A group structure on the golden triples

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1. *Introduction*

Some favourite topics of both mathematics teachers and mathematics students are the golden ratio, Fibonacci numbers, groups, and Pythagorean triples. The material of this paper involves the first three, and there are some analogies with the fourth.

We make the following defintion. A *golden triple* is a triple of integers (a, b, c) satisfying the Diophantine equation

$$a^2 + ab - b^2 = c^2 \qquad (c > 0). \tag{1}$$

The significance of the name will become apparent shortly. Golden triples are similar to Pythagorean triples (a, b, c), which satisfy the equation

$$a^2 + b^2 = c^2.$$

and Eisenstein triples (a, b, c), also called 60-degree triples [1, 2], which satisfy the equation

$$a^2 - ab + b^2 = c^2 \qquad (c > 0).$$

The purpose of this paper is to show that there is a group structure on the set of golden triples. We will also examine a subgroup of the golden triples, and another set of triples similar to the golden triples. This involves an interesting blend of number theory and algebra, and the mathematics is fairly elementary.

2. *Pythagorean triples*

A Gaussian integer is a complex number whose real and imaginary parts are integers. It is well known that the square of a Gaussian integer is another Gaussian integer which is associated with a Pythagorean triple [3]. Thus, for example, $(2 + 1i)^2 = 3 + 4i$, which has a magnitude of 5. It is therefore associated with the Pythagorean triple (3, 4, 5). More generally,

$$(m + ni)^2 = \left(m^2 - n^2\right) + (2mn)\, i \tag{2}$$

and this has magnitude $m^2 + n^2$. We have thus generated the parametric form of Pythagorean triples, $\left(m^2 - n^2,\ 2mn,\ m^2 + n^2\right)$.

Now, given two Pythagorean triples, one can generate yet another by multiplying the associated Gaussian integers. For instance, if we take (3, 4, 5) and (12, 5, 13), we can multiply:

$$(3 + 4i) \cdot (12 + 5i) = 16 + 63i.$$

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We multiply the magnitudes of $3 + 4i$ and $12 + 5i$, namely 5 and 13, and get a magnitude of 65. We thus obtain the Pythagorean triple (16, 63, 65).

It is known [4, 5] that the set of Pythagorean triples (allowing the a and b components to be integers, including negative integers) is a group under the multiplication just described.

3. *Golden triples*

We are now interested in integer solutions of (1).

We claim that, as Pythagorean triples are associated with Gaussian integers $a + bi$, so the solutions of (1) are associated with numbers of the form $a + b\varphi$, where

$$\varphi = \frac{1 + \sqrt{5}}{2} \approx 1.618$$

is the golden ratio, and a and b are integers.

Numbers of the form $m + n\varphi$ are called *golden integers* [6 (p. 153), 7]. They are used by physicists in the study of quasicrystals [7, 8]. They have also been used in coding theory [9]. It seems appropriate to name triples of integers (a, b, c) *golden triples*, where the magnitude of the golden integer $a + b\varphi$ is c. (We will define the magnitude of a golden integer shortly.) Then a, b and c satisfy (1), as we will show below.

We will need some basic facts about φ:

$$\varphi = \frac{1}{2} + \frac{\sqrt{5}}{2} \tag{3}$$

$$\varphi^2 = 1 + \varphi \tag{4}$$

$$\overline{\varphi} = \frac{1}{2} - \frac{\sqrt{5}}{2} \text{ (the conjugate)} \tag{5}$$

$$\varphi + \overline{\varphi} = 1 \tag{6}$$

$$\varphi \cdot \overline{\varphi} = -1. \tag{7}$$

We let a and b be integers, and define the magnitude of $a + b\varphi$ to be

$$c = |a + b\varphi| = \sqrt{(a + b\varphi)(a + b\overline{\varphi})}.$$

Then we have, using (6) and (7),

$$\begin{aligned} c^2 &= (a + b\varphi)(a + b\overline{\varphi}) \\ &= a^2 - b^2 + ab \end{aligned}$$

and this is (1).

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If z and w are golden integers, then the magnitude is well-behaved in that it satisfies

$$|z \cdot w| = |z| \cdot |w|.$$

The proof is left as an exercise for the reader.

