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Constructing equilibrium states for some partially hyperbolic attractors via densities

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Abstract. We describe a new construction of equilibrium states for a class of partially hyperbolic systems. This generalizes our construction for Gibbs measures in the uniformly hyperbolic setting. This more general setting introduces new issues that we need to address carefully, in particular requiring additional assumptions on the transformation. We treat two cases: either the centre-stable manifold satisfies a bounded expansion condition; or the centre-unstable manifold satisfies a subexponential contraction condition which appears new in the context of equilibrium state constructions. The problem of constructing equilibrium states was previously raised by Pesin and Sinai and by Dolgopyat for the particular case of u-Gibbs measures, and by Climenhaga, Pesin and Zelerowicz for other equilibrium states.

Key words: partial hyperbolicity, equilibrium states, topological dynamics, smooth dynamics
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1. *Introduction*

In this paper we describe a new construction of equilibrium states for certain partially hyperbolic diffeomorphisms. In particular, we extend the ideas in [**12**] to the setting of partially hyperbolic diffeomorphisms where we further assume some control on the action in either the centre-stable or centre-unstable direction (see Definitions 2.2 and 5.3 respectively for details).

Given a continuous map on a compact metric space $f : X \to X$ and a continuous function $G : X \to \mathbb{R}$, we say that an f-invariant probability measure μ_G is an equilibrium state for G if

$$h(\mu_G) + \int G \, d\mu_G = \sup \left\{ h(\mu) + \int G \, d\mu : \mu \in \mathcal{M}_f(X) \right\} \tag{1.1}$$

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where $\mathcal{M}_f(X)$ denotes the space of f-invariant probability measures, that is, μ_G is a measure which maximizes the sum of the entropy $h(\mu)$ and the integral of G, over all f-invariant measures μ [13, 15]. The value attained by the supremum in (1.1) is called the pressure and denoted $P(G)$.

In this paper we consider a construction of equilibrium states for partially hyperbolic diffeomorphisms $f : X \to X$ by using sequences of measures supported on a piece of unstable manifold $W^u_\delta(x)$ that are absolutely continuous with respect to the induced volume. Let $\Phi : X \to \mathbb{R}$ be the geometric potential defined by $\Phi(x) = -\log|\det(Df \mid E^u_x)|$, where E^u_x is the unstable bundle for the diffeomorphism.

THEOREM 1.1. *Let $f : X \to X$ be a $C^{1+\alpha}$ topologically mixing partially hyperbolic attracting diffeomorphism satisfying Lyapunov stability (Definition 2.2) and let $G : X \to \mathbb{R}$ be a continuous function. Given $x \in X$ and $\delta > 0$, consider the sequence of probability measures $(\lambda_n)_{n=1}^\infty$ supported on $W^u_\delta(x)$ and absolutely continuous with respect to the induced volume $\lambda = \lambda_{W^u_\delta(x)}$,*

$$\lambda_n(A) = \frac{\int_A \exp\{\sum_{i=0}^{n-1}(G-\Phi)(f^i y)\}\, d\lambda(y)}{\int_{W^u_\delta(x)} \exp\{\sum_{i=0}^{n-1}(G-\Phi)(f^i z)\}\, d\lambda(z)} \quad \text{for Borel } A \subset W^u_\delta(x). \tag{1.2}$$

Then the weak limit points of the averages*

$$\mu_n := \frac{1}{n}\sum_{k=0}^{n-1} f^k_* \lambda_n, \quad n \geq 1 \tag{1.3}$$

(where $f_*\lambda_n(A) = \lambda_n(f^{-1}A)$ for Borel sets $A \subset X$), are equilibrium states for G.

From the change of variables for $f^n : W^u_\delta(x) \to f^n(W^u_\delta(x))$, we can reformulate (1.2) as

$$\lambda_n(A) = \frac{\int_{f^n(A)} \exp\{\sum_{i=1}^{n} G(f^{-i} y)\}\, d\lambda_{f^n W^u_\delta}(y)}{\int_{f^n W^u_\delta(x)} \exp\{\sum_{i=1}^{n} G(f^{-i} z)\}\, d\lambda_{f^n W^u_\delta}(z)} \tag{1.4}$$

which is often more convenient in the proofs.

In the setting of Theorem 1.1, existence of equilibrium states is guaranteed by abstract methods (h-expansivity) and our contribution is the concrete construction. The proof of Theorem 1.1 relies on the following characterization of the pressure which is of independent interest.

