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Almost hypoelliptic differential operators with variable coefficients

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1. *Introduction.* We have called a linear partial differential operator P almost hypoelliptic in some of the variables if the distribution u is strongly regular in those variables in every open subset where this is true of Pu. In an earlier paper (1) we have characterised almost hypoelliptic operators with constant coefficients. Following techniques of Hörmander (2) and Yoshikawa (5) we obtain in this paper some *a priori* inequalities for almost hypoelliptic operators and derive results about the solvability of the adjoint operator P^*. Our notation for various standard distribution spaces and multi-indices follows Hörmander (2).

The definitions are now given more carefully.

Notation 1·1. We shall work in $M+N$ dimensional real space R^{M+N} and consider the variables as belonging to the two distinct subsets R^M, whose points will be denoted by $x = (x_1 \dots x_M)$, and R^N, whose points will be denoted by $y = (y_1 \dots y_N)$. Thus (x, y) denotes a point of R^{M+N}. In the same way we shall write points in the $M+N$ dimensional complex space as (ξ, η), where $\xi \in C^M$ and $\eta \in C^N$, and ξ and η will be written as sums of real and imaginary vectors as $\xi = \xi' + i''\xi$, $\eta = \eta' + i\eta''$ respectively.

Definition 1·2. If K is a compact subset of R^{M+N} with non-empty interior and r, s are any real numbers then $H^{r,s}(K)$ is the completion of $C_0^\infty(K)$ with respect to the norm $||\ ||_{r,s}$, where

$$||f||_{r,s}^2 = \int_{R^{M+N}} (1+|\xi'|^2)^r (1+|\eta'|^2)^s |\hat{f}(\xi', \eta')|^2 d\xi' d\eta',$$

for $f \in C_0^\infty(K)$, and $\hat{f}$ is the Fourier transform of f.

Notation 1·3. If U, V are open sets in R^{M+N}, by $U \subset\subset V$ we shall mean that the closure $\overline{U}$ of U is compact and contained in V.

If U is an open set in R^{M+N} then there is a sequence of open sets U_i, $i = 1, 2, \dots$ such that

$$U_i \subset\subset U_{i+1} \subset\subset \dots \subset\subset U$$

and

$$U = \bigcup_i U_i.$$

There is then a corresponding sequence of functions $\chi_i \in C_0^\infty(U)$ such that

$$\chi_i \equiv 1 \quad \text{on} \quad \overline{U}_i$$

and

$$\chi_i \equiv 0 \quad \text{outside } U_{i+1}.$$

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For any $\mu \in \mathscr{D}'(U)$ and any χ_i there are real numbers r, s such that

$$\chi_i \mu \in H^{r,s}(\overline{U}_{i+1}).$$

If r and s are independent of i we say that

$$\mu \in H^{r,s}_{\text{loc}}(U).$$

$H^{r,s}_{\text{loc}}(U)$ is a Fréchet space with the semi-norms $||\chi_i \mu||_{r,s}$.

DEFINITION 1·4. Suppose U is an open subset of R^{M+N}. A distribution $f \in \mathscr{D}'(U)$ is said to be strongly regular in x if to every subset $V \subset\subset U$ there is an integer k such that for every M-tuple $p = (p_1 \dots p_M)$, $D^p_x f$ is of order $\leqslant k$ in V. Schwartz (4) calls such distributions 'intégralement sémi-régulière' and obtains the following characterization:

LEMMA 1·5. *A distribution $f \in \mathscr{D}'(U)$ is strongly regular in x if and only if for each $\phi \in C^\infty_0(U)$ there are functions $g_q(x, y)$ such that*

$$\phi f = \sum_{|q| \leqslant k} D^q_y g_q(x, y),$$

where the g_q are functions of $(x, y) \in U \subset R^{M+N}$ with compact support, arbitrarily differentiable in x, and such that each derivative in an x-variable is a continuous function of x and y.

LEMMA 1·6. *Suppose $f \in \mathscr{E}'(U)$ is a distribution with compact support in $U \subset R^{M+N}$. Then f is strongly regular in x if and only if there is a fixed real number t such that*

$$f \in H^{s,t},$$

(the completion of $C^\infty_0(R^{M+N})$ with respect to the norm $||\ ||_{s,t}$*), for all $s \in R$.*

Proof. This follows immediately from the description of such an f in Lemma 1·5.