One example of a golden triple is (10, 7, 11). Substituting into (1), we get

$$10^2 + 10 \cdot 7 - 7^2 = 11^2,$$

which is a true statement.

Another golden triple is (2, 3, 1), since $2^2 + 2 \cdot 3 - 3^2 = 1^2$.

As with Pythagorean triples and Gaussian integers, we can generate golden triples by squaring golden integers. For instance, using (4),

$$(1 + 1\varphi)^2 = 1 + \varphi^2 + 2\varphi = 1 + (1 + \varphi) + 2\varphi = 2 + 3\varphi$$

and we thus obtain the golden triple (2, 3, 1).

We will now derive a parametric form for golden triples, analogous to (2) for Pythagorean triples. Let $m + n\varphi$ be a golden integer. Then, using (4),

$$\begin{aligned}(m + n\varphi)^2 &= m^2 + n^2\varphi^2 + 2mn\varphi \\ &= (m^2 + n^2) + (n^2 + 2mn)\varphi. \end{aligned} \tag{8}$$

Now, using (6) and (7), $(m + n\varphi)^2$ has magnitude

$$\begin{aligned}|(m + n\varphi)^2| &= |m + n\varphi|^2 = (m + n\varphi)(m + n\overline{\varphi}) \\ &= m^2 - n^2 + mn. \end{aligned} \tag{9}$$

We restrict c to be positive, so we take the absolute value of (9).

We have thus generated a parametric form of golden triples:

$$(m^2 + n^2, n^2 + 2mn, |m^2 - n^2 + mn|). \tag{10}$$

As with Gaussian integers, it is easy to show that the golden integers are closed under multiplication. Furthermore, given two golden triples, the magnitudes (c) of their associated golden integers are positive integers, so multiplying them gives yet another positive integer. Thus, given two golden triples, one can generate yet another by multiplying the associated golden integers. For example, we start with the golden triples (10, 7, 11) and (2, 3, 1). Using (4),

$$\begin{aligned}(10 + 7\varphi) \cdot (2 + 3\varphi) &= 20 + 21\varphi^2 + 44\varphi \\ &= 41 + 65\varphi. \end{aligned}$$

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Multiplying the magnitudes of $10 + 7\varphi$ and $2 + 3\varphi$, namely 11 and 1, we get a magnitude of 11. We thus obtain the golden triple (41, 65, 11). It is easy to check that it satisfies (1).

4. *A group of golden triples*

We have essentially defined a multiplication of the golden triples, via multiplication of their associated golden integers. Using (4), we have

$$(a_1 + b_1\varphi)\cdot(a_2 + b_2\varphi) = (a_1a_2 + b_1b_2) + (b_1b_2 + a_1b_2 + a_2b_1)\varphi$$

and so the rule for multiplication of golden triples is

$$(a_1, b_1, c_1)\cdot(a_2, b_2, c_2) = (a_1a_2 + b_1b_2,\ b_1b_2 + a_1b_2 + a_2b_1,\ c_1c_2). \quad (11)$$

We will define an equivalence relation on the golden triples, such that any golden triple for which a, b and c share a common factor is defined to be equal to the 'primitive' golden triple in which a, b and c are relatively prime. That is, if a and b are not both 0, and k is not 0, then

$$(ka, kb, kc) = (a, b, c).$$

Theorem 1: The set of golden triples is an abelian group under multiplication.

Proof: As explained above, the golden triples are closed under multiplication. This multiplication is obviously associative and commutative, since it corresponds to ordinary multiplication of golden integers.

Next we look for an identity element. The identity for multiplication of golden integers is obviously 1, that is, $1 + 0\varphi$. Thus the golden triple (1, 0, 1) is the identity for multiplication of golden triples.

Finally we try to find a multiplicative inverse of (a, b, c). Because of the equivalence relation we defined, the product of (a, b, c) and its inverse does not have to be (1, 0, 1), but merely $(k, 0, k)$, where k is some non-zero integer. We make the following claim.

$$\text{The inverse of } (a, b, c) \text{ is } (a + b, -b, c). \quad (12)$$

To show this, we multiply

$$(a, b, c)\cdot(a + b, -b, c) = (a^2 + ab - b^2, -b^2 - ab + b(a + b), c^2),$$

by (11). And so

$$(a, b, c)\cdot(a + b, -b, c) = (c^2, 0, c^2) = (1, 0, 1)$$

by the equivalence relation.