PROPOSITION 1.2. *Let $f : X \to X$ be a topologically mixing partially hyperbolic attracting diffeomorphism satisfying Lyapunov stability. For any continuous function $G : X \to \mathbb{R}$ we have*

$$P(G) = \limsup_{n\to+\infty} \frac{1}{n} \log \int_{W^u_\delta(x)} \exp\left\{\sum_{k=0}^{n-1}(G-\Phi)(f^k y)\right\} d\lambda_{W^u_\delta(x)}(y).$$

Partially hyperbolic systems satisfying Lyapunov stability have lately received growing attention. In particular, Climenhaga, Pesin and Zelerowicz [8] construct unique

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equilibrium states for C^2 systems and potentials satisfying a leafwise version of the Bowen property by a Carathéodory-type construction using fixed reference measures on pieces of local unstable manifold. Additionally, in the setting of partially hyperbolic diffeomorphisms with isometric action along the centre, Carrasco and Rodriguez-Hertz [5] provide a geometric construction of equilibrium states for Hölder potentials that are either the geometric potential or are constant along centre leaves. Another consideration of such systems can be found in [2].

We finish this paper with an extension of our construction to systems with a stable, centre-unstable splitting. In this setting we require integrability of the centre-unstable bundle and a subexponential contraction hypothesis (Definition 5.3) which is less restrictive than Lyapunov stability.

2. *Definitions*

In this section we introduce the notion of partially hyperbolic dynamics and provide useful background results and classical examples. We then briefly recall the definition of Kolmogorov–Sinai entropy and state a useful result due to Misiurewicz.

2.1. *Partial hyperbolicity.*

We begin with the definition of partial hyperbolicity.

Definition 2.1. (Partially hyperbolic set) Let M be a compact Riemannian manifold and $f : M \to M$ be a diffeomorphism. A closed f-invariant subset $X \subset M$ is said to be partially hyperbolic (in the broad sense) if:

- there exists a continuous invariant splitting of the tangent bundle into subbundles E^u and E^{cs} such that $T_X M = E^u \oplus E^{cs}$; and
- there is a Riemannian metric $\|\cdot\|$ on M and constants $C_1 > 1$ and $0 < C_2 < C_1$ such that for every $x \in X$,

$$\|Df_x v\| \geq C_1 \|v\| \quad \text{for } v \in E^u(x),$$
$$\|Df_x v\| \leq C_2 \|v\| \quad \text{for } v \in E^{cs}(x).$$

We shall always assume that the map is topologically mixing.

If $C_2 < 1$ then f is uniformly hyperbolic. Definition 2.1 also covers the more traditional and narrower definition of partial hyperbolicity involving a splitting into an expanding, centre and contracting direction.

There is a continuous cone family K^{cs} defined on X which is Df^{-1}-invariant [9, Proposition 2.9]. A curve γ is a cs-curve if all its tangent vectors lie in K^{cs}. We can now give the formal definition of Lyapunov stability.

Definition 2.2. (Lyapunov stability) A partially hyperbolic diffeomorphism $f : X \to X$ has *Lyapunov stability* if for every $\epsilon > 0$ there exists an $\epsilon_0 > 0$ such that if γ is a curve in X with length at most ϵ_0 and $n \geq 0$ is such that $f^n\gamma$ is a cs-curve then the length of $f^n\gamma$ is at most ϵ.

This condition is labelled (C1) in [8]. For an interesting consideration of systems satisfying Lyapunov stability, see [1].

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2.2. *Local manifolds.* We begin with the definition of a local unstable manifold.

Definition 2.3. (Local unstable manifold) Choose $\rho > 0$ such that $C_2 < 1/\rho < C_1$. For $x \in X$ and sufficiently small $\delta > 0$ we define the local unstable manifold through x as follows:

$$W^u_\delta(x) = \{y \in M : d(f^{-n}x, f^{-n}y) \leq \delta \text{ for all } n \geq 0 \\ \text{and } d(f^{-n}x, f^{-n}y)/\rho^n \to 0 \text{ as } n \to \infty\}.$$

The following result is the unstable manifold theorem.

THEOREM 2.4. [14, Theorem IV.1] *For $f \in C^r(M)$, $r > 1$, and $\delta > 0$ sufficiently small, there exist a $\lambda \in (C_1^{-1}, 1)$ and $C > 0$ such that for any $x \in X$ there is a C^r local unstable manifold $W^u_{\text{loc}}(x) \subset M$ such that:*

- *$T_y W^u_\delta(x) = E^u(y)$ for every $y \in W^u_\delta(x) \cap X$;*
- *$W^u_\delta(x)$ is a C^r embedded disk of dimension $\dim(E^u)$; and*
- *for every $n \geq 0$ and $y \in W^u_\delta(x)$, $d(f^{-n}x, f^{-n}y) \leq C\lambda^n d(x, y)$.*

In the uniformly hyperbolic setting we are able to define a similar local stable manifold focusing on the forward orbits of points, but it is not always possible in the partially hyperbolic setting. The following theorem states that with the assumption of f satisfying Definition 2.2, there is a local centre-stable manifold that plays the role of the stable manifold from the uniform hyperbolic setting.

