to constant values of the poloidal angle $\theta$, and thus coincide with magnetic field lines.

If the pressure increases away from the axis, $\hat{z}(r)$ is no longer monotonic. Starting from small values of $r$, $|\hat{z}(r)|$ first decreases and then increases again as $r$ goes to $\infty$. In other words, the pressure surfaces start close to the axis, then increase their distance from it until at some point $z = z_0$ they turn around and, still increasing in distance from the axis, continue back in the direction from which they started. This gives a kind of arcade-like surface with a specific 'head' or turnaround region. Figure 5 shows an example of such a configuration. It has $\hat{P} = 0.8$ and $a_0(\theta) = 1 + 0.2\cos\theta + 0.2\cos 2\theta$. The distance to the axis extends from 0.38 to 2.1. Again, the longitudinal grid lines correspond to magnetic field lines.

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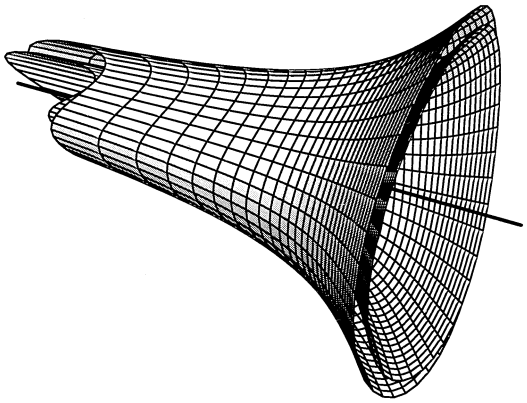


**Figure 4.** Nested pressure surfaces $P(r,\theta,z) = -r^2 a_0(\theta)[r^2 a_0(\theta) + 4z]$, with $z$ axis. Longitudinal grid lines are magnetic field lines. Here $a_0(\theta) = -1 + 0.1\cos\theta + 0.1\cos 2\theta$, $P = -0.4$ (inner surface) and $P = -0.9$ (outer surface).

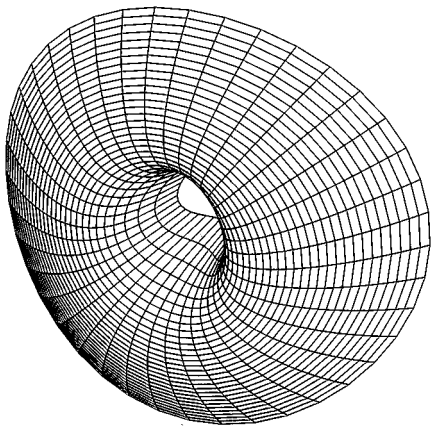


**Figure 5.** Pressure surface $P = 0{\cdot}8$ with arcade geometry. Here $a_0(\theta) = 1 + 0.2\cos\theta + 0.2\cos 2\theta$. $P$ is defined as in Fig. 4.

It remains to discuss the case $\lambda(\theta) \neq 0$. Here the regular solution of (43) is

$$\left.\begin{aligned} \alpha(\rho,\theta) &= \rho^{1/2} I_1(\lambda(\theta)(2\rho)^{1/2}) \,, \\ \beta(z,\theta) &= c_1(\theta)\,\sin[\lambda(\theta)\,z] \;+\; c_2(\theta)\,\cos[\lambda(\theta)\,z] \,, \end{aligned}\right\} \tag{53}$$

for $\lambda^2 > 0$. The coefficients $c_1(\theta)$ and $c_2(\theta)$ are arbitrary functions, and $I_1$ is a modified Bessel function. For $\lambda^2 < 0$, $\lambda$ and $I_1$ of (53) are replaced by $|\lambda|$ and $J_1$ respectively. Also, the trigonometric functions are replaced by hyperbolic ones. $H(\rho,\theta,z)$ is the sum of $H_h$, (42), and $H_i$, (45). It turns out, however, that the compatibility conditions (39) can only be satisfied provided that $a_0(\theta)$, $\lambda(\theta)$, $c_1(\theta)$ and $c_2(\theta)$ are constant. This implies an axisymmetric and hence, in the present context, irrelevant solution.

*2.2.2. Case II:* $a_1(z) \neq 0$. In this case an inhomogeneous solution of (36) is $H_i = -a_0(\theta)/a_1(\theta)$. Again, as in the last section, nothing is lost if $H_i = H_i(\theta)$ is omitted

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by putting $a_0(\theta) = 0$. Thus $H = H_h$ is given by the ansatz (42) or, as will again prove useful, as the sum of two such product terms. A case for which (39) can readily be analyzed is $\lambda = 0$. It follows from (43) and (44) that

$$\partial^2_{\rho\rho}\alpha \;=\; a_1(\theta)\,\alpha\;, \tag{54}$$

and $\beta(\theta, z) = b(\theta) + c(\theta)z$, where $b$ and $c$ are arbitrary functions of $\theta$. The equations (39), however, can only be satisfied provided that $H$ (and consequently the total configuration) is independent of $\theta$ or $z$. A three-dimensional solution, however, can in fact be found from the two-term ansatz

$$H \;=\; \beta_1(\theta, z)\,\sin(\kappa\rho) \;+\; \beta_2(\theta, z)\,\cos(\kappa\rho)\;, \tag{55}$$

where $\kappa = (-a_1)^{1/2}$, for $a_1 < 0$, and

$$H \;=\; \beta_1(\theta, z)\,\sinh(|\kappa|\rho) \;+\; \beta_2(\theta, z)\,\cosh(|\kappa|\rho)\;, \tag{56}$$

for $a_1 > 0$. Here

$$\beta_i(\theta, z) \;=\; b_i(\theta) \;+\; c_i z\;, \qquad i \;=\; 1,\; 2\;. \tag{57}$$

