Combinatorics, Probability and Computing (2005) 14, 311–318. © 2005 Cambridge University Press
doi:10.1017/S0963548304006376 Printed in the United Kingdom

Construction of a Maximum Stable Set with k-Extensions

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Received 11 March 2002; revised 14 May 2003

A stable set I of a graph G is called *k-extendable*, $k \geqslant 1$, if there exists a stable set $X \subseteq V(G) \setminus I$ such that $|X| \leqslant k$ and $|N(X) \cap I| < |X|$. A graph G is called *k-extendable* if every stable set in G, which is not maximum, is k-extendable. Let us denote by $\mathrm{E}(k)$ the class of all k-extendable graphs.

We present a finite forbidden induced subgraph characterization of the maximal hereditary subclass $\mathrm{PE}(k)$ in $\mathrm{E}(k)$ for every $k \geqslant 1$.

Thus, we define a hierarchy $\mathrm{PE}(1) \subset \mathrm{PE}(2) \subset \cdots \subset \mathrm{PE}(k) \subset \cdots$ of hereditary classes of graphs, in each of which a maximum stable set can be found in polynomial time. The hierarchy covers all graphs, and all its classes can be recognized in polynomial time.

1. Introduction

A set $I \subseteq V(G)$ in a graph G is *stable* (or *independent*) if S does not contain adjacent vertices. A stable set of a graph G is called *maximal* if it is not contained in another stable set of G. A stable set of a graph G is called *maximum* if G does not have a stable set containing more vertices. The cardinality of a maximum stable set in G is the *stability number* of G, and it is denoted by $\alpha(G)$.

Decision Problem 1 (Stable Set Problem).

Instance: *A graph G and an integer k.*
Question: *Is there a stable set S in G with $|S| \geqslant k$?*

Computing $\alpha(G)$ is polynomial-time equivalent to the Stable Set Problem, which is known to be NP-complete (Karp [6]; see also Garey and Johnson [3]).

† Supported by the Office of Naval Research (Grant N00014-92-J-1375) and the National Science Foundation (Grant DMS 98-06389). The second author is also supported by DIMACS Winter 2002/2003 Award.

https://doi.org/10.1017/S0963548304006376 Published online by Cambridge University Press

We shall define a hierarchy

$$\mathrm{PE}(1) \subset \mathrm{PE}(2) \subset \cdots \subset \mathrm{PE}(k) \subset \cdots$$

of hereditary classes of graphs, in each of which a maximum stable set can be found in polynomial time. Note that the hierarchy covers all graphs. We characterize these classes in terms of forbidden induced subgraphs (Theorem 4.2), thus providing a polynomial-time recognition algorithm of graphs belonging to each of these classes. Specifically, it is shown that minimal forbidden induced subgraphs for PE(k) constitute the class BM_k of bipartite matchable graphs. Theorem 3.2 gives a structural characterization of graphs in BM_k as well as a simple way to construct these graphs from the set of all trees of order $k+1$ with subdivisions. Explicit characterizations of PE(2) (Corollary 4.8) and PE(3) (Corollary 4.9) imply several known results.

2. Classes of k-extendable graphs

The neighbourhood of a vertex x is denoted by $N(x)$. If $X \subseteq V(G)$ then $N(X) = \bigcup_{x \in X} N(x)$.

Definition. A stable set I of a graph G is called *k-extendable*, $k \geqslant 1$, if there exists a stable set $X \subseteq V(G) \setminus I$ such that

(EX1) $|X| \leqslant k$, and
(EX2) $|N(X) \cap I| < |X|$.

A graph G is called *k-extendable* if every stable set in G, which is not maximum, is k-extendable.

We let E(k) denote the class of all k-extendable graphs. The class E(1) coincides with the known class [9] of *well covered graphs*, *i.e.*, graphs having the property that every maximal stable set is maximum.

Proposition 2.1. *For every $k \geqslant 1$, there is a polynomial-time algorithm for finding a maximum stable set in any graph in* E(k).