THEOREM 2.5. (Local product structure [10]) *Let $f : M \to M$ be a diffeomorphism and $X \subset M$ be a compact f-invariant subset admitting a splitting $E^{cs} \oplus E^u$ satisfying Lyapunov stability. For every $x \in X$ there is a local manifold $W^{cs}_{\text{loc}}(x) \subset M$ satisfying:*

- *$T_y W^{cs}_{\text{loc}}(x) = E^{cs}(y)$ for every $y \in W^{cs}_{\text{loc}}(x) \cap X$; and*
- *$W^{cs}_{\text{loc}}(x)$ is a C^1 embedded disk of dimension $\dim(E^{cs})$.*

Moreover, there exists an $\epsilon > 0$ such that if $x, y \in X$ and $d(x, y) < \epsilon$ then $W^u_\delta(x) \cap W^{cs}_{\text{loc}}(y)$ is exactly one point which we denote $[x, y]$.

2.3. *Entropy.* Let $(X, \mathcal{B}, \mu, f)$ be a measure-preserving probability space. We now describe some results on the entropy with respect to μ. For the duration of this subsection we will only require the weaker assumption that $f : X \to X$ is a homeomorphism.

We begin with some standard definitions, [15].

Definition 2.6. Given a finite measurable partition $\mathcal{P} = \{P_1, \ldots, P_k\}$ and a probability measure ν, we can associate the entropy of the partition defined by

$$H_\nu(\mathcal{P}) = -\sum_{i=1}^{k} \nu(P_i) \log \nu(P_i).$$

Given $n \geq 1$, we let $\bigvee_{i=0}^{n-1} f^{-i}\mathcal{P} = \{P_{i_0} \cap f^{-1}P_{i_1} \cap \cdots \cap f^{-(n-1)}P_{i_{n-1}} : 1 \leq i_0, \ldots, i_{n-1} \leq k\}$ be the refinement of the partitions $\mathcal{P}, f^{-1}\mathcal{P}, \ldots, f^{-(n-1)}\mathcal{P}$.

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Definition 2.7. We can define the entropy associated to the partition $\mathcal{P}$ by

$$h_\nu(\mathcal{P}) = \lim_{n\to+\infty} \frac{1}{n} H_\nu\left(\bigvee_{i=0}^{n-1} f^{-i}\mathcal{P}\right).$$

Finally, the entropy with respect to the measure is defined by

$$h(\nu) = \sup_{\mathcal{P}}\{h_\nu(\mathcal{P}): \mathcal{P} \text{ is a countable partition with } H_\nu(\mathcal{P}) < \infty\}.$$

In the case that $\mathcal{P}$ is a generating partition ($\bigvee_{i=-\infty}^{\infty} f^{-i}\mathcal{P} = \mathcal{B}$) we have that $h(\nu) = h_\nu(\mathcal{P})$ is the entropy of the measure ν.

We require the following application of a lemma due to Misiurewicz [11], relating the entropy with respect to λ_n and the averaging of pushforwards μ_n (defined in (1.2) and (1.3), respectively).

LEMMA 2.8. *For any $n \geq 2$ and $0 < q < n$,*

$$q H_{\lambda_n}\left(\bigvee_{h=0}^{n-1} f^{-h}\mathcal{P}\right) \leq n H_{\mu_n}\left(\bigvee_{i=0}^{q-1} f^{-i}\mathcal{P}\right) + 2q^2 \log \text{ Card } (\mathcal{P}).$$

We shall not provide a proof here, but a detailed proof can be found above Lemma 4.5 in [12].

3. *Pressure and growth*

The pressure $P(G)$ has various different interpretations in terms of the growth of appropriate quantities. We begin with the standard definition of the pressure $P(G)$ written in terms of the growth rates of sets of spanning sets and separated sets due to Bowen.

Recall that, given $\epsilon > 0$ and $n \geq 1$, an (n, ϵ)-spanning set $S \subset X$ is such that $\bigcup_{x\in S} B(x, n, \epsilon)$ covers X, where $B(x, n, \epsilon) := \bigcap_{k=0}^{n-1} f^{-k}B(f^k x, \epsilon)$ is called a Bowen ball. On the other hand, an (n, ϵ)-separated set $\Sigma \subset X$ is such that $d_n(x, y) > \epsilon$ for $x, y \in \Sigma$ (and, in particular, $B(x, n, \epsilon/2)$, $x \in \Sigma$, are disjoint in X).

