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Connected fundamental domains for congruence subgroups

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Abstract. We give explicit sets of right coset representatives for the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, and $\Gamma(N)$, and prove that the corresponding unions of standard modular triangles are connected fundamental domains. The construction is based on a study of the projective line $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$. For every residue class $j \in \mathbb{Z}/N\mathbb{Z}$, the number of representatives above j is governed by the simple function

$$W_j = \min\{m \in \mathbb{Z}_{>0} \mid mj - 1 \in (\mathbb{Z}/N\mathbb{Z})^*\}.$$

We also include examples illustrating how the connected domains make cusp and boundary data visible.

1 Introduction

Let $\Gamma(1) = SL_2(\mathbb{Z})$ act on the upper-half plane $\mathbb{H} = \{z \,|\, \mathrm{Im} z > 0\}$ by the Möbius transformation

$$\gamma \cdot z = \frac{az + b}{cz + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1).$$

Two points in $\mathbb{H}$ are called *equivalent* under $\Gamma < \Gamma(1)$ if they are in the same orbit of Γ. We have the following definition from [Shi71, p. 15].

Definition 1.1 For a discrete subgroup $\Gamma < \Gamma(1)$, we call a set $F \subset \mathbb{H}$ a *fundamental domain* for Γ if

(1) F is open and connected;
(2) no two points of F are equivalent under Γ;
(3) every point of $\mathbb{H}$ is equivalent to a point in $\overline{F}$, the closure of F.

It is well known that a fundamental domain for $\Gamma(1)$ in the above sense is

$$D = \left\{ z \in \mathbb{H} \,\middle|\, |z| > 1, |\mathrm{Re} z| < \frac{1}{2} \right\}. \tag{1.2}$$

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Since $-I$ is in the center of $\Gamma(1)$, we see that $(\pm I)\Gamma$ is also a subgroup of $\Gamma(1)$. When $-I \in \Gamma$, $(\pm I)\Gamma = \Gamma$. Note that the Möbius transformation for $-I$ is the identity. If we have a choice of right coset representatives

$$(\pm I)\Gamma\backslash\Gamma(1) = \{(\pm I)\Gamma\gamma_1, \ldots, (\pm I)\Gamma\gamma_n\}, \tag{1.3}$$

where $n = [\Gamma(1) : (\pm I)\Gamma]$, we will show in Lemma 2.1 that

$$\mathrm{Int}\Big(\bigcup_{i=1}^{n} \overline{\gamma_i D}\Big) \tag{1.4}$$

is a fundamental domain for Γ provided that it is *connected*, where the closure is, as always in this article, taken in $\mathbb{H}$. We will then call such a list of right coset representatives $\{\gamma_1, \ldots, \gamma_n\}$ *fundamental*.

Consider the congruence subgroups [DS05, p. 13]

$$\Gamma(N) < \Gamma_1(N) < \Gamma_0(N) < \Gamma(1), \quad N > 1.$$

The main purpose of this article is to provide explicit and canonical lists of fundamental right coset representatives such that the corresponding domains are connected.

There are several classical and computational approaches to fundamental domains for congruence subgroups. Chuman gave an explicit Schreier system of representatives for $\Gamma_0(N)\backslash\Gamma(1)$, with applications to generators and relations of $\Gamma_0(N)$ [Chu73]. More generally, Kulkarni's Farey symbol method gives an arithmetic-geometric framework for constructing special polygons for finite-index subgroups of the modular group [Kul91]. There are also computer programs for finding and drawing connected fundamental domains, including Verrill's program [Ver01]. The emphasis here is different. We give a closed-form family of representatives whose connectedness is built into the construction. The resulting domains provide a concrete entry point to the geometry of modular curves and complement the cusp-class and Farey symbol approaches of Chuman and Kulkarni.

It is also useful to view our domains at a slightly coarser level. Each standard triangle has three distinguished sides, which in the figures may be called the left side, the right side, and the base side. If γD is one of our standard triangles, then the triangle across its base side is γSD. Thus, whenever both γ and γS occur in our representative list, the common base side is an internal edge and the two triangles form a two-triangle cell, or kite. In this sense, the domains constructed below may be regarded first as triangulated domains and then, after internal base-side gluings, as partially decomposed into kites. This gives a visual bridge between our explicit standard-tile domains and the coarser polygonal regions appearing in Farey symbol constructions. We do not claim that this provides a new construction of Kulkarni special polygons, as the Farey symbols have become well established and easy to use [KL08].