The functions $b_1$, $b_2$, $c_1$ and $c_2$ are arbitrary. From (39) and (55), one obtains

$$\left.\begin{aligned} &\partial_\theta\{\,(\,c_1^2 - c_2^2 + 4c_1c_2\kappa\rho\,)\cos 2\kappa\rho \;+\; 2[\,(c_1^2 - c_2^2)\kappa\rho - 2c_1c_2\,]\sin 2\kappa\rho \\ &\qquad -\,(c_1^2 + c_2^2)\,\}\;=\;0\;, \\ &\partial_\theta\{\,a_1\,[\,b_1c_1 + b_2c_2 + (c_1^2 + c_2^2)\,z\,]\,\}\;=\;0\;. \end{aligned}\right\} \tag{58}$$

As a consequence, $\kappa$, $a_1$ and $c_1$, $c_2$ have to be constant, and $b_1(\theta)$ and $b_2(\theta)$ are related by

$$c_1\,b_1'(\theta) \;+\; c_2\,b_2'(\theta) \;=\; 0\;. \tag{59}$$

The pressure is found from (37) and (38) to be

$$P \;=\; P_0 \;+\; \tfrac{1}{2}\kappa^2\,\left[\,(\,\beta_1\sin\kappa\rho + \beta_2\cos\kappa\rho\,)^2 - b_1^2 - b_2^2\,\right]\;, \tag{60}$$

where $P_0$ is an arbitrary constant. Since $P - P_0$ is proportional to $H^2 - b_1^2(\theta) - b_2^2(\theta)$, the condition $\mathbf{B}\cdot\nabla P = 0$, using (32), is again confirmed. The magnetic field components are

$$\left.\begin{aligned} B_r \;&=\; -\,\frac{1}{r}(\,c_1\sin\Phi \;+\; c_2\cos\Phi\,)\;, \\ B_\theta \;&=\; 0\;, \\ B_z \;&=\; \kappa\;(\,\beta_1\cos\Phi \;-\; \beta_2\sin\Phi\,)\;, \end{aligned}\right\} \tag{61}$$

where $\Phi(r) = \frac{1}{2}\kappa r^2$. The current density components are

$$\left.\begin{aligned} j_r \;&=\; \frac{\kappa}{r}(\,b_1'\cos\Phi \;-\; b_2'\sin\Phi\,)\;, \\ j_\theta \;&=\; \kappa^2 r\,(\,\beta_1\sin\Phi \;+\; \beta_2\cos\Phi\,)\;, \\ j_z \;&=\; 0\;. \end{aligned}\right\} \tag{62}$$

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The field is non-singular on the axis for $c_2 = 0$. In this case (59) implies $b_1'(\theta) = 0$. The remaining free function is $b_2(\theta)$. The pressure on the axis is then $P_{00} = P_0 - \frac{1}{2}\kappa^2 b_1^2$.

For $a_1 > 0$, the plus sign in (59) changes into a minus sign, and the pressure assumes the form

$$P = P_0 - \tfrac{1}{2}\kappa^2 [\, (\beta_1 \sinh |\kappa|\rho + \beta_2 \cosh |\kappa|\rho\,)^2 - b_1^2 + b_2^2 \,] \,. \tag{63}$$

The magnetic field is analogous to (61), with the minus sign in $B_z$ changed into a plus sign, and $\kappa$ replaced by $|\kappa|$.

The surfaces of constant pressure $z = z(r, \theta)$ are easily discussed in terms of $Z \equiv (b_1 + c_1 z)/b_2(\theta)$. From (60) and $c_2 = 0$, one obtains

$$Z = \frac{U - \cos\Phi}{\sin\Phi} \,, \tag{64}$$

where

$$U(\theta) = \left[1 + 2\,\frac{P - P_{00}}{\kappa^2 b_2^2(\theta)}\right]^{1/2} \,. \tag{65}$$

Similarly to the situation in (31), the periodicity of the trigonometric functions brings about an infinity of solution branches. Considering the branch that comes close to the axis, $\Phi \to 0$, it is evident from $\cos^2\Phi \leqslant 1$ that, for fixed $\theta$, the solution extends from $z = -\infty$ to $+\infty$ provided that $0 < |U| < 1$. These limits are reached for $r \to 0$ and $r \to (2\pi/\kappa)^{1/2}$ respectively (depending on the sign of $U - 1$). At fixed $z$ = const the distance $r$ to the axis is single-valued within the branch, and for periodic $b_2(\theta)$ the curves $r(\theta)$ are closed around the $z$ axis. Thus the surfaces $P$ = const wrap around the axis. They are more and more radially compressed in one $z$ direction, but reach a constant extent in the opposite direction.

An example of this configuration is given in Fig. 6. The normalized pressure $\hat{P} \equiv 2(P - P_{00})/(\kappa b_1)^2$ on the two surfaces shown has the values $-0.1$ and $-0.5$ on the inner and the outer surfaces respectively. The normalized free function $\hat{b}_2(\theta) \equiv b_2(\theta)/b_1$ is chosen as $\hat{b}_2(\theta) = 1 + 0.15\cos\theta + 0.1\cos 2\theta$, and $\kappa = 2^{1/2}$ is assumed. $\hat{z} = 1 + c_1 z/b_1$, on the axis, extends from $-2.6$ to 1.3. Again, the longitudinal grid lines represent magnetic field lines.