LEMMA 3.1. *Given $n \geq 1$ and $\epsilon > 0$, let*

$$Z_0(n, \epsilon) = \inf\left\{\sum_{y\in S} \exp\{G^n(y)\} : S \text{ is an } (n, \epsilon)\text{-spanning set}\right\}$$

and

$$Z_1(n, \epsilon) = \sup\left\{\sum_{y\in \Sigma} \exp\{G^n(y)\} : \Sigma \text{ is an } (n, \epsilon)\text{-separated set}\right\},$$

where $G^n(x) = \sum_{j=0}^{n-1} G(f^j x)$. Then the following limits exist and are equal:

$$P_{\text{top}}(G) := \lim_{\epsilon\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log Z_0(n, \epsilon) = \lim_{\epsilon\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log Z_1(n, \epsilon).$$

We call $P_{\text{top}}(G)$ the topological pressure of G.

(See [15, Ch. 9]).

By the variational principle [15] for any continuous G, $P_{\text{top}}(G) = P(G)$.

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We will need the following characterization of the pressure in terms of growth rates of appropriate densities defined on a piece of unstable manifold.

PROPOSITION 3.2. *Let $f : X \to X$ be a mixing partially hyperbolic attracting diffeomorphism satisfying Lyapunov stability. For any continuous function $G : X \to \mathbb{R}$ we have*

$$P(G) = \limsup_{n\to+\infty} \frac{1}{n} \log \int_{W^u_\delta(x)} \exp\left\{ \sum_{k=0}^{n-1} (G - \Phi)(f^k y) \right\} d\lambda_{W^u_\delta(x)}(y). \tag{3.1}$$

We can use the change of variables formula to rewrite this in the equivalent form

$$P(G) = \limsup_{n\to+\infty} \frac{1}{n} \log \int_{f^n W^u_\delta(x)} \exp\left\{ \sum_{k=1}^{n} G(f^{-k} y) \right\} d\lambda_{f^n W^u_\delta(x)}(y). \tag{3.2}$$

Example 3.3. If we consider the potential $G = 0$ then Proposition 3.2 gives

$$P(0) = \limsup_{n\to+\infty} \frac{\log \lambda(f^n W^u_\delta(x))}{n}.$$

Therefore, the topological entropy $h_{\text{top}}(f) = P(0)$ is the exponential growth rate of the volume of a small piece of unstable manifold.

Example 3.4. Another important potential is the geometric potential $G = \Phi = -\log |\det(Df|E^u_x)|$. Then Proposition 3.2 gives

$$\begin{aligned} P(\Phi) &= \limsup_{n\to+\infty} \frac{1}{n} \log \int_{W^u_\delta(x)} \exp\left\{ \sum_{k=0}^{n-1} (\Phi - \Phi)(f^k y) \right\} d\lambda_{W^u_\delta(x)}(y) \\ &= \limsup_{n\to+\infty} \frac{1}{n} \log \lambda(W^u_\delta(x)) = 0. \end{aligned}$$

This is a well-known result in the uniformly hyperbolic setting [**3**, Theorem 4.11], and we provide a new elementary proof of this fact for partially hyperbolic diffeomorphisms.

Example 3.5. More generally, we can consider the geometric q-potential defined by

$$\Phi_q = -q \log |\det(Df \mid E^u_x)| \quad \text{for } q \in \mathbb{R}.$$

Then the pressure has the following characterization:

$$P(\Phi_q) = \limsup_{n\to\infty} \frac{1}{n} \log \int_{W^u_\delta(x)} |\det(Df^n \mid E^u_z)|^{1-q} \, d\lambda_{W^u_\delta(x)}(z).$$

The proof of Proposition 3.2 is analogous to the proof of Proposition 4.6 in [**12**]. The proof in [**12**] relies on the local product lemma and the requirement that points on local unstable manifolds contract in the past. We describe the adjustments to the partially hyperbolic setting using the local product structure in Lemma 2.5 and the assumption that we have some control over the expansion of points under f^n in the centre-stable direction.

Proof of Proposition 3.2. We begin with the following standard result, which can be compared with Lemma 4.7 in [**12**].

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LEMMA 3.6. *For any $\epsilon_0 > 0$ there exists an $m > 0$ such that $f^m W^u_\delta(x)$ is ϵ_0-dense in X. In particular, we can assume that $X = \bigcup_{y \in f^m W^u_\delta(x)} W^{cs}_{\mathrm{loc}}(y)$.*

Proof. For $\epsilon_1 > 0$ small enough, consider an ϵ_1-fattening, W^u_{δ,ϵ_1}, of $W^u_\delta(x)$. By this we mean $W^u_{\delta,\epsilon_1} = \cup\{B(y, \epsilon_1) : y \in W^u_\delta(x)\}$. For any $z \in W^u_{\delta,\epsilon_1}$, using Theorem 2.5, we have that $W^u_\delta(x) \cap W^{cs}_{\mathrm{loc}}(z)$ is a single point which we call z_1. Provided $\epsilon_1 > 0$ is small enough, by Lyapunov stability we deduce that $d(f^m z_1, f^m z) < \epsilon_0/2$ for any $m \in \mathbb{N}$. Thus we have shown, for any $z \in W^u_{\delta,\epsilon_1}$, that there exists a $z_1 \in W^u_\delta(x)$ such that $d(f^m z_1, f^m z) < \epsilon_0/2$ for any $m \in \mathbb{N}$.