Therefore the golden triples form an abelian group. This completes the proof.

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5. *The Fibonacci subgroup of the golden triples*

Next we examine the subset of the golden triples for which $c = 1$.

Let $F = \{(a, b, 1)\}$.

Theorem 2: F is a subgroup of the group of golden triples.

The proof is simple, and is left as an exercise for the reader.

We next try to identify the subgroup F, by finding a known group with which it is isomorphic. Remember that the Fibonacci numbers are numbers of the form F_n, satisfying $F_1 = 1, F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$. The first nine Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21 and 34.

Theorem 3: For all even n, if F_n represents the nth Fibonacci number, then $(F_{n-1}, F_n, 1)$ is a golden triple.

Proof: By (1), we wish to show that

$$F_{n-1}^2 + F_{n-1}F_n - F_n^2 = 1. \tag{13}$$

The formula

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n. \tag{14}$$

is called Cassini's identity [10, p. 74]. Now $F_{n+1} = F_{n-1} + F_n$. Substituting into (14), we get

$$F_{n-1}(F_{n-1} + F_n) - F_n^2 = (-1)^n,$$

and if n is even then (14) follows immediately. This completes the proof.

We can call F the *Fibonacci subgroup of the golden triples*. Some golden triples in F are (1, 1, 1), (2, 3, 1), (5, 8, 1), and (13, 21, 1).

According to [11, p. 405], François Lucas was the first to prove (13), and J. Wasteels proved the converse, that any positive integer solutions of (1) for which $c = 1$ must be consecutive Fibonacci numbers.

In what follows, we use the well-known [12, p. 52] identity

$$\varphi^n = F_{n-1} + F_n\varphi. \tag{15}$$

Theorem 4: If m and n are even integers, then golden triples satisfy

$$(F_{m-1}, F_m, 1)\cdot(F_{n-1}, F_n, 1) = (F_{m+n-1}, F_{m+n}, 1).$$

Proof: If we take two triples for which $c = 1$, we now know they must be of the form $(F_{m-1}, F_m, 1)$ and $(F_{n-1}, F_n, 1)$, where m and n are even integers. Their corresponding golden integers are $F_{m-1} + F_m\varphi$ and $F_{n-1} + F_n\varphi$.

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Using (15), $\varphi^m = F_{m-1} + F_m\varphi$ and $\varphi^n = F_{n-1} + F_n\varphi$. Multiplying, we get

$$\varphi^m\varphi^n = \varphi^{m+n} = (F_{m-1} + F_m\varphi)\cdot(F_{n-1} + F_n\varphi).$$

But by (15), $\varphi^{m+n} = F_{m+n+1} + F_{m+n}\varphi$. Therefore

$$(F_{m-1} + F_m\varphi)\cdot(F_{n-1} + F_n\varphi) = F_{m+n-1} + F_{m+n}\varphi \qquad (16)$$

and the theorem follows immediately.

The elements of the subgroup F are of the form $(F_{n-1}, F_n, 1)$, where n is even, and the multiplication rule is

$$(F_{m-1}, F_m, 1)\cdot(F_{n-1}, F_n, 1) = (F_{m+n-1}, F_{m+n}, 1).$$

Thus, to multiply the golden triples, we add the subscripts of the middle (that is, b) components.

The identity (1, 0, 1) is equal to $(F_{-1}, F_0, 1)$. The subscript of the middle component is 0.

By (12), the inverse of $(F_{n-1}, F_n, 1)$ is

$$(F_{n-1} + F_n, -F_n, 1) = (F_{n+1}, -F_n, 1).$$

It is easy to show that, for Fibonacci numbers, if k is even, then $F_{-k} = -F_k$, and if k is odd, then $F_{-k} = F_k$. Then the inverse of $(F_{n-1}, F_n, 1)$ is $(F_{n+1}, -F_n, 1) = (F_{-n-1}, F_{-n}, 1)$. If the middle component of the given triple has subscript n, the middle component of the inverse has subscript $-n$.

We see that, under multiplication, F is isomorphic to the set of even integers under addition, which is itself isomorphic to the set of integers under addition.

