FLEXURE WITH SHEAR CENTRES: A GENERAL TREATMENT WITH COMPLEX VARIABLE

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This work is consequent on a forthcoming joint paper by A. C. Stevenson and the present writer on the use of the torsion function in the solution of flexure problems. The treatment is general, but special attention is paid to doubly-connected cross-sections which can be mapped conformally on a circular annulus. The eccentric circles cross-section (4) is solved in order to illustrate the power of the method, and results are obtained for the cross-section bounded by confocal limaçons. This is of interest, since it has been customary to regard these problems as largely determined by the cross-section of the cylinder; but these results show that the terms involving the elastic constants are now no longer negligible. The method is capable of giving useful information to aeronautical engineers, and the writer hopes to present in a later communication a treatment of an aerofoil-like profile.

1. *The complex equations.* With the notation

$$\Theta = \widehat{xx}+\widehat{yy}, \quad \Phi = \widehat{xx}-\widehat{yy}+2i\widehat{xy}, \quad \Psi = \widehat{xz}+i\widehat{yz}$$

the body-stress equations of equilibrium under no body forces are

$$\frac{\partial\Phi}{\partial z}+\frac{\partial\Theta}{\partial\bar{z}}+\frac{\partial\Psi}{\partial Z} = 0 = \frac{\partial\Psi}{\partial z}+\frac{\partial\bar{\Psi}}{\partial\bar{z}}+\frac{\partial\widehat{zz}}{\partial Z}, \tag{1·01}$$

where z, $\bar{z}$ are the usual complex coordinates and Z represents the third space variable.

The complex stress-displacement equations are

$$(1-2\eta)\,\Theta = 2\mu\left\{\frac{\partial D}{\partial z}+\frac{\partial\bar{D}}{\partial\bar{z}}+2\eta\frac{\partial w}{\partial Z}\right\}, \quad \Psi = \mu\frac{\partial D}{\partial Z}+2\mu\frac{\partial w}{\partial\bar{z}}, \tag{1·02}$$

$$(1-2\eta)\,\widehat{zz} = 2\mu\left\{\eta\left(\frac{\partial D}{\partial z}+\frac{\partial\bar{D}}{\partial\bar{z}}\right)+(1-\eta)\frac{\partial w}{\partial Z}\right\}, \quad \Phi = 4\mu\frac{\partial D}{\partial\bar{z}}, \tag{1·03}$$

where μ is the rigidity, η is Poisson's ratio, u, v, w the cartesian displacements, and $D = u+iv$.

The Beltrami equations are

$$\nabla^2\Theta+4(1-\sigma)\frac{\partial^2\Delta}{\partial z\,\partial\bar{z}} = 0 = \nabla^2\widehat{zz}+(1-\sigma)\frac{\partial^2\Delta}{\partial Z^2}, \tag{1·04}$$

$$\nabla^2\Phi+4(1-\sigma)\frac{\partial^2\Delta}{\partial\bar{z}^2} = 0 = \nabla^2\Psi+2(1-\sigma)\frac{\partial^2\Delta}{\partial\bar{z}\,\partial Z}, \tag{1·05}$$

where $$\Delta = \Theta+\widehat{zz}, \quad (1+\eta)(1-\sigma) = 1, \quad \nabla^2 \equiv 4\frac{\partial^2}{\partial z\,\partial\bar{z}}+\frac{\partial^2}{\partial Z^2}.$$

We assume the anti-plane stress

$$\Theta = 0 = \Phi, \quad \widehat{zz} = (Z-l)(pz+\bar{p}\bar{z}+q), \tag{1·06}$$

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where q is a real constant, p is complex and l the length of the cylinder. Equation (1·01) now becomes

$$\frac{\partial \Psi}{\partial \bar{Z}} = 0 = \frac{\partial \Psi}{\partial z} + \frac{\partial \bar{\Psi}}{\partial \bar{z}} + pz + \bar{p}\bar{z} + q. \tag{1·07}$$

The only Beltrami equation not identically satisfied gives

$$4\frac{\partial^2 \Psi}{\partial z \partial \bar{z}} + 2(1-\sigma)\bar{p} = 0.$$

Integrating and using (1·07) we find, with r a real constant,

$$\frac{\partial \Psi}{\partial z} = ir - \tfrac{1}{2}q - \tfrac{1}{2}(1+\sigma)pz - \tfrac{1}{2}(1-\sigma)\bar{p}\bar{z}, \tag{1·08}$$

whence $$\Psi = \bar{\Omega}'(\bar{z}) + (ir - \tfrac{1}{2}q)z - \tfrac{1}{4}(1+\sigma)pz^2 - \tfrac{1}{2}(1-\sigma)\bar{p}z\bar{z},$$

and $\Omega(z)$ is a complex potential in differentiated form for later convenience.

Solving for D and w we find

$$D = (Z-l)\left\{\frac{irz}{\mu} - \frac{\eta}{E}(pz^2 + qz)\right\} - \frac{\bar{p}}{2E}(Z-l)^3, \tag{1·09}$$

$$2\mu w = \Omega(z) + \bar{\Omega}(\bar{z}) - \tfrac{1}{2}qz\bar{z} - \tfrac{1}{4}(1-\sigma)z\bar{z}(pz + \bar{p}\bar{z}) + \tfrac{1}{2}(Z-l)^2(1-\sigma)(pz + \bar{p}\bar{z} + q), \tag{1·10}$$

where $E = 2\mu(1+\eta)$ is Young's modulus.