Proof. Let $G \in \mathrm{E}(k)$ and let us choose any stable set I in G. We consider all stable subsets $X \subseteq V(G) \setminus I$ of cardinality $|X| \leqslant k$. For a fixed k, the total number of such subsets is polynomial in $|V(G)|$. If there exists a stable set X satisfying (EX2) then the stable set $I' = (I \setminus N(X)) \cup X$ satisfies

$$|I'| = |I| - |N(X) \cap I| + |X| > |I|.$$

We apply the above procedure to I', and repeat as many times as possible.

If there is no $X \subseteq V(G) \setminus I$ satisfying both (EX1) and (EX2), then $G \in \mathrm{E}(k)$ implies that I is a maximum stable set. □

The efficient recognition of the classes E(k) is unlikely, and even the case $k = 1$ is hard. Indeed, it was shown in [2] and [10] that the recognition of well-covered graphs

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is co-NP-complete. Therefore we shall restrict our attention to the maximal hereditary subclass PE(k) of E(k).

Definition. A graph G is called *perfectly k-extendable* if every induced subgraph of G is k-extendable. For $k \geqslant 1$, we let PE(k) denote the class of all perfectly k-extendable graphs.

Let Z be a set of graphs. We let FIS(Z) denote the hereditary class of all graphs defined by Z as a set of Forbidden Induced Subgraphs.

It is easy to see that the path P_3 does not belong to E(1). Therefore PE(1) = FIS(P_3) is the class of all graphs which have the property that each of their components is a complete graph.

3. Bipartite matchable graphs

Definition. A bipartite graph G is called *matchable* if there exists a bipartition $I \cup J = V(G)$ such that $G - v$ has a perfect matching for every vertex $v \in J$.

Clearly, $|J| = |I| + 1$. Thus, every matchable graph has an odd order. We let BM_k, $k \geqslant 1$, denote the set of all bipartite matchable graphs of order $2k + 1$.

Let T be an arbitrary tree having $k + 1$ vertices. We subdivide every edge of T by a single new vertex, and we denote the resulting tree by T^*. Since the number of edges in T is k, the order T^* is $2k + 1$. We let TM_k denote the set of all those trees T^* which are constructed in this way from the set of all trees T of order $k + 1$. We show that TM_k consists of matchable graphs.

Proposition 3.1. *$TM_k \subseteq BM_k$ for every $k \geqslant 1$.*

Proof. Let $T^* \in TM_k$, *i.e.*, assume that T^* is obtained from a tree T with $k + 1$ vertices by subdividing every edge by a vertex. Let $J = V(T)$ and $I = V(T^*) \setminus J$.

We show that $T^* \in BM_k$, *i.e.*, that $T^* - v$ has a perfect matching for every vertex $v \in J$. Let us fix a vertex $v \in J$. For every vertex $v' \in J \setminus \{v\}$, the unique (v, v')-path in T^* is of the form $(v = v_0, u_1, v_1, u_2, v_2, \ldots, u_t, v_t = v')$, where $t \geqslant 1$, $u_i \in I$ and $v_0, v_i \in J$ $(i = 1, 2, \ldots, t)$. The set of edges $M(v, v') = \{u_i v_i : i = 1, 2, \ldots, t\}$ constitutes a matching in T^*. The union of matchings $M(v, v')$ taken over all $v' \in J \setminus \{v\}$ is a perfect matching in $T^* - v$. □

Theorem 3.2. *A bipartite graph G belongs to BM_k $(k \geqslant 1)$ if and only if G contains a spanning tree $T^* \in TM_k$.*

Proof (Necessity). Necessity. Since $G \in BM_k$, G is a bipartite graph, having parts I and J of cardinalities $|I| = k$ and $|J| = k + 1$, respectively.

Let $v_0 \in J$. The graph $G - v_0$ has a perfect matching

$$M = \{u_i v_i : i = 1, 2, \ldots, k\}.$$

We may assume that $J = \{v_0, v_1, \ldots, v_k\}$ and $I = \{u_1, u_2, \ldots, u_k\}$. Let G' be a subtree in G which has maximum cardinality among all subtrees satisfying the following two conditions:

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(T1) $v_0 \in V(G')$, and
(T2) every vertex $u_i \in I' = I \cap V(G')$ has degree 2 in G', and $u_i v_i \in E(G')$.