For any $\epsilon_0 > 0$, by using compactness we can choose a finite set C such that $\{B(c, \epsilon_0/4) : c \in C\}$ forms a cover of X. Since W^u_{δ,ϵ_1} is open, by topological mixing provided $m \in \mathbb{N}$ is sufficiently large, for any $c \in C$, $f^m W^u_{\delta,\epsilon_1} \cap B(c, \epsilon_0/4) \neq \emptyset$. Let $y_c \in f^m W^u_{\delta,\epsilon_1} \cap B(c, \epsilon_0/4)$; then for any $y \in X$ there is a $c \in C$ such that $y \in B(c, \epsilon_0/4)$ and, in particular, $d(y, y_c) < \epsilon_0/2$. We have $f^{-m} y_c \in W^u_{\delta,\epsilon_1}$ and therefore by the preceding paragraph there is a $z \in W^u_\delta(x)$ such that $d(y_c, f^m z) < \epsilon_0/2$.

By the triangle inequality, for any $y \in X$, we have that there is a $z \in W^u_\delta(x)$ such that $d(y, f^m z) \leq d(y, y_c) + d(y_c, f^m z) < \epsilon_0$. Taking ϵ_0 small enough (and making ϵ_1 smaller if necessary), we can apply the local product structure (Lemma 2.5) again to conclude. □

To get a lower bound on the growth rate in Proposition 3.2 we proceed as follows. Given $\epsilon > 0$ and $n \geq 1$, we want to construct an $(n, 2\epsilon)$-spanning set. We begin by choosing a covering of $f^{n+m} W^u_\delta(x)$ by ϵ-balls

$$B_{d_u}(x_i, \epsilon) : i = 1, \ldots, N := N(n + m, \epsilon)$$

contained within the unstable manifold with respect to the induced metric denoted by d_u and let $A_\epsilon := f^{n+m} W^u_\delta(x) \setminus \bigcup_{y \in \partial f^{n+m} W^u_\delta(x)} B_{d_u}(y, \epsilon/2)$, where $\partial f^{n+m} W^u_\delta(x)$ is the boundary of $f^{n+m} W^u_\delta(x)$. We can choose a maximal set $S = \{x_1, \ldots, x_{N(n+m,\epsilon)}\}$ with the additional property that $d_u(x_i, x_j) > \epsilon/2$ for $i \neq j$ and $x_i \in A_\epsilon$. By our choice of S we have that

$$A_\epsilon \subset \bigcup_{i=1}^{N(n+m,\epsilon)} B_{d_u}(x_i, \epsilon/2).$$

By the triangle inequality we have that

$$f^{n+m} W^u_\delta(x) \subset \bigcup_{i=1}^{N(n+m,\epsilon)} B_{d_u}(x_i, \epsilon).$$

We have the additional property $B_{d_u}(x_i, \epsilon/4) \cap B_{d_u}(x_j, \epsilon/4) = \emptyset$ for $i \neq j$, so then the disjoint union satisfies

$$\bigcup_{i=1}^{N(n+m,\epsilon)} B_{d_u}(x_i, \epsilon/4) \subset f^{n+m} W^u_\delta(x).$$

We assume without loss of generality that

$$f^n : f^m W^u_\delta(x) \to f^{n+m} W^u_\delta(x)$$

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locally expands distance along the unstable manifold (which is achieved by our choice of the Riemannian metric being adapted in Definition 2.1). In particular, we will use the local expansion to show that the preimages $y_i := f^{-n}x_i \in f^m(W^u_\delta(x))$ $(i = 1, \dots, N)$ form an $(n, 2\epsilon)$-spanning set.

By Lemma 3.6, for any point $z \in X$ we can choose a point $y \in f^m(W^u_\delta(x))$ with $z \in W^{cs}_{\text{loc}}(y)$ and $d(y, z) < \epsilon_0$. Then by Lyapunov stability for $\epsilon_0 > 0$ sufficiently small we have $d(f^j z, f^j y) < \epsilon$ for $0 \le j \le n$. We can then choose a y_i such that $d_n(y, y_i) < \epsilon$ since f^n is locally expanding along unstable manifolds. In particular, by the triangle inequality,

$$d(f^j z, f^j y_i) \le d(f^j z, f^j y) + d(f^j y, f^j y_i) \le 2\epsilon$$

for $0 \le j \le n$. Therefore, $\{y_1, \dots, y_{N(n+m,\epsilon)}\}$ is an $(n, 2\epsilon)$-spanning set.