If unit normal $\hat{n}$ to the surface makes angle θ with the x-direction, we find

$$\hat{nz} = \tfrac{1}{2}\{\Psi e^{-i\theta} + \bar{\Psi} e^{i\theta}\}.$$

Again, if ds be an element of arc along the boundary,

$$dz = ds\, e^{i(\theta + \frac{1}{2}\pi)},$$

$$\hat{nz} = \tfrac{1}{2}i\left(\Psi \frac{d\bar{z}}{ds} - \bar{\Psi}\frac{dz}{ds}\right). \tag{1·11}$$

The boundary condition requires $\hat{nz} = 0$. This gives

$$\Psi\, d\bar{z} = \bar{\Psi}\, dz. \tag{1·12}$$

If we reduce the forces across any cross-section to force and couple resultants X, Y, T and L, M, N at the origin of the cross-section, we find readily

$$X + iY = \int \Psi\, dS, \quad T = \int \hat{zz}\, dS, \tag{1·13}$$

$$L + iM = -i\int z\hat{zz}\, dS, \quad N = \int (x\hat{yz} - y\hat{xz})\, dS. \tag{1·14}$$

We require the total tension to be zero. Hence evaluating (1·13) we find

$$pZ_G + \bar{p}\bar{Z}_G + q = 0, \tag{1·15}$$

where Z_G is the coordinate of the centroid G of the cross-section.

Multiplying by z the second equation of (1·01) and integrating we obtain

$$\int_S \left\{\frac{\partial}{\partial z}(z\Psi) - \Psi + \frac{\partial}{\partial \bar{z}}(z\bar{\Psi}) + z\frac{\partial \hat{zz}}{\partial \bar{Z}}\right\} dS = 0,$$

or

$$\int \Psi\, dS = \int z\frac{\partial}{\partial \bar{Z}}\hat{zz}\, dS + \int \left\{\frac{\partial}{\partial z}(z\Psi) + \frac{\partial}{\partial \bar{z}}(z\bar{\Psi})\right\} dS,$$

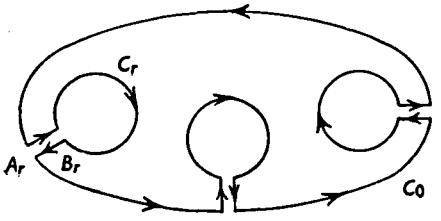
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or from (1·13) and the complex form of Stokes's theorem

$$X+iY=\int_S z(pz+\overline{p}\bar{z}+q)\,dS-\tfrac{1}{2}i\int_C z(\overline{\Psi}\,dz-\Psi\,d\bar{z}),$$

where the boundary C consists of the outer containing boundary C_0, the inner boundaries C_r, and the twice described line cuts A_rB_r. Using (1·11) and (1·12) we find

$$\tfrac{1}{2}i\int_C z(\overline{\Psi}\,dz-\Psi\,d\bar{z})=0.$$



Hence if (W, W') are the resolutes of the applied load

$$W+iW'=\int_S z(pz+\overline{p}\bar{z}+q)\,dS. \tag{1·16}$$

Write A_0, B_0, H_0 as inertial constants of cross-section at origin O; A, B, H the constants at G; S the area of cross-section. Substituting for q in (1·16) from (1·15) we obtain

$$\begin{aligned}W+iW'&=p(B_0-A_0+2iH_0-Sz_G^2)+\overline{p}(A_0+B_0-Sz_G\bar{z}_G)\\&=p(B-A+2iH)+\overline{p}(A+B).\end{aligned} \tag{1·17}$$

From the external loading $N=W'x_L-Wy_L$, or $N=\Re iz_L\overline{F}$, where $z_L=x_L+iy_L$ is the load point and $F=W+iW'$. Hence we may write

$$N+iN^*=iz_L\overline{F} \tag{1·18}$$

and

$$N+iN^*=i\int_S z\overline{\Psi}\,dS. \tag{1·19}$$

We now introduce the torsion function $\omega(z)$ (3). This function satisfies the boundary condition

$$\omega(z)-\overline{\omega}(\bar{z})=iz\bar{z}+iC_r, \tag{1·20}$$

where C_r, a real constant, is the boundary constant for the curve C_r. This double use of C_r to represent a curve as well as its boundary constant should help rather than confuse. The function $\omega=\phi+i\psi$ is single-valued, since ϕ appears in the displacement w and ψ is single-valued from the boundary condition.

Equation (1·19) can be written

$$\begin{aligned}N+iN^*&=i\int_S\left\{\frac{\partial}{\partial\bar{z}}(z\bar{z}\overline{\Psi})-z\bar{z}\frac{\partial\overline{\Psi}}{\partial\bar{z}}\right\}dS\\&=\frac{1}{2}\int_C z\bar{z}\overline{\Psi}\,dz-i\int_S z\bar{z}\frac{\partial\overline{\Psi}}{\partial\bar{z}}\,dS,\end{aligned}$$

by Stokes's theorem.

The value of N^* is immaterial to this problem. We shall change it by writing the complex conjugate of an integral whenever convenient. This procedure leaves N unaltered.

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Since $z\bar{z}\bar{\Psi}$ is single-valued,

$$\frac{1}{2}\int_C z\bar{z}\bar{\Psi}\,dz = \left\{\frac{1}{2}\int_{C_0} - \frac{1}{2}\sum_{r=1}^{n}\int_{C_r}\right\} z\bar{z}\bar{\Psi}\,dz.$$

Using (1·20) we find

$$\int_{C_r} z\bar{z}\bar{\Psi}\,ds = -i\int_{C_r}\{\omega(z)-\bar{\omega}(\bar{z})-iC_r\}\bar{\Psi}\,dz.$$

Hence $$N+iN^*+i\int_S z\bar{z}\frac{\partial\bar{\Psi}}{\partial\bar{z}}\,dS = -\tfrac{1}{2}i\left\{\int_{C_0} - \sum_{r=1}^{n}\int_{C_r}\right\}\{\omega(z)-\bar{\omega}(\bar{z})-iC_r\}\bar{\Psi}\,dz,$$

or using the boundary condition (1·12)

$$N+iN^*+i\int_S z\bar{z}\frac{\partial\bar{\Psi}}{\partial\bar{z}}\,dS = -\tfrac{1}{2}i\left\{\int_{C_0} - \sum_{r=1}^{n}\int_{C_r}\right\}(\omega\bar{\Psi}\,dz-\bar{\omega}\Psi\,d\bar{z}-iC_r\bar{\Psi}\,dz).$$