Note that an obvious greedy-type algorithm shows the existence of G'. Let $|I'| = t$. We denote $J' = J \cap V(G')$. The condition (T2) implies that G' is obtained from a tree with $t+1$ vertices by subdividing every edge by a vertex. Therefore $|J'| = t+1$. If $t = k$ then $G' \in TM_k$, thus proving the necessity of the condition. If $t < k$, we denote $I' = \{u_1, u_2, \ldots, u_t\}$. Correspondingly, $J' = \{v_0, v_1, \ldots, v_t\}$. A graph $H = G - v_{t+1}$ has a perfect matching. By Hall's theorem (see [4]), $|N_H(X)| \geqslant |X|$ for every $X \subseteq I$. In particular, $|N_H(I \setminus I')| \geqslant |I \setminus I'| = k - t$. Since $|J \setminus (\{v_{t+1}\} \cup \setminus J')| = k - t - 1$, there exists a vertex $v_i \in J'$ which is adjacent to a vertex $u_j \in I \setminus I'$.

If we add to G' the vertices u_j and v_j with the edges $v_i u_j$ and $u_j v_j$, we obtain a subgraph G'', satisfying (T1) and (T2), but having more vertices than G', in contradiction to the maximum cardinality assumption.

Sufficiency. A tree $T^* \in TM_k$ has a unique bipartition $I \cup J = V(T^*)$, where $|I| = k$ and $|J| = k+1$. Since G is a bipartite graph, G can be obtained from T^* by adding a set of edges uv with $u \in I$ and $v \in J$. By Proposition 3.1, T^* is a matchable graph. Therefore G is also a matchable graph, *i.e.*, $G \in BM_k$. □

4. Minimal forbidden induced subgraphs for PE(k)

We let Z_k denote the set of all minimal induced subgraphs forbidden for PE(k), *i.e.*, $G \in Z_k$ if and only if $G \notin \mathrm{E}(k)$ and $H \in \mathrm{E}(k)$ for every proper induced subgraph H of G.

Lemma 4.1. *$BM_k \subseteq Z_k$ for every $k \geqslant 1$.*

Proof. Let $G \in BM_k$. By definition, G is a bipartite graph with parts I and J ($|I| = k$ and $|J| = k+1$) such that $G - v$ has a perfect matching for every vertex $v \in J$. Since $|I| < |J|$, I is not a maximum stable set.

Let $X \subseteq V(G) \setminus I = J$ and $|X| \leqslant k$. Since $|J| = k+1$, there is a vertex $v \in J \setminus X$. The graph $G - v$ has a perfect matching, therefore $|N(X)| \geqslant |X|$. Hence I is not k-extendable. Thus, $G \notin \mathrm{E}(k)$.

We shall prove the minimality of G. Let H be an arbitrary proper induced subgraph of G. We shall show that $H \in \mathrm{E}(k)$. Let I' be a maximal stable set in H which is not maximum, and let J' be a maximum stable set. If $|I' \setminus J'| \geqslant k$ then

$$|J' \setminus I'| \geqslant |I' \setminus J'| + 1 \geqslant k + 1.$$

It follows that

$$|V(H)| \geqslant |I' \setminus J'| + |J' \setminus I'| \geqslant 2k + 1 = |V(G)|,$$

in contradiction to the assumption that $H \neq G$. Therefore $|I' \setminus J'| < k$.

Let $X \subseteq J' \setminus I'$ such that $|X| = |I' \setminus J'| + 1$. The set X is stable, $|X| \leqslant k$ and $|N(X) \cap I'| \leqslant |I' \setminus J'| < |X|$. Thus, I' is k-extendable in H. Therefore $H \in \mathrm{E}(k)$. □

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Theorem 4.2. *For every $k \geqslant 1$, the class* PE(k) *has a finite forbidden induced subgraph characterization, and $Z_k = BM_k$.*

Proof. For a fixed $k \geqslant 1$, the set BM_k is finite, because every graph in BM_k has $2k+1$ vertices. Therefore it is sufficient to prove the equality $Z_k = BM_k$.