DEFINITION 1·7. For an open set $U \subset R^{M+N}$ we consider U as a union of open subsets U_i with associated functions χ_i as in 1·3.

The space $H^{\infty,t}_{\text{loc}}(U)$ is then the space of distributions on U strongly regular in x and of fixed order t in y in every compact subset $V \subset\subset U$.

$f \in H^{\infty,t}_{\text{loc}}(U)$ means in particular, therefore, that the distribution $D^p_x(\chi_i f)$ is always of order t, for all i and all derivatives D^p_x in the x-variables.

Alternatively

$$f \in H^{\infty,t}_{\text{loc}}(U) \Rightarrow (\chi_i f) \in H^{s,t} \text{ for all } i \text{ and all } s,$$

$$\Rightarrow f \in H^{s,t}_{\text{loc}}(U) \text{ for all } s.$$

$H^{\infty,t}_{\text{loc}}(U)$ is a Fréchet space with the semi-norms $||\chi_i f||_{s,t}$, $i = 1, 2, \dots$, $s = 1, 2, \dots$.

DEFINITION 1·8. A linear partial differential operator $P(x, y, D_x, D_y)$, with coefficients C^∞ functions of $(x, y) \in R^{M+N}$, is said to be almost hypoelliptic in x on the open set $U \subset R^{M+N}$ if the distribution $u \in \mathscr{D}'(U)$ is strongly regular in x for every open subset of U where $P(x, y, D_x, D_y)u$ is strongly regular in x.

Finally we need

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THEOREM 1·9. *For any real numbers p, s, t and any compact set $K \subset R^{M+N}$ with non-empty interior, the natural injection*

$$H^{t,p}(K) \to H^{s,p}(K)$$

is compact if $t > s$.

Proof (adapted from Malgrange (3)). It is sufficient to show that any sequence $\{\phi_m\} \subset H^{t,p}(K)$ such that $||\phi_m||_{t,p} \leqslant 1$ and which converges weakly to zero in $H^{s,p}(K)$ converges strongly to zero in $H^{s,p}(K)$.

However, given $\epsilon > 0$, as $t > s$ there is a number $M > 0$ such that $(1+|\xi'|^2)^{s-t} \leqslant \epsilon$ if $|\xi'| > M$. Therefore

$$\begin{aligned}\int_{|\xi'|\geqslant M}\int_{|\eta'|\geqslant M} &|\hat{\phi}_m(\xi',\eta')|^2(1+|\xi'|^2)^s(1+|\eta'|^2)^p\,d\xi'\,d\eta' \\ &= \int_{|\xi'|\geqslant M}\int_{|\eta'|\geqslant M} |\hat{\phi}_m(\xi',\eta')|^2(1+|\xi'|^2)^t(1+|\xi'|^2)^{s-t}(1+|\eta'|^2)^p\,d\xi'\,d\eta' \\ &\leqslant \epsilon\cdot||\phi_m||^2_{t,p} \leqslant \epsilon.\end{aligned}$$

Because the sequence ϕ_m converges weakly to zero in $H^{s,p}(K)$, the ϕ_m have their support in the fixed compact set K, and so the ϕ_m converge strongly to zero in $\mathscr{E}'(R^{M+N})$. For $|\xi'| \leqslant M$, $|\eta'| \leqslant M$ the functions $\exp 2\pi i(x\cdot\xi' + y\cdot\eta')$ lie in a bounded set of $\mathscr{E}(R^{M+N})$, so the functions

$$\psi_m(\xi',\eta') = |\hat{\phi}_m(\xi',\eta')|^2(1+|\xi'|^2)^s(1+|\eta'|^2)^p$$

converge uniformly to zero as $m \to \infty$, when $|\xi'| \leqslant M$, $|\eta'| \leqslant M$. Therefore the integrals

$$\int_{|\xi'|\leqslant M}\int_{|\eta'|\leqslant M} \psi_m(\xi',\eta')\,d\xi'\,d\eta'$$

converge to zero as $m \to \infty$, and so

$$||\phi_m||_{s,p} \to 0.$$

2. *Some* a priori *inequalities.* DEFINITION 2·1. Following Yoshikawa (5), if P is a linear operator from the vector space E to the vector space F and if E_1, F_1 are subspaces of E and F respectively, then we write $\hat{P}: E_1 \to F_1$ for the restriction of P to $E_1 \cap P^{-1}(F_1)$.

