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ARTICLE

Antidirected subgraphs of oriented graphs

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Abstract

We show that for every $\eta > 0$ every sufficiently large n-vertex oriented graph D of minimum semidegree exceeding $(1+\eta)\frac{k}{2}$ contains every balanced antidirected tree with k edges and bounded maximum degree, if $k \geq \eta n$. In particular, this asymptotically confirms a conjecture of the first author for long antidirected paths and dense digraphs.

Further, we show that in the same setting, D contains every k-edge antidirected subdivision of a sufficiently small complete graph, if the paths of the subdivision that have length 1 or 2 span a forest. As a special case, we can find all antidirected cycles of length at most k.

Finally, we address a conjecture of Addario-Berry, Havet, Linhares Sales, Reed, and Thomassé for antidirected trees in digraphs. We show that this conjecture is asymptotically true in n-vertex oriented graphs for all balanced antidirected trees of bounded maximum degree and of size linear in n.

Keywords: oriented graph; tree; semidegree; cycle; path; subdivision; digraph

2020 MSC Codes: Primary: 05C20; Secondary: 05C07, 05C35

1. Introduction

A typical question in extremal graph theory is whether information on the degree sequence of a graph G can be used to find some specific subgraph in G. A famous example is Dirac's theorem [9], which states that any n-vertex graph G of minimum degree $\delta(G) \geq \frac{n}{2}$ contains a spanning cycle. A minimum degree of at least $\frac{n-1}{2}$ guarantees a spanning path. Such a result also holds for shorter paths: Already Dirac, as well as Erdős and Gallai, observed that if a graph G has a connected component with at least $k+1$ vertices and $\delta(G) \geq \frac{k}{2}$, then G contains a k-edge path (see [10]). Note that we need to require that G has a sufficiently large component as otherwise G could be the disjoint union of complete graphs of order k.

Considering the same question for oriented graphs D, we replace the minimum degree with the minimum semidegree $\delta^0(D)$, which is defined as the minimum over all the in- and all the out-degrees of the vertices. Note that any oriented graph of minimum semidegree $\delta^0(D) > \frac{k}{2}$ has an underlying connected graph on more than k vertices (and therefore, no extra condition on large components will be necessary). Jackson [15] showed that every oriented graph D with $\delta^0(D) > \frac{k}{2}$ contains the directed path on k edges, which is best possible. The first author conjectured that this result extends to any orientation of the path.

Conjecture 1.1. [29] *Every oriented graph D with $\delta^0(D) > \frac{k}{2}$ contains every oriented path on k edges.*

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This conjecture is best possible for odd k, as a regular tournament on k vertices does not contain any oriented path with k edges. A different example works for *antidirected paths*, that is, oriented paths whose edges alternate directions (more generally, an *antidirected graph* is an oriented graph having no 2-edge directed path). Note that in an ℓ-blow-up of the directed triangle (where each vertex is replaced with ℓ independent vertices), any longest antidirected path can only cover 2ℓ vertices.

It is known [22] that if $\delta^0(D) > \frac{3k-2}{4}$, then the oriented graph D contains each antipath with k edges. Further, if we replace the bound with $\delta^0(D) \geq k$, Conjecture 1.1 becomes very easy (a greedy embedding strategy suffices). This strategy also works if we wish to find a k-edge oriented tree instead of an k-edge oriented path. Moreover, in this case we cannot do any better: In the $(k-1)$-blow-up of a directed triangle, each vertex has semidegree $k-1$, but no antidirected star with k edges is present. However, the antidirected star is very unbalanced. The situation might be different for *balanced* antidirected trees T, that is, those that have as many vertices of in-degree 0 as of out-degree 0. We suspect that these trees appear whenever the host oriented graph D has minimum semidegree greater than $\frac{k}{2}$.

We show that this is asymptotically true if D is dense and T is large and has bounded maximum degree, where the *maximum degree* $\Delta(T)$ of an oriented tree T is defined as the maximum degree of the underlying undirected tree.

Theorem 1.2. *For all $\eta \in (0,1)$ and $c \in \mathbb{N}$ there is n_0 such that for all $n \geq n_0$ and $k \geq \eta n$, every oriented graph D on n vertices with $\delta^0(D) > (1+\eta)\frac{k}{2}$ contains every balanced antidirected tree T with k edges and with $\Delta(T) \leq (\log(n))^c$.*

Instead of Theorem 1.2 we will prove a slightly stronger version, namely Theorem 5.2. This is necessary for a later application of the result in the proof of Theorem 1.9 below. Theorem 5.2 allows us to specify a subset of $V(D)$ where the root of T will be mapped, a feature that might be of independent interest.

Theorem 1.2 can be interpreted as a version for smaller trees of a recent result by Kathapurkar and Montgomery [17] (and the preceding result in [27]). In [17] it is shown that for each $\eta > 0$, there is some $c > 0$ such that every sufficiently large n-vertex directed graph with minimum semidegree at least $(\frac{1}{2}+\eta)n$ contains a copy of every n-vertex oriented tree whose underlying maximum degree is at most $c\frac{n}{\log n}$. This generalises a well-known theorem of Komlós, Sárközy, and Szemerédi [23] for graphs. The result from [17] cannot hold for smaller trees and accordingly smaller semidegree without adding further conditions, because of the example of the antidirected star from above.

Turning back to our original question for oriented paths, note that Theorem 1.2 immediately implies that Conjecture 1.1 holds asymptotically for large antidirected paths in large graphs. (If the path is not balanced, we can extend it by one vertex and apply Theorem 1.2 with a sufficiently smaller η.)

Corollary 1.3. *For all $\eta \in (0,1)$ there is n_0 such that for all $n \geq n_0$ and $k \geq \eta n$ every oriented graph D on n vertices with $\delta^0(D) > (1+\eta)\frac{k}{2}$ contains every antidirected path with k edges.*

How about asking for oriented cycles instead of oriented paths or trees? Keevash, Kühn, and Osthus [18] showed that every sufficiently large n-vertex oriented graph G of minimum semidegree $\delta^0(G) \geq \frac{3n}{8}$ contains a directed Hamilton cycle, and Kelly [19] showed that asymptotically, the same bound guarantees any orientation of a Hamilton cycle. Kelly, Kühn, and Osthus [20] extended this to a pancyclicity result, proving that every oriented graph D of minimum semidegree $\delta^0(D) \geq \frac{3n}{8} + o(n)$ contains every orientation of every cycle (of any length between 3 and n). These authors also show that for cycles of constant length the bounds can be improved, but the improvement depends on the *cycle type*: the number of forward edges minus the number of

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backwards edges of the cycle. For some cycle types and lengths, a semidegree exceeding $\frac{n}{3}$ is necessary: for instance, if 3 does not divide k, a blowup of the directed triangle has minimum semidegree $\frac{n}{3}$ but does not contain a directed C_k. For cycle type 0 the bound on the minimum semidegree can be much lower: For each $k \geq 3$ and $\eta > 0$ every large enough n-vertex oriented graph of minimum semidegree at least ηn contains all oriented cycles of length at most k and cycle type 0 [20].

As antidirected cycles have cycle type 0, the discussion from the previous paragraph implies that we can find short antidirected cycles with a minimum semidegree of ηn and antidirected cycles of any length with a minimum semidegree exceeding $\delta^0(D) \geq \frac{3n}{8} + o(n)$. In light of Theorem 1.2 it seems natural to suspect that an intermediate bound on the minimum semidegree could suffice for antidirected cycles of medium length. This is indeed possible, as our next result shows.

Theorem 1.4. *For all $\eta \in (0,1)$ there is n_0 such that for all $n \geq n_0$ and $k \geq \eta n$, every oriented graph D on n vertices with $\delta^0(D) > (1+\eta)\frac{k}{2}$ contains any antidirected cycle of length at most k.*

We will in fact prove a more general result, Theorem 1.7 below. Theorem 1.7 focuses on embedding antidirected subdivisions of complete graphs. Before stating Theorem 1.7, let us review the history of (oriented) subdivisions in (oriented) graphs of high minimum (semi)degree.

For undirected graphs, Mader [26] proved that there is a function $g(h)$ such that every graph with minimum degree at least $g(h)$ contains a subdivision of the complete graph K_h. Thomassen [32] showed that a direct translation of this result to digraphs is not true: For every function $g(h)$ there is a digraph of minimum outdegree $g(h)$ that does not contain a subdivision of the complete digraph on $h \geq 3$ vertices. (A subdivision of a digraph D substitutes each edge of D with a directed path.) Mader [25] modified Thomassen's construction showing that the same is true if we replace the minimum outdegree with the minimum semidegree. Crucially, the constructed digraphs have no even directed cycles (which appear in subdivisions of K_3). Following these discoveries, Mader suggested to replace the subdivision of the complete digraph with the *transitive tournament*, that is, the tournament without directed cycles. He conjectured the following.

Conjecture 1.5 (Mader [25]). *There is a function $f(h)$ such that every digraph of minimum outdegree at least $f(h)$ contains a subdivision of the transitive tournament of order h.*

This conjecture is open even for $h = 5$. Recently, there has been much activity on variants of Conjecture 1.5, see for instance [1, 8, 11, 12, 13, 16, 24, 28, 30]. Aboulker, Cohen, Havet, Lochet, Moura, and Thomassé [1] observed that Conjecture 1.5 is equivalent to the following conjecture.

Conjecture 1.6. [1] *There is a function $f(h)$ such that every digraph of minimum semidegree at least $f(h)$ contains a subdivision of a transitive tournament of order h.*

We will show a result along the lines of Conjectures 1.5 and 1.6 for oriented graphs. Namely, we will prove that any oriented graph of large minimum semidegree contains an antidirected orientation of a subdivision of a complete graph K_h, where h cannot be too large. We can even choose the lengths of the antidirected paths in the subdivision, as long as the very short paths do not span a cycle. To make this more precise, we need some notation.

For $h, k \in \mathbb{N}$, consider a subdivision of K_h where each edge $e \in E(K_h)$ is substituted by a path of length $g(e)$, with $\sum_{e \in E(K_h)} g(e) = k$. Call any antidirected orientation of this graph a *k-edge antisubdivision of K_h*. If furthermore the edges of K_h with $g(e) < 3$ induce a forest in K_h, we say the k-edge antisubdivision is *long*. Note that in particular, if all edges of K_h are substituted with paths of length at least 3, the resulting antisubdivision is long.