Changing the value of N^* by replacing $i\bar{\omega}\Psi\,d\bar{z}$ by its complex conjugate,

$$N+iN^*+i\int_S z\bar{z}\frac{\partial\bar{\Psi}}{\partial\bar{z}}\,dS = -\tfrac{1}{2}i\left\{\int_{C_0} - \sum_{r=1}^{n}\int_{C_r}\right\}(2\omega-iC_r)\bar{\Psi}\,dz.$$

Hence, using the complex Stokes's theorem,

$$N+iN^* = \int_S\{2\omega-iz\bar{z}\}\frac{\partial\bar{\Psi}}{\partial\bar{z}}\,dS-\frac{1}{2}\left\{\int_{C_0} C_0 - \sum_{r=1}^{n}\int_{C_r} C_r\right\}\bar{\Psi}\,dz. \tag{1·21}$$

Now $\bar{\Psi}$ contains $\Omega'(z)$. Hence $\int_{C_r}\Omega'(z)\,dz = Cy\Omega$, where $Cy\Omega$ = change in value of Ω in going round a simple closed curve once. However, as the real part of Ω contributes to the displacement w, it is necessarily single-valued. Hence this integral makes no contribution to N.

Writing in the value for $\frac{\partial\bar{\Psi}}{\partial\bar{z}}$ and $\bar{\Psi}$ we find

$$N+iN^* = -\int_S\{2\omega-iz\bar{z}\}\{(ir+\tfrac{1}{2}q)+\tfrac{1}{2}(1+\sigma)\,\bar{p}\bar{z}+\tfrac{1}{2}(1-\sigma)\,pz\}\,dS$$
$$+\frac{1}{2}\left\{\int_{C_0} C_0 - \sum_{r=1}^{n} C_r\int_{C_r}\right\}[(ir+\tfrac{1}{2}q)\,\bar{z}+\tfrac{1}{4}(1+\sigma)\,\bar{p}\bar{z}^2+\tfrac{1}{2}(1-\sigma)\,pz\bar{z}]\,dz. \tag{1·22}$$

We have thus expressed the moment N in terms of the torsion function ω only.

2. *The centres of flexure and least strain.* The average angle of rotation in the (x, y) plane is

$$\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) = -\tfrac{1}{2}i\left(\frac{\partial D}{\partial z}-\frac{\partial\bar{D}}{\partial\bar{z}}\right).$$

The rate of change of this angle per unit length of the cylinder is termed the local twist τ_P at the point P. Hence

$$\tau_P = \tfrac{1}{2}i\frac{\partial}{\partial Z}\left(\frac{\partial\bar{D}}{\partial\bar{z}}-\frac{\partial D}{\partial z}\right) = \frac{r}{\mu}+\frac{i\eta}{E}(pz-\bar{p}\bar{z}).$$

The average twist over the cross-section

$$= \int\tau_P\,dS/S = \frac{\tau}{\mu}+\frac{i\eta}{E}(pz_G-\bar{p}\bar{z}_G) = \text{local twist } \tau_G \text{ at } G.$$

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The associated twist is the mean twist when the load point is G. Hence (1·18) is written

$$N+iN^* = iz_G\overline{F}. \tag{2·01}$$

Again

$$\mu\tau_G = r+\tfrac{1}{2}i\sigma(pz_G-\overline{p}\overline{z}_G). \tag{2·02}$$

Equations (2·01), (2·02) and (1·22) now give a solution for τ_G. Writing S_r = area bounded by C_r, z_{G_r} = centroid of S_r ($r = 0, 1, \ldots, n$), and remembering that equality holds for real parts only, we find

$$\begin{aligned}\mu\tau_G\Big\{\sum_{r=1}^{n} C_rS_r - C_0S_0 - i\int_S\{2\omega-iz\bar{z}\}\,dS\Big\}&\\ = -\int\{\tfrac{1}{2}(1-\sigma)\,pz_G+\tfrac{1}{2}(1+\sigma)\,\overline{p}\overline{z}_G-\tfrac{1}{2}(1-\sigma)\,pz-\tfrac{1}{2}(1+\sigma)\,\overline{p}\overline{z}\}\{2\omega-iz\bar{z}\}\,dS&\\ +iz_G\overline{F}+i\sigma p\sum_{r=1}^{n} C_r(z_G-z_{G_r})\,S_r - i\sigma pC(z_G-z_{G_0})\,S_0.&\end{aligned} \tag{2·03}$$

This shows that the associated twist can be expressed in terms of the torsion function only.

The centre of flexure is the load-point for which $\tau_G = 0$. Hence combining (2·02) with (1·15) we obtain

$$(ir+\tfrac{1}{2}q)+\tfrac{1}{2}(1-\sigma)\,pz_G+\tfrac{1}{2}(1+\sigma)\,\overline{p}\overline{z}_G = 0. \tag{2·04}$$

Integrating this over the cross-section S and transforming by Stokes's theorem,

$$\Big[\int_{C_0} - \sum_{r=1}^{n}\int_{C_r}\Big]\{(ir+\tfrac{1}{2}q)\,\bar{z}+\tfrac{1}{4}(1+\sigma)\,\overline{p}\bar{z}^2+\tfrac{1}{2}(1-\sigma)\,pz\bar{z}\}\,dz = 0. \tag{2·05}$$