For $|U| > 1$, there is no sign change of $Z$ in the $r$ interval quoted. $|Z|$, as a function of $r$, first decreases and then increases again as $r$ goes from small values to its final value. This gives an arcade-like surface, as is also encountered in case I solutions. Figure 7 shows an example of this configuration. It corresponds to the choices $\hat{P} = 0.2$, $\kappa = 2^{1/2}$ and $\hat{b}_2(\theta) = 1 + 0.15\cos\theta + 0.1\sin 2\theta$. The region $r = 0.3$ to 1.6 is shown.

For $a_1 > 0$, the trigonometric functions in (64) are replaced by hyperbolic ones. The pressure surfaces, instead of approaching the axis, blow up towards $r \to \infty$ at a finite value of $z$. An example is shown in Fig. 8. It corresponds to $\hat{P} = 0.2$, $\kappa = 2^{1/2}$ and $\hat{b}_2(\theta) = 1 + 0.15\cos\theta + 0.05\cos 3\theta$. The region shown extends from $r = 0.3$ to 2.6.

It remains to consider the postponed case $\lambda(\theta) \neq 0$ as well. $H(\rho, \theta, z)$ assumes the form

$$H = \alpha_1(\rho, \theta)\, \sin[\lambda(\theta)\, z] + \alpha_2(\rho, \theta)\, \cos[\lambda(\theta)\, z] \,, \tag{66}$$

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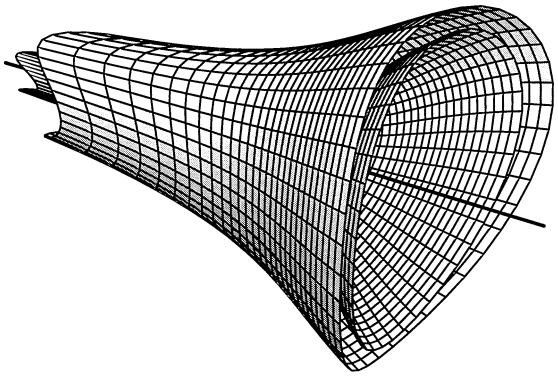


**Figure 6.** Nested pressure surfaces $P(r,\theta,z) = [z\sin r^2 + b_2(\theta)\cos r^2]^2 - b_2^2(\theta)$, with $z$ axis. Longitudinal grid lines are magnetic field lines. Here $b_2(\theta) = 1 + 0.15\cos\theta + 0.1\cos 2\theta$, $P = -0.1$ (inner surface) and $P = -0.5$ (outer surface).

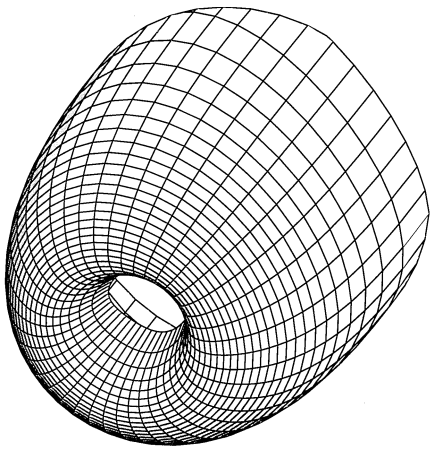


**Figure 7.** Pressure surface $P = 0.2$ with arcade geometry. Here $b_2(\theta) = 1 + 0.15\cos\theta + 0.1\sin 2\theta$. $P$ is defined as in Fig. 6.

where $\alpha_{1,2}(\rho,\theta)$, from (43) and (44), are confluent hypergeometric functions of $\rho$, with $\theta$ as a parameter. The analysis of (39), focussed in particular on terms with secular $z$ dependence, shows that only axisymmetric solutions, i.e. $\partial_\theta(\alpha_1,\ \alpha_2,\ \lambda) = 0$, survive the compatibility conditions, just as with the Bessel function analogue of case I.

## 3. Generalized Clebsch representation

In this section the ansatz

$$\mathbf{B} = \nabla \times (\nabla K \times \nabla z) \tag{67}$$

is investigated. We are again working in Cartesian coordinates $(x,y,z)$, i.e. $K = K(x,y,z)$ and $P = P(x,y,z)$, and $\Delta$ and $\Delta_{xy}$ denote the 3D Laplacian and the 2D Laplacian in the $(x,y)$ plane respectively. The Cartesian components of the

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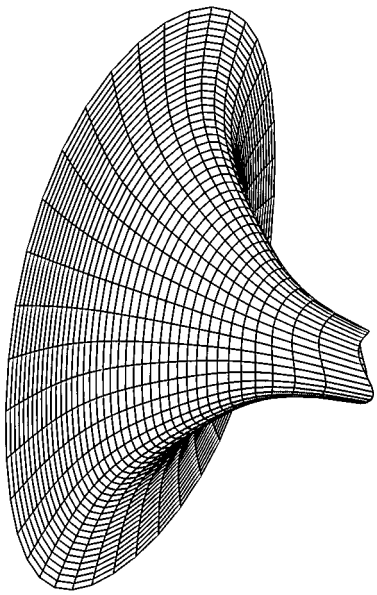


