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# Generalized Neumann expansions involving hypergeometric functions

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1. Making use of the familiar abbreviation

$$(\lambda)_m = \frac{\Gamma(\lambda+m)}{\Gamma(\lambda)} = \lambda(\lambda+1)(\lambda+2)\dots(\lambda+m-1),$$

let us adopt a contracted notation for the generalized hypergeometric function ${}_AF_B[x]$ and write

$${}_AF_B[z] = {}_AF_B\left[\begin{matrix}(a);\\(b);\end{matrix}\ z\right] = \sum_{m=0}^{\infty} \frac{((a))_m}{((b))_m}\frac{z^m}{m!}$$
$$= \sum_{m=0}^{\infty} \frac{(a_1)_m (a_2)_m \dots (a_A)_m}{(b_1)_m (b_2)_m \dots (b_B)_m}\frac{z^m}{m!},$$

where $(a)$ denotes the sequence of parameters

$$a_1, a_2, \dots, a_A.$$

It will be assumed thoughout the present paper that there are $A$ of the $a$ parameters, $A'$ of the $a'$ parameters, and so on. Thus $((a))_m$ is to be interpreted as

$$\prod_{j=1}^{A} (a_j)_m,$$

with similar interpretations for $((a'))_m$, etc.

Recently in these proceedings we gave four generalized expansions for the product (see (9))

$${}_AF_B\left[\begin{matrix}(a);\\(b);\end{matrix}\ x\right] {}_CF_D\left[\begin{matrix}(c);\\(d);\end{matrix}\ y\right]$$

in terms of Neumann series. For instance, we proved that ((9), p. 245 (1·2)), for all values of $\lambda$ with the possible exception of negative integers and zero,

$$(\tfrac{1}{4}z)^{\lambda}\, {}_AF_B\left[\begin{matrix}(a);\\(b);\end{matrix}\ xz\right] {}_CF_D\left[\begin{matrix}(c);\\(d);\end{matrix}\ yz\right]$$
$$= \frac{\Gamma(\lambda)}{\Gamma(2\lambda)} e^{\frac{1}{2}z} \sum_{n=0}^{\infty} (-)^n \frac{(\lambda+n)\,\Gamma(2\lambda+n)}{n!} I_{\lambda+n}(\tfrac{1}{2}z)$$
$$\times F\left[\begin{matrix}-n, 2\lambda+n: (a); (c);\\ \lambda+\frac{1}{2}: (b); (d);\end{matrix}\ x, y\right], \qquad (1\cdot1)$$

where $A \leqslant B$, $C \leqslant D$ (or $A = B+1$ with $|xz| < 1$, and $C = D+1$ with $|yz| < 1$), and the notation for the double hypergeometric function is due to Burchnall and Chaundy

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(see (3), p. 112) in preference, for the sake of brevity, to an earlier one introduced by Kampé de Fériet (see (1), p. 150).

When $y = 0$, (1·1) is reduced to ((8), p. 895, formula (1·3)) which includes, as its 'confluent' case $A = B = 1$, Dr L. J. Slater's well-known expansion ((6), p. 628)

$$M(n+t-\tfrac{1}{2}; 2n+t; x) = 2^{2n}\, n!\, e^{\frac{1}{2}x} x^{-n} \sum_{r=0}^{\infty} A_{n+r}(t)\, I_{n+r}(\tfrac{1}{2}x),$$

where

$$A_{n+r}(t) = (-)^r \frac{(n+r)\,(2n)_r\,(1-t)_r}{nr!\,(2n+t)_r},$$

for all values of $n$ and $t$, real or complex (see also (7), p. 32).