THEOREM 2·2. *If U is an open subset of R^{M+N} and $P = P(x, y, D_x, D_y)$ is a differential operator of order m, then for any real numbers s, p the restriction*

$$\hat{P}: H^{s,p}_{\mathrm{loc}}(U) \to H^{\infty,p-m}_{\mathrm{loc}}(U)$$

is closed and $$G = \{(f, Pf) : f \in H^{s,p}_{\mathrm{loc}}(U),\ Pf \in H^{\infty,p-m}_{\mathrm{loc}}(U)\}$$

is a Fréchet space.

Proof. If $x_n \to x$ in $H^{s,p}_{\mathrm{loc}}(U)$ and

$$Px_n \to y \text{ in } H^{\infty,p-m}_{\mathrm{loc}}(U) = \bigcap_{r\in R} H^{r,p-m}_{\mathrm{loc}}(U),$$

then certainly $$Px_n \to Px \text{ in } H^{s-m,p-m}_{\mathrm{loc}}(U) \Rightarrow Px = y \in H^{r,p-m}_{\mathrm{loc}}(U).$$

Therefore P is closed, and so its graph G is closed. As G is a closed subspace of the Cartesian product of the Fréchet spaces $H^{s,p}_{\mathrm{loc}}(U)$ and $H^{\infty,p-m}_{\mathrm{loc}}(U)$, G is itself a Fréchet space.

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THEOREM 2·3. *In the situation of Theorem 2·2 suppose P is almost hypoelliptic in x and suppose Q is any second differential operator of order n.*

Again consider the restrictions

$$\hat{P}: H^{s,p}_{\mathrm{loc}}(U) \to H^{\infty,p-m}_{\mathrm{loc}}(U),$$

$$\hat{Q}: H^{s,p}_{\mathrm{loc}}(U) \to H^{\infty,p-n}_{\mathrm{loc}}(U).$$

Then for any $\chi_i \in C^\infty_0(U)$ and $t \in R$ there exist χ_j, χ_k, $t' \in R$ and $C < \infty$ such that

$$||\chi_i Q\phi||_{t,p-n} \leqslant C\{||\chi_j \phi||_{s,p} + ||\chi_k P\phi||_{t',p-m}\} \tag{2·1}$$

for all $\phi \in H^{\infty,p}_{\mathrm{loc}}(U)$ (and so certainly for all $\phi \in C^\infty(U)$).

Proof. Because P is almost hypoelliptic in x the domain of $\hat{P}$ is contained in $H^{\infty,p}_{\mathrm{loc}}(U) \subset H^{s,p}_{\mathrm{loc}}(U)$, so the domain of $\hat{P}$ is contained in the domain of $\hat{Q}$. It is then easy to check that the map from the Fréchet space G into the Fréchet space $H^{\infty,p-n}_{\mathrm{loc}}(U)$ given by

$$(f, Pf) \to Qf$$

is closed, and so continuous by the closed graph theorem. In terms of the semi-norms this continuity is just the inequality (2·1).

COROLLARY 2·4. *If V is any open set $V \subset\subset U$ and if P is almost hypoelliptic in x and of order m, then for any $p, s, t \in R$ there is a $t' \in R$ and a (new) constant $C < \infty$ such that*

$$||\phi||_{t,p} \leqslant C\{||\phi||_{s,p} + ||P\phi||_{t',p-m}\} \tag{2·2}$$

for all $\phi \in C^\infty_0(V)$.

Proof. We take the Q of Theorem 2·3 as the identity map, and so of order zero, and choose χ_i so that $\chi_i \equiv 1$ on $\overline{V}$. Recalling that multiplication by χ_j (resp. χ_k) $\in C^\infty_0(U)$ is a continuous operation in $H^{s,p}_{\mathrm{loc}}(U)$ (resp. $H^{t',p-m}_{\mathrm{loc}}(U)$), inequality (2·2) is obtained immediately from inequality (2·1), with possibly a new constant C.

3. *Some results on solvability for almost hypoelliptic operators.* The results of this section extend those of Yoshikawa (5). We shall continue to investigate an operator P almost hypoelliptic in x defined on an open set $U \subset R^{M+N}$. For $V \subset\subset U$ and any given real numbers s, p let t run through the integers greater than s, and for each such t let $\tau(t) \geqslant t'$ be a positive integer such that inequality (2·2) is satisfied, and such that $\tau(t+1) > \tau(t)$.