**Figure 8.** Pressure surface $P(r,\theta,z) = [z\sinh r^2 + b_2(\theta)\cosh r^2]^2 - b_2^2(\theta)$. Here $b_2(\theta) = 1 + 0.15\cos\theta + 0.05\cos 3\theta$ and $P = 0.2$. Field lines tend towards $r = \infty$ at finite $z$.

magnetic field $\mathbf{B}$ and current density $\mathbf{j}$ read

$$\mathbf{B} = (\partial^2_{xz}K,\ \partial^2_{yz}K,\ -\Delta_{xy}K)^{\mathrm{T}}, \tag{68}$$

$$\mathbf{j} = (-\Delta\partial_y K,\ \Delta\partial_x K,\ 0)^{\mathrm{T}}. \tag{69}$$

The divergence constraint (2) is, of course, satisfied by the ansatz and the equilibrium equations (1) reduce to

$$\left.\begin{aligned}(\Delta_{xy}K)\partial_x(\Delta_{xy}K + \partial^2_{zz}K) + \partial_x P = 0,\\ (\Delta_{xy}K)\partial_y(\Delta_{xy}K + \partial^2_{zz}K) + \partial_y P = 0,\end{aligned}\right\} \tag{70}$$

$$(\partial^2_{xz}K)\partial_x(\Delta_{xy}K + \partial^2_{zz}K) + (\partial^2_{yz}K)\partial_y(\Delta_{xy}K + \partial^2_{zz}K) + \partial_z P = 0. \tag{71}$$

The integrability condition $\partial_x\partial_y P - \partial_y\partial_x P = 0$ for the pressure $P$ together with (70) furnish, similarly to the Clebsch cases in the last section, a relation between $\Delta_{xy}K$ and $\partial^2_{zz}K$ with an arbitrary profile function $f$:

$$\Delta_{xy}K = f(\partial^2_{zz}K, z). \tag{72}$$

Here $\partial^2_{zz}K \not\equiv 0$ is assumed, the case $\partial^2_{zz}K \equiv 0$ being dealt with later. Using (72), one can integrate (70) and obtain

$$f^2(\partial^2_{zz}K, z) + 2\,F(\partial^2_{zz}K, z) + 2\,P = C(z), \tag{73}$$

with $C(z)$ being an arbitrary function of $z$, and $F$ an integral function of $f$,

$$F(u,z) := \int_0^u f(\tilde{u}, z)\,d\tilde{u}. \tag{74}$$

Equation (71) can be put in the form

$$(1 + \partial_u f|_{u=\partial^2_{zz}K})\partial_z[(\partial^2_{xz}K)^2 + (\partial^2_{yz}K)^2] + 2\,\partial_z P = 0. \tag{75}$$

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Differentiating (73) with respect to $z$, one can eliminate $\partial_z P$ from (73) and (75):

$$(1+\partial_u f|_{u=\partial^2_{zz}K})\partial_z[(\partial^2_{xz}K)^2+(\partial^2_{yz}K)^2]-\partial_z(f^2+2F)=-C'. \tag{76}$$

Analogously to the last section, (70) and (71) are replaced by (72) and (76), which determine the functions $C$ and $K$ once the profile function $f$ is given. The pressure $P$ is then given by (73).

In order to proceed, a linear profile function

$$f=a_0(z)+a_1(z)\,\partial^2_{zz}K \tag{77}$$

is chosen once more. In the case $a_1=0$, (72) is again a 2D elliptic equation, and (76) can be integrated once with respect to $z$:

$$(\partial^2_{xz}K)^2+(\partial^2_{yz}K)^2-2\,a_0(z)\,\partial^2_{zz}K=\bar{C}(z)+Q(x,y). \tag{78}$$

Here $\bar{C}(z):=a_0^2-C$ and $Q(x,y)$ are arbitrary functions. The pressure function is now given by

$$P=-a_0(z)\,\partial^2_{zz}K-\tfrac{1}{2}\bar{C}(z). \tag{79}$$

The general solution of (72) has the form

$$K=a(z)\,x^2+c(z)\,y^2+K_{\rm harm}(x,y;z), \tag{80}$$

with $2(a+c)=a_0$ and $K_{\rm harm}$ being a harmonic function in $x$ and $y$ that, additionally, may depend on $z$. For $K_{\rm harm}$, second- and some higher-order harmonic polynomials in $x$ and $y$ with $z$-dependent coefficients have been tried. However, no 3D solutions satisfying (78) with poloidally closed pressure surfaces have been found.

In the case $a_1\neq 0$ the coefficient $a_0$ can again be made zero; furthermore, $a_1$ is assumed to be independent of $z$ and $\neq -1$. In that case (76) can again be integrated,

$$(\partial^2_{xz}K)^2+(\partial^2_{yz}K)^2-a_1\,(\partial^2_{zz}K)^2=\tilde{C}(z)+Q(x,y), \tag{81}$$

and the pressure function $P$ takes the form

$$P=-\tfrac{1}{2}(1+a_1)[a_1\,(\partial^2_{zz}K)^2+\tilde{C}(z)], \tag{82}$$

where $\tilde{C}(z)=-C(z)/(1+a_1)$.