On a closer examination we observe that the formula (1·1) admits of a 'mild' generalization in the form

$$\tfrac{1}{2}z^{\lambda} F\left[\begin{matrix}(e),(f):(a);(c);\\(g),(h):(b);(d);\end{matrix} xz, yz\right]$$
$$= \frac{e^{\frac{1}{2}z}}{(\frac{1}{2})_{\lambda}} \sum_{n=0}^{\infty} (-)^n \frac{(\lambda+n)\,\Gamma(2\lambda+n)}{n!} I_{\lambda+n}(\tfrac{1}{2}z)$$
$$\times F\left[\begin{matrix}-n, 2\lambda+n, (e), (f):(a);(c);\\ \lambda+\tfrac{1}{2}, (g), (h):(b);(d);\end{matrix} x, y\right], \qquad (1\cdot2)$$

where $\lambda$ is unrestricted, $A+E+F \leqslant B+G+H$ and $C+E+F \leqslant D+G+H$.

In order to give a direct proof of (1·2) we substitute the known formula ((4), p. 98)

$$(\tfrac{1}{2}z)^{\nu} = \Gamma(\nu)\, e^{-xz} \sum_{m=0}^{\infty} (\nu+m)\, C_m^{\nu}(x)\, I_{\nu+m}(z)$$

with $x = -1$, in the power series expansion of the first member, and then collect the terms involving the same Bessel function.

When $e = g$ and $f = h$, the double series on the left-hand side of (1·2) breaks up into the product of two generalized hypergeometric functions, and we thus arrive at the formula (1·1).

Similar consequences of the known formulae (see (10), p. 138 (1); p. 141 (7) and p. 151 (1)) are the generalized Neumann expansions

$$z^{\lambda} F\left[\begin{matrix}(e),(f):(a);(c);\\(g),(h):(b);(d);\end{matrix} x^2z^2, y^2z^2\right]$$
$$= \sum_{n=0}^{\infty} (-)^n \frac{(\lambda+2n)\,\Gamma(\lambda+n)}{n!} I_{\lambda+2n}(2z)$$
$$\times F\left[\begin{matrix}-n, \lambda+n, (e), (f):(a);(c);\\ (g), (h):(b);(d);\end{matrix} x^2, y^2\right], \qquad (1\cdot3)$$

$$z^{\lambda} F\left[\begin{matrix}(e),(f):(a);(c);\\(g),(h):(b);(d);\end{matrix} x^2z^2, y^2z^2\right]$$
$$= \Gamma(\lambda+1) \sum_{n=0}^{\infty} \frac{(-\frac{1}{2}z)^n}{n!} I_{\lambda+n}(2z)$$
$$\times F\left[\begin{matrix}-n, \lambda+1, (e), (f):(a);(c);\\ (g), (h):(b);(d);\end{matrix} x^2, y^2\right] \qquad (1\cdot4)$$

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and

$$(\tfrac{1}{2}z)^{\mu+\nu}F\left[\begin{matrix}(e),(f):(a);(c);\\(g),(h):(b);(d);\end{matrix}x^2z^2,y^2z^2\right]$$
$$=\frac{\Gamma(\mu+1)\,\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)}\sum_{n=0}^{\infty}(-)^n\frac{(\mu+\nu+2n)\,\Gamma(\mu+\nu+n)}{n!}I_{\mu+n}(z)\,I_{\nu+n}(z)$$
$$\times F\left[\begin{matrix}-n,\mu+1,\nu+1,\mu+\nu+n,(e),(f):(a);(c);\\\tfrac{1}{2}(\mu+\nu+1),\tfrac{1}{2}(\mu+\nu+2),(g),(h):(b);(d);\end{matrix}x^2,y^2\right] \quad (1{\cdot}5)$$

respectively, the special cases $e=g$ and $f=h$ of which being already known (see (9), sections 2 and 3; see also (2) for various confluent forms of (1·3) and (1·4)).

2. *Extensions.* Since

$$I_\nu(z)=\frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}e^{-z}\,{}_1F_1[\nu+\tfrac{1}{2};\,2\nu+1;\,2z],$$

in terms of confluent hypergeometric functions, (1·2) gives us

$$F\left[\begin{matrix}(e),(f):(a);(c);\\(g),(h):(b);(d);\end{matrix}xz,yz\right]$$
$$=\sum_{n=0}^{\infty}\frac{(2\lambda)_n}{n!\,(\lambda)_n}(-\tfrac{1}{4}z)^n\,{}_1F_1[\lambda+n+\tfrac{1}{2};\,2\lambda+2n+1;\,z]$$
$$\times F\left[\begin{matrix}-n,2\lambda+n,(e),(f):(a);(c);\\\lambda+\tfrac{1}{2},(g),(h):(b);(d);\end{matrix}x,y\right]. \quad (2{\cdot}1)$$