Using this in (1·22) we find

$$\begin{aligned}N+iN^* = -\int_S\{2\omega-iz\bar{z}\}\{(ir+\tfrac{1}{2}q)+\tfrac{1}{2}(1+\sigma)\,\overline{p}\overline{z}+\tfrac{1}{2}(1-\sigma)\,pz\}\,dS&\\ +\frac{1}{2}\sum_{r=1}^{n}(C-C_r)\int_{C_r}\{(ir+\tfrac{1}{2}q)\,\bar{z}+\tfrac{1}{4}(1+\sigma)\,\overline{p}\bar{z}^2+\tfrac{1}{2}(1-\sigma)\,pz\bar{z}\}\,dz,&\end{aligned} \tag{2·06}$$

or, substituting for r and q from (2·02) and (1·15),

$$\begin{aligned}N+iN^* = \frac{1}{2}\int[(1-\sigma)\,pz_G+(1+\sigma)\,\overline{p}\overline{z}_G+\tfrac{1}{2}(1-\sigma)\,pz+\tfrac{1}{2}(1+\sigma)\,\overline{p}\overline{z}]\,[2\omega-iz\bar{z}]\,dS&\\ +\frac{1}{2}\sum_{r=1}^{n}(C_0-C_r)\int_{C_r}[-\{(1-\sigma)\,pz_G+(1+\sigma)\,\overline{p}\overline{z}_G\}\tfrac{1}{2}\bar{z}+\tfrac{1}{4}(1+\sigma)\,\overline{p}\bar{z}^2+\tfrac{1}{2}(1-\sigma)\,pz\bar{z}]\,dz.&\end{aligned} \tag{2·07}$$

For the sake of abbreviation we write

$$P = -\int_S 2\omega\,dS, \quad Q = -\int_S 2z\omega(z)\,dS, \quad R = -\int_S 2\bar{z}\omega(z)\,dS, \tag{2·08}$$

$$\Gamma_1+i\Gamma_2 = \sum_{r=1}^{n}(C_0-C_r)\int_{C_r}\{\tfrac{1}{2}(1-\sigma)\,p(z\bar{z}-\bar{z}z_G)+\tfrac{1}{2}(1+\sigma)\,\overline{p}(\tfrac{1}{2}z^2-\overline{z}\overline{z}_G)\}\,dz. \tag{2·09}$$

Using Stokes's theorem and changing the value of Γ_2 we find

$$\Gamma_1+i\Gamma_2 = -2i\sigma p\sum_{r=1}^{n}(C_0-C_r)\int_{S_r}(z-z_G)\,dS = \sigma p\Gamma,$$

where

$$\Gamma = 2i\sum_{r=1}^{n}(C_0-C_r)\,(z_G-z_{G_r})\,S_r. \tag{2·10}$$

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Substituting from (2·08) and (2·10) into (2·07), and altering the value of N^*, we find eventually

$$N+iN^* = -\tfrac{1}{2}\{(1-\sigma)\,pz_G+(1+\sigma)\,\overline{pz}_G\}\,P+\tfrac{1}{2}(1-\sigma)\,pQ+\tfrac{1}{2}(1+\sigma)\,\bar{p}R+\tfrac{1}{2}p\sigma(\Gamma+V), \tag{2·11}$$

where
$$V=\int_S 2i(z_G-z)\,z\bar{z}\,dS. \tag{2·12}$$

Write (1·17) as
$$W+iW' = pM_1+\bar{p}M_2, \tag{2·13}$$

where
$$M_1 = B_0-A_0+2iH_0-Sz_G^2, \quad M_2 = B_0+A_0-Sz_G\bar{z}_G. \tag{2·14}$$

From (2·12) and its complex conjugate we find

$$p = \varpi_1 W'+\varpi_2 W, \quad \text{where} \quad \varpi_2 = \frac{\bar{M}_1-M_2}{M_1\bar{M}_1-M_2^2}, \quad \varpi_1 = \frac{i(\bar{M}_1+M_2)}{M_1\bar{M}_1-M_2^2}. \tag{2·15}$$

Taking (x_F, y_F) as the centre of flexure we have

$$N = W'x_F - Wy_F.$$

Hence x_F is the coefficient of W' and y_F the coefficient of W in (2·11). This gives at once

$$x_F = -\tfrac{1}{2}\{(1-\sigma)\,\varpi_1 z_G+(1+\sigma)\,\overline{\varpi}_1\bar{z}_G\}\,P+\tfrac{1}{2}(1-\sigma)\,\varpi_1 Q+\tfrac{1}{2}(1+\sigma)\,\overline{\varpi}_1 R+\tfrac{1}{2}\sigma\varpi_1(V+\Gamma), \tag{2·16}$$

$$y_F = \tfrac{1}{2}\{(1-\sigma)\,\varpi_2 z_G+(1+\sigma)\,\overline{\varpi}_2\bar{z}_G\}\,P-\tfrac{1}{2}(1-\sigma)\,\varpi_2 Q-\tfrac{1}{2}(1+\sigma)\,\overline{\varpi}_2 R-\tfrac{1}{2}\sigma\varpi_2(V+\Gamma). \tag{2·17}$$

The centre of least strain, (x_s, y_s), is obtained by putting $\sigma = 0$ in these formulae:

$$\left.\begin{aligned} x_s &= -\tfrac{1}{2}(\varpi_1 z_G+\overline{\varpi}_1\bar{z}_G)\,P+\tfrac{1}{2}\varpi_1 Q+\tfrac{1}{2}\overline{\varpi}_1 R,\\ y_s &= \tfrac{1}{2}(\varpi_2 z_G+\overline{\varpi}_2\bar{z}_G)\,P-\tfrac{1}{2}\varpi_2 Q-\tfrac{1}{2}\overline{\varpi}_2 R. \end{aligned}\right\} \tag{2·18}$$

3. *Simply-connected cross-sections.* The work, so far, has been general. We shall here deal with simply-connected cross-sections which can be mapped on the unit circle. It is obvious, however, that the preceding work can be applied to any other form of mapping. For our simply-connected cross-section, we shall take the boundary constant C_0 to be zero.

Suppose that the transformation

$$z = z(\sigma) = \sum_0^\infty a_r\sigma^r, \quad z'(\sigma)\neq 0 \text{ or } \infty \text{ within } S, \tag{3·01}$$

maps the domain S in the z-plane on the unit circle $\sigma\bar{\sigma} = 1$ in the σ-plane. Whenever a possibility of confusion with the elastic constant σ arises we shall write the constant as σ_0.