The ansatz for $K$,

$$K=a(z)\,g(x)+c(z)\,h(y), \tag{83}$$

is now found to be successful. Equation (72) furnishes the equations

$$\left.\begin{aligned}\frac{g''(x)}{g}=a_1\frac{a''(z)}{a}&=-\kappa^2,\\ \frac{h''(y)}{h}=a_1\frac{c''(z)}{c}&=-\lambda^2,\end{aligned}\right\} \tag{84}$$

which for $a_1<0$, for example, are solved by

$$\left.\begin{aligned}g(x)=g_0\cos\kappa x,\qquad a(z)=\exp(\bar{\kappa}z),\qquad a_1\,\bar{\kappa}^2&=-\kappa^2,\\ h(y)=h_0\cos\lambda y,\qquad c(z)=\exp(-\bar{\lambda}z),\qquad a_1\,\bar{\lambda}^2&=-\lambda^2.\end{aligned}\right\} \tag{85}$$

The condition (81) is satisfied if $\bar{\kappa}=\bar{\lambda}$ (and hence $\kappa=\pm\lambda$) and if one takes

$$\left.\begin{aligned}\tilde{C}(z):=\kappa^2\bar{\kappa}^2\left[g_0^2\exp(2\bar{\kappa}z)+h_0^2\exp(-2\bar{\kappa}z)\right]+\tilde{C}_0,\\ Q(x,y):=2\,\kappa^2\bar{\kappa}^2\,g_0h_0\cos\kappa x\cos\kappa y-\tilde{C}_0,\end{aligned}\right\} \tag{86}$$

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where $\tilde{C}_0$ is an arbitrary constant. Inserting (85) in (83), one obtains

$$K = g_0 \exp(\bar{\kappa} z) \cos \kappa x + h_0 \exp(-\bar{\kappa} z) \cos \kappa y, \tag{87}$$

and the pressure takes the form

$$\begin{aligned} P &= \tfrac{1}{2}(1 + a_1) \left( \kappa^2 \bar{\kappa}^2 K^2 - \tilde{C} \right) \\ &= P_0 - \tfrac{1}{2} \kappa^2 (\bar{\kappa}^2 - \kappa^2) [g_0^2 \exp(2\bar{\kappa} z) \sin^2 \kappa x + h_0^2 \exp(-2\bar{\kappa} z) \sin^2 \kappa y \\ &\qquad + 2 g_0 h_0 (1 - \cos \kappa x \cos \kappa y)], \end{aligned} \tag{88}$$

where $P_0 = (\bar{\kappa}^2 - \kappa^2)(g_0 h_0 \kappa^2 - \tilde{C}_0 / 2\bar{\kappa}^2)$ is the pressure on the axis.

For $|\kappa x|, |\kappa y| \ll 1$ the cross-sections $z = \text{const}$ of the pressure surfaces are elliptical, provided that $g_0 h_0 > 0$; otherwise, they are hyperbolical. As $z \to \pm\infty$, they shrink to a vertical line on one side and a horizontal line on the other.

The magnetic field components are

$$\left.\begin{aligned} B_x &= -\kappa \bar{\kappa} g_0 \exp(\bar{\kappa} z) \sin \kappa x, \\ B_y &= \kappa \bar{\kappa} h_0 \exp(-\bar{\kappa} z) \sin \kappa y, \\ B_z &= \kappa^2 \left[ g_0 \exp(\bar{\kappa} z) \cos \kappa x + h_0 \exp(-\bar{\kappa} z) \cos \kappa y \right], \end{aligned}\right\} \tag{89}$$

while the components of the current density are

$$\left.\begin{aligned} j_x &= \kappa (\bar{\kappa}^2 - \kappa^2) h_0 \exp(-\bar{\kappa} z) \sin \kappa y, \\ j_y &= -\kappa (\bar{\kappa}^2 - \kappa^2) g_0 \exp(\bar{\kappa} z) \sin \kappa x, \\ j_z &= 0. \end{aligned}\right\} \tag{90}$$

Similarly to the configuration in (61) and (62), the current is purely poloidal. The magnetic field lines, however, are no longer plane. This follows from, for example, the integration of the field-line equations $dx/B_x = dy/B_y = dz/B_z$, which, for $|\bar{\kappa} z| \gg 1$, can be done analytically. As $\bar{\kappa} z \to \infty$ and $\bar{\kappa} z \to -\infty$ the field lines approach $x \to 0$, $y \to \text{const}$ and $y \to 0$, $x \to \text{const}$ respectively. The connection between these asymptotic values requires a field-line twist.

Figure 9 shows two nested pressure surfaces. Also shown as thicker lines are three field lines: two at the symmetry positions $x = 0$ and $y = 0$ and one in between. Note that the 'longitudinal' grid lines in *this* figure are generated by some auxiliary construction, and do not correspond to constant polar angles in the $(x,y)$ plane or to field lines. Figure 9 is drawn with $g_0 = h_0 = 1$, $\kappa = 1$, $\bar{\kappa} = 0.5$ and $z \in [-2.5, 2.5]$. The normalized pressure $\hat{P} = 2(P - P_0)/[\kappa^2(\bar{\kappa}^2 - \kappa^2)]$ has the values $\hat{P} = -0.2$ and $-0.7$.