Since G is continuous we have the following bound (cf. [12, Lemma 4.9]).

LEMMA 3.7. *For all $\tau > 0$ there exists $\epsilon > 0$ sufficiently small such that for all $n \ge 1$ and points $y, z \in X$ satisfying $d(f^j y, f^j z) \le \epsilon$ for $0 \le j \le n-1$ we have $|G^n(y) - G^n(z)| \le n\tau$.*

It remains to relate $Z_0(n, 2\epsilon)$ to an integral over $f^{n+m}W^u_\delta(x)$. By the properties of our choice of ϵ-cover for $f^{n+m}W^u_\delta(x)$ we have that for all $n \ge 1$,

$$\begin{aligned} Z_0(n, 2\epsilon) &\le \sum_{i=1}^{N} \exp\{G^n(y_i)\} \le \sum_{i=1}^{N} \frac{1}{\lambda(B_{d_u}(x_i, \epsilon/4))} \int_{B_{d_u}(x_i,\epsilon/4)} \exp\{G^n(f^{-n}x_i)\}\, d\lambda(z) \\ &\le \frac{1}{M} e^{n\tau} \int_{f^{n+m}W^u_\delta(x)} \exp\{G^n(f^{-n}z)\}\, d\lambda(z) \end{aligned} \tag{3.3}$$

where $M = M(\epsilon) = \inf_z \lambda(B_{d_u}(z, \epsilon/4)) > 0$. Finally, we can bound

$$\int_{f^{n+m}W^u_\delta(x)} \exp\{G^n(f^{-n}z)\}\, d\lambda(z) \le e^{m\|G\|_\infty} \int_{f^{n+m}W^u_\delta(x)} \exp\{G^{n+m}(f^{-(n+m)}z)\}\, d\lambda(z). \tag{3.4}$$

Comparing equations (3.3) and (3.4), we see that

$$P(G) = \lim_{\epsilon\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log Z_0(n, 2\epsilon) \le \limsup_{n\to+\infty} \frac{1}{n} \log \int_{f^n W^u_\delta(x)} \exp\{G^n(f^{-n}z)\}\, d\lambda(z) + \tau.$$

Since $\tau > 0$ can be chosen arbitrarily small the lower bound follows.

To prove the reverse inequality, given $\epsilon > 0$ and $n \ge 1$, we want to create an $(n, \kappa\epsilon)$-separated set for some constant $\kappa > 0$. To this end, we can choose a maximal number of points $x_i \in f^n W^u_\delta(x)$ $(i = 1, \dots, N = N(n, \epsilon))$ so that $d_u(x_i, x_j) > \epsilon$ whenever $i \ne j$. We can again assume without loss of generality that $f^n : W^u_\delta(x) \to f^n W^u_\delta(x)$ is locally distance expanding and thus, in particular, the points $y_i = f^{-n}x_i$ $(i = 1, \dots, N = N(n, \epsilon))$ form an $(n, \kappa\epsilon)$-separated set, for some $\kappa > 0$ independent of n and ϵ.

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The balls $B_{d_u}(x_i, \epsilon)$ $(i = 1, \dots, N = N(n, \epsilon))$ form a cover for $f^n W^u_\delta(x)$, since otherwise we could choose an extra point $z \in f^n W^u_\delta(x)$ with $\inf_i\{d(z, x_i)\} \geq \epsilon$ contradicting the maximality of the x_i. We can therefore use Lemma 3.7 to bound

$$\begin{aligned} Z_1(n, \kappa\epsilon) &\geq \sum_{i=1}^{N} \frac{e^{-n\tau}}{\lambda(B_{d_u}(x_i, \epsilon))} \int_{B_{d_u}(x_i,\epsilon)} \exp\{G^n(f^{-n}z)\}\, d\lambda(z) \\ &\geq \frac{e^{-n\tau}}{L} \int_{f^n W^u_\delta(x)} \exp\{G^n(f^{-n}z)\}\, d\lambda(z), \end{aligned}$$

where $L = L(\epsilon) = \sup_z \lambda(B_{d_u}(z, \epsilon)) > 0$. In particular, we see that

$$\begin{aligned} P(G) &= \lim_{\epsilon\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log Z_1(n, \kappa\epsilon) \\ &\geq \limsup_{n\to+\infty} \frac{1}{n} \log \int_{f^n W^u_\delta(x)} \exp\{G^n(f^{-n}z)\}\, d\lambda(z) - \tau. \end{aligned}$$

Since $\tau > 0$ is arbitrary. this inequality completes the proof of Proposition 3.2. □

4. *Proof of Theorem 1.1*

The proof of Theorem 1.1 relies on the growth rate result, Proposition 3.2.