Let
$$\omega(\sigma) = \omega[z(\sigma)] = \sum_0^\infty \alpha_r\sigma^r. \tag{3·02}$$

The boundary condition can now be written, since $\sigma\bar{\sigma} = 1$,

$$\begin{aligned} \omega(\sigma)-\bar{\omega}(\bar{\sigma}) = iz\bar{z} &= iz(\sigma)\,\bar{z}(\sigma^{-1})\\ &= i\sum_{r=0}^\infty a_r\sigma^r\sum_{s=0}^\infty \bar{a}_s\sigma^{-s}. \end{aligned} \tag{3·03}$$

Hence
$$\omega(\sigma)-\bar{\omega}(\bar{\sigma}) = i\sum_{-\infty}^\infty b_r\sigma^r, \quad \text{where} \quad b_r = \sum_{s=0}^\infty a_{r+s}\bar{a}_s, \tag{3·04}$$

and
$$\bar{b}_r = \text{coefficient of } (\bar{\sigma})^r = \text{coefficient of } \sigma^{-r} = b_{-r} \quad (r = 0, 1, 2, \ldots, \infty). \tag{3·05}$$

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Again using (3·02) and $\sigma\overline{\sigma} = 1$ we have

$$\omega(\sigma) - \overline{\omega}(\overline{\sigma}) = \sum_0^\infty \alpha_r \sigma^r - \sum_0^\infty \overline{\alpha}_r \sigma^{-r} = i \sum_{-\infty}^{\infty} b_r \sigma^r$$

from (3·04), or

$$\alpha_r = ib_r, \quad \overline{\alpha}_r = -ib_{-r}, \quad \alpha_0 - \overline{\alpha}_0 = ib_0 \quad (r = 1, 2, \ldots, \infty).$$

This verifies $\overline{b}_r = b_{-r}$ and shows that (3·05) can give no information on the value of the real part of α_0. This, however, is immaterial to our problem, as it provides, physically, a longitudinal rigid body displacement of the cylinder along the Z-axis.

Hence we take
$$\omega(\sigma) = \tfrac{1}{2}ib_0 + \sum_{r=1}^{\infty} ib_r \sigma^r. \tag{3·06}$$

We write down here for convenience combinations of the coefficients a_r which occur in the work immediately following:

$$b_r = \sum_{s=0}^{\infty} a_{r+s}\overline{a}_s, \quad d_r = \sum_{s=0}^{\infty} (s+r)\,\overline{a}_s a_{s+r}, \quad c_r = \sum_{s=1}^{\infty} s\overline{a}_{s+r} a_s, \tag{3·07}$$

$$j_r = \sum_{n=1}^{r} b_n d_{r-n}, \quad h_r = \sum_{n=1}^{\infty} b_{n+r} c_n, \quad f_r = \sum_{n=1}^{\infty} b_n c_{n+r}. \tag{3·08}$$

The integrals P, Q, R, V defined in (2·08) and (2·12) are easily computed in terms of these coefficients.

$$P = -2\pi i(f_0 + \tfrac{1}{2}b_0 c_0), \quad R = -\pi i\left\{\overline{a}_0(\tfrac{1}{2}b_0 c_0 + f_0) + \sum_{r=1}^{\infty} \overline{a}_r(\tfrac{1}{2}b_0 d_r + j_r + h_r)\right\}, \tag{3·09}$$

$$Q = -2\pi i \sum_0^\infty a_r(\tfrac{1}{2}b_0 c_r + f_r), \quad V = \pi i z_G\left\{\sum_0^\infty b_{-r} d_r + \sum_1^\infty b_r c_r\right\} - \pi i \sum_{r=1}^{\infty} r g_{-r} a_r. \tag{3·10}$$

The centre of flexure, from (2·16) and (2·17), is now given by

$$x_F = -\tfrac{1}{2}\{(1-\sigma)\,\varpi_1 z_G + (1+\sigma)\,\overline{\varpi}_1 \overline{z}_G\} P + \tfrac{1}{2}(1-\sigma)\,\varpi_1 Q + \tfrac{1}{2}(1+\sigma)\,\overline{\varpi}_1 R + \sigma\tfrac{1}{2}\varpi_1 V, \tag{3·11}$$

$$y_F = \tfrac{1}{2}\{(1-\sigma)\,\varpi_2 z_G + (1+\sigma)\,\overline{\varpi}_2 \overline{z}_G\} P - \tfrac{1}{2}(1-\sigma)\,\varpi_2 Q - \tfrac{1}{2}(1+\sigma)\,\overline{\varpi}_2 R - \sigma\tfrac{1}{2}\varpi_2 V. \tag{3·12}$$

As an example, we evaluate the centre of flexure of the elliptic limaçon, results for which have already been obtained by Stevenson (2).

Consider the system of curvilinear coordinates given by

$$\zeta = \xi + i\eta, \quad \sigma = e^{\xi}, \quad z = c\sigma(1 + \lambda\sigma). \tag{3·13}$$

Writing $z_1 = z + \lambda c$, we have $z_1 = c\sigma\{1 + \lambda(\sigma + \sigma^{-1})\}$. Putting $z_1 = re^{i\theta}$, this gives $r = c(1 + 2\lambda\cos\theta)$ for $\xi = 0$. This is the equation $r = a + b\cos\theta$, $a = c$, $b/a = 2\lambda$. The area

$$S = -\tfrac{1}{2}i\int_C \overline{z}\,dz = -\tfrac{1}{2}i\int_C c^2(1 + \lambda\sigma^{-1})(1 + 2\lambda\sigma)\frac{d\sigma}{\sigma}, \quad \sigma\overline{\sigma} = 1.$$