The case $a_1 = -1$ has been excluded so far. In fact, (87) with $\kappa = \bar{\kappa}$ describes a 3D vacuum field, which exhibits, moreover, 3D magnetic surfaces $S(x, y, z) = \text{const}$, i.e.

$$\mathbf{B} \cdot \nabla S = 0, \tag{91}$$

where $S$ is given by

$$\begin{aligned} S &= \kappa^4 [K^2 - g_0^2 \exp(2\kappa z) - h_0^2 \exp(-2\kappa z)] \\ &= g_0^2 \exp(2\bar{\kappa} z) \sin^2 \kappa x + h_0^2 \exp(-2\bar{\kappa} z) \sin^2 \kappa y \\ &\quad + 2 g_0 h_0 (1 - \cos \kappa x \cos \kappa y). \end{aligned} \tag{92}$$

Analogously to the equilibrium problem, 3D vacuum fields generically show field-

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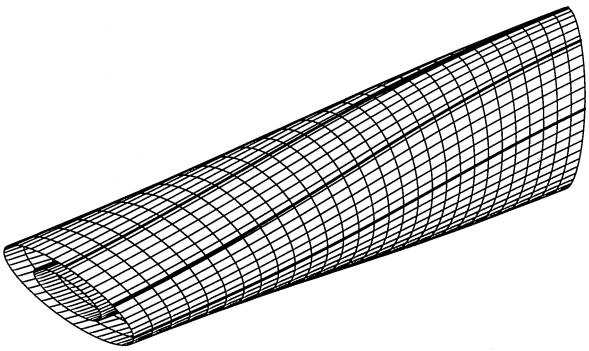


**Figure 9.** Nested pressure surfaces $P(x, y, z) = -[\exp(z)\,\sin^2 x + 2\,(1 - \cos x \cos y) + \exp(-z)\sin^2 y]$ with three magnetic field lines (thick). The middle one has finite torsion. Here $P = -0.2$ (inner surface) and $P = -0.7$ (outer surface).

line chaos, which prevents the formation of magnetic surfaces. 3D vacuum fields with magnetic surfaces are therefore exceptions as rare as 3D equilibria. The vacuum field can furthermore be obtained as a continuous limit ($a_1 \to -1$) of a family of equilibria, a limit that is otherwise often singular (Grad 1984, 1985).

It remains to consider the case $\partial^2_{zz} K \equiv 0$. In that case (70) can immediately be integrated to yield

$$(\Delta_{xy} K)^2 + 2\,P = C(z). \tag{93}$$

Differentiating (93) with respect to $z$ leads together with (71) to

$$(\partial^2_{xz} K)\,\Delta_{xy}\partial_x K + (\partial^2_{yz} K)\,\Delta_{xy}\partial_y K - (\Delta_{xy} K)\,\Delta_{xy}\partial_z K = -C'. \tag{94}$$

The general solution of (94) under the constraint $\partial^2_{zz} K \equiv 0$ has the form

$$\left.\begin{aligned} K &= K_0(x, y) + K_1(x, y)\,z, \\ C' &= C_0 + C_1\,z, \end{aligned}\right\} \tag{95}$$

which, inserted into (94), leads to

$$(\partial_x K_1)\partial_x \Delta_{xy} K_0 + (\partial_y K_1)\partial_y \Delta_{xy} K_0 - (\Delta_{xy} K_1)\Delta_{xy} K_0 = -C_0, \tag{96}$$

$$(\partial_x K_1)\partial_x \Delta_{xy} K_1 + (\partial_y K_1)\partial_y \Delta_{xy} K_1 - (\Delta_{xy} K_1)^2 = -C_1. \tag{97}$$

Equation (97) is a 2D nonlinear equation for $K_1$. Once $K_1$ is determined and inserted in (96), this is a 2D linear equation for $\Delta_{xy} K_0$. On switching to cylindrical coordinates, (96) and (97) read

$$\left.\begin{aligned} (\partial_r K_1)\partial_r \Delta_{r\theta} K_0 + \frac{1}{r^2}(\partial_\theta K_1)\partial_\theta \Delta_{r\theta} K_0 - (\Delta_{r\theta} K_1)\Delta_{r\theta} K_0 &= -C_0, \\ (\partial_r K_1)\partial_r \Delta_{r\theta} K_1 + \frac{1}{r^2}(\partial_\theta K_1)\partial_\theta \Delta_{r\theta} K_1 - (\Delta_{r\theta} K_1)^2 &= -C_1, \end{aligned}\right\} \tag{98}$$

with $\Delta_{r\theta}$ being the 2D Laplacian in the $(r,\theta)$ plane,

$$\Delta_{r\theta} = \frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\partial^2_{\theta\theta}. \tag{99}$$

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An implicit solution of (98) is given by

$$\left.\begin{aligned} K_1(r) &= -\int^r (c_1 \sin \tfrac{1}{2}k\tilde{r}^2 + c_2 \cos \tfrac{1}{2}k\tilde{r}^2)\frac{d\tilde{r}}{\tilde{r}}, \\ \Delta_{r\theta} K_0(r,\theta) &= -k[b_1(\theta)\cos \tfrac{1}{2}kr^2 - b_2(\theta)\sin \tfrac{1}{2}kr^2], \end{aligned}\right\} \tag{100}$$

where $c_1$, $c_2$, $b_1(\theta)$ and $b_2(\theta)$ obey the relation

$$c_1 b_1' + c_2 b_2' = 0. \tag{101}$$

Comparing the magnetic field (68) of the solution (100) with (61), one realizes that the solution of Sec. 2.2, case II, has been regained in a different representation. This shows that the 'simplicity' of an equilibrium depends strongly on the representation chosen. Solutions of (96) and (97) (or (98)) representing new equilibria have not been found so far.