LEMMA 3·1. *In the notation of definition 2·1 the operator*

$$P_t = \hat{P}: H^{t,p}(\overline{V}) \to H^{\tau(t),p-m}(\overline{V})$$

is closed.

Proof. If $t - m \geqslant \tau(t)$ the proof is trivial, whilst if $t - m < \tau(t)$ the proof is like Theorem 2·2.

If $D_t \subset H^{t,p}(\overline{V})$ denotes the domain of P_t then passing to the limit in inequality (2·2) we have

$$||\phi||_{t,p} \leqslant C_t\{||\phi||_{s,p} + ||P\phi||_{\tau(t)\,p-m}\} \tag{3·1}$$

for $C_t = C < \infty$ and all $\phi \in D_t$.

From the closure of P_t we have that $D_{t+1} \subset D_t$ and $\phi \in D_{t+1} \Rightarrow P_{t+1}\phi = P_t\phi$.

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LEMMA 3·2. (i) *N_t, the null space of P_t, is finite dimensional, and so we can write*

$$H^{t,p}(\bar{V}) = N_t \oplus M_t.$$

(ii) *There is a constant C, $0 < C < \infty$, such that for all $t > s$ and all $\phi \in D \cap M_t$*

$$||\phi||_{t,p} \leqslant C||P\phi||_{\tau(t),p-m}.$$

(iii) *R_t, the range of P_t, is closed.*

Proof. (i) For $\phi \in N_t$ we have $||\phi||_{t,p} \leqslant C_t||\phi||_{s,p}$,

and as $t > s$ the injection $H^{t,p}(\bar{V}) \to H^{s,p}(\bar{V})$ is compact by Theorem 1·9, so N_t is finite dimensional.

(ii) Suppose no inequality of this form holds, so for each integer k there is a $\phi_k \in M_t \cap D_t$ such that

$$||\phi_k||_{t,p} \geqslant k||P_t\phi_k||_{\tau(t),p-m} \tag{3·2}$$

and

$$||\phi_k||_{t,p} = 1.$$

Because the injection $H^{t,p}(\bar{V}) \to H^{s,p}(\bar{V})$ is compact there is a subsequence $\{\phi_{k'}\}$ of $\{\phi_k\}$, such that $\phi_{k'} \to \psi$ in $H^{s,p}(\bar{V})$, for some $\psi \in H^{s,p}(\bar{V})$. But (3·2) implies $||P_t\phi_{k'}||_{\tau(t),p-m} \to 0$, so we see from (3·1) that $\psi \in H^{t,p}(\bar{V})$ and $\phi_{k'} \to \psi$ in $H^{t,p}(\bar{V})$. As P_t is closed, $\psi \in D_t$ and $P_t\psi = 0$. But M_t closed implies that $\psi \in M_t$, and

$$||\phi_{k'}||_{t,p} = 1 \Rightarrow \psi = 0.$$

Therefore, $\psi \in M_t \cap D_t$, $\psi \neq 0$ and $P_t\psi = 0$, a contradiction, so an inequality of the desired form must hold.

(iii) The closure of R_t follows from this inequality.

COROLLARY 3·3. *We note that if $\phi \in C_0^\infty(\bar{V}) \subset H^{t,p}(\bar{V})$ is such that*

$$P_t\phi = 0 \in H^{\tau(t),p-m}(\bar{V})$$

then $\phi \in N_t$. So the null space N of P acting on $C_0^\infty(\bar{V})$ is finite dimensional.

DEFINITION 3·4. The adjoint operator of a linear differential operator P is the operator P^* which is such that

$$\int (Pu)v = \int u(P^*v)$$

for all $u, v \in C_0^\infty(\bar{V})$.

THEOREM 3·5. *If P is almost hypoelliptic in x of degree m and $V \subset\subset U \subset R^{M+N}$, suppose $f \in H^{-t,-p}(\bar{V})$ is such that $\langle f, v\rangle = 0$ for all $v \in N$. Then there is a $g \in H^{-\tau(t),m-p}$ such that*

$$P^*g = f \text{ in } V.$$

Proof. Consider the functional on $P(C_0^\infty(\bar{V}))$ given by $\langle g, Pw\rangle = \langle f, w\rangle$ for $w \in C_0^\infty(\bar{V})$. As the null space N is finite dimensional we can express $C_0^\infty(\bar{V})$ as a direct sum