In this article, we mainly work in the ring $\mathbb{Z}/N\mathbb{Z}$. We write the elements as if they are just integers as $a, b \in \mathbb{Z}/N\mathbb{Z}$. Note that the greatest common divisors $\gcd(a, N)$ and $\gcd(a, b, N)$ are well defined as positive integers between 1 and N.

To make our final pictures more centered and hence more visualizable, we define our symmetric choice of residue classes mod N. The usual choice of classes from 0 to $N - 1$ mathematically would work the same (with suitable small modifications), but the pictures would not be as nice when drawn out.

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Notation 1.5 *For $N > 1$, let $N_1 = \left\lfloor \frac{N-1}{2} \right\rfloor$ and $N_2 = \left\lfloor \frac{N}{2} \right\rfloor$, where $\lfloor \cdot \rfloor$ is the usual floor function.*

For $x \in \mathbb{Z}$, we define $\widetilde{x}$ to be the unique integer such that $\widetilde{x} \equiv x \mod N$ and $-N_1 \le \widetilde{x} \le N_2$.

To formulate the construction, we study the projective line over the finite ring

$$\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) = \{(a,b) \in (\mathbb{Z}/N\mathbb{Z})^2 \mid \gcd(a,b,N) = 1\}/\sim,$$

where $(a,b) \sim (a',b')$ if $(a,b) = u(a',b')$ for some unit $u \in (\mathbb{Z}/N\mathbb{Z})^*$. We write the class of (a,b) as $(a:b)$.

For $x = (a:b) \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$, define

$$M(x) = M(a:b) = \min\{m \in \mathbb{Z}_{\ge 0} \mid ma - b \in (\mathbb{Z}/N\mathbb{Z})^*\}. \tag{1.6}$$

The condition $ma - b \in (\mathbb{Z}/N\mathbb{Z})^*$ says that $(a:b)$ is distant from the affine point $(1:m)$ in the usual terminology for projective lines over rings (see, e.g., [BH01, Section 2]). This observation is only a useful way to recognize the condition and will not be used later.

Proposition 1.7 *The function M in (1.6) is well defined and finite on $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$.*

Let $m = M(a:b)$ and put

$$c = ma - b \in (\mathbb{Z}/N\mathbb{Z})^*. \tag{1.8}$$

Then, $(a:b) = (ac^{-1} : bc^{-1})$ and

$$m(ac^{-1}) - (bc^{-1}) = 1 \quad \text{in } \mathbb{Z}/N\mathbb{Z}. \tag{1.9}$$

This gives a distinguished section of the natural projection from admissible pairs to $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$:

$$\sigma : \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \longrightarrow (\mathbb{Z}/N\mathbb{Z})^2, \qquad \sigma(a:b) = c^{-1}(a,b). \tag{1.10}$$

This is clearly independent of the choices of a, b in $(a:b)$, and we call $\sigma(a:b)$ the M-normalized representative of $(a:b)$. If $\sigma(x) = (j,\ell)$, then

$$M(x)j - \ell = 1 \quad \text{in } \mathbb{Z}/N\mathbb{Z}. \tag{1.11}$$

Thus, every M-normalized representative with first coordinate j has the form

$$(j, M(x)j - 1).$$

We introduce some notation:

$$\mathcal{S} = \sigma(\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})) \subset (\mathbb{Z}/N\mathbb{Z})^2,$$
$$\mathcal{S}(j) = \mathcal{S} \cap \pi_1^{-1}(\{j\}),$$

where $\pi_1 : (\mathbb{Z}/N\mathbb{Z})^2 \to \mathbb{Z}/N\mathbb{Z}$ is projection to the first component. Therefore, $\mathcal{S}(j)$ is the fiber of $\pi_1 : \mathcal{S} \to \mathbb{Z}/N\mathbb{Z}$ over j, and we aim to characterize it completely now.