Theorem 1.7. *For all $\eta \in (0,1)$ there are $n_0 \in \mathbb{N}$ and $\gamma > 0$ such that for each $n \geq n_0$, each $h \leq \gamma\sqrt{n}$ and each $k \geq \eta n$ the following holds. Every oriented graph D on n vertices with $\delta^0(D) > (1+\eta)\frac{k}{2}$ contains each long k-edge antisubdivision of K_h.*

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As indicated above, Theorem 1.7 quickly implies Theorem 1.4.[1]

Let us now turn back to oriented trees. Looking for parameters related to the appearance of certain oriented trees in a digraph, a natural alternative to the minimum semidegree is the edge density. In 1970, Graham [14] confirmed a conjecture he attributes to Erdős: for every antidirected tree T there is a constant c_T such that every sufficiently large directed graph D on n vertices and with at least $c_T n$ edges contains T. A similar statement does not hold for other oriented trees: A bipartite graph on sets A and B, with every edge oriented from A to B, has $|A||B|$ edges, and only has antidirected subgraphs [2, 7].

In 1982, Burr [7] gave an improvement of Graham's result: Every n-vertex digraph D with more than $4kn$ edges contains each antidirected tree T on k edges. He obtains this result by greedily embedding T into a suitable bipartite subgraph of D. Burr states that the bound $4kn$ on the number of edges can 'almost certainly be made rather smaller' and provides an example where $(k-1)n$ edges are not sufficient: The k-edge star with all edges directed outwards, and the complete bipartite graph $K_{k-2,k-2}$ with half of the edges oriented in either direction in an appropriate way (one can also take the $(k-1)$-blow-up of the directed triangle). In 2013, Addario-Berry, Havet, Linhares Sales, Reed, and Thomassé [2] formulated a conjecture which states that $(k-1)n$ is the correct bound.

Conjecture 1.8. [2] *Every n-vertex digraph D with more than $(k-1)n$ edges contains each antidirected tree on k edges.*

The authors of [2] prove this conjecture for antidirected trees of diameter at most 3 and point out that it implies the well-known Erdős–Sós conjecture for graphs. If the digraph D from Conjecture 1.8 is an oriented graph, it is known [22] that about $\frac{3}{2}(k-1)n$ edges are sufficient to find all antidirected paths on k edges.

We show that Conjecture 1.8 is approximately true in oriented graphs for all balanced antidirected trees of bounded maximum degree.

Theorem 1.9. *For all $\eta \in (0,1)$ and $c \in \mathbb{N}$, there is $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and every $k \geq \eta n$, every n-vertex oriented graph D with more than $(1+\eta)(k-1)n$ edges contains each balanced antidirected tree T with k edges and $\Delta(T) \leq (\log(n))^c$.*

In particular, Conjecture 1.8 holds asymptotically for long antidirected paths in oriented graphs. Moreover, Theorem 1.9 implies the asymptotic bound $2(k-1)n$ for digraphs (as any digraph can be turned into an oriented graph by deleting at most half the edges).

The paper is organised as follows. We start with a sketch of the proofs of Theorems 1.4, 1.7 and 1.9 in Section 2. We then go through some preliminaries in Section 3. We give the proof of Theorem 1.4 in Section 5. We prove Theorem 1.7 in Section 6, and Theorem 1.9 in Section 7. In Section 8, we discuss some open questions.

2. Proof sketches

2.1. Connected antimatchings

Fundamental to our proofs is the concept of a *connected antimatching*: this is a set M of disjoint edges in an oriented graph D such that every pair of edges in M is connected by an antidirected walk in D (see Definition 4.5). Section 4 is dedicated to connected antimatchings. First, we will

[1] Indeed, it suffices to note that any antidirected cycle with more than 4 edges can be interpreted as a long antisubdivision of K_3, and so, Theorem 1.4 follows directly from Theorem 1.7 if the antidirected cycle we are looking for is larger than the minimum semidegree of D. If we are looking for a shorter antidirected cycle C of length at least 6, we can complete C to a long k-edge antisubdivision of K_4, which we can find with the help of Theorem 1.7. Finally, an antidirected C_4 exists by the results of [21].

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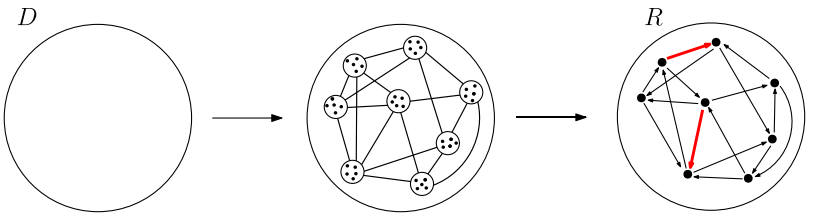


Figure 1. Regularise D to obtain R, with a connected antimatching marked in bold.

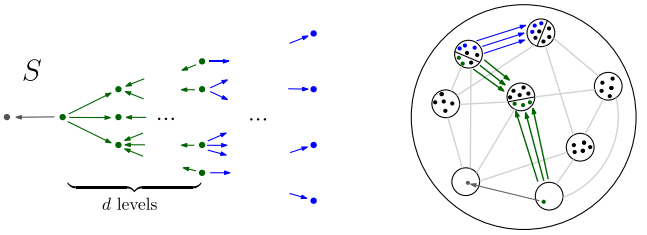


Figure 2. Embedding a small antidirected tree S in D.

prove that any oriented graph of large minimum semidegree contains a large connected antimatching M. Second, we will show that we can choose such an antimatching M in a way that the edges of M lie at bounded distance d from each other.

2.2. Sketch of the proof of Theorem 1.4

Given an oriented graph D and an antidirected tree T fulfilling the conditions of the theorem, we apply the digraph regularity lemma to D to find a partition into a bounded number of clusters. The reduced oriented graph R maintains the condition on the minimum semidegree and thus has a large connected antimatching whose edges lie at bounded distance from each other. (In particular, the clusters covered by M can accommodate all of T.)

Now we turn to our antidirected tree T. We decompose T into a family $\mathcal{T}$ of small subtrees (for details see Section 3.2). As we prove in Section 3.3, it is possible to assign the trees in $\mathcal{T}$ to edges of M in a way that they will fit comfortably into the corresponding clusters.

We now embed T as follows. We go through the decomposition of T, embedding one small tree at a time, always keeping the embedded part connected in (the underlying tree of) T. We embed the first d levels of each small tree S into the clusters of an antiwalk in the reduced oriented graph that starts at the already embedded parent of the root of S and leads to the edge CD of M that was assigned to S. All later levels of S are embedded into clusters C and D. Since T has bounded maximum degree, the union of first d levels of the trees in $\mathcal{T}$ is very small, and therefore it is not a problem if the first d levels of S are embedded outside the edge CD. After going through all S, we have finished the embedding of T. See Figures 1 and 2 for an illustration.

2.3. Sketch of the proof of Theorem 1.7

Let a long k-antisubdivision of K_h be given, together with an oriented graph D satisfying the conditions of the theorem. Since the antisubdivision is long, we can choose a set $\mathcal{P}$ of long antidirected paths of the antisubdivision such that deleting two inner vertices of each of the antidirected paths in $\mathcal{P}$ leaves us with an antidirected tree T.

As in the proof of Theorem 1.2, we find a connected antimatching M in the reduced oriented graph of D (see Section 4 for details). We embed T with the help of M as follows. Fixing one edge e of M, we embed the branch vertices of the antisubdivision into the clusters corresponding to

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e. We embed the long antipaths into clusters corresponding to e and to other edges of M, with the first and last vertices on these antipaths going to clusters on antidirected paths in the reduced oriented graph that connect the edges of M.

We finish the embedding by adding the two deleted vertices on the paths from $\mathcal{P}$. Their neighbours were embedded into the clusters corresponding to the edge e and regularity allows us to connect them by a 3-edge path on unused vertices. This finishes the embedding of the antisubdivision.

2.4. Sketch of the proof of Theorem 1.9

Given the antidirected tree T and the oriented graph D as in the theorem, we start by finding an oriented subgraph D' of D where each vertex has either out-degree at least $\frac{k}{2}$ or out-degree 0, and either in-degree at least $\frac{k}{2}$ or in-degree 0. Moreover, D' has at least one edge.

We then transform D' into an oriented graph D'' which has minimum semidegree exceeding $\frac{k}{2}$. More precisely, D'' consists of four copies of D', two of them with all edges reversed, glued together appropriately.

We now need Theorem 5.2, a variation of Theorem 1.2 which allows us to embed T into D'', guaranteeing that the root of T is embedded in a small set of $V(D)$ which we can choose before the embedding. We choose to have the root v of T embedded in a vertex w of one of the original copies of D (i.e. a copy with edges in the original orientation). We can further ensure that one of the edges at v is embedded into an edge having the original orientation of D. Then it is not hard to deduce that all of T was embedded into an original copy of D, and therefore, T is contained in D.

3. Preliminaries

3.1. Basic digraph notation

A *digraph* is a pair of sets (V, E), where the elements of V are called *vertices* and E is a set of ordered pairs of distinct vertices in V, called *edges*. An *oriented graph* is a digraph that allows at most one edge between a pair of vertices.

The edges of a digraph D are directed, and we will write uv for an edge that is directed from u to v. While a digraph may have both edges uv, vu for any pair of vertices u, v, an *oriented graph* has at most one of these edges. The *out-degree* $\delta^+(v)$ of a vertex v is the number of edges coming out of v, and the *in-degree* $\delta^-(v)$ is defined analogously. The *minimum semidegree* $\delta^0(D)$ of a digraph is the minimum over the out-degrees and the in-degrees of all vertices in D.

We denote by $V_{\text{in}}(D)$ the set of all vertices of D that have no outgoing edges, while $V_{\text{out}}(D)$ is the set of all vertices of D that have no incoming edges. A digraph D is *antidirected* if $V(D) = V_{\text{in}}(D) \cup V_{\text{out}}(D)$. If the underlying undirected graph of D is a tree, we also call D an *antitree*. *Antipaths* and *anticycles* are defined analogously. Note that anticycles are even. The *length* of an antipath or anticycle is the number of edges in it.

An *antiwalk* (or *antidirected walk*) is a sequence of edges that alternate directions, but we usually denote it by writing the sequence of the corresponding vertices. Note that an antiwalk may repeat vertices and edges, and a repeated vertex v may have both outgoing and incoming edges in the antiwalk. The *length* of an antiwalk is the length of the corresponding sequence of edges. For instance, the antiwalk *abab* has length 3, while having only one edge.

Finally, an antitree T is *balanced* if $|V_{\text{in}}(T)| = |V_{\text{out}}(T)|$. Note that antipaths of odd length are balanced.