$\langle F, \cdot\rangle$ is isomorphic to $\langle Z, +\rangle$. This isomorphism maps the golden triple $(F_{n-1}, F_n, 1)$ of F, with n even, to the integer $\frac{1}{2}n$.

6. *Parametric form of golden triples with* $c = 1$

Using (10), let us select some golden triples for which $c = 1$. See Table 1.

m	n	a	b	c
1	0	1	0	1
0	1	1	1	1
1	1	2	3	1
1	2	5	8	1
2	3	13	21	1
3	5	34	55	1

TABLE 1: Some golden triples generated by the parametric form $(a, b, c) = (m^2 + n^2, n^2 + 2mn, |m^2 - n^2 + mn|)$, where $c = 1$.

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It is evident that not only are a and b consecutive Fibonacci numbers, but so are m and n. Specifically, if $m = F_{k-1}$ and $n = F_k$ (where k is any integer), then $a = F_{2k-1}$ and $b = F_{2k}$. The reason becomes clear from the following theorem.

Theorem 5: $(F_{k-1} + F_k\varphi)^2 = F_{2k-1} + F_{2k}\varphi$.

Proof:

By (15),

$$F_{k-1} + F_k\varphi = \varphi^k.$$

Then, by (15),

$$\begin{aligned}(F_{k-1} + F_k\varphi)^2 &= \left(\varphi^k\right)^2 = \varphi^{2k} \\ &= F_{2k-1} + F_{2k}\varphi.\end{aligned}$$

This completes the proof.

7. *Aureate triples*

We will now modify (1) slightly, changing the coefficient of ab from 1 to -1:

$$a^2 - ab - b^2 = c^2 \qquad (c > 0). \tag{17}$$

Triples (a, b, c) of integers satisfying this equation will be called *aureate triples*. The word 'aureate' is a synonym of the word 'golden'. Aureate triples are very similar to golden triples.

We will call numbers of the form $m - n\overline{\varphi}$, where $\overline{\varphi} = \frac{1}{2} - \frac{\sqrt{5}}{2}$, *aureate integers*. If the magnitude of the aureate integer $a - b\overline{\varphi}$ is c, then (a, b, c) is an aureate triple, where we define the magnitude of $a - b\overline{\varphi}$ to be

$$c = \left|a - b\overline{\varphi}\right| = \sqrt{\left(a - b\overline{\varphi}\right)(a - b\varphi)}.$$

The proof is left as an exercise for the reader.

As with Pythagorean triples and golden triples, we can generate aureate triples by squaring aureate integers. For instance, using (4),

$$\left(1 - 1\overline{\varphi}\right)^2 = 1 + \overline{\varphi}^2 - 2\overline{\varphi} = 1 + \overline{\varphi^2} - 2\overline{\varphi} = 1 + \overline{(1 + \varphi)} - 2\overline{\varphi} = 2 - 1\overline{\varphi}$$

and we thus obtain the aureate triple (2, 1, 1).

By a computation analogous to (8), (9), and (10), we can generate a parametric form of aureate triples:

$$\left(m^2 + n^2,\ 2mn - n^2,\ \left|m^2 - n^2 - mn\right|\right). \tag{18}$$

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As with golden integers, it is easy to show that the aureate integers are closed under multiplication, and that given two aureate triples one can generate yet another by multiplying the associated aureate integers. For instance, we start with the aureate triples (2, 1, 1) and (13, 3, 11). Using (4),

$$\begin{aligned}\left(2 - 1\overline{\varphi}\right)\cdot\left(13 - 3\overline{\varphi}\right) &= 26 + 3\overline{\varphi}^2 - 19\overline{\varphi} \\ &= 29 - 16\overline{\varphi}.\end{aligned}$$

Multiplying the magnitudes of $2 - 1\overline{\varphi}$ and $13 - 3\overline{\varphi}$, namely 1 and 11, we get a magnitude of 11.

We thus obtain the aureate triple (29, 16, 11). It is easy to check that it satisfies (17).

8. *A group of aureate triples*

Theorem 6: The set of aureate triples is an abelian group under multiplication. The rule for multiplication of aureate triples is:

$$\left(a_1, b_1, c_1\right)\cdot\left(a_2, b_2, c_2\right) = \left(a_1a_2 + b_1b_2, -b_1b_2 + a_1b_2 + a_2b_1, c_1c_2\right). \quad (19)$$

The aureate triple (1, 0, 1) is the identity for multiplication of aureate triples.