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Definition 1.12 For $j \in \mathbb{Z}/N\mathbb{Z}$, define

$$W_j = \min\{r \in \mathbb{Z}_{>0} \mid rj - 1 \in (\mathbb{Z}/N\mathbb{Z})^*\}. \tag{1.13}$$

The set in (1.13) is nonempty, since $Nj - 1 = -1$ in $\mathbb{Z}/N\mathbb{Z}$. We regard W_j as the first break in the vertical string

$$(j, -1), \quad (j, j-1), \quad (j, 2j-1), \quad \ldots.$$

The following proposition says that $\mathcal{S}(j)$ is exactly the initial segment of this string before the break.

Proposition 1.14 *For each $j \in \mathbb{Z}/N\mathbb{Z}$, we have*

$$\mathcal{S}(j) = \{(j, mj - 1) \mid 0 \le m < W_j\}.$$

We define

$$M_j = W_j - 1.$$

Thus, M_j is the last height in the normalized vertical string over j, while W_j is the next height, where the string stops.

We now translate these projective-line representatives into matrices. Recall that $\Gamma(1)$ is generated by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{1.15}$$

with $S^2 = -I$.

The main result is the following theorem, where a fundamental list for Γ means, as in (1.3), a set of representatives for $(\pm I)\Gamma\backslash\Gamma(1)$.

Theorem 1.16 *Let $\Gamma(N) < \Gamma_1(N) < \Gamma_0(N) < \Gamma(1)$.*

(1) *For $\Gamma_0(N)$, a fundamental list of right coset representatives is Θ_0 given by*

$$\begin{cases} ST^i, & -N_1 \le i \le N_2, \\ ST^jST^m, & -N_1 \le j \le N_2,\ \gcd(j, N) > 1,\ 0 \le m \le M_j. \end{cases} \tag{1.17}$$

(2) *For $\Gamma_1(N)$, a fundamental list of right coset representatives is Θ_1 given by*

$$\begin{cases} ST^i, & -N_1 \le i \le N_2, \\ ST^jST^m, & -N_1 \le j \le N_2,\ \gcd(j, N) > 1,\ 0 \le m \le M_j, \\ ST^kST^i, & -N_1 \le k \le -2,\ \gcd(k, N) = 1,\ -N_1 \le i \le N_2, \\ ST^kST^{\widetilde{k^{-1}+j}}ST^m, & -N_1 \le k \le -2,\ \gcd(k, N) = 1, \\ & \qquad -N_1 \le j \le N_2,\ \gcd(j, N) > 1,\ 0 \le m \le M_j, \end{cases} \tag{1.18}$$

where k^{-1} denotes an inverse of k modulo N, and $\widetilde{x}$ is defined in Notation 1.5.

(3) *For $\Gamma(N)$, a fundamental list of right coset representatives is*

$$\{T^\ell \gamma \mid -N_1 \le \ell \le N_2,\ \gamma \in \Theta_1\}.$$

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Part (1), for $\Gamma_0(N)$, is the main point. The representatives for $\Gamma_1(N)$ and $\Gamma(N)$ are then obtained by the standard quotient maps between congruence subgroups.

In Section 2, we provide proofs for the results mentioned above. In Section 3, we give some examples and pictures.

2 Proofs of the results

Proof of Proposition 1.7 If (a,b) is replaced by $u(a,b)$ with $u \in (\mathbb{Z}/N\mathbb{Z})^*$, then $ma-b$ is replaced by $u(ma-b)$. Thus, the condition $ma-b \in (\mathbb{Z}/N\mathbb{Z})^*$ is independent of the chosen representative of $(a:b)$, so M is well defined.

It remains to show that the set in (1.6) is nonempty. Let p be a prime divisor of N. If $p \mid a$, then $p \nmid b$ by $\gcd(a,b,N)=1$, so $ma-b \not\equiv 0 \pmod p$ for every m. If $p \nmid a$, then the condition $ma-b \not\equiv 0 \pmod p$ excludes exactly one residue class of m modulo p. By the Chinese remainder theorem, choosing m to avoid these finitely many bad residue classes modulo the primes dividing N, we get $ma-b$ prime to N. Hence, M is finite. ∎

Proof of Proposition 1.14 For $(j,\ell) \in \mathcal{S}(j)$, we denote $M(j:\ell)=m$. Then, by (1.11), $\ell = mj-1$.