3.2. Tree decompositions

In our proof of the main theorem, we will have to partition the given antitree into smaller trees. It will be enough to define this decomposition for undirected trees, as we can apply it later to the

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underlying graph of our antitree. This type of decomposition appeared in the literature already in the 1990's, for instance in [3]. Here we use a version from [6]:

Definition 3.1 (β-decomposition). *Let $\beta \in (0,1)$, and let T be a tree on $k+1$ vertices, rooted at r. If there are a set $W \subseteq V(T)$ and a family $\mathcal{T}$ of disjoint rooted trees such that*

(i) $r \in W$;
(ii) $\mathcal{T}$ *consists of the components of* $T - W$, *and each* $S \in \mathcal{T}$ *is rooted at the vertex closest to the root of* T;
(iii) $|S| \leq \beta k$ *for each* $S \in \mathcal{T}$; *and*
(iv) $|W| \leq \frac{1}{\beta} + 2$,

then the pair $(W, \mathcal{T})$ is called a β-decomposition of T.

Lemma 3.2.[6] *Let $\beta \in (0,1)$, and let T be a tree on $k+1$ vertices, rooted at r. Then there is a β-decomposition $(W, \mathcal{T})$ of T.*

Let us see what happens to a balanced tree T if we remove some vertices from the upper levels of the trees $S \in \mathcal{T}$ of the tree-decomposition of T. We first define what exactly we wish to remove. As usual, the jth level of a tree S contains every vertex in S that has distance j to the root of S.

Definition 3.3 ($Lev_m(S)$, $L_m(T)$). *Given $j \in \mathbb{N}$ and a β-decomposition $(W, \mathcal{T})$ of a tree T, define $Lev_j(S)$ as the union of the first j levels of $S \in \mathcal{T}$. Set $L_j(T;(W,\mathcal{T})) = \bigcup_{S\in\mathcal{T}} Lev_j(S)$, and write shorthand $L_j(T)$ if $(W, \mathcal{T})$ is clear from the context.*

We show now that after removing $L_j(T)$ from T for some bounded j, the remainder of T is still relatively well balanced.

Lemma 3.4. *Let $k, j \in \mathbb{N}^+$, $\alpha, \beta \in (0, \frac{1}{2})$. Let T be a balanced rooted antitree with k edges such that $\Delta(T) \leq (\frac{\alpha\beta k}{8})^{\frac{1}{j+1}}$. Let $(W, \mathcal{T})$ be a β-decomposition of T. Then, $(1-\alpha)p_T \leq q_T \leq (1+\alpha)p_T$, where*

$$p_T := |V_{in}(T) - L_j(T)| \text{ and } q_T := |V_{out}(T) - L_j(T)|.$$

Proof. Note that $L_j(T)$ is contained in the union of the balls of radius j centred at the vertices of W. Thus

$$|L_j(T)| \leq |W| \cdot \Delta^{j+1} \leq \frac{2\Delta^{j+1}}{\beta} \leq \frac{\alpha k}{4}.$$

As T is balanced, we easily deduce the desired inequalities. □

3.3. Packing trees into edges

Once the given antitree is cut into small pieces S, and we shaved off their first levels L_j, the details of which are described in Section 3.2, we will have to decide how to pack the remainder of the small pieces into the edges of the connected antimatching of the reduced oriented graph. The following lemma shows how to allocate all of the $S \setminus L_j(T)$.

Lemma 3.5. *Let $m, t \in \mathbb{N}$, $\alpha > 0$ and let $(p_i, q_i)_{i\in I} \subseteq \mathbb{N}^2$ be a family such that:*

(a) $(1-\alpha)\sum_{i\in I} p_i \leq \sum_{i\in I} q_i \leq (1+\alpha)\sum_{i\in I} p_i$,
(b) $p_i + q_i \leq \alpha m$, *for all $i \in I$, and*
(c) $\max\{\sum_{i\in I} p_i, \sum_{i\in I} q_i\} < (1-10\alpha)mt$.

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Then there is a partition $\mathcal{J}$ of I of size t such that for every $J \in \mathcal{J}$,

$$\sum_{j\in J} p_j \le (1-7\alpha)m \;\; \textit{and} \;\; \sum_{j\in J} q_j \le (1-7\alpha)m.$$

Proof. For every $i \in I$, set $\delta_i := p_i - q_i$ and for every $S \subseteq I$, define $\delta_S := \sum_{i\in S} \delta_i$. Let $A_1, \ldots, A_t \subseteq I$ be disjoint sets such that for every $j \in [t]$,

$$\textstyle\sum_{i\in A_j} p_i \le (1-9\alpha)m \text{ and } \delta_{A_j} \in [-\alpha m, \alpha m], \tag{1}$$

and such that, under these conditions, $R := I \setminus (A_1 \cup \ldots \cup A_t)$ is minimised. Note that for each $j \in R$ there is an index $k(j) \in [t]$ such that

$$p_j + \textstyle\sum_{i\in A_{k_p(j)}} p_i \le (1-9\alpha)m, \tag{2}$$

since otherwise, using the fact that $p_j \le \alpha m$ by (b), we have

$$\sum_{i\in I} p_i \ge \sum_{k\in[t]} \sum_{i\in A_k} p_i > t\cdot((1-9\alpha)m - p_j) \ge t\cdot(1-10\alpha)m,$$

a contradiction to (c). Next, we claim that

$$\delta_i\delta_j \ge 0 \text{ for every } i,j \in R. \tag{3}$$

To see (3), suppose there are $a, b \in R$ with $\delta_a\delta_b < 0$. Say $p_a \ge p_b$. Then there is an index $k(a)$ such that (2) holds with $j = a$, and since $p_a \ge p_b$, the inequality also holds when substituting p_a with p_b. Moreover, since $a \in R$, we know that neither a nor b can be added to $A_{k(a)}$ without violating (1), and thus

$$\delta_{A_{k(a)}} + \delta_a, \delta_{A_{k(a)}} + \delta_b \notin [-\alpha m, \alpha m].$$

However, $\delta_{A_{k(a)}} \in [-\alpha m, \alpha m]$ by (ii). So, since we assumed that $\delta_a\delta_b < 0$, it must be that $|\delta_a| + |\delta_b| > 2\alpha m$. But this contradicts (b). We proved (3), which enables us to split the rest of the proof into two cases.

Case 1: $\delta_i \ge 0$ for all $i \in R$. For $k \in [t]$, we let $A'_k \supseteq A_k$ be disjoint sets such that $\sum_{i\in A'_k} p_i \le (1-9\alpha)m$ for every $k \in [t]$, and thus also,

$$\sum_{i\in A'_k} q_j = \sum_{i\in A'_k} (p_j - \delta_{A'_k}) \le \sum_{i\in A'_k} (p_j - \delta_{A_k}) \le (1-9\alpha)m + \alpha m < (1-7\alpha)m. \tag{4}$$

Similarly as for (2), we can show that for each $j \in R' := I \setminus (A'_1 \cup \ldots \cup A'_t)$ there is an index $k'(j) \in [t]$ with $p_j + \sum_{i\in A'_{k_p(j)}} p_i \le (1-9\alpha)m$, and thus, by (4), $q_j + \sum_{i\in A'_{k_p(j)}} q_i \le (1-9\alpha)m$. So $R' = \emptyset$, and we are done, taking $\mathcal{J} = \{A'_1, \ldots, A'_t\}$.

Case 2: $\delta_i \le 0$ for all $i \in R$. We proceed as in Case 1, but choose sets $A'_k \supseteq A_k$ such that $\sum_{i\in A'_k} q_i \le (1-9\alpha)m$ for every $k \in [t]$. The rest of the argument is similar. □

3.4. Diregularity

We will use the regularity method which goes back to Szemerédi's work from the 1970's [31]. We will need regularity for oriented graphs, but start introducing the concept for undirected graphs. Let G be a graph, let $\varepsilon > 0$ and let A, B be two disjoint subsets of $V(G)$. The *density* of the pair (A, B) is $d(A,B) = \frac{|E(A,B)|}{|A|\cdot|B|}$, with $|E(A,B)|$ denoting the number of edges between A and B. The pair (A, B) is *ε-regular* if $|d(X,Y) - d(A,B)| < \varepsilon$ for every $X \subseteq A$ and $Y \subseteq B$ satisfying $|X| > \varepsilon|A|$ and $|Y| > \varepsilon|B|$.

Let (A, B) be an ε-regular pair of density d and let $Y \subseteq B$ be such that $|Y| > \varepsilon|B|$. A vertex $x \in A$ is called *ε-typical* (or simply *typical*) with respect to Y if it has more than $(d-\varepsilon)|Y|$ neighbours

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in Y. It is well known and easy to prove that A has at most $\varepsilon|A|$ vertices that are not typical with respect to Y.

A partition $\{V_0, \ldots, V_k\}$ of $V(G)$ is called an *ε-regular* partition of G if

(i) $|V_0| \leq \varepsilon|V|$,
(ii) $|V_1| = \ldots = |V_k|$,
(iii) all but at most εk^2 of the pairs (V_i, V_j), with $1 \leq i < j \leq k$, are ε-regular.

Let D be a digraph and let $A, B \subseteq V(D)$ disjoint. We denote by (A, B) the oriented subgraph of D with vertex set $A \cup B$ and every edge directed from A to B in D. In this case, we say the pair (A, B) is *ε-regular* if the underlying graph is ε-regular. With these notions and based on the Szemerédi's Regularity Lemma for graphs without orientation [31], we show its version for digraphs as stated by Alon and Shapira in [4]:

Lemma 3.6 (Degree form of the Diregularity Lemma [4]). *For all $\varepsilon \in (0, 1)$, $m_0 \in \mathbb{N}$, there are $M_0, n_0 \in \mathbb{N}$ such that for each $d \in [0, 1]$ and for each digraph D on $n \geq n_0$ vertices there are a partition of $V(D)$ into sets $V_0, V_1, \ldots, V_k$ and a spanning subdigraph D' of D, called the regularised digraph, such that the following holds:*

- $m_0 \leq k \leq M_0$,
- $|V_0| \leq \varepsilon n$ *and* $|V_1| = \ldots = |V_k| =: m$,
- $d^+_{D'}(x) > d^+_D(x) - (d + \varepsilon)n$ *for all vertices* $x \in D$,
- $d^-_{D'}(x) > d^-_D(x) - (d + \varepsilon)n$ *for all vertices* $x \in V(D)$,
- *for all* $1 \leq i < j \leq k$ *and* $i \neq j$, *the bipartite graph* $(V_i, V_j)_{D'}$ *whose vertex classes are* V_i *and* V_j *and whose edge set consists of all the* V_i-V_j *edges in* D' *is* ε-*regular and has density either 0 or at least* d,
- *for all* $1 \leq i \leq k$ *the digraph* $D'[V_i]$ *is empty.*

Given a regularised digraph D' with clusters $V_1, \ldots, V_t$, the *reduced digraph* R is a digraph with vertices $V_1, \ldots, V_t$ such that the edge V_iV_j exists only if D' contains a V_i-V_j edge. Observe that, R need not be an oriented graph, even if D' is. However, it is possible [21] to discard appropriate edges from the reduced digraph to find an oriented spanning subgraph R' of R that preserves the minimum semidegree of the original oriented graph D (proportionally to the order of the reduced digraph). The new oriented graph R' is called the *reduced oriented graph*. This result is formalised in the following lemma.