The form of (2·1) obviously suggests the existence of a more general expansion involving a generalized hypergeometric function in place of Whittaker's ${}_1F_1$, and I am indebted to Dr A. Verma for letting me read an unpublished manuscript in which he employs the Laplace transform technique and derives from our formula (1·1) by the method of mathematical induction that

$$F\left[\begin{matrix}(e),(f):(a);(c);\\(g),(h):(b);(d);\end{matrix}xz,yz\right]$$
$$=\sum_{n=0}^{\infty}\frac{(2\lambda)_n\,((e))_n}{n!\,(\lambda)_n\,((g))_n}(-\tfrac{1}{4}z)^n\,{}_{E+1}F_{G+1}\left[\begin{matrix}\lambda+n+\tfrac{1}{2},(e)+n;\\2\lambda+2n+1,(g)+n;\end{matrix}z\right]$$
$$\times F\left[\begin{matrix}-n,2\lambda+n,(f):(a);(c);\\\lambda+\tfrac{1}{2},(h):(b);(d);\end{matrix}x,y\right], \quad (2{\cdot}2)$$

provided $A+E+F\leqslant B+G+H$, $C+E+F\leqslant D+G+H$, and $E\leqslant G$ (or $E=G+1$ with $|z|<1$).

It is not difficult to give an extension of (2·2) in the form

$$F^{(3)}\left[\begin{matrix}(a),(c)::(e);(e');(e''):(f);(f');(f'');\\(b),(d)::(g);(g');(g''):(h);(h');(h'');\end{matrix}xt,yt,zt\right]$$
$$=\sum_{n=0}^{\infty}\frac{(2\lambda)_n\,((a))_n}{n!\,(\lambda)_n\,((b))_n}(-\tfrac{1}{4}t)^n\,{}_{A+1}F_{B+1}\left[\begin{matrix}\lambda+n+\tfrac{1}{2},(a)+n;\\2\lambda+2n+1,(b)+n;\end{matrix}t\right]$$
$$\times F^{(3)}\left[\begin{matrix}-n,2\lambda+n,(c)::(e);(e');(e''):(f);(f');(f'');\\\lambda+\tfrac{1}{2},(d)::(g);(g');(g''):(h);(h');(h'');\end{matrix}x,y,z\right], \quad (2{\cdot}3)$$

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where $A+C+E+E''+F \leqslant B+D+G+G''+H$, etc., $A \leqslant B$ (or $A = B+1$ and $|t| < 1$), and the triple series

$$F^{(3)}\left[\begin{matrix}(a)::\ (b);\ (b');\ (b'');\ (c);\ (c');\ (c'');\\ (e)::\ (f);\ (f');\ (f'');\ (g);\ (g');\ (g'');\end{matrix}\ x, y, z\right]$$
$$= \sum_{m,n,p=0}^{\infty} \frac{((a))_{m+n+p}((b))_{m+n}((b'))_{n+p}((b''))_{p+m}((c))_m((c'))_n((c''))_p}{((e))_{m+n+p}((f))_{m+n}((f'))_{n+p}((f''))_{p+m}((g))_m((g'))_n((g''))_p} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!};$$

nor does it seem worthwhile to record the various particular cases of (2·3) which would express one or the other of the Lauricella's hypergeometric functions of three variables (see e.g. (1), p. 114 and (5), p. 111) as generalized Neumann series.