Hence
$$S = \pi c^2(1 + 2\lambda^2). \tag{3·14}$$

Again
$$z_G S = \tfrac{1}{2}i\int_C z\overline{z}_k\,dz = -\tfrac{1}{2}i\int_C c^3[1 + \lambda^2 + \lambda(\sigma^{-1} + \sigma)][\sigma + 2\lambda\sigma^2]\frac{d\sigma}{\sigma}.$$

Whence
$$z_G = \frac{\lambda c}{1 + 2\lambda^2}. \tag{3·15}$$

Similarly
$$B_0 - A_0 + 2iH_0 = -\tfrac{1}{2}i\int_C \overline{z}z^2\,dz = 0, \tag{3·16}$$

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and $$B_0 + A_0 = -\tfrac{1}{4}i\int_C z\bar{z}^2 dz = \tfrac{1}{2}\pi c^4(2\lambda^4 + 6\lambda^2 + 1). \tag{3·17}$$

This gives $$B_0 = A_0 = \tfrac{1}{4}\pi c^4(2\lambda^4 + 6\lambda^2 + 1). \tag{3·18}$$

Using these results we readily find from (2·15) that ϖ_2 is real and $\varpi_1 = -i/2A_0$. The torsion function ω round the boundary behaves as

$$\omega(\sigma) - \bar{\omega}(\bar{\sigma}) = iz(\sigma)\bar{z}(\bar{\sigma}) \quad \text{or} \quad \omega(\sigma) - \bar{\omega}(\sigma^{-1}) = ic^2(1 + \lambda^2 + \lambda\sigma^{-1} + \lambda\sigma).$$

This gives $$\omega(\sigma) = \tfrac{1}{2}ic^2(1 + \lambda^2) + ic^2\lambda\sigma. \tag{3·19}$$

The integral $P = i\int_C \omega\bar{z}(\bar{\sigma})z'(\sigma)\,d\sigma$,

or $$P = i\int_C \tfrac{1}{2}ic^4(1 + \lambda^2 + 2\lambda\sigma)(\sigma^{-1} + \lambda\sigma^{-2})(\sigma + 2\lambda\sigma)\frac{d\sigma}{\sigma} = -\pi ic^4\{2\lambda^4 + 5\lambda^2 + 1\}.$$

Similarly $$Q = -\pi ic^5(\lambda + \lambda^3), \quad R = -\pi ic^5(2\lambda + 5\lambda^3),$$

$$V = \frac{\pi ic^5\lambda}{1 + 2\lambda^2}(2\lambda^4 + 6\lambda^2 + 1) - 2\pi ic^5(\lambda + 2\lambda^3).$$

We note that P, Q, R, V are imaginary while z_G and ϖ_2 are real. This makes (3·12) imaginary, i.e. $y_F = 0$. This result we expected from considerations of symmetry. Using these results in (3·11) we obtain

$$x_F = \frac{c(4\lambda^3 + \lambda)}{2\lambda^4 + 6\lambda^2 + 1} + \sigma\frac{2\lambda^5 c}{(1 + 2\lambda^2)(2\lambda^4 + 6\lambda^2 + 1)},$$

or $$x_F = \frac{4b^3a^2 + 4ba^4}{b^4 + 12b^2a^28a^4} + \sigma\frac{b^5a^2}{(2a^2 + b^2)(b^4 + 12b^2a^2 + 8a^4)}.$$

This answer reveals an error in the σ term in Stevenson's result (2), but agrees with Holl and Rock (1) for the centre of least strain. On putting $b = a$ we get the result for the cardioid

$$x_F = a\left(\frac{24 + \sigma}{63}\right).$$

This agrees with Stevenson's result for the cardioid.

4. *Doubly-connected cross-sections.* We shall discuss doubly-connected cross-sections in the z-plane, which can be mapped on an annulus in the σ-plane. We follow here very closely Stevenson's paper (3) on the torsion problem. We take the radii of the annulus as r_1 and r_2 with $r_2 > r_1$. We now write our two boundary constants as $C(r)$, $r = r_1, r_2$, where $C(r)$ is a function of r and its form is at our disposal. With the transformation

$$z = z(\sigma) = \sum_{-\infty}^{\infty} a_n\sigma^n, \quad z'(\sigma) \neq 0 \text{ or } \infty \text{ within } S, \tag{4·00}$$

the torsion function $\omega(\sigma)$ is given by

$$\omega(\sigma) = i\sum_{-\infty}^{\infty}{}' \alpha_n\sigma^n, \quad \alpha_n = \frac{\bar{b}_{-n}(r_2) - \bar{b}_{-n}(r_1)}{r_2^{2n} - r_1^{2n}}, \quad b_n(r) = \sum_{-\infty}^{\infty} a_{n+s}\bar{a}_s r^{2s}, \tag{4·01}$$

where Σ' omits the term $n = 0$.

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As in § 3, we write down certain combinations of the coefficients a_n which can be easily evaluated:

$$c_n(r) = \sum_{-\infty}^{\infty} (m+n)\, a_{m+n}\bar{a}_m r^{2m}, \quad j_n(r) = \sum_{-\infty}^{\infty} c_m(r)\, a_{n-m}, \tag{4·02}$$

$$f_n(r) = \sum_{-\infty}^{\infty} c_{n+m}(r)\, \bar{a}_m r^{2m}, \quad g_n(r) = \sum_{-\infty}^{\infty} b_m(r)\, b_{n-m}(r). \tag{4·03}$$

The coordinates of the centre of flexure are given by (2·16) and (2·17), where Γ is now $[2iC(r)]_{r_1}^{r_2}(z_G - z_{G_1})\, S_1$ from (2·10). The integrals P, Q, R, V in terms of these coefficients are

$$P = -2\pi i\left[\sum_{-\infty}^{\infty} \alpha_{-n} c_n(r)\right]_{r_1}^{r_2}, \quad Q = -2\pi i\left[\sum_{-\infty}^{\infty} \alpha_{-n} j_n(r)\right]_{r_1}^{r_2}, \quad R = -\pi i\left[\sum_{-\infty}^{\infty} \alpha_{-n} f_n(r)\right]_{r_1}^{r_2}, \tag{4·04}$$

$$V = iz_G\left[\sum_{-\infty}^{\infty} c_n(r)\, b_{-n}(r)\right]_{r_1}^{r_2} - \pi i\left[\sum_{-\infty}^{\infty} n g_{-n}(r)\, a_n\right]_{r_1}^{r_2}. \tag{4·05}$$

Combining these according to (2·16) and (2·17) we obtain the centre of flexure in terms of the coefficients a_n.