The inverse of (a, b, c) is

$$\left(a - b, -b, c\right). \quad (20)$$

The proof is analogous to that of Theorem 1, and is left as an exercise for the reader.

It turns out that the group of golden triples is isomorphic to the group of aureate triples, under the map

$$M\left((a, b, c)\right) = (a + b, b, c). \quad (21)$$

The inverse map sends the aureate triple (d, b, c) to the golden triple $(d - b, b, c)$.

Lemma 1: The map M is a homomorphism.

Proof: We wish to show that

$$M\left((a_1, b_1, c_1)\cdot(a_2, b_2, c_2)\right) = M\left((a_1, b_1, c_1)\right)\cdot M\left((a_2, b_2, c_2)\right).$$

By (11), the left-hand side equals

$$M\left(\left(a_1a_2 + b_1b_2, b_1b_2 + a_1b_2 + a_2b_1, c_1c_2\right)\right),$$

which by (21) equals

$$\left(a_1a_2 + 2b_1b_2 + a_1b_2 + a_2b_1, b_1b_2 + a_1b_2 + a_2b_1, c_1c_2\right).$$

And similarly for the right-hand side.

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Lemma 2: The map M is one-to-one and onto.

The proof is left as an exercise for the reader.

Theorem 7: The group of golden triples is isomorphic to the group of aureate triples, under a map that sends the golden triple (a, b, c) to the aureate triple $(a + b, b, c)$.

Proof: It follows immediately from Lemmas 1 and 2.

In fact the aureate integer associated with the aureate triple $(a + b, b, c)$ is actually equal to the golden integer associated with the golden triple (a, b, c).

Theorem 8: The aureate integer associated with the aureate triple $(a + b, b, c)$ is equal to the golden integer associated with the golden triple (a, b, c).

Proof: The aureate integer associated with the aureate triple $(a + b, b, c)$ is $(a + b) - b\overline{\varphi}$. By (6), this is equal to $(a + b) + b(-1 + \varphi) = a + b\varphi$, and this is the golden integer associated with the golden triple (a, b, c). This completes the proof.

9. *The Fibonacci subgroup of the aureate triples*

Next, we examine the subset of the aureate triples for which $c = 1$.

Let $F_{\text{aur}} = \{(a, b, 1)\}$.

We state the following results without proof, since they are analogous to those above for golden triples.

Theorem 9: F_{aur} is a subgroup of the group of aureate triples.

Theorem 10: For all even n, if F_n represents the nth Fibonacci number, then $(F_{n+1}, F_n, 1)$ is an aureate triple.

We can call F_{aur} the Fibonacci subgroup of the aureate triples. Some aureate triples in F_{aur} are (2, 1, 1), (5, 3, 1), (13, 8, 1) and (34, 21, 1).

Theorem 11: If m and n are even integers, then aureate triples satisfy

$$(F_{m+1}, F_m, 1) \cdot (F_{n+1}, F_n, 1) = (F_{m+n+1}, F_{m+n}, 1).$$

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Under multiplication, F_{aur} is isomorphic to the set of integers under addition. That is, $\langle F_{\text{aur}}, \cdot \rangle$ is isomorphic to $\langle Z, + \rangle$. This isomorphism maps the aureate triple $(F_{n+1}, F_n, 1)$ of F_{aur}, with n even, to the integer $\frac{1}{2}n$.

10. *Parametric form of aureate triples with* $c = 1$

Using (18), let us select some aureate triples for which $c = 1$. See Table 2.

m	n	a	b	c
1	0	1	0	1
1	1	2	1	1
2	1	5	3	1
3	2	13	8	1
5	3	34	21	1

TABLE 2: Some aureate triples generated by the parametric form $(a, b, c) = (m^2 + n^2, 2mn - n^2, |m^2 - n^2 - mn|)$ where $c = 1$.

It is evident that, as with golden triples, not only are a and b consecutive Fibonacci numbers, but so are m and n. Specifically, if $m = F_{k+1}$ and $n = F_k$, then $a = F_{2k+1}$ and $b = F_{2k}$.

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