$$C_0^\infty(\bar{V}) = N \oplus M,$$

where $M \subset D_t \cap M_t$ for all t. Any $w \in C_0^\infty(\bar{V})$ can be written in a unique way as $w = u + v$, $u \in M$, $v \in N$. By hypothesis $\langle f, v\rangle = 0$, and certainly $Pv = 0$, so

$$\begin{aligned}
|\langle g, Pw\rangle| &= |\langle f, w\rangle| = |\langle f, u\rangle| \\
&\leqslant ||f||_{-t,-p}||u||_{t,p} \\
&\leqslant C||f||_{-t,-p}||Pu||_{\tau(t),p-m} \quad \text{by Theorem 3·2 (ii).} \\
&= C||f||_{-t,-p}||Pw||_{\tau(t),p-m}
\end{aligned}$$

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Therefore g is a continuous linear functional on $P(C_0^\infty(\bar{V}))$ and so, by the Hahn–Banach theorem, g can be extended to a continuous linear functional on $H^{\tau(t),p-m}(\bar{V})$. In other words, there is a $g \in H^{-\tau(t),m-p}$ with $||g||_{-\tau(t),m-p} \leqslant C||f||_{-t,-p}$ such that

$$P^*g = f \text{ in } V.$$

THEOREM 3·6. *Suppose P is almost hypoelliptic in x on $U \subset R^{M+N}$ as above. Then for each $s \geqslant -\frac{1}{2}n$ each point $a \in U$ has a sufficiently small compact neighbourhood N such that*

$$||\phi||_{s,p} \leqslant C||P\phi||_{t',p-m},$$

for some $t' \in R$, $C < \infty$, and for all $\phi \in C_0^\infty(N)$.

Proof. Consider a compact neighbourhood $V \subset\subset U$ of a. Then Corollary 2·4 tells us there is a constant $C < \infty$ and $t' \in R$ such that

$$||\phi||_{t,p} \leqslant C\{||\phi||_{s,p} + ||P\phi||_{t',p-m}\},$$

with $t > s$, for all $\phi \in C_0^\infty(V)$. However, as $t > s \geqslant -\frac{1}{2}n$ we have by an immediate adaptation of a Lemma of Malgrange ((3) Lemma 1·2·3) that each point $a \in U$ has a compact neighbourhood K such that

$$||\phi||_{s,p} \leqslant (1+C)^{-1}||\phi||_{t,p}$$

for all $\phi \in C_0^\infty(K)$. Putting $N = K \cap V$ we have for all $\phi \in C_0^\infty(N)$ that:

$$(1+C)||\phi||_{s,p} \leqslant ||\phi||_{t,p} \leqslant C||\phi||_{s,p} + C||P\phi||_{t',p-m}.$$

Therefore

$$||\phi||_{s,p} \leqslant C||P\phi||_{t',p-m}.$$

THEOREM 3·7. *Suppose f is any distribution in $H_{\text{loc}}^{-s,-p}(U)$, where $s \geqslant 0 > -\frac{1}{2}n$, and suppose P is almost hypoelliptic in x as above. Then for every $a \in U$ there is a neighbourhood N, $t' \in R$ and a distribution $g \in H_{\text{loc}}^{-t',m-p}(U)$ such that*

$$P^*g = f \text{ in } N.$$

Proof. Given $a \in U$, as $s \geqslant 0 > -\frac{1}{2}n$ we take N to be the neighbourhood of a obtained in the above theorem. So for $\phi \in C_0^\infty(N)$, $P\phi = 0 \Rightarrow \phi = 0$. Therefore, as in Theorem 3·5, we can define a linear functional on $P((C_0^\infty(N))$ by

$$\langle g, P\phi \rangle = \langle f, \phi \rangle. \tag{3·3}$$

So

$$|\langle g, P\phi \rangle| = |\langle f, \phi \rangle| \leqslant ||\chi f||_{-s,-p}||\phi||_{s,p},$$

where $\chi \in C_0^\infty(U)$ is $\equiv 1$ on N,

$$\leqslant C||\chi f||_{-s,-p}||P\phi||_{t',p-m}$$

by Theorem 3·6.

Thus g is continuous on $P(C_0^\infty(N))$ and so can be extended to a continuous linear functional on $H^{t',p-m}(N)$. Also (3·3) implies that

$$P^*g = f \text{ in } N.$$

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