On the other hand, for $m \in \mathbb{Z}_{\geq 0}$, consider $(j, mj-1) \in (\mathbb{Z}/N\mathbb{Z})^2$. Suppose that $\forall\, 0 \leq t < m,\ tj-(mj-1) \notin (\mathbb{Z}/N\mathbb{Z})^*$; then by

$$m \cdot j - (mj-1) = 1,$$

we know from (1.11) and (1.13) that

$$M(j:mj-1)=m \quad \text{and} \quad \sigma(j:mj-1)=(j,mj-1).$$

Therefore, we have the following equivalence:

$$\begin{aligned} & (j,mj-1) \in \mathcal{S}(j) \text{ with } M(j:mj-1)=m \\ \iff\ & \forall\, 0 \leq t < m,\ tj-(mj-1) \notin (\mathbb{Z}/N\mathbb{Z})^* \\ \iff\ & \forall\, 1 \leq r \leq m,\ rj-1 \notin (\mathbb{Z}/N\mathbb{Z})^* \\ \iff\ & m < W_j. \end{aligned}$$

∎

We now record two elementary facts that convert the algebraic representative lists into fundamental domains.

Lemma 2.1 *The set*

$$\operatorname{Int}\Big(\bigcup_{i=1}^{n} \overline{\gamma_i D}\Big)$$

described in (1.4) *is a fundamental domain for* Γ*, provided that it is connected.*

Proof Write

$$E = \operatorname{Int}\Big(\bigcup_{i=1}^{n} \overline{\gamma_i D}\Big), \qquad \mathcal{D} = \bigcup_{i=1}^{n} \gamma_i D. \tag{2.2}$$

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Since the tiles are ideal geodesic triangles,

$$\overline{E} = \bigcup_{i=1}^{n} \overline{\gamma_i D} = \overline{\mathcal{D}},$$

so every nonempty open subset of E meets $\mathcal{D}$.

We check the conditions in Definition 1.1. For (1), openness and connectedness of E are immediate from the hypothesis. For (2), suppose $x, y \in E$ and $y = \gamma x$ for some $\gamma \in \Gamma$. Let $U = E \cap \gamma^{-1}E$, a nonempty open set since it contains x. Since $\mathcal{D}$ is dense and open in E, and $\gamma^{-1}\mathcal{D}$ is dense in $\gamma^{-1}E$, we may choose

$$x' \in U \cap \mathcal{D} \cap \gamma^{-1}\mathcal{D}.$$

Then $y' = \gamma x' \in \mathcal{D}$. By the definition (2.2), write $x' = \gamma_i x''$ and $y' = \gamma_j y''$, with $x'', y'' \in D$. Since D is a fundamental domain for $\Gamma(1)$, we get $x'' = y''$ and $\gamma\gamma_i = \pm\gamma_j$. As the γ_i are representatives for $(\pm I)\Gamma\backslash\Gamma(1)$, this gives $i = j$, $\gamma = \pm I$, and hence $x = y$.

To prove (3), for any $x \in \mathbb{H}$, choose $\theta \in \Gamma(1)$ and $y \in \overline{D}$ with $x = \theta y$. Writing $\theta = \pm\gamma\gamma_i$ with $\gamma \in \Gamma$ by (1.3), we get $x = \gamma(\gamma_i y)$ and $\gamma_i y \in \overline{E}$. Thus, every Γ-orbit meets $\overline{E}$. ∎

Definition 2.3 For a finite set $\Theta \subset PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/(\pm I)$, let G_Θ be the graph with vertex set Θ, where θ and θ' are adjacent if

$$\theta' = \theta S, \qquad \theta' = \theta T, \qquad \text{or} \qquad \theta' = \theta T^{-1}.$$

Lemma 2.4 *If G_Θ is connected, then*

$$\mathrm{Int}\Big(\bigcup_{\theta\in\Theta} \overline{\theta D} \Big)$$

is connected.