Now replace $B$ and $C$ by $B+1$ and $C+1$ respectively and set

$$b_{B+1} = c_{C+1} = \lambda + \tfrac{1}{2}.$$

The special case $x = y = z = 0$ of (2·3) will then give us

$$1 = \sum_{n=0}^{\infty} (-)^n \frac{((a))_n}{(\lambda+n)_n((b))_n}\frac{t^n}{n!}\,{}_{A}F_{B+1}\left[\begin{matrix}(a)+n;\\ \lambda+2n+1, (b)+n;\end{matrix}\ t\right]$$

which leads to

$$F^{(3)}\left[\begin{matrix}(a')::\ (e);\ (e');\ (e'');\ (f);\ (f');\ (f'');\\ (b')::\ (g);\ (g');\ (g'');\ (h);\ (h');\ (h'');\end{matrix}\ xt, yt, zt\right]$$
$$= \sum_{n=0}^{\infty} (-)^n \frac{((a))_n}{(\lambda+n)_n((b))_n}\cdot\frac{t^n}{n!}\,{}_{A}F_{B+1}\left[\begin{matrix}(a)+n;\\ \lambda+2n+1,\ (b)+n;\end{matrix}\ t\right]$$
$$\times F^{(3)}\left[\begin{matrix}-n, \lambda+n, (a'), (b)\ ::\ (e);\ (e');\ (e'')::\ (f);\ (f');\ (f'');\\ (a),\ (b')::\ (g);\ (g');\ (g'')::\ (h);\ (h');\ (h'');\end{matrix}\ x, y, z\right],$$

and therefore finally

$$F^{(3)}\left[\begin{matrix}(a'),\ (a'')::\ (e);\ (e');\ (e'')::\ (f);\ (f');\ (f'');\\ (b'),\ (b'')::\ (g);\ (g');\ (g'')::\ (h);\ (h');\ (h'');\end{matrix}\ xt, yt, zt\right]$$
$$= \sum_{n=0}^{\infty} (-)^n \frac{((a))_n((a'))_n}{(\lambda+n)_n((b))_n((b'))_n}\cdot\frac{t^n}{n!}\,{}_{A+A'}F_{B+B'+1}\left[\begin{matrix}(a)+n, (a')+n;\\ \lambda+2n+1,\ (b)+n,\ (b')+n;\end{matrix}\ t\right]$$
$$\times F^{(3)}\left[\begin{matrix}-n, \lambda+n, (a''), (b)\ ::\ (e);\ (e');\ (e'')::\ (f);\ (f');\ (f'');\\ (a),\ (b'')::\ (g);\ (g');\ (g'')::\ (h);\ (h');\ (h'');\end{matrix}\ x, y, z\right], \qquad (2\cdot4)$$

which generalizes our earlier formulae already referred to.

## REFERENCES


(1) APPELL, P. and KAMPÉ DE FÉRIET, J. *Fonctions hypergéométriques et hypersphériques. Polynomes d'Hermite* (Gauthier-Villars; Paris, 1926).

(2) BAILEY, W. N. Some expansions in Bessel functions involving Appell's function $F_4$. *Quart. J. Math. Oxford Series* **6** (1935), 233–238.

(3) BURCHNALL, J. L. and CHAUNDY, T. W. Expansions of Appell's double hypergeometric functions. II. *Quart. J. Math. Oxford Series* **12** (1941), 112–128.

(4) ERDÉLYI, A., MAGNUS, W., OBERHETTINGER, F. and TRICOMI, F. G. *Higher transcendental functions. Vol. II* (McGraw-Hill; New York, 1953).


https://doi.org/10.1017/S0305004100041359 Published online by Cambridge University Press

(5) Lauricella, G. Sulle funzioñi ipergeometriche a piu variabili. *Rend. Circ. Mat. Palermo*, **7** (1893), 111–158.
(6) Slater, L. J. Expansions of generalized Whittaker functions. *Proc. Cambridge Philos. Soc.* **50** (1954), 628–631.
(7) Slater, L. J. Confluent hypergeometric functions (Cambridge, 1960).
(8) Srivastava, H. M. Some expansions of generalized Whittaker functions. *Proc. Cambridge Philos. Soc.* **61** (1965), 895–896.
(9) Srivastava, H. M. Some expansions in products of hypergeometric functions. *Proc. Cambridge Philos. Soc.* **62** (1966), 245–247.
(10) Watson, G. N. *Theory of Bessel functions* (Cambridge, 1944).

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