## 4. Outlook

Our primary interest in this study of 3D MHD equilibria is not so much in applications but is theoretical. It relates to the supposed non-existence of 3D equilibria in toroidal geometry with sheared field lines, a problem that arises in fusion-oriented MHD. Simplified equilibria such as those considered here may help to distinguish the essential features from the irrelevant ones. Basically, it is found from the examples given in this paper that 3D equilibria are not ruled out *per se*. Closed-field-line equilibria such as those of the first class or equilibria with infinitely extended field lines such as those of the second and third classes do indeed exist. A discrete symmetry like mirror symmetry is not necessary, nor do field lines have to be plane. All three classes include representatives with poloidally closed pressure surfaces. The closing in the toroidal direction, however, seems to cause trouble. For the purely poloidal equilibria, for example, pressure surfaces can indeed be chosen to be periodic in the toroidal direction; but actual closing requires distorting the straight axis, and this is ruled out (Salat 1995). In the purely toroidal case not even periodic solutions have been found so far. To construct such equilibria (whether or not of Clebsch or generalized Clebsch type) would be of considerable interest. The main shortcoming of all known 3D equilibria is, of course, the missing field line twist. This property combined with double periodicity carries the potential of field line chaos, which is at the heart of the problem. So, the construction of a twisted 3D equilibrium (periodic or not) or proof of its impossibility would be an important step ahead. Note in this context that any 3D equilibrium based on a Clebsch representation with $k = r$ (see Appendix A) would be at least locally twisted.

Nevertheless, we want to mention some possible applications of the new equilibria. The open equilibria in Sec. 2.1 (see Fig. 2), for example, might be of some interest in the reconnection context. They generalize the well-known 2D geometry for reconnection via X-points in that they allow variation of the magnetic field in the third direction. In particular, rotation of the X-point and variation of the angle of the intersecting field lines along the third direction are allowed. Whether these new features do affect the reconnection process remains to be explored.

Other possible applications concern the modelling of magnetic arcade structures in the solar corona (Priest 1982) or the geomagnetic tail (Schindler and Birn 1978). The former application is supported by Figs 5 and 7, which can be interpreted as

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3D generalizations of an ordinary 2D arcade configuration. As far as the second application is concerned, 3D equilibria are useful for modelling magnetotail configurations under quiescent conditions. In Birn (1979), for example, approximate 3D equilibria were constructed on the basis of a Clebsch representation for the magnetic field and a Taylor series expansion in one variable and an asymptotic expansion in another variable.

Finally, it should be mentioned that for astrophysical purposes Low investigated 3D equilibria in a series of papers (Low 1982, 1984, 1985, 1991, 1992, 1993; see also Neukirch 1995). The basic difference from our investigation is the inclusion of a gravitational field. This additional freedom is crucial for his construction of 3D equilibria. In fact, for vanishing gravitational field, these equilibria have no well-defined limit.

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## Appendix A

This appendix investigates the ansatz $\mathbf{B} = \nabla H \times \nabla r$. The problem is again reduced to a 2D elliptic equation on the cylinder $r = \text{const}$ and two compatibility conditions for the $r$ dependence; 3D solutions, however, have not been found.

Equations (1) here take the form

$$\partial_r \left( \frac{1}{r} \partial_\theta H \right)^2 + \frac{1}{r^2} \partial_r (r \partial_z H)^2 + 2\, \partial_r P = 0, \tag{A 1}$$

$$\left. \begin{aligned} (\Delta_{\theta z} H) \partial_\theta H + \partial_\theta P = 0, \\ (\Delta_{\theta z} H) \partial_z H + \partial_z P = 0, \end{aligned} \right\} \tag{A 2}$$

where $\Delta_{\theta z}$ denotes the 2D Laplacian on the $(\theta,z)$ cylinder,

$$\Delta_{\theta z} = \frac{1}{r^2} \partial_{\theta\theta}^2 + \partial_{zz}^2. \tag{A 3}$$

From the integrability condition $\partial_z \partial_\theta P - \partial_\theta \partial_z P = 0$ for $P$ and (A 2), one deduces in the usual way a 2D elliptic equation for $H$ with an arbitrary profile function $f(H, r)$:

$$\Delta_{\theta z} H = f(H, r). \tag{A 4}$$

Note that, contrary to the previously considered cases, (A 1) is no longer a total $r$ derivative. The compatibility conditions are therefore derived from the remaining integrability conditions for $P$: $\partial_r \partial_\theta P - \partial_\theta \partial_r P = 0$ and $\partial_r \partial_z P - \partial_z \partial_r P = 0$. One obtains

$$\left. \begin{aligned} \partial_r [(\partial_\theta H)\, \partial_{zz}^2 H - (\partial_z H)\, \partial_{\theta z}^2 H] - \frac{2}{r} (\partial_z H)\, \partial_{\theta z}^2 H = 0, \\ \partial_r \left\{ \frac{1}{r^2} [(\partial_z H)\, \partial_{\theta\theta}^2 H - (\partial_\theta H)\, \partial_{\theta z}^2 H] \right\} - \frac{2}{r} (\partial_z H)\, \partial_{zz}^2 H = 0. \end{aligned} \right\} \tag{A 5}$$

Assuming, as usual, a linear profile function,

$$f = a_0(r) + a_1(r)\, H, \tag{A 6}$$

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one can again be distinguish the cases $a_1 = 0$ and $a_1 \neq 0$. In the latter case

$$H_i = -\frac{a_0(r)}{a_1(r)} \tag{A 7}$$

is an inhomogeneous solution, and the separation ansatz

$$H_h = \alpha(r,\theta)\,\beta(r,z) \tag{A 8}$$

for the homogeneous part leads to

$$\left.\begin{aligned} \partial^2_{\theta\theta}\alpha &= -m^2\,\alpha, \\ \partial^2_{zz}\beta &= -\lambda^2(r)\,\beta, \end{aligned}\right\} \tag{A 9}$$

with $m^2/r^2 + \lambda^2 + a_1 = 0$. This suggests a general ansatz for $H_h$:

$$\begin{aligned} H_h = {} & u_1(r)\sin(m\theta + \lambda z) + u_2(r)\sin(m\theta - \lambda z) \\ & + u_3(r)\cos(m\theta + \lambda z) + u_4(r)\cos(m\theta - \lambda z). \end{aligned} \tag{A 10}$$

However, non-trivial (i.e. 3D) solutions of the type (A 10) that additionally satisfy the compatibility conditions (A 5) have not been found.