Lemma 3.7 (see [21]). *Let $\varepsilon, d \in [0, 1]$, $m_0 \in \mathbb{N}$, let G be a large enough oriented graph and let R' be the reduced digraph obtained by applying the Lemma 3.6 to G with parameters ε, m_0 and d. Then, R' has a spanning oriented subgraph R with*

(i) $\delta^+(R) \geq \left(\frac{\delta^+(G)}{|G|} - (3\varepsilon + d)\right)|R|$,

(ii) $\delta^-(R) \geq \left(\frac{\delta^-(G)}{|G|} - (3\varepsilon + d)\right)|R|$.

This oriented graph R is called the (ε, d)-reduced oriented graph.

3.5. Embedding small trees in antiwalks

We now show how to embed a small tree into an antiwalk of the reduced oriented graph. This will be useful for embedding the first levels of a piece S of the decomposition of our antitree T

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into an antiwalk leading to a suitable edge a_ib_i of the connected antimatching. The edge a_ib_i will accommodate the rest of S.

We start with a convenient definition, expressing that the orientations of the small tree S and the antiwalk coincide.

Definition 3.8 (Consistent antiwalk and antitree). *Let P be a non-trivial antiwalk and let T be a rooted antitree. We say P and T are consistent if one of the following holds:*

(i) $r(T) \in V_{\text{out}}(T)$ *and P starts with an out-edge, or*

(ii) $r(T) \in V_{\text{in}}(T)$ *and P starts with an in-edge.*

Now we embed an antitree into a consistent antiwalk of a reduced digraph. A very similar result for undirected graphs was proved in [6].

Lemma 3.9. *Let $\varepsilon \in (0, \frac{1}{4})$, $m, h \in \mathbb{N}$. Let $P = Q_0 \dots Q_{h-1}Q_h$ be an antiwalk in an $(\varepsilon, 2\sqrt{\varepsilon})$-reduced oriented graph R of an oriented graph D, where clusters have size m. For $0 \le i \le h$, let $Z_i \subseteq Q_i$ be such that $|Z_i| \ge 3\sqrt{\varepsilon}m$ if $i \ne 0$ and $|Z_0| \ge 3\varepsilon m$. For $i = h-1, h$, let $X_i \subseteq Q_i \setminus Z_i$ with $|X_i| > 3\sqrt{\varepsilon}m$. Let S be a rooted antitree with $|S| < \frac{\varepsilon}{10}m$ such that P and S are consistent. Then there is an embedding φ of S into D such that*

(a) for each $i \le h$, and for each v from the ith level of S, $\varphi(v) \in Z_i$ and $\varphi(v)$ is typical with respect to Z_{i+1} (or with respect to Z_{i-1} if $i = h$), and

(b) for each $i > h$, and for each v from the ith level of S, $\varphi(v) \in X_{h-1}$ is typical with respect to X_h if $i - h$ is odd, and $\varphi(v) \in X_h$ is typical with respect to X_{h-1} if $i - h$ is even.

Proof. Let $r(S)$ be the root of S and denote the embedding by $\phi : V(S) \longrightarrow \cup_{i=0}^{h} Q_i$. Suppose that $Q_0Q_1 \in E(R)$, and $r(S) \in V_{\text{out}}(S)$, the other case is analogous.

We embed $r(S)$ in a typical vertex of Z_0 with respect to Z_1, and then embed successively, for $i < h$, the vertices from the ith level into vertices from Z_i that are typical to Z_{i+1}. (This is possible since we embed at most $|S| < \frac{\varepsilon}{10}m$ vertices in total, and since $|Z_i| \ge 3\sqrt{\varepsilon}m$ for all i, while there are less than εm atypical vertices in each Z_i.) For the vertices from the hth level we take care that their images are typical with respect to both Z_{h-1} and X_{h-1}. The vertices from later levels are embedded alternatingly into X_{h-1} and X_h, and we take care that their images are typical with respect to both Z_h and X_h, or both Z_{h-1} and X_{h-1}, respectively. □

4. Connected antimatchings

In this section we introduce the concept of a connected antimatching, which is central to our proof. We also prove two lemmas (Lemmas 4.6 and 4.8) relating the minimum semidegree of an oriented graph D to the existence and size of a connected antimatching in D, whose edges are not too distant from each other.

Before we can introduce the notion of a connected antimatching, we need some preliminary definitions.

Definition 4.1 (in-vertex and out-vertex of an antiwalk). *Let P be an antiwalk, and let $v \in V(P)$. We call v an out-vertex of P if $P = v$ or P has one or more edges of the form vx. We call v an in-vertex of P if $P = v$ or P has one or more edges of the form xv.*

Note that also in non-trivial walks a vertex can be both an in- and an out-vertex.

Definition 4.2 (out-walk; out-in-walk; out-out-walk). *Let C be an oriented graph with $a, z \in V(C)$. A non-trivial antiwalk P starting in a and ending in z whose first edge is directed away from a is called an out-walk from a to z. Call P an out-in-walk if its last edge is directed towards z, and an out-out-walk otherwise. Also call a trivial walk an out-out-walk.*

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Definition 4.3 (*In*(C, a); *Out*(C, a)). *Let C be an oriented graph with $a \in V(C)$. Then* $\mathrm{In}(C, a)$ *is the set of all $z \in V(C)$ such that there is an out-in-walk from a to z, and* $\mathrm{Out}(C, a)$ *is the set of all $z \in V(C)$ such that there is an out-out-walk from a to z.*

Observe that $\mathrm{In}(C, a)$ and $\mathrm{Out}(C, a)$ are not necessarily disjoint.

Definition 4.4 (anticonnected oriented graph). *Let C be an oriented graph with $a \in V(C)$. We say (C, a) is anticonnected if $V(C) = \mathrm{In}(C, a) \cup \mathrm{Out}(C, a)$.*

Definition 4.5 (connected antimatching). *Call $M = \{a_i b_i\}_{1 \le i \le m}$ a connected antimatching of size m in an oriented graph D if M is a matching in the underlying graph of D and if $a_i \in \mathrm{Out}(D, a_1)$ for every $1 \le i \le m$. We write $V(M)$ for the set of all vertices covered by M.*

Our first lemma of this section links the size of a connected antimatching in an oriented graph with its minimum semidegree. We can even choose a vertex to be included in the antimatching, as an outvertex of a matching edge (of course, by inverting all direction we could also choose it as an invertex).

Lemma 4.6. *Let $t \in \mathbb{N}^+$, let D be an oriented graph with $\delta^0(D) \ge t$, and let $w \in V(D)$. Then D has a connected antimatching $M = \{a_i b_i\}_{1 \le i \le t}$ of size t, with $w = a_1$.*

Proof. Let $M = \{a_i b_i\}_{1 \le i \le m}$ be a connected antimatching of maximum size in D with the property that $w = a_1$. Note that $m \ge 1$, because any single edge constitutes a connected antimatching. For contradiction, we assume that $m < t$. Let C be the largest induced subdigraph of D such that

(a) $M \subseteq E(C)$
(b) $a_i \in \mathrm{Out}(C, a_1)$ for every $1 \le i \le m$ and
(c) (C, a_1) is anticonnected.

We claim that

$$\text{if } v \in \mathrm{In}(C, a_1) \text{ then } N^-(v) \subseteq V(C), \text{ and if } v \in \mathrm{Out}(C, a_1) \text{ then } N^+(v) \subseteq V(C). \tag{5}$$

Indeed, let $v \in \mathrm{In}(C, a_1)$ and suppose that there is a vertex $x \in N^-(v) \setminus V(C)$. By definition of $\mathrm{In}(C, a_1)$, the digraph C contains an out-in-walk P_v from a_1 to v. Then P_v and the edge xv form an out-out-walk from a_1 to x. Thus, adding the vertex x and the edge xv to C, we obtain a subdigraph C' of D that is larger than C and fulfils (i)-(iii), a contradiction. The proof for $v \in \mathrm{Out}(C, a_1)$ is analogous. This proves (5).

Now, we show that

$$\text{if } v \in \mathrm{In}(C, a_1) \setminus V(M) \text{ then } N^-(v) \subseteq V(M), \text{ and if } v \in \mathrm{Out}(C, a_1) \setminus V(M) \text{ then } N^+(v) \subseteq V(M). \tag{6}$$

In order to see (6), let $v \in \mathrm{In}(C, a_1) \setminus V(M)$, and let $x \in N^-(v)$. By (5), we know that $x \in V(C)$. For contradiction, assume $x \notin V(M)$. Define $a_{m+1} := x$, $b_{m+1} := v$ and set $M' := \{a_i b_i\}_{1 \le i \le m+1}$. Consider an out-in-walk from a_1 to v and add the edge xv at the end to see that $x \in \mathrm{Out}(C, a_1)$. So M' is a connected antimatching of size $m + 1$, with $w = a_1$, a contradiction to the choice of M. We can proceed similarly for $v \in \mathrm{Out}(C, a_1) \setminus V(M)$. This proves (6).

Next, we claim that

$$|V(C) \setminus V(M)| \le 1. \tag{7}$$

To prove (7) by contradiction, we start by considering a vertex u in $V(C) \setminus V(M)$. Since $\delta^0(D) \ge t$ and because of (6), we know that if $u \in \mathrm{In}(C, a_1)$, then u has at least t in-neighbours in $V(M)$, and if $u \in \mathrm{Out}(C, a_1)$, then u at least t out-neighbours in $V(M)$. Now suppose there are two distinct vertices in $V(C) \setminus V(M)$. Since we assume that $|M| < t$, there are distinct $u_1, u_2 \in V(C) \setminus V(M)$,

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edges $e_1, e_2 \in E(C)$, an index $j \leq t-1$ and an edge $v_1v_2 := a_jb_j \in M$ such that for each $i = 1, 2$ one of the following holds:

- $u_i \in \mathrm{In}(C, a_1)$ and $e_i = v_iu_i$, or
- $u_i \in \mathrm{Out}(C, a_1)$ and $e_i = u_iv_i$, or

If possible, choose $j \neq 1$, and if that is not possible try to choose u_1 in a way that $e_1 = a_1u_1$.