We now include as an example the centre of flexure of a hollow shaft whose cross-section boundaries are eccentric circles. This problem has been solved by Stevenson (4) using Cartesian flexure functions. We solve this now using the complex torsion function only. The results obtained agree with those of Stevenson.

Stevenson (3) has solved the torsion problem for this particular cross-section. We, therefore, follow him and write down as our transformation equation

$$z = \frac{c}{1-\sigma}, \quad |\sigma| < 1, \quad c \text{ real}. \tag{4·06}$$

This maps eccentric ring-space bounded by circles

$$(x-h_n)^2 + y^2 = k_n^2 \quad (n = 1, 2;\ h > k;\ k_2 > k_1;\ h_2 > h_1) \tag{4·07}$$

into annulus in the σ-plane whose boundaries are concentric circles with radii r_1, r_2 $(r_2 > r_1)$. The torsion function (1) is easily found to be

$$\omega(\sigma) = ic \sum_{-\infty}^{\infty}{}' \frac{h_2 r_2^{2n} - h_1 r_1^{2n}}{r_2^{2n} - r_1^{2n}}, \tag{4·08}$$

where we have written $h_n = h(r_n)$, $k_n = k(r_n)$ $(n = 1, 2)$. We can easily compute the various values required, i.e.

$$S = \pi(k_2^2 - k_1^2), \quad z_G = \frac{k_2^2 h_2 - k_1^2 h_1}{k_2^2 - k_1^2}, \quad A_0 = \tfrac{1}{4}\pi(k_2^4 - k_1^4), \tag{4·09}$$

$$B_0 = \tfrac{1}{4}\pi(k_2^4 - k_1^4) + \pi(h_2^2 k_2^2 - h_1^2 k_1^2), \quad \varpi_2 \text{ real}, \quad \varpi_1 = \frac{-i}{\tfrac{1}{4}\pi(k_2^4 - k_1^4)}. \tag{4·10}$$

We further write down the various combinations of a_n, where $n \geqslant 0$,

$$b_n(r) = ch, \quad b_{-n}(r) = chr^{2n}, \quad c_n(r) = nch + k^2, \quad c_{-n}(r) = r^{2n}k^2, \tag{4·11}$$

$$\alpha_n = \frac{c(h_2 r_2^{2n} - h_1 r_1^{2n})}{r_2^{2n} - r_1^{2n}}, \quad \alpha_{-n} = \frac{cd}{r_2^{-2n} - r_1^{-2n}}, \quad d = h_2 - h_1, \tag{4·12}$$

$$j_n(r) = \tfrac{1}{2}n(n+1)\, c^2 h + nck^2 + k^2 h, \quad j_{-n}(r) = k^2 h r^{2n}, \tag{4·13}$$

$$f_n(r) = nch^2 + 2k^2 h, \quad f_{-n}(r) = (nck^2 + 2k^2 h)\, r^{2n}, \tag{4·14}$$

$$g_{-n}(r) = 2ch^3 r^{2n} + (n-1)\, c^2 h^2 r^{2n}. \tag{4·15}$$

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Again
$$\int_S iz^2\bar{z}\,dS = \pi i[h^3k^2 + hk^4]_{r_1}^{r_2}. \tag{4·16}$$

Since the coefficients $\alpha_n, c_n(r), j_n(r), f_n(r)$ are all real, P, Q, R are imaginary. Hence, remembering that ϖ_2 is real, we see that the real part of (2·17) is zero. Hence $y_F = 0$, a result to be expected from symmetry. Also as $\varpi_1 = -\overline{\varpi}_1$, (2·16) gives

$$x_F = \tfrac{1}{2}\varpi_1\{\sigma(2Pz_G - Q - R + V + \Gamma) + (Q - R)\}. \tag{4·17}$$

If we write $\alpha_0 = 0$ in the following series we readily find

$$\left[\sum_{-\infty}^{\infty}\alpha_{-n}c_n(r)\right]_{r_1}^{r_2} = (k_2^4 - k_1^4) + c^2d^2\sum_{n=1}^{\infty}\frac{n}{r_2^{-2n} - r_1^{-2n}}, \tag{4·18}$$

$$\left[\sum_{-\infty}^{\infty}\alpha_{-n}j_n(r)\right]_{r_1}^{r_2} = (h_2k_2^4 - h_1k_1^4) + c^2d\sum_{n=1}^{\infty}\frac{\tfrac{1}{2}n(n+1)\,cd + n(k_2^2 - k_1^2)}{r_2^{-2n} - r_1^{-2n}}, \tag{4·19}$$

$$\left[\sum_{-\infty}^{\infty}\alpha_{-n}f_n(r)\right]_{r_1}^{r_2} = 3(h_2k_2^4 - h_1k_1^4) + c^2d\sum_{n=1}^{\infty}\frac{nd(h_2 + h_1) - n(k_2^2 - k_1^2)}{r_2^{-2n} - r_1^{-2n}}. \tag{4·20}$$

From (4·0) we obtain with some simplification

$$Q - R = \pi i\left\{(h_2k_2^4 - h_1k_1^4) - c^2d\sum_{n=1}^{\infty}\frac{n^2cd + 2n(k_2^2 - k_1^2)}{r_2^{-2n} - r_1^{-2n}}\right\}. \tag{4·21}$$