By Lemma 4.1, $BM_k \subseteq Z_k$. In order to prove the reverse inclusion, let us consider a graph $G \in Z_k$. Since $G \notin \mathrm{E}(k)$, there exists a stable set I in G which is neither maximum nor k-extendable. □

Claim 4.3.

(i) *$V(G) \setminus I$ is a unique maximum stable set in G.*
(ii) $|V(G) \setminus I| = |I| + 1$.

Proof. **(i)** Let J be a maximum stable set in G. It is sufficient to show that $J = V(G) \setminus I$.

Suppose that there exists a vertex $u \notin I \cup J$. The set I is stable in $G - u$, but it is neither maximum (since $|I| < |J|$) nor k-extendable (for otherwise I would be k-extendable in G). Hence $G - u \notin \mathrm{E}(k)$, in contradiction to the minimality of G. Thus, $V(G) = I \cup J$.

Suppose now that there exists a vertex $u \in I \cap J$. Since $V(G) = I \cup J$, u is an isolated vertex. Again, $I \setminus \{u\}$ is a stable set in $G - u$, but it is neither maximum (since $|I \setminus \{u\}| < |J \setminus \{u\}|$) nor k-extendable (for otherwise the set I would be k-extendable in G). Hence $G - u \notin \mathrm{E}(k)$, in contradiction to the minimality of G. Thus, $I \cap J = \emptyset$.

(ii) Suppose that $|J| > |I| + 1$. Let us choose a vertex $u \in J$. The set I is stable in $G - u$, but it is not maximum (since $|I| < |J \setminus \{u\}|$), and it is not k-extendable. Hence $G - u \notin \mathrm{E}(k)$, in contradiction to the minimality of G. □

Claim 4.3 implies that G is a bipartite graph which has parts I and $J = V(G) \setminus I$, with $|J| = |I| + 1$.

Claim 4.4. *For every non-empty set $X \subseteq I$, the inequality $|N(X)| > |X|$ holds.*

Proof. The set $J' = (J \setminus N(X)) \cup X$ is stable, since $X \subseteq I$ stable. It follows from $X \neq \emptyset$ that $J' \neq J$. By Claim 4.3, J is a unique maximum stable set. Therefore $|J'| < |J|$ and $|N(X)| > |X|$. □

Claim 4.5. *$G - v$ has a perfect matching for every vertex $v \in J$.*

Proof. By Claim 4.4, for every non-empty set $X \subseteq I$ the inequality $|N_G(X)| > |X|$ holds. Therefore $|N_{G-v}(X)| \geqslant |N_G(X)| - 1 \geqslant |X|$. By Hall's theorem (see [4]) there exists a matching covering I in the graph $G - v$. Claim 4.3(ii) implies that it is a perfect matching in $G - v$. □

Claim 4.6. $|I| \geqslant k$.

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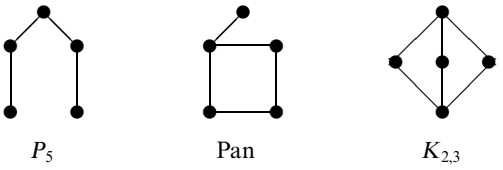


Figure 1. Minimal forbidden induced subgraphs for PE(2).

Proof. If $|I| < k$, then $|X| \leqslant k$ and $|N(X)| \leqslant |I| < |X|$ for every $X \subseteq J$ of cardinality $|X| = |I| + 1$. Therefore I is a k-extendable set, a contradiction. Thus, $|I| \geqslant k$. □

Claims 4.3, 4.5 and 4.6 imply that $G \in BM_l$, where $l = |I| \geqslant k$. By Theorem 3.2, G contains a spanning tree $T^* \in TM_l$.

The tree T^* is obtained from a tree T of order $l+1$ by subdividing every edge by a vertex. We construct a subtree T_o of order $k+1$ in T by subsequently deleting end-vertices. Subdividing every edge in T_o by a vertex produces a subtree of T^*. This subtree is denoted by T_o^*.

