But then the triangles BAQ and ABP are congruent by the congruence criterion SSS, thus the triangle ABC has $\angle A = \angle B$, whence $AC = BC$.

You have perhaps noticed that the assumption that AP and BQ are the bisectors of the internal angles at A and B is needed only to obtain the following:

(a) AP and BQ intersect in the incentre I;

(b) CI bisects the internal angle at C;

(c) $g = s - c$.

But after that we need only (b) in order to prove that $b_A = a_B$. We have therefore proved a more general result than the Steiner-Lehmus theorem:

Theorem 3: (Generalised Steiner-Lehmus theorem). In a triangle ABC let CR be the bisector of the internal angle at the vertex C, with the end-point R on the side AB. If $X \neq R$ is a point of the line segment CR and the triangle's cevians through X from the vertices A and B are equal, then $AC = BC$.

This theorem is essentially just a rephrasing of Archibald Henderson's generalised theorem (7b) in his 'essay on the internal bisector problem to end all essays on the internal bisector problem' [1].

Reference

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100.08 Pythagoras theorem by sliding a triangle

The sliding argument

If we move a right triangle in the plane in a direction parallel to one of its legs then the area swept by the triangle is equal to the area left behind. By 'area swept' (AS) we mean the set of points in the plane which the triangle occupies during the translation, but not including the *initial* position. Analogously, by 'area left behind' (AL) we mean the set of points in the plane that the triangle occupies during the translation, but not including the *final* position.

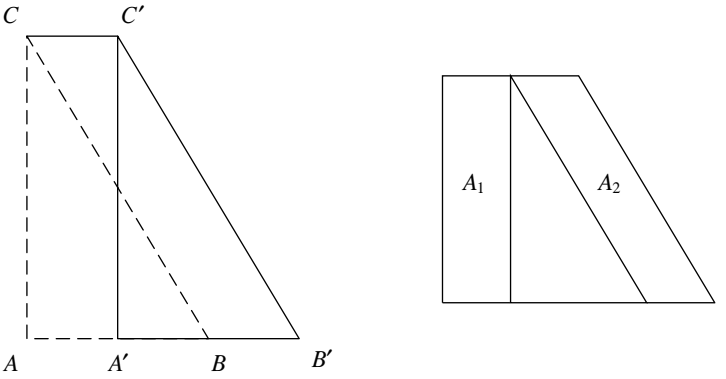
Of course, the *sliding argument* reduces to the fact that parallelograms which are on the same base and between the same parallels have equal area, but we prefer to interpret it as a consequence of the *area conservation* of the triangle, as we shall explain now.

In Figure 1(a), a triangle with vertices A, B and C is translated through a vector $\mathbf{AA'}$. Let A_T be the area of the triangle before the translation, and

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$A_{T'} = A'B'C'$ be the area of the triangle after the translation. The total area in Figure 1(a), $AB'C'C$, is equal to AL plus $A_{T'}$. On the other hand, the same area is equal to A_T plus AS. So we get $\mathrm{AL} + A_{T'} = A_T + \mathrm{AS}$ and, since $A_{T'} = A_T$, we get AL = AS which is the statement of the sliding argument. The same proof also works when the final triangle does not overlap the initial triangle, or it is a consequence of the previous argument by decomposing the translation into several *small* translations.

Of course, the area of AL is equal to the area of its reflection through the leg AC of the triangle, so the sliding argument might be interpreted as giving equality between the two areas A_1 and A_2 in Figure 1(b). We shall use this interpretation in what follows. We shall also use the corresponding version of the sliding argument when the triangle is translated parallel to the other leg AC.



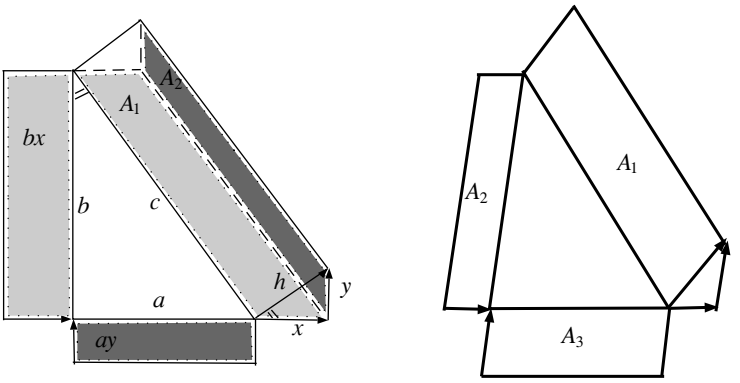
(a) (b)

FIGURE 1

Pythagoras theorem

In Figure 2(a), a right triangle with legs a and b is moved a distance h in the direction perpendicular to its hypotenuse c. The area swept by the triangle is then a rectangle with area ch. On the other hand, the same area is swept by the triangle if we slide it horizontally a distance x and then slide it vertically a distance y, i.e. $ch = A_1 + A_2$ (adding and subtracting a small right-angled triangle). Now we use the sliding argument twice (one for the horizontal translation and the other one for the vertical translation of the triangle) to give $A_1 = bx$ and $A_2 = ay$, so $ch = bx + ay$. Noticing the similarity between two right triangles, when $h = c$ we have $x = b$ and $y = a$, and we get the Pythagoras formula $c^2 = a^2 + b^2$. See also Figure 3, where the areas with the same colour are seen to be equal by the sliding argument (or, if one wishes, because they correspond to parallelograms which are on the same base and between the same parallels).

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(a) $ch = bx + ay$ (b) $A_1 = A_2 + A_3$

FIGURE 2

The same reasoning applies to give a direct proof of a particular, but interesting, case of Pappus Area Theorem (see [1] or [2]). The triangle need not be right-angled but the second and third movements of the triangle are in directions parallel to two of its sides. See Figure 3 for an illustration.

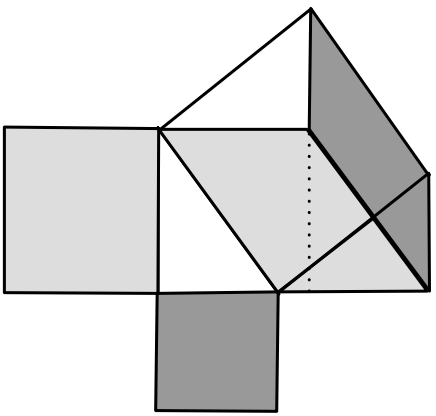


FIGURE 2(b)

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