Proof of Theorem 1.1. Consider a weak* accumulation point, $\mu = \lim_{k\to\infty} \mu_{n_k}$. For any continuous $F : X \to \mathbb{R}$,

$$\begin{aligned} \left| \int F\, d\mu_n - \int F \circ f\, d\mu_n \right| &= \left| \frac{1}{n} \sum_{k=0}^{n-1} \int F \circ f^k\, d\lambda_n - \frac{1}{n} \sum_{k=0}^{n-1} \int F \circ f^{k+1}\, d\lambda_n \right| \\ &\leq \frac{2\|F\|_\infty}{n} \to 0 \quad \text{as } n \to +\infty, \end{aligned}$$

and therefore μ is f-invariant.

For convenience we denote

$$\begin{aligned} Z_n^G &= \int_{W^u_\delta(x)} \exp\left\{ \sum_{k=0}^{n-1} (G - \Phi)(f^k y) \right\} d\lambda_{W^u_\delta(x)}(y) \\ &= \int_{f^n W^u_\delta(x)} \exp\left\{ \sum_{k=1}^{n} G(f^{-k} y) \right\} d\lambda_{f^n W^u_\delta(x)}(y). \end{aligned}$$

We want to show that μ is an equilibrium state for G.

Definition 4.1. Given a finite partition $\mathcal{P} = \{P_i\}_{i=1}^N$, we say that it has size $\epsilon > 0$ if $\sup_i\{\operatorname{diam}(P_i)\} < \epsilon$.

By Lemma 3.7, for any $\tau > 0$ and $n \geq 1$ we can choose a partition $\mathcal{P}$ of size ϵ such that for all $x, y \in A \in \vee_{i=0}^{n-1} T^{-i}\mathcal{P}$ we have that

$$|G^n(x) - G^n(y)| \leq n\tau. \tag{4.1}$$

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Proceeding with the proof of Theorem 1.1, for each $A \in \bigvee_{h=0}^{n-1} f^{-h}\mathcal{P}$ we fix a choice of an $x_A \in A$. By definition of λ_n, for each $0 \le j \le n-1$,

$$\int_{f^j W^u_\delta(x)} G d(f^j_* \lambda_n) = \frac{1}{Z_n^G} \int_{f^n(W^u_\delta(x))} \exp\{G^n(f^{-n}y)\} G(f^{-(n-j)}y)\, d\lambda_{f^n W^u_\delta}(y).$$

Hence, using the definition of μ_n in (1.3) and the lower bound in equation (4.1), we have that

$$\int G(y)\, d\mu_n(y) \ge \frac{1}{nZ_n^G} \sum_{A \in \bigvee_{h=0}^{n-1} f^{-h}\mathcal{P}} (G^n(x_A) - n\tau) K_{n,A}, \tag{4.2}$$

where $K_{n,A} = \int_{f^n(A \cap W^u_\delta(x))} \exp\{G^n(f^{-n}y)\}\, d\lambda_{f^n W^u_\delta}(x)$.

Using Lemma 3.7 again, we have

$$\begin{aligned} \log K_{n,A} &\le G^n(x_A) + n\tau + \log \lambda_{f^n W^u_\delta(x))}(f^n(A)) \\ &\le G^n(x_A) + n\tau \end{aligned} \tag{4.3}$$

where in the last inequality we use that the diameters of elements in the partition are arbitrarily small so that $\log \lambda_{f^n W^u_\delta(x))}(f^n(A))$ is negative.

Consider the entropy

$$\begin{aligned} H_{\lambda_n}\left(\bigvee_{h=0}^{n-1} f^{-h}\mathcal{P}\right) &= -\sum_{A \in \bigvee_{h=0}^{n-1} f^{-h}\mathcal{P}} \lambda_n(A) \log \lambda_n(A) \\ &= -\sum_{A \in \bigvee_{h=0}^{n-1} f^{-h}\mathcal{P}} \frac{K_{n,A}}{Z_n^G} \log \frac{K_{n,A}}{Z_n^G} \\ &= \log Z_n^G - \sum_{A \in \bigvee_{h=0}^{n-1} f^{-h}\mathcal{P}} \frac{K_{n,A}}{Z_n^G} \log K_{n,A}, \end{aligned} \tag{4.4}$$

where the last equality uses $\sum_{A \in \bigvee_{h=0}^{n-1} f^{-h}\mathcal{P}} K_{n,A} = Z_n^G$. Therefore, comparing (4.3) and (4.4) gives

$$H_{\lambda_n}\left(\bigvee_{h=0}^{n-1} f^{-h}\mathcal{P}\right) \ge \log Z_n^G - \sum_{A \in \bigvee_{h=0}^{n-1} f^{-h}\mathcal{P}} \frac{K_{n,A}}{Z_n^G} (G^n(x_A) + n\tau). \tag{4.5}$$