Obtain M' from M by adding e_1 and e_2 to M and removing e_3. Note that for $i = 1, 2$, if $e_i = u_iv_i$ then $u_i \in \mathrm{Out}(C, a_1)$, and furthermore, if $e_i = v_iu_i$ then $u_i \in \mathrm{In}(C, a_1)$, which means that u_i is reachable by an out-in-walk from a_1, to which we can add the edge v_iu_i to obtain an out-out-walk from a_1 to v_i, implying that $v_i \in \mathrm{Out}(C, a_1)$. So, in the case that $j \neq 1$, and in the case that $j = 1$ and $e_1 = v_1u_1$, we know that M' is a connected antimatching which is larger than M, with w being the outvertex of a matching edge we can call a_1b_1. This is a contradiction to the choice of M. Thus $j = 1$ and $e_1 = u_1a_1$.

Note that the fact that $j = 1$ implies that both u_1 and u_2 have only $t-2$ out- or in-neighbours in $V(M) \setminus \{a_1, b_1\}$, where we count out-neighbours for u_i if $u_i \in \mathrm{Out}(C, a_1)$ and in-neighbours otherwise. Thus both are connected by edges to both of a_1, b_1. Since we would have chosen $e_1 = a_1u_1$ if that was possible for a choice of u_1, we know that both u_1, u_2 are in-neighbours of a_1, in fact, a_1 has no out-neighbours outside $V(M)$. So, because of our bound on the minimum semidegree, a_1 has at least $t-1$ out-neighbours in $V(M) \setminus \{a_1, b_1\}$. As the outdegree of u_1 into $V(M) \setminus \{a_1, b_1\}$ is at least $t-2$, we find an edge $a_jb_j \in M$, with $j \neq 1$, such that $a_1a_j, u_1b_j \in E(D)$ or $a_1b_j, u_1a_j \in E(D)$. Adding these two edges and e_2 to M while deleting a_1b_1 and a_jb_j gives a connected antimatching M'', which is larger than M, and where $a_1 = w$ (choosing the corresponding edge as a_1b_1). Therefore, inequality (7) holds.

By (7), and since we assume M to have at most $t-1$ edges, we see that

$$|V(C)| < 2t. \tag{8}$$

Now, by (5), the out-neighbourhood of a_1 is contained in $V(C)$ and thus is a subset of $\mathrm{In}(C, a_1)$. Similarly, the in-neighbourhood of b_1 is a subset of $\mathrm{Out}(C, a_1)$. Considering the minimum semidegree of D, we deduce that

$$|\mathrm{In}(C, a_1)|, |\mathrm{Out}(C, a_1)| \geq t.$$

By (8), we conclude $\mathrm{In}(C, a_1) \cap \mathrm{Out}(C, a_1) \neq \emptyset$ and thus contains a vertex v. Because of (5), both the in-neighbourhood and the out-neighbourhood of v are contained in C. As each of these two disjoint sets has at least t elements, C has at least $2t$ vertices, a contradiction to (8). □

For the next lemma, we need another definition.

Definition 4.7 (out-out-distance $\mathrm{ood}(a, a')$; out-in-distance $\mathrm{oid}(a, a')$). *In an oriented graph D, the out-out-distance $\mathrm{ood}(a, a')$ between two vertices $a, a' \in V(D)$ is the length of the shortest out-out-walk from a to a', if such a walk exists. Otherwise $\mathrm{ood}(a, a') = \infty$.*

Similarly, the out-in-distance $\mathrm{oid}(a, a')$ between two vertices $a, a' \in V(D)$ is the length of the shortest out-in-walk from a to a', if such a walk exists, and otherwise $\mathrm{oid}(a, a') = \infty$.

Observe that $\mathrm{ood}(a, a) = 0$ and for edges $ab, a'b'$ of a connected antimatching, $\mathrm{ood}(a, a')$ is finite by definition. For use in the proof of the following lemma, we define $\mathrm{ood}(M) = \sum_{i=2}^{d} \mathrm{ood}(a_1, a_i)$, where $M = \{a_ib_i\}_{1 \leq i \leq t}$ is a connected antimatching.

Lemma 4.8. *Let $t \in \mathbb{N}$ and let D be an oriented graph with $\delta^0(D) \geq t$. Let $w \in V(D)$. Then D contains a connected antimatching $M = \{a_ib_i\}_{1 \leq i \leq t}$, with $w = a_1$, and such that for every $1 \leq i \leq t$, $\mathrm{ood}(a_1, a_i) \leq 8t$.*

Proof. By Lemma 4.6, we know that D contains a connected antimatching $M' = \{a'_ib'_i\}_{1 \leq i \leq t}$ with $w = a'_1$. Among all such antimatchings choose $M = \{a_ib_i\}_{1 \leq i \leq t}$ such that $\mathrm{ood}(M)$ is minimised.

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For the sake of contradiction, suppose there exists k with $2 \leq k \leq t$ such that $\text{ood}(a_1, a_k) > 8d$. Let P be a shortest out-out-walk from a_1 to a_k. We claim that

each $v \in V(D)$ appears at most once as an in-vertex and at most once as an out-vertex on P. (9)

Indeed, suppose vertex v appears at least twice as an in-vertex on P. Then there are distinct vertices $x, y \in V(D)$ such that $P = P_1 x v P_2 y v P_3$ (where the each of paths P_i is allowed to be empty). The antiwalk $P_1 x v P_3$ is shorter than P, a contradiction to the choice of P. We can argue similarly if v appears twice on P as an out-vertex. This proves (9).

From (9) it follows that P does not repeat edges. Therefore, and since we assumed that P has length greater than $8t$, we know that

$$|E(P)| > 8t. \tag{10}$$

By (9), every a_i is incident to at most 4 edges in P. The same holds for every b_i. So at most $8t$ edges have one of their extremes on M, and hence, by (10), there is an edge xy on P such that $x, y \notin V(M)$. Replacing $a_k b_k$ with xy in M, we obtain a connected antimatching M' of size t with $\text{ood}(M') < \text{ood}(M)$, a contradiction to the choice of M. □

5. Antitrees: The proof of Theorem 1.2

Instead of Theorem 1.2, we will prove a slightly stronger result, namely Theorem 5.2 below. The additional properties of Theorem 5.2 will be necessary for the proof of Theorem 1.9.

Let us define the following shorthand notation.

Definition 5.1. *For digraphs A and D, we write $A \subseteq_\gamma D$ if for each set $V^* \subseteq V(D)$ of size at least $\gamma|V(D)|$ and for each $x \in V(A)$, there is an embedding of A in D with x mapped to V^*.*

Here is the result that immediately implies Theorem 1.2.

Theorem 5.2. *For all $\eta \in (0, 1)$, $c \in \mathbb{N}$ there is n_0 such that for all $n \geq n_0$ and $k \geq \eta n$ the following holds for every oriented graph D on n vertices and every balanced antidirected tree T with k edges. If $\delta^0(D) > (1 + \eta)\frac{k}{2}$ and $\Delta(T) \leq (\log(n))^c$, then $T \subseteq_\eta D$.*

In the remainder of this section, we prove Theorem 5.2.

Proof of Theorem 5.2. We will first define our constants, then prepare the given digraph D and the antitree T and finally proceed to embed T in D.

Setting the constants. We define our constants ε, β and n_0 so that

$$\frac{1}{n_0} \ll \beta \ll \varepsilon \ll \eta < 1.$$

More precisely, given η and c, we set $\varepsilon := \frac{\eta^2}{10^5}$. Lemma 3.6 with input ε and $m_0 := \lceil \frac{1}{\varepsilon} \rceil$ gives constants n_0' and M_0 such that we can apply the lemma to digraphs on $n \geq n_0'$ vertices. Set $\beta := \frac{\varepsilon}{100 M_0}$, and choose $n_0 \geq n_0'$ such that

$$(\log n_0)^{18 M_0 c} \leq \frac{\beta^3}{10} n_0. \tag{11}$$

Finally, let $n \geq n_0$ and let $k \geq \eta n$.

Objective. Let D be an oriented graph D on n vertices, with $\delta^0(D) > (1 + \eta)\frac{k}{2}$. Let T be a balanced antidirected tree T with k edges, with $\Delta(T) \leq (\log(n))^c$. We need to show that for every $V^* \subseteq V(D)$ with $|V^*| \geq \eta n$, the antitree T can be embedded in D, with x embedded in V^*.

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So let such $V^* \subseteq V(D)$ and $x \in V(T)$ be given. Note that we can assume that $x \in V_{\text{out}}(T)$, as otherwise we could switch the orientations of D and of T, which would move x to Out(T). Once T is embedded, we switch all orientations back to normal.

Preparing the oriented graph D. We apply Lemma 3.6 with $d = 2\sqrt{\varepsilon}$ to obtain D', a digraph with $r \leq M_0$ clusters $C_1, \dots, C_r$ of the same size m, and with $\delta^0(D') > (1 + \frac{\eta}{2})\frac{k}{2}$. We divide each cluster of D' into two slices, C_i^1 and C_i^2 of sizes

$$|C_i^1| = \lfloor 10\sqrt{\varepsilon}|C_i| \rfloor \text{ and } |C_i^2| = \left\lceil (1 - 10\sqrt{\varepsilon})|C_i| \right\rceil. \tag{12}$$

Note that at least one of the clusters C_i contains at least $\eta|C_i|$ vertices from V^*. Let C^* be such a cluster.

Let R be the reduced oriented graph of D' given by Lemma 3.7, on vertices $C_1, \dots, C_r$, of minimum semidegree greater than

$$t := \left\lceil \left(1 + \frac{\eta}{2}\right) \frac{kr}{2n} \right\rceil.$$

By Lemma 4.8, R has a connected antimatching $M = \{a_i b_i\}_{i=1}^t$ such that

$$\text{ood}(a_1, a_j) \leq 8t \text{ for each } j \in [t], \tag{13}$$

and such that $C^* = C_{a_1}$, where here and later we write C_{a_i} (resp. C_{b_i}) for the cluster corresponding to a_i (resp. b_i) in our reduced oriented graph R, for every antimatching edge $a_i b_i \in M$.