Proof If two vertices of G_Θ are adjacent, the corresponding closed triangles share a full edge: after applying the inverse of one vertex, this is just the fact that $\overline{D}$ shares its three edges with $\overline{SD}$, $\overline{TD}$, and $\overline{T^{-1}D}$. Choose a spanning tree for G_Θ. Starting from one tile and adding the remaining tiles along this tree, each new tile is attached along a full edge, so the interior of the union remains connected. Hence, the full tile union has a connected interior. ∎

Proof of Theorem 1.16 *Part (1),* $\Gamma_0(N)$. From [Cre97, Section 2.2], the right cosets in $\Gamma_0(N)\backslash\Gamma(1)$ are in bijection with $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$. The bijection is induced by the bottom-row map

$$\mathcal{R} : \Gamma(1) \longrightarrow \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (c, d), \tag{2.5}$$

followed by passage to the projective class.

Let

$$A = \{(a : b) \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \mid a \in (\mathbb{Z}/N\mathbb{Z})^*\} = \{(1 : r) \mid r \in \mathbb{Z}/N\mathbb{Z}\},$$
$$H = \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})\backslash A.$$

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Define

$$\text{pr} : \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \longrightarrow \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}, \qquad \text{pr}(x) = \begin{cases} (1, r), & x = (1 : r) \in A, \\ \sigma(x), & x \in H. \end{cases} \tag{2.6}$$

Thus, pr keeps the usual affine representative on A and uses the M-normalized representative on H as in (1.10).

We compute

$$ST^i = \begin{pmatrix} 0 & -1 \\ 1 & i \end{pmatrix}$$

and

$$ST^jST^m = \begin{pmatrix} -1 & -m \\ j & mj-1 \end{pmatrix}. \tag{2.7}$$

Therefore,

$$\mathcal{R}(ST^i) = (1, i), \qquad \mathcal{R}(ST^jST^m) = (j, mj-1).$$

The first family represents the affine part A. The global fibers of σ are described by Proposition 1.14 for every residue class j. The auxiliary map pr deliberately treats the affine part differently: it keeps these points in the usual affine coordinate, for the purpose of keeping the corresponding domain connected. The remaining non-affine points are exactly the fibers with nonunit first coordinate, and the second family with $0 \le m \le M_j$ represents them in their M-normalized form. Therefore, the bottom rows of the matrices in Θ_0 are exactly the pairs $\text{pr}(x)$ as x ranges through $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$.

It remains to prove connectedness. In G_{Θ_0}, the vertex $S = ST^0$ belongs to Θ_0, and the vertices ST^i form the path

$$(ST^{-N_1}, \dots, ST^{-1}, S, ST, \dots, ST^{N_2}).$$

For each nonunit j, the vertex ST^jS is adjacent to ST^j, and the path

$$(ST^jS, ST^jST, \dots, ST^jST^{M_j})$$

connects every vertex above j to ST^jS. Hence, every vertex of G_{Θ_0} is connected to S. Lemmas 2.1 and 2.4 show that Θ_0 is a fundamental list for $\Gamma_0(N)$.

Part (2), $\Gamma_1(N)$. We use the exact sequence [DS05, p. 14]

$$1 \to (\pm I)\Gamma_1(N) \to \Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^*/(\pm I) \to 1,$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \{\pm d\} (\text{mod } N).$$

For a unit $k \in \mathbb{Z}/N\mathbb{Z}$, choose an inverse k^{-1} modulo N. Then,

$$ST^kST^{k^{-1}}S = \begin{pmatrix} -k^{-1} & 1 \\ k^{-1}k - 1 & -k \end{pmatrix} \in \Gamma_0(N),$$

and, as k runs through representatives for $(\mathbb{Z}/N\mathbb{Z})^*/(\pm I)$, these matrices give representatives for $(\pm I)\Gamma_1(N)\backslash\Gamma_0(N)$. For the class of $k = 1$, we use I instead of $STSTS = -T^{-1}$. Thus,

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$$\{I, ST^k ST^{k^{-1}} S \mid -N_1 \le k \le -2,\ \gcd(k, N) = 1\} \tag{2.8}$$

is a set of representatives for $(\pm I)\Gamma_1(N)\backslash\Gamma_0(N)$.