Let G_o be a subgraph of G induced by $V(T_o^*)$. Since G_o is a bipartite graph and $T_o^* \in TM_k$ is a spanning subgraph in G_o, it follows from Theorem 3.2 that $G_o \in BM_k$. By Lemma 4.1, $G_o \in Z_k$. By the minimality of G, $G = G_o$, and hence $l = k$.

Corollary 4.7. *For every $k \geqslant 1$, there exists a polynomial-time algorithm for recognizing* PE(k).

Proof. It follows from the existence of a finite forbidden induced subgraph characterization for PE(k) (Theorem 4.2). □

Corollary 4.8. PE(2) = FIS(P_5, Pan, $K_{2,3}$)*: see Figure 1.*

Proof. The path P_3 is the unique tree with 3 vertices. Therefore $TM_2 = \{P_5\}$. According to Theorem 3.2, $BM_2 = \{P_5, \text{Pan}, K_{2,3}\}$. By Theorem 4.2, PE($k$) = FIS($BM_2$). □

Corollary 4.9. PE(3) = FIS($G_1, G_2, \ldots, G_{20}$)*: see Figure 2.*

Proof. There are exactly two trees with four vertices, namely P_4 and $K_{1,3}$. Therefore $TM_3 = \{G_1, G_2\}$ (Figure 2). According to Theorem 3.2, every graph in BM_3 is bipartite and it contains a spanning subgraph isomorphic either to G_1 or to G_2. Using this fact, it can be easily checked that $BM_3 = \{G_1, G_2, \ldots, G_{20}\}$ (Figure 2). By Theorem 4.2, PE(3) = FIS(BM_3). □

The following proposition is an extension of a result of [1].

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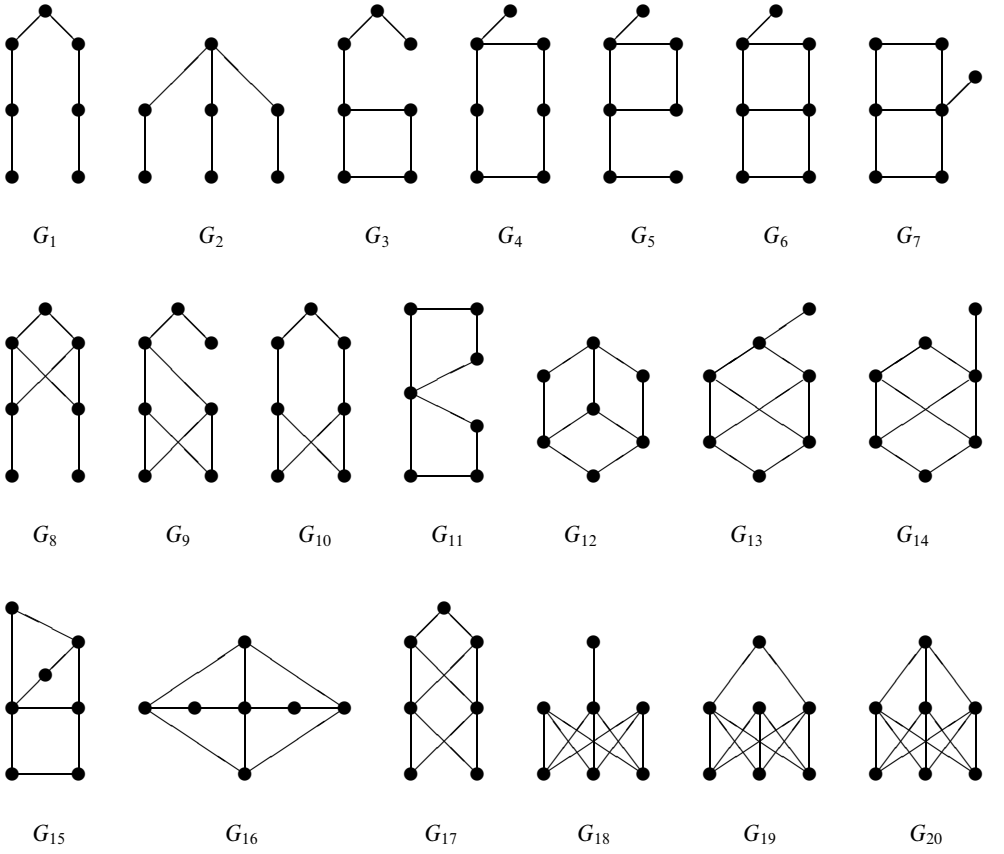


Figure 2. Minimal forbidden induced subgraphs for PE(3).