Preparing the antitree T. Note that because of (11), and since we assume that $\Delta(T) \leq \log n$,

$$k \geq \eta n \geq 10\beta^{-2}(\Delta(T))^{18M_0}. \tag{14}$$

We root T at the x. By Lemma 3.2, there is a β-decomposition $(W, \mathcal{T}')$ of the underlying undirected tree of T, with $|W| < \frac{1}{\beta} + 2 \leq \frac{2}{\beta}$. Let $\mathcal{T}$ denote the set of oriented subtrees of T corresponding to $\mathcal{T}'$. For each $S \in \mathcal{T}$, we define

$$(p_S, q_S) := \left(|V_{\text{in}}(S) - Lev_{16t+2}(S)|, |V_{\text{out}}(S) - Lev_{16t+2}(S)|\right).$$

We claim that the family $(p_S, q_S)_{S \in \mathcal{T}}$ satisfies the conditions of Lemma 3.5 with $\sqrt{\varepsilon}$ playing the role of α. Indeed, Lemma 3.5(a) holds by Lemma 3.4, because of (14), and since $t \geq M_0$. As $(W, \mathcal{T}')$ is a β-decomposition, and by our choice of β, we know that

$$p_S + q_S \leq |S| \leq \beta k \leq \beta n \leq \frac{\sqrt{\varepsilon}}{2} m \leq \sqrt{\varepsilon}|C_{a_1}^2|$$

for each $S \in \mathcal{T}$, and thus, Lemma 3.5(b) holds. Finally Lemma 3.5(c) is true because T is balanced, and thus, by our choice of t, and since $|W| \geq 1$,

$$\begin{aligned} \max\left\{\sum_{S \in \mathcal{T}} p_S, \sum_{S \in \mathcal{T}} q_S\right\} &\leq \frac{k}{2} \leq (1 - 10\sqrt{\varepsilon})^2(1 - \varepsilon)(1 + \frac{\eta}{2})\frac{k}{2} \\ &\leq (1 - 10\sqrt{\varepsilon})^2(1 - \varepsilon)\frac{n}{r}t \leq (1 - 10\sqrt{\varepsilon})|C_{a_1}^2|t. \end{aligned}$$

So Lemma 3.5, gives a partition $\{P_j\}_{j=1}^t$ of $\mathcal{T}$, such that for every $j \in [t]$,

$$\max\left\{\sum_{i \in P_j} p_i, \sum_{i \in P_j} q_i\right\} \leq (1 - 7\sqrt{\varepsilon})|C_{a_1}^2|. \tag{15}$$

For convenience, let us also define, for each $S \in \mathcal{T}$, the set W_S. This set contains all vertices from W whose path to $r(T) = x$ passes through S before passing through any other $S' \in \mathcal{T}$. That

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is, W_S contains all children of S in W, as well as their children in W, etcetera. Similarly, we define $W_{r(T)}$ as the set of all vertices from W whose path to $r(T)$ does not meet any tree from $\mathcal{T}$.

Idea of the embedding procedure. The process starts with embedding $r(T) = x$ into C^*. Then we embed the vertices from $W_{r(T)}$ into $C^1_{a_1} \cup C^1_{b_1}$.

After that, in every step of the process, we will embed some $S \in \mathcal{T}$, say with $S \in P_j$, together with the set W_S. We choose S such that the parent of its root $r(S)$ is already embedded, say in cluster C. As C itself or a neighbouring cluster C' lies at out-out-distance at most $8t$ to C_{a_1} in R, there is a short antiwalk P from C or from CC' to C_{a_j}. We embed the first $16t + 2$ levels of $T[S \cup W_S]$ into the Q^1-slices of clusters Q of P, and the rest into $C^2_{a_j}, C^2_{b_j}$.

The bound on the maximum degree of T ensures that only few vertices in total go to clusters from connecting antiwalks P. Therefore, T will be mainly embedded into the edges from M, where we have control on how space is allocated.

The embedding. Let us make this sketch more precise. Let $V_1 = \bigcup_{i=1}^r C^1_i$ and $V_2 = \bigcup_{i=1}^r C^2_i$. Embed $r(T) = x$ in a vertex of $C^* \cap W = C_{a_1} \cap W$ that is typical to $C^1_{b_1}$. We next embed all of $W_{r(T)}$ into $C^1_{a_1} \cup C^1_{b_1}$. This can be done levelwise, in each step choosing vertices that are typical with respect to $C^1_{a_1}$ or $C^1_{b_1}$, respectively. As $W_{r(T)}$ has at most $\frac{2}{\beta}$ vertices, we can embed all of $W_{r(T)}$ without a problem.

We now go through the antitrees $S \in \mathcal{T}$ and embed S together with W_S in W. We do this in an ordered way, so that when starting to work with S, the parent of $r(S)$ is already embedded. We will show that at every step of the process, that is, for each $S \in \mathcal{T}$, the following conditions are met:

(a) $Lev_{16t+2}(T[S \cup W_S])$ is embedded into V_1, and the rest of $T[S \cup W_S]$ is embedded into V_2,

(b) if $S \in P_j$, then $V_{\text{out}}(T[S \cup W_S]) \setminus Lev_{16t+2}(T[S \cup W_S])$ is embedded into $C^2_{a_j}$ and $V_{\text{in}}(T[S \cup W_S]) \setminus Lev_{16t+2}(T[S \cup W_S])$ is embedded into $C^2_{b_j}$,

(c) every vertex in $V_{\text{out}}(T[S \cup W_S])$ is embedded in a cluster C_v corresponding to a vertex v with $\text{ood}(a_1, v) \leq 8t$,

(d) every vertex in $V_{\text{in}}(T[S \cup W_S])$ is embedded in a cluster C_v corresponding to a vertex v with $\text{oid}(a_1, v) \leq 8t$, and

(e) for every $w \in W$, if the image of w lies in cluster C then it is typical with respect to $(C')^1$ for some neighbour C' of C.

Assume now we are in a step of the process and about to embed $T[S \cup W_S]$ for some $S \in \mathcal{T}$. Let $j \in [t]$ be such that $S \in P_j$. Observe that the parent of $r(S)$ is already embedded in a vertex w of a cluster C so that w is typical to $(C')^1$ for some neighbour C' of cluster C, by (e).

First assume that $r(S) \in V_{\text{out}}(S)$. Then the parent of $r(S)$ is an in-neighbour of $r(S)$. Because of (c) and (13), there is an out-out-walk P'' of (even) length at most $16t$ starting at C and ending at C_{a_j}. We add the cluster C_{b_j} at the end and the cluster C' at the beginning to obtain an antiwalk P' of length at most $16t + 2$. We obtain an antiwalk P of length exactly $16t + 2$ by repeating two subsequent vertices of P' an appropriate number of times. Note that P and $T[S \cup W_S]$ are consistent.

Now, if $r(S) \in V_{\text{in}}(S)$, we proceed analogously, only that here, P'' is an in-in-walk of length at most $16t - 1$, and when constructing P', we add the antiwalk $C_{b_j}C_{a_j}$ at the end. That gives an in-out-walk P of length exactly $16t + 2$ that is consistent with $T[S \cup W_S]$.

Observe that $|S \cup W| < \frac{\varepsilon}{10}m$. Set $h := 16t + 2$, and let X_{h-1} and X_h be the sets of all unoccupied vertices of the last two clusters on P. By (a), the only vertices embedded in $C^2_{a_j} \cup C^2_{b_j}$ are vertices from W and antitrees from P_j without their first $16t + 2$ levels. By (12) and (15), we conclude that

$$|X_{h-1}|, |X_h| > 3\sqrt{\varepsilon}m. \tag{16}$$

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We will use Lemma 3.9 to embed $T[S \cup W_S]$ into P. For this, let Z_0 be the set of all unused out-neighbours (if $r(S) \in V_{\text{out}}(S)$) or in-neighbours (if $r(S) \in V_{\text{out}}(S)$) of w in C'. For $1 \le i \le h$, let Z_i be the set of all unused vertices of Q^1, where Q is the $(i+1)$th vertex on P. By (a), any used vertex in Q^1 lies in $L_{16t+2}(T) \cup W$, and this set contains at most

$$|W| \cdot \Delta(T)^{18t} \le \frac{2}{\beta} \cdot (\Delta(T))^{18r\frac{k}{n}} \le \frac{2}{\beta} \cdot (\Delta(T))^{18M_0} \le \frac{\beta}{5}k \le \frac{\varepsilon}{500M_0}n \le \varepsilon m$$

vertices, where we used (14) for the third inequality. So, since by (12), the neighbourhood of w in $(C')^1$ is at least $d \cdot |(C')^1| \ge 2\sqrt{\varepsilon} \cdot 9\sqrt{\varepsilon}|C'| \ge 18\varepsilon m$, we conclude that $|Z_1| > 3\varepsilon m$ and similarly, we see that $|Z_i| > 3\sqrt{\varepsilon}m$ for $1 \le i \le h$. Thus, we can apply Lemma 3.9 to embed $T[S \cup W_S]$. Observe that conditions (a)–(e) are satisfied after embedding S.

After embedding $T[S \cup W_S]$ for each $S \in \mathcal{T}$ in this way, we have embedded all of T, with x embedded in V^*, which finishes the proof. □

6. Antisubdivisions of K_h: The proof of Theorem 1.7

The proof of Theorem 1.7 is similar to the proof of Theorem 1.2, with the difference that we only need the connected antimatching in the reduced oriented graph, and then do the embedding 'by hand'.

6.1. Preparation

We choose our constants as in the proof of Theorem 1.2, in particular ensuring that $n_0 \ge 100\varepsilon^{-1}M_0^3$. In addition, we set

$$\gamma := \frac{\varepsilon}{10M_0}.$$

We now prepare the oriented graph in the same way as for Theorem 1.2. In the reduced oriented graph of D, we use Lemma 4.6 to find a connected antimatching $M = \{a_ib_i\}_{1\le i\le t}$ of convenient size t. We split each cluster C into two slices C^1 and C^2, where C^1 only contains about a $10\sqrt{\varepsilon}$ portion of the vertices of C.

Let a long k-edge antisubdivision H of K_h be given, that is, a set $X = x_1, \dots, x_h$ of vertices, and antipaths $P_{i,j}$ connecting x_i with x_j, for each $1 \le i < j \le h$. For each two-edge path $P_{i,j}$ we add the middle vertex $x_{i,j}$ to X. Call the obtained set X'. By assumption, X' induces a forest in K_h. Choose a subset of

$$\{P_{i,j} : 1 \le i < j \le h \wedge e(P_{i,j}) \ge 3\}$$

so that together with X', they induce a tree in K_h. For all other paths $P_{i,j}$ we choose a subpath $Q_{i,j} \subseteq P_{i,j}$ of length 3. Let $\mathcal{Q}$ be the set of all such paths $Q_{i,j}$, and let Y be the set of their endvertices. Note that $T = K_h - (\bigcup_{Q_{i,j}\in\mathcal{Q}} V(Q_{i,j}) \setminus Y)$ is an antitree. We root T at x_1.