Multiplying these representatives by the elements of Θ_0 gives the list Θ_1 in (1.18). The first two lines are $I \cdot \Theta_0$. The third and the fourth lines follow from

$$(ST^k ST^{k^{-1}} S)(ST^i) = -ST^k ST^{k^{-1}+i},$$
$$(ST^k ST^{k^{-1}} S)(ST^j ST^m) = -ST^k ST^{k^{-1}+j} ST^m,$$

where we replace the exponents $k^{-1} + i$ and $k^{-1} + j$ by their symmetric representatives, as in Notation 1.5. We can do this, since the right cosets represented by ST^i and $ST^j ST^m$ in Θ_0 (1.17) depend only on the residue classes of i and j modulo N.

The graph G_{Θ_1} contains G_{Θ_0}. Since the allowed k's lie in the range of the i's, $ST^k S$ is adjacent to ST^k, and

$$(ST^k ST^{-N_1}, \ldots, ST^k S, \ldots, ST^k ST^{N_2})$$

connects every vertex $ST^k ST^i$ to S. Similarly, $\widetilde{k^{-1} + j}$ lies in the same range, and

$$(ST^k ST^{\widetilde{k^{-1}+j}} S, ST^k ST^{\widetilde{k^{-1}+j}} ST, \ldots, ST^k ST^{\widetilde{k^{-1}+j}} ST^{M_j})$$

connects every remaining vertex to S. Hence, G_{Θ_1} is connected, and Lemmas 2.1 and 2.4 show that Θ_1 is a fundamental list for $\Gamma_1(N)$.

Part (3), $\Gamma(N)$. The matrices T^ℓ, with $-N_1 \le \ell \le N_2$, give representatives for $\Gamma(N)\backslash\Gamma_1(N)$. Therefore, the products $T^\ell \gamma$, $\gamma \in \Theta_1$, give representatives for $(\pm I)\Gamma(N)\backslash\Gamma(1)$. Since $I = ST^0 ST^0$ belongs to Θ_1 in $PSL_2(\mathbb{Z})$, the vertices T^ℓ occur in the list and form the path

$$(T^{-N_1}, \ldots, T^{-1}, I, T, \ldots, T^{N_2}).$$

For each fixed ℓ, left multiplication by T^ℓ gives a copy of the connected graph G_{Θ_1} attached at T^ℓ. Hence, the full graph is connected. Lemmas 2.1 and 2.4 finish the proof. ∎

Remark 2.9 We consider a special case when $N = p^r q^s$ has two prime factors p and q. For $\gcd(a, N) > 1$, $\gcd(b, N) > 1$, and $\gcd(a, b, N) = 1$, we show that $M(a : b) = 1$, that is, $a - b$ is a unit mod N. This is true since under our assumption, $\gcd(a, N)$ is one of the prime factors to some positive power and $\gcd(b, N)$ is the other. Therefore, $a - b$ is not divisible by either of the prime factors, so a unit mod N. Hence, M can be at most 1 in this case.

On the other hand, if $N = p^r$ is a prime power, then $\gcd(a, N) > 1$ and $\gcd(a, b, N) = 1$ would require $\gcd(b, N) = 1$, and then $M(a : b) = 0$.

Meanwhile, the values M_j are not bounded by 1 in general; the first example with larger values is illustrated for $N = 30$ in Example 3.2 below.

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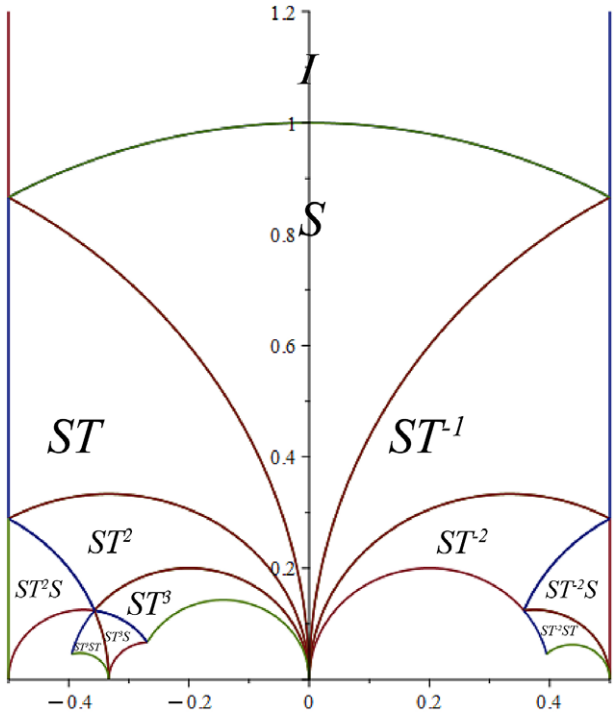


Figure 1: Our fundamental domain for $\Gamma_0(6)$ labeled.