Corollary 4.10. *There is a polynomial-time algorithm for finding a maximum stable set for all graphs in the class* $\mathrm{FIS}(P_5, K_{1,4})$.

Proof. The graphs $G_1, G_2, \ldots, G_{13}$ (Figure 2) contain P_5 as an induced subgraph, while $G_{14}, G_{15}, \ldots, G_{20}$ (Figure 2) contain $K_{1,4}$ as an induced subgraph. In view of this remark and Corollary 4.9, we have

$$\mathrm{FIS}(P_5, K_{1,4}) \subseteq \mathrm{FIS}(G_1, G_2, \ldots, G_{20}) = \mathrm{PE}(3).$$

Finally, by Proposition 2.1, there is a polynomial-time algorithm for finding a maximum stable set in all the graphs belonging to $\mathrm{PE}(3) \subseteq \mathrm{E}(3)$. □

The following result is a generalization of theorems of Mahadev [7] and Hertz [5].

Corollary 4.11 (Mosca [8]). *There is a polynomial-time algorithm for finding a maximum stable set for the graphs in the class* $\mathrm{FIS}(P_5, K_{2,3})$.

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Proof. The graphs $G_1, G_2, \ldots, G_{13}$ and G_{16} (Figure 2) contain P_5 as an induced subgraph, while the graphs $G_{14}, G_{15}, G_{17}, G_{18}, G_{19}, G_{20}$ (Figure 2) contain $K_{2,3}$ as an induced subgraph. It follows from Corollary 4.9 that

$$\mathrm{FIS}(P_5, K_{2,3}) \subseteq \mathrm{FIS}(G_1, G_2, \ldots, G_{20}) = \mathrm{PE}(3).$$

By Proposition 2.1, there is a polynomial-time algorithm for finding a maximum stable set in all the graphs belonging to $\mathrm{PE}(3) \subseteq \mathrm{E}(3)$. □

Acknowledgement

We thank the anonymous referees, whose suggestions helped to improve the presentation of the paper.

References

[1] Branstädt, A. and Hammer, P. L. (1999) On the stability number of claw-free P_5-free and more general graphs. *Discrete Appl. Math.* **95** 163–167.
[2] Chvátal, V. and Slater, P. J. (1993) A note on well-covered graphs. In *Quo Vadis, Graph Theory?*, North-Holland, Amsterdam, pp. 179–181.
[3] Garey, M. R. and Johnson, D. S. (1979) *Computers and Intractability: A Guide to the Theory of NP-Completeness*, Freeman, San Francisco, CA.
[4] Hall, M. (1967) *Combinatorial Theory*, Blaisdell, Waltham.
[5] Hertz, A. (1995) Polynomially solvable cases for the maximum stable set problem. *Discrete Appl. Math.* **60** 195–210.
[6] Karp, R. M. (1972) Reducibility among combinatorial problems. In *Complexity of Computer Computations*, Plenum Press, New York, pp. 85–103.
[7] Mahadev, N. V. R. (1990) Vertex deletion and stability number. Technical Report ORWP 90/2, Swiss Federal Institute of Technology.
[8] Mosca, R. (1997) Polynomial algorithms for the maximum stable set problem on particular classes of P_5-free graphs. *Inform. Process. Lett.* **61** 137–144.
[9] Plummer, M. D. (1993) Well-covered graphs: A survey. *Quaestiones Math.* **16** 253–287.
[10] Sankaranarayana, R. S. and Stewart, L. K. (1992) Complexity results for well-covered graphs. *Networks* **22** 247–262.

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