Let Z be the set of all vertices on paths in T between vertices of $X \cup Y$ that have length at most M_0. Set $W = X' \cup Y \cup Z$, and note that W naturally partitions into two sets: $W_{in} = W \cap V_{\text{in}}(H)$ and $W_{out} = W \cap V_{\text{out}}(H)$. Observe that there are at most $\frac{h^2}{2}$ paths $P_{i,j}$ of length at least 3, and thus

$$|W| \le 2h + h^2 + M_0h^2 \le 4M_0h^2. \tag{17}$$

Also note that each component of $T - W$ is a long path (longer than M_0). Let $\mathcal{P}$ be the set of all these components.

6.2. Embedding of T

We first give quick overview of the embedding of T, the details are given further below. First, we embed the root x_1 into $C_{a_1} \cup C_{b_1}$. Then at each step, we embed either a vertex from W or a path

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from $\mathcal{P}$ whose parent is already embedded. All vertices of W are embedded into $C^1_{a_1} \cup C^1_{b_1}$, and the paths from $\mathcal{P}$ are embedded mainly into the C^2-slices of clusters corresponding to edges of M. For this, we use the edges of the antimatching M (including a_1b_1 although this is not crucial) in an ordered way, and when one edge is sufficiently used, we go to the next edge of M. For the connections between $C_{a_i} \cup C_{b_i}$ and $C_{a_j} \cup C_{b_j}$ we use the slices C^1. At all times, vertices are embedded into typical vertices with respect to the unused part of the slice C^2 to be used in the next step, or with respect to the slice C^1 to be used in the future for children. The details of the embedding of W are given in the next paragraph, and the details for the paths from $\mathcal{P}$ are given in the subsequent paragraph.

The vertices of W are embedded into $C^1_{a_1} \cup C^1_{b_1}$, as mentioned above. Namely, each $w \in W$ (in particular x_1) is embedded into $C^1_{a_1}$ if $w \in W_{out}$ and w is embedded into $C^1_{b_1}$ if $w \in W_{in}$. Every time we embed a vertex from W into $C^1_{a_1}$, we make sure its image is typical with respect to $C^1_{b_1}$, and every time we embed a vertex from W into $C^1_{b_1}$, we make sure its image is typical with respect to $C^1_{a_1}$. By (17), the set W is small enough to easily fit into $C^1_{a_1} \cup C^1_{b_1}$. It is also much smaller than a typical neighbourhood in $C^1_{a_1}$ or in $C^1_{b_1}$, and therefore, it is not a problem if a vertex from W is embedded much earlier than some of its children (as long as we ensure the children are embedded into C^1-slices). For a more precise analysis, see below.

Let us now turn to the paths from $\mathcal{P}$. As mentioned above, these paths are embedded mainly into the C^2-slices of clusters corresponding to edges of the antimatching M. However, the first vertex s of such a path S goes to either $C^1_{a_1}$ or $C^1_{b_1}$ (depending on the image of its parent), and we choose for the image of s a vertex that is typical with respect to C^1_0 where C_0 is the first cluster on a path P from C_{a_1}, or from C_{b_1}, to the clusters C_{a_j} and C_{b_j} we plan to use for S. We embed the next vertices of S along P, always into C^1-slices, always typical to the next slice until we reach $C_{a_j} \cup C_{b_j}$ (after at most M_0 steps). Then we switch to the slices $C^2_{a_j}$ and $C^2_{b_j}$. Now, every vertex is embedded into a typical vertex with respect to the unused part of $C^2_{a_j}$ or of $C^2_{b_j}$. If necessary, we move from $C_{a_j} \cup C_{b_j}$ to $C_{a_{j+1}} \cup C_{b_{j+1}}$ (using at most M_0 vertices which are embedded into C^1-slices on a suitable path). We keep moving to subsequent $C_{a_i} \cup C_{b_i}$'s if necessary. When we reach the last vertices on S, we start moving back to $C_{a_1} \cup C_{b_1}$, through the C^1-slices on a path of length at most M_0. We reach $C_{a_1} \cup C_{b_1}$ and embed the last vertex of S into a typical vertex with respect to $C^1_{a_1}$ or to $C^1_{b_1}$. If S has a child in W, this child can be embedded now, or later in the process.

Let us analyse the embedding procedure to see that T can indeed be embedded in this way. First, we note that the size of M is sufficient, and the C^2-slices are large enough so that all the paths from $\mathcal{P}$ can be embedded without a problem. Second, we observe that we used the C^1-slices exclusively for W, for the first $\leq M_0$ and last $\leq M_0$ vertices on a long path, and for moving from one edge a_jb_j to the next edge $a_{j+1}b_{j+1}$. So the number of vertices embedded into the C^1-slices is at most $|W| + 2M_0 \cdot h^2 + M_0 \cdot t$. Because of (17) and since $t \leq M_0$, this number is bounded from above by

$$6M_0h^2 + M_0^2 \leq \varepsilon \frac{n}{8M_0} \leq \frac{\varepsilon}{2}m.$$

So, whenever are about to embed a vertex into a C^1-slice, it is enough to know that the image of its parent was chosen typical with respect to the whole slice (as a typical neighbourhood in the slice has at least $18\varepsilon m$ vertices). We can thus embed all of T.

6.3. *Embedding interior vertices on paths from* $\mathcal{Q}$

It only remains to embed the two middle vertices $q_{i,j}, q'_{i,j}$ of the paths $Q_{i,j}$, which we do successively, in any order. Say the neighbour of $q_{i,j}$ was embedded in $v \in C^1_{a_1}$, and the neighbour of $q'_{i,j}$ was

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embedded in $v' \in C^1_{b_1}$. We consider the neighbourhood N_1 of the image of v in the unused part of $C^1_{b_1}$ and the neighbourhood N_2 of the image of v' in the unused part of $C^1_{a_1}$. Since at most $\frac{\varepsilon}{2}m + h^2 \leq \varepsilon |C^1_{a_1}|$ vertices of $C^1_{a_1}$ and $C^1_{b_1}$ have been used so far, we know that N_1 and N_2 are sufficiently large to ensure that there is an edge ww' we can use. We complete the embedding by mapping v and v' to w and w'.

7. Edge density: The proof of Theorem 1.9

In order to prove Theorem 1.9 we will need a lemma that allows us to rewrite the condition on the edge density as a condition on a parameter that is very similar to the semidegree, and which we will call the minimum pseudo-semidegree. Define the *minimum pseudo-semidegree* $\bar{\delta}^0(D)$ of a digraph D with at least one edge as the minimum d such that for each vertex $v \in V(D)$ we have $\delta^+(v), \delta^-(v) \in \{0\} \cup [d, \infty)$. The minimum pseudo-semidegree of an empty digraph is 0.

Lemma 7.1. *Let $k \in \mathbb{N}^+$. If a digraph D has more than $(k-1)|V(D)|$ edges, then it contains a digraph D' with $\bar{\delta}^0(D') \geq \frac{k}{2}$.*

Proof. Note that the vertices of D have, on average, in-degree greater than $k-1$ and out-degree greater than $k-1$. Consider the following folklore construction of an auxiliary bipartite graph B associated to the digraph D: first, divide each vertex $v \in V(D)$ into two vertices v_{in} and v_{out}, letting v_{in} be adjacent to all edges ending at v, and letting v_{out} be adjacent to all edges starting at v; second, omit all directions on edges.

Then the average degree of B is greater than $k-1$, and a standard argument shows that B has a non-empty subgraph B' of minimum degree at least $\frac{k}{2}$ (for this, it suffices to successively delete edges of degree at most $\frac{k-1}{2}$ and to calculate that we have not deleted the entire graph). Translating B' back to the digraph setting, we see that D has a subdigraph D', such that D' minimum pseudo-semidegree at least $\frac{k}{2}$, which is as desired. □

The next auxiliary result states that the minimum semidegree and the minimum pseudo-semidegree are practically equivalent for the purposes of finding antidirected subgraphs A in oriented graphs D, if there is some control over the placement of A in D.

Say an oriented graph is *weakly connected* if its underlying graph is connected. Recall that the notation $\subseteq_\gamma$ is given Definition 5.1.

Lemma 7.2. *For any weakly connected antidirected graph A and for any $\ell, n_0 \in \mathbb{N}$, the following holds. If for each oriented graph D on at least n_0 vertices with $\delta^0(D) \geq \ell$ we have $A \subseteq_{1/8} D$, then each oriented graph D' on at least n_0 vertices with $\bar{\delta}^0(D') \geq \ell$ contains A.*

Proof. Given A, ℓ, and n_0, let D' be an oriented graph on at least n_0 vertices with $\bar{\delta}^0(D') \geq \ell$. Assume that $|V_{\text{out}}(D')| \geq |V_{\text{in}}(D')|$ (the other case is analogous). Then

$$|D' \setminus V_{\text{in}}(D')| \geq |D' \setminus V_{\text{out}}(D')|.$$

We will construct an auxiliary oriented graph D to which the hypothesis of the lemma can be applied. Take four copies D_1, D_2, D_3, D_4 of D'. Let $A_i = V_{\text{out}}(D_i)$ and $B_i = V_{\text{in}}(D_i)$ for $i = 1, \ldots 4$. For $i = 1, 3$, identify the vertices of A_i with the vertices of A_{i+1}, and the vertices of B_i with the vertices of B_{i-1} (addition modulo 4). Finally, reverse all directions on edges in copies D_2 and D_4. For an illustration, see Fig. 3.

Note that the obtained digraph D is an oriented graph and has more than n_0 vertices. Also note that $\delta^0(D) \geq \ell$, since all vertices in $D_i \setminus (A_i \cup B_i)$ maintain their semidegree, the vertices in $A_i = A_{i+1}$ have at least ℓ out-neighbours in D_i and at least ℓ in-neighbours in D_{i+1}, for $i = 1, 3$, and a similar statement holds for the vertices in the sets B_i.

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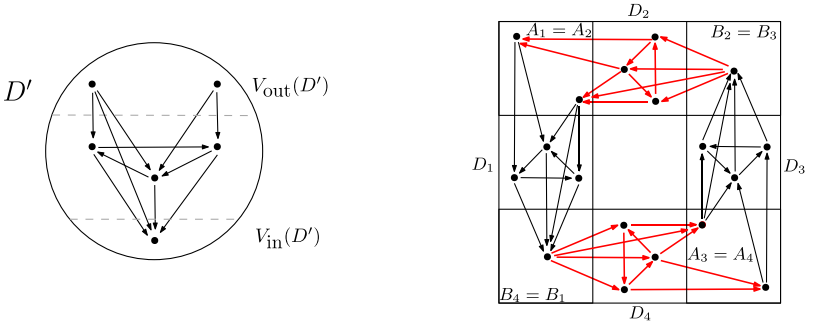


Figure 3. Construction of D from D' in the proof of Lemma 7.2.