3 Examples

We implemented the domains in Maple by drawing the ideal geodesic triangles. In the figures, Maple colors the edges red, blue, and green, with overlap priority red > blue > green.

Example 3.1 We show a labeled picture of our fundamental domain for $\Gamma_0(6)$ (Figure 1). Since

$$[\Gamma_0(N) : (\pm I)\Gamma_1(N)] = \frac{\phi(N)}{2}$$

and $\phi(6)/2 = 1$, this is also the domain for $\Gamma_1(6)$.

The functions M_j and W_j are defined for every residue class $j \in \mathbb{Z}/6\mathbb{Z}$. The nonunit columns, which are the columns entering the second family of representatives in Theorem 1.16, are

j	-2	0	2	3
M_j	1	0	0	1
W_j	2	1	1	2

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The unit columns are part of the same globally defined functions, but the corresponding points are represented through the affine part of the list.

The connected picture makes several features visible at once. Internal base-side pairs (ST^jD, ST^jSD) form kites whenever both representatives occur; for instance, the $j = 0$ pair is (SD, D) because $S^2 = -I$ acts trivially on $\mathbb{H}$. Apart from the pairing of the left and right vertical sides, the bottom boundary arcs labeled $1, \dots, 8$ are paired by

$$1 \sim 8, \qquad 2 \sim 7, \qquad 3 \sim 4, \qquad 5 \sim 6,$$

as in [Nie26, Theorem 1.21]. The cusps are represented by $\infty, 0, 1/2, 1/3$, with widths $1, 6, 3, 2$; there are no elliptic points of orders 2 or 3, and the genus formula gives $g(X_0(6)) = 0$.

Example 3.2 The case $N = 30 = 2 \cdot 3 \cdot 5$ is the first one in which values $M_j > 1$ occur. Here,

$$[\Gamma(1) : \Gamma_0(30)] = \psi(30) = 30 \prod_{p|30} \left(1 + \frac{1}{p}\right) = 72.$$

The affine part of $\mathbb{P}^1(\mathbb{Z}/30\mathbb{Z})$ has 30 elements. The classes $(j : -1)$ with nonunit j contribute 22 more, and the remaining 20 non-affine classes account for the higher values of M_j.

For nonunit $j \in \mathbb{Z}/30\mathbb{Z}$, the values are as follows. These are again only the nonunit columns of the globally defined functions M_j and W_j.

M_j	j	W_j
0	$-12, -10, -6, 0, 2, 8, 12, 14$	1
1	$-14, -9, -8, -5, -3, 4, 6, 9, 10, 15$	2
2	$-4, -2$	3
3	$3, 5$	4

Thus, the number of representatives of the form ST^jST^m with j nonunit and $0 \le m \le M_j$ is

$$1 \cdot 8 + 2 \cdot 10 + 3 \cdot 2 + 4 \cdot 2 = 42 = \psi(30) - 30.$$

For ST^jST^m in (2.7), the associated cusp is

$$(ST^jST^m)(\infty) = -\frac{1}{j},$$

and it occurs $M_j + 1$ times. Grouping the above values by cusp equivalence gives the following widths; the contributing j's are counted with multiplicity $M_j + 1$.

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Cusp rep	Width	Contributing j's
$\frac{1}{2}$	15	$2, 4, 8, 14, -14, -8, -4, -2$
$\frac{1}{3}$	10	$3, 9, -9, -3$
$\frac{1}{5}$	6	$5, -5$
$\frac{1}{6}$	5	$6, 12, -12, -6$
$\frac{1}{10}$	3	$10, -10$
$\frac{1}{15}$	2	15
∞	1	Affine cusp at ∞
0	30	Affine cusp at 0

These widths agree with the standard computations in [DS05, Proposition 3.8.3] and [Ste07, Algorithm 1.19]. This example is meant only to record the first occurrence of the higher values $M_j > 1$; the general pattern is proved in the companion paper [Nie26].

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