In the case $a_1 = 0$ an ansatz of the form

$$\left.\begin{aligned} H_i &= \lambda(r) + \mu(r)\,z + \nu(r)\,z^2, \\ H_h &= [u_1(r)\sinh\Phi + u_2(r)\cosh\Phi]\,\sin m\theta \\ &\quad + [u_3(r)\sinh\Phi + u_4(r)\cosh\Phi]\,\cos m\theta, \end{aligned}\right\} \tag{A 11}$$

with $2\nu = a_0$ and $\Phi := mz/r$ was tried. However, non-trivial solutions satisfying (A 5) could not be found either. Whether more general ansatzes are successful or 3D equilibria of this Clebsch type do not exist at all remains an open problem.

## Appendix B

This appendix shows that the remaining two generalized Clebsch cases cannot be treated with the method that has been successful so far. More precisely, none of the integrability conditions for the pressure can be resolved.

For the ansatz $\mathbf{B} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} K \times \boldsymbol{\nabla}\theta)$, (1) give

$$\left.\begin{aligned} \frac{1}{r^2}(\Delta_{rz}K)\left[\partial_r(\Delta_{rz}K) + \frac{1}{r^2}\partial^2_{\theta\theta}\partial_r K\right] - \frac{2}{r^5}(\partial^2_{\theta z}K)^2 + \partial_r P = 0, \\ \frac{1}{r^2}(\Delta_{rz}K)\left[\partial_z(\Delta_{rz}K) + \frac{1}{r^2}\partial^2_{\theta\theta}\partial_z K\right] + \frac{2}{r^5}(\partial^2_{\theta z}K)\,\partial^2_{r\theta}K + \partial_z P = 0, \end{aligned}\right\} \tag{B 1}$$

$$\frac{1}{r^2}(\partial^2_{r\theta}K)\left[\partial_r(\Delta_{rz}K) + \frac{1}{r^2}\partial^2_{\theta\theta}\partial_r K\right] + \frac{1}{r^2}(\partial^2_{\theta z}K)\left[\partial_z(\Delta_{rz}K) + \frac{1}{r^2}\partial^2_{\theta\theta}\partial_z K\right] + \partial_\theta P = 0, \tag{B 2}$$

with $\Delta_{rz}$ being the 2D Stokes operator in the $(r,z)$ plane,

$$\Delta_{rz} = r\,\partial_r\left(\frac{1}{r}\,\partial_r\right) + \partial^2_{zz}. \tag{B 3}$$

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For the ansatz $\mathbf{B} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} K \times \boldsymbol{\nabla} r)$, we obtain

$$\left.\begin{aligned}
&(\Delta_{\theta z} K)\left\{\partial_\theta(\Delta_{\theta z} K) + \partial_r\left[r\partial_r\left(\frac{1}{r}\partial_\theta K\right)\right]\right\} + \frac{2}{r^5}(\partial^2_{\theta z} K)\,\partial_r(r\partial_z K) + \partial_\theta P = 0,\\
&(\Delta_{\theta z} K)\left\{\partial_z(\Delta_{\theta z} K) + \partial_r\left[\frac{1}{r}\partial_r(r\partial_z K)\right]\right\} - \frac{2}{r^5}(\partial^2_{\theta z} K)\,\partial_r\left(\frac{1}{r}\partial_\theta K\right) + \partial_z P = 0,
\end{aligned}\right\} \tag{B 4}$$

$$\begin{aligned}
&\frac{1}{r}\left[\partial_r\left(\frac{1}{r}\partial_\theta K\right)\right]\left\{\partial_\theta(\Delta_{\theta z} K) + \partial_r\left[r\partial_r\left(\frac{1}{r}\partial_\theta K\right)\right]\right\}\\
&\quad + \frac{1}{r}\left[\partial_r(r\partial_z K)\right]\left\{\partial_z(\Delta_{\theta z} K) + \partial_r\left[\frac{1}{r}\partial_r\left(r\partial_z K\right)\right]\right\} + \partial_r P = 0,
\end{aligned} \tag{B 5}$$

where $\Delta_{\theta z}$ is the 2D Laplacian on the $(\theta,z)$ cylinder given in (A 3). If one tries to proceed as in Sec. 3, one finds that unfortunately the integrability condition $\partial_r\partial_z P - \partial_z\partial_r P = 0$ applied to (B 1) does not lead to a simple relation between $\Delta_{rz} K$ and $\partial^2_{\theta\theta} K$, and the condition $\partial_\theta\partial_z P - \partial_z\partial_\theta P = 0$ applied to (B 4) does not work either.

For these generalized Clebsch ansatzes, one therefore has to solve (B 1), (B 2) or (B 4), (B 5) directly. This is a formidable task, which is not followed up here.

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