We choose V^* as the set of all vertices in $V(D_1) \setminus B_1$. Note that $|V^*| \geq \frac{|V(D_1)|}{2}$ and therefore $|V^*| \geq \frac{|V(D)|}{8}$. We let xx' be any edge of A (directed from x to x'). By the hypothesis of the lemma, we find an embedding of A into D, with x embedded into some vertex $v^* \in V^*$.

We claim that

$$\text{all vertices of } A \text{ are embedded into } V(D_1). \tag{18}$$

Then, as D_1 is a copy of D' (with the original directions on the edges), we can conclude that A is a subdigraph of D'.

So all that remains is to prove (18). Proceeding by contradiction, we assume otherwise, that is, we assume there is a vertex $z \in V(A)$ whose image does not lie in $V(D_1)$. Then the underlying graph of A, which is connected by assumption, contains a path from x to z. So A, which is antidirected, contains an antipath $xy_1y_2 \dots y_hz$ from x to z. Since xx' is an edge of A, we know that the edge xy_1 is directed from x to y_1.

Recall that x is embedded in $v^* \in V^* = V(D_1) \setminus B_1$. This location of v^* ensures that all out-neighbours of v^* are in D_1. More specifically, by the definition of A_1, all out-neighbours of v^* are in $D_1 \setminus A_1$. In particular, y_1 is embedded in a vertex w_1 of $D_1 \setminus A_1$. This in turn ensures that all in-neighbours of w_1 are in $D_1 \setminus B_1$. In particular, y_2 is embedded in a vertex w_2 of $D_1 \setminus B_1$. Continuing to argue in this manner, we see that all y_i and also z are embedded in $V(D_1)$. This proves (18), and thus completes the proof of the lemma. □

Remark 7.3. *We remark that Lemma 7.2 also holds if both D and D' are directed graphs instead of oriented graphs. However, since Theorem 5.2 only holds for oriented graphs of high minimum semidegree, we need the lemma in the form it is written.*

Remark 7.4. *Observe that in Lemma 7.2 it is essential to require that a vertex of A can be mapped to a specific set in D: If we left out this requirement, then we might find a copy of A in D_2 instead of in D_1. Thus in D' we would only get a copy of A with all directions reversed.*

We are ready to prove Theorem 1.9.

Proof of Theorem 1.9. Given η and c from Theorem 1.9, we note that we may assume that $\eta \leq 1/8$. Let n_0 be given by Theorem 5.2 for input η and c.

Now, let $n \geq n_0$ and $k \geq \eta n$, and let D be an oriented graph on n vertices with $|E(D)| > (1 + \eta)(k-1)|V(D)|$. Let T be a balanced antitree with k edges and $\Delta(T) \leq (\log n)^c$. We need to show that $T \subseteq D$.

Apply Lemma 7.1 to D to obtain an oriented graph $D' \subseteq D$ with $\bar{\delta}^0(D) \geq (1 + \eta)\frac{k}{2}$. Use Lemma 7.2 and Theorem 5.2 to see that T can be embedded in D'. As $D' \subseteq D$, we see that $T \subseteq D$. □

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8. Conclusion

8.1. Oriented trees in digraphs

Any digraph D with a minimum semidegree of at least k contains every orientation of every tree with k edges, by a greedy embedding argument. This bound cannot be lowered, not even if we are only looking for oriented paths, as the disjoint union of complete digraphs of order k has minimum semidegree $k-1$ but no k-edge oriented path. Perhaps an additional condition, for instance on the maximum semidegree of D, or requiring D to be weakly connected and sufficiently large, combined with a lower bound on the semidegree of D could give a variant of Corollary 1.3 for digraphs.

8.2. Oriented trees in oriented graphs

In the introduction, we saw the example of a k-edge star with all edges directed outwards, which is not contained in, for instance, the $(k-1)$-blow-up of the directed triangle, although this graph has minimum semidegree $k-1$. We overcame this difficulty in Theorem 1.2 by concentrating on balanced antitrees. Another possibility, which was suggested in [29], could be to add an extra condition on the oriented graph D, for instance a condition on the maximum semidegree of D, with the hope of guaranteeing all k-edge antitrees, or even all k-edge oriented trees. Such a condition has been successfully used in the undirected graph setting (see e.g. [5]).

8.3. Oriented subdivisions

Perhaps Theorem 1.7 extends to other orientations of subdivisions of the complete graph. For instance, one could ask whether there is a function $f(h)$ such that every oriented graph of minimum semidegree at least $f(h)$ contains an orientation of a subdivision of K_h, where each path changes directions only a bounded number of times. This would be implied by Conjecture 1.6.

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References

[1] Aboulker, P., Cohen, N., Havet, F., Lochet, W., Moura, P. F. S. and Thomassé, S. (2019) Sub- divisions in digraphs of large out-degree or large dichromatic number. *Elect. J. Comb.* **26**(3) P3.19.

[2] Addario-Berry, L., Havet, F., Linhares Sales, C., Reed, B. and Thomassé, S. (2013) Oriented trees in digraphs. *Discr. Math.* **313**(8) 967–974.

[3] Ajtai, M., Komlós, J. and Szemerédi, E. (1995) On a Conjecture of Loebl. In: Proceedings of the 7th International Conference on Graph Theory, Combinatorics, and Algorithms. Wiley, New York, pp. 1135–1146.

[4] Alon, N. and Shapira, A. (2004) Testing subgraphs in directed graphs. *J. Comput. Syst. Sci.* **69**(3) 354–382.

[5] Besomi, G., Pavez-Signé, M. and Stein, M. (2019) Degree conditions for embedding trees. *SIAM J. Discr. Math.* **33**(3) 1521–1555.

[6] Besomi, G., Pavez-Signé, M. and Stein, M. (2020) Maximum and minimum degree conditions for embedding trees. *SIAM J. Discr. Math.* **34**(4) 2108–2123.

[7] Burr, S. (1980) Subtrees of Directed Graphs and Hypergraphs. In:Proceedings of the Eleventh South- eastern Conference on Combinatorics, Graph Theory and Combinatorics Florida Atlantic University, Boca Raton, FL, pp. 227–239.

https://doi.org/10.1017/S0963548324000038 Published online by Cambridge University Press

[8] Cohen, N., Havet, F., Lochet, W. and Nisse, N. (2018) Subdivisions of oriented cycles in digraphs with large chromatic number. *J. Graph Theor.* **89**(4) 439–456.
[9] Dirac, G. A. (1952) Some theorems on abstract graphs. *Proc. London Math. Soc.* **s3-2**(1) 69–81.
[10] Erdős, P. and Gallai, T. (1959) On maximal paths and circuits of graphs. *Act Math Acad Sci Hung* **10**(3) 337–356.
[11] Girão, A., Popielarz, K. and Snyder, R. (2021) Subdivisions of digraphs in tournaments. *J. Comb. Theory, Ser. B* **146** 266–285.
[12] Gishboliner, L., Steiner, R. and Szabó, T. (2022) Dichromatic number and forced subdivisions. *J. Comb. Theory, Ser. B* **153** 1–30.
[13] Gishboliner, L., Steiner, R. and Szabó, T. (2022) Oriented cycles in digraphs of large outdegree. *Combinatorica* **42**(1) 1145–1187.
[14] Graham, R. L. (1970) On subtrees of directed graphs with no path of length exceeding one. *Canad. Math. Bull.* **13**(3) 329–332.
[15] Jackson, B. (1981) Long paths and cycles in oriented graphs. *J. Graph Theor.* **5**(2) 145–157.
[16] Jagger, C. (1998) Extremal digraph results for topological complete tournaments. *Eur. J. Combin.* **19** 687–694.
[17] Kathapurkar, A. and Montgomery, R. (2022) Spanning trees in dense directed graphs. *J. Comb. Theory Ser. B* **156** 223–249.
[18] Keevash, P., Kühn, D. and Osthus, D. (2009) An exact minimum degree condition for Hamilton cycles in oriented graphs. *J. Lond. Math. Soc.* **79**(1) 144–166.
[19] Kelly, L. (2011) Arbitrary Orientations of Hamilton Cycles in Oriented Graphs. *Electron. J. Comb.* **18**(1) P186.
[20] Kelly, L., Kühn, D. and Osthus, D. (2008) A Dirac type result on Hamilton cycles in oriented graphs. *Comb. Prob. Comp.* **17**(5) 689–709.
[21] Kelly, L., Kühn, D. and Osthus, D. (2010) Cycles of given length in oriented graphs. *J. Comb. Theory Ser. B* **100**(3) 251–264.
[22] Klimošová, T. and Stein, M. (2022) Antipaths in oriented graphs(In preparation).
[23] Komlós, J., Sárközy, G. N. and Szemerédi, E. (1995) Proof of a packing conjecture of Bollobás. *Comb. Prob. Comp.* **4**(3) 241–255.
[24] Kühn, D., Osthus, D. and Young, A. (2008) A note on complete subdivisions in digraphs of large outdegree. *J. Graph Theor.* **57**(1) 1–6.
[25] Mader, W. (1967) Homomorphieeigenschaften und mittlere Kantendichte von Graphen. *Math. Ann.* **174**(4) 265–268.
[26] Mader, W. (1985) Degree and local connectivity in digraphs. *Combinatorica* **5**(2) 161–165.
[27] Mycroft, R. and Naia, T. (2020) Trees and tree-like structures in dense digraphs. arXiv preprint: 2012.09201.
[28] Scott, A. (2000) Subdivisions of Transitive Tournaments. *Eur. J. Combin.* **21**(8) 1067–1071.
[29] Stein, M. (2021) Tree containment and degree conditions. In: *Discrete Mathematics and Applications* (Raigoroskii, A. and Rassias, M., eds), Springer, pp. 459–486.
[30] Steiner, R. (2024) Subdivisions with congruence constraints in digraphs of large chromatic number. *J. Graph Thr.* **105**(1) 136–143.
[31] Szemerédi, E. (1978) Regular partitions of graphs. *Problémes Combinatoires et Théorie des Graphes Colloques Internationaux CNRS* **260**, 399–401.
[32] Thomassen, C. (1985) Even cycles in directed graphs. *Eur. J. Combin.* **6**(1) 85–89.

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