Again from (2·10) we find
$$\Gamma = 2\pi icdk_1^2(h_1 - z_G). \tag{4·22}$$

We now adopt the following abbreviations:

$$c^2\sum_1^{\infty}\frac{n}{r_1^{-2n} - r_2^{-2n}} = \Sigma_1, \quad c^3\sum_1^{\infty}\frac{n^2}{r_1^{-2n} - r_2^{-2n}} = \Sigma_2. \tag{4·23}$$

After some straightforward simplification with the use of a few radical axis properties we find that the centre of flexure (4·17) can be written

$$x_F = x_0 - \sigma(x_1 + x_2), \tag{4·24}$$

where
$$x_0 = \{d^2\Sigma_2 + 2d(k_2^2 - k_1^2)\,\Sigma_1 + (h_2k_2^4 - h_1k_1^4)\}/(k_2^4 - k_1^4),$$
$$x_1 = \{(k_2^4 - k_1^4) - 4d^2\Sigma_1\}\,(z_G - \tfrac{1}{2}c)/(k_2^4 - k_1^4),$$
$$x_2 = \{d^2\Sigma_2 + 2d(h_2^2 - k_1^2)\,\Sigma_1 - (k_2^4h_2' - k_1^4h_1')\}/(k_2^4 - k_1^4),$$
$$h_2' = h_2 - \tfrac{1}{2}c, \quad h_1' = h_1 - \tfrac{1}{2}c,$$

in agreement with Stevenson (4), after making the necessary transfer of origin and identification of corresponding quantities.

As a further example of centre of flexure of doubly-connected cross-sections we solve for elliptic limaçons. Consider $\xi = \log_e r$ in the system of curvilinear coordinates

$$\zeta = \xi + i\eta, \quad \sigma = e^{\zeta}, \quad z = c\sigma(1 + \sigma), \quad c \text{ real}. \tag{4·25}$$

The equation to this boundary in the σ-plane is $\sigma = re^{i\eta}$. Writing $z_1 = z + cr^2 = Re^{i\theta}$ we find as in § 3

$$R = a + b\cos\theta, \quad \text{where} \quad a = cr, \quad b = \frac{2a^2}{c}. \tag{4·26}$$

The critical point of this system is given by $\sigma = -\tfrac{1}{2}$. Hence if we take $\tfrac{1}{2} > r_2 > r_1$, the annulus bounded by concentric circles with centres at the origin and radii r_2, r_1 in the σ-plane is mapped by an annulus bounded by elliptic limaçons in the z-plane.

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The analysis is now similar to that carried out in § 3. We quote the following results, which are easily obtained by the method of residues:

$$S = [\pi c^2(r^2+2r^4)]_{r_1}^{r_2}, \quad z_{GS} = \pi c^3(r_2^4 - r_1^4), \tag{4·27}$$

$$H_0 = 0, \quad B_0 = A_0 = [\tfrac{1}{4}\pi c^4 r^4(1+6r^2+2r^4)]_{r_1}^{r_2}, \tag{4·28}$$

$$\varpi_2 \text{ is real}, \quad \varpi_1 = -\frac{i}{2A_0}. \tag{4·29}$$

The transformation constants a_n on comparison with (4·00) give $a_1 = c$, $a_2 = c$ as the only non-vanishing terms. The only non-vanishing constants for $b_n(r)$, α_n, are

$$b_0(r) = c^2(r^2+r^4), \quad b_{-1}(r) = c^2r^4, \quad b_1(r) = c^2r^2, \tag{4·30}$$

$$\alpha_1 = c^2(r_2^2+r_1^2), \quad \alpha_{-1} = -c^2r_1^2r_2^2, \tag{4·31}$$

and the torsion function $$\omega(\sigma) = i\alpha_1\sigma + i\alpha_{-1}\sigma^{-1}. \tag{4·32}$$

The integrals P, Q, R and V are best evaluated directly

$$P = [-2\pi ic^2r^2(\alpha_1 r^2 + 2\alpha_{-1})]_{r_1}^{r_2}, \quad Q = [-2\pi ic^3r^2(1+3r^2)\alpha_{-1}]_{r_1}^{r_2},$$

$$R = [-\pi ic^3r^4(\alpha_1 + 4\alpha_1 r^2)]_{r_1}^{r_2}, \quad V = 4iz_G A_0 + [-2\pi ic^5r^6(1+3r^2)]_{r_1}^{r_2}.$$

The boundary constant $C(r) = -b_0(r) = -c^2(r^2+r^4)$. Hence

$$\Gamma = \frac{\pi c^3}{S}[-2ic^2(r^2+r^4)]_{r_1}^{r_2}(S_1r_2^4 - S_2r_1^4).$$

Following an argument similar to that used for the eccentric circles, or from symmetry, $y_F = 0$ and

$$x_F = \tfrac{1}{2}\varpi_1\{\sigma(2P_1z_G - Q_1 - R_1 + V + \Gamma) + (Q_1 - R_1)\}.$$

We have computed results for two particular cases of this type of cross-section:

(i) $c = 100$, $r_2 = 0{\cdot}25$, $r_1 = 0{\cdot}2$. This leads to equations

$$R_2 = 25 + 12{\cdot}5\cos\theta, \quad R_1 = 20 + 8\cos\theta,$$

where R_1, R_2 denote the radii vectors for the inner and outer limaçons respectively:

$$x_F = 13{\cdot}400 - 3{\cdot}755\sigma.$$

(ii) $c = 100$, $r_2 = 0{\cdot}3$, $r_1 = 0{\cdot}2$. This leads to equations

$$R_2 = 30 + 18\cos\theta, \quad R_1 = 20 + 8\cos\theta, \quad x_F = 15{\cdot}930 - 4{\cdot}197\sigma.$$

An interesting point arises here. $\sigma = 0$ gives centre of least strain. But σ has maximum $\frac{1}{3}$, and the contribution of the term in σ is now no longer negligible, as has been the case in other centres which, so far as the writer knows, have been calculated.

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