By (4.2) we can also bound

$$n \int_X G\, d\mu_n \ge \frac{1}{Z_n^G} \sum_{A \in \bigvee_{h=1}^{n} f^{-h}\mathcal{P}} (G^n(x_A) - n\tau) K_{n,A}. \tag{4.6}$$

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Comparing (4.5) and (4.6), we can write

$$\begin{aligned} & H_{\lambda_n}\Big(\bigvee_{h=0}^{n-1} f^{-h}\mathcal{P}\Big) + n\int_X G\, d\mu_n \\ & \geq \log Z_n^G - \sum_{A\in\bigvee_{h=0}^{n-1} f^{-h}\mathcal{P}} \frac{K_{n,A}}{Z_n^G}(G^n(x_A) + n\tau) \\ & \quad + \frac{1}{Z_n^G} \sum_{A\in\bigvee_{h=0}^{n-1} f^{-h}\mathcal{P}} (G^n(x_A) - n\tau)K_{n,A} \\ & \geq \log Z_n^G - 2n\tau. \end{aligned} \tag{4.7}$$

We use (4.7) and Lemma 2.8 to write, for $0 < q < n$,

$$\begin{aligned} q\log Z_n^G - qn\int_X G\, d\mu_n - 2qn\tau &\leq q H_{\lambda_n}\Big(\bigvee_{h=0}^{n-1} f^{-h}\mathcal{P}\Big), \\ &\leq n H_{\mu_n}\Big(\bigvee_{i=0}^{q-1} f^{-i}\mathcal{P}\Big) + 2q^2|\mathcal{P}|, \end{aligned}$$

which we can rearrange to get

$$\frac{\log Z_n^G}{n} - 2\tau - \frac{2q|\mathcal{P}|}{n} \leq \frac{H_{\mu_n}(\bigvee_{i=0}^{q-1} f^{-i}\mathcal{P})}{q} + \int_X G\, d\mu_n.$$

Letting $n_k \to +\infty$ gives that

$$\begin{aligned} P(G) &= \lim_{k\to\infty} \frac{\log Z_{n_k}^G}{n_k} \\ &\leq \lim_{k\to\infty} \left(\frac{H_{\mu_{n_k}}(\bigvee_{i=0}^{q-1} f^{-i}\mathcal{P})}{q} + \int_X G\, d\mu_{n_k}\right) + 2\tau \\ &= \frac{H_{\mu}(\bigvee_{i=0}^{q-1} f^{-i}\mathcal{P})}{q} + \int_X G\, d\mu + 2\tau. \end{aligned}$$

Here we use Portmanteau's theorem, assuming without loss of generality that the boundaries of the partition have zero measure. Letting $q \to \infty$,

$$P(G) \leq h_\mu(\mathcal{P}) + \int_X G\, d\mu + 2\tau. \tag{4.8}$$

Finally, we recall that τ is arbitrary. Therefore, since μ is an f-invariant probability measure, we see from the variational principle (1.1) that the inequalities in (4.8) are actually equalities (since $h_\mu(\mathcal{P}) \leq h(\mu)$) and therefore we conclude that the measure μ is an equilibrium state for G. □

Remark 4.2. If an equilibrium state corresponding to a potential G were known to be unique then we would have that $\mu_n \to \mu_G$ as $n \to +\infty$.

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5. *Partially hyperbolic diffeomorphisms with subexponential contraction in centre-unstable manifolds*

In this section we explore constructions of equilibrium states for partially hyperbolic diffeomorphisms with a stable, centre-unstable splitting. Our motivation is to weaken the Lyapunov stability condition in Theorem 1.1 in this context. Again we construct equilibrium states using a suitable sequence of reference measures but this time supported on a local centre-unstable manifold. We still require additional conditions, this time assuming integrability of the centre-unstable bundle and restricting the contraction of orbits in centre-unstable manifolds.

Definition 5.1. (Partially hyperbolic set with stable, centre-unstable splitting) Let M be a compact Riemannian manifold and $f : M \to M$ be an attracting C^1 diffeomorphism. A closed f-invariant subset $X \subset M$ is said to be partially hyperbolic if:

- there exists a continuous invariant splitting of the tangent bundle into subbundles E^{cu} and E^s such that $T_X M = E^{cu} \oplus E^s$; and
- there exist a Riemannian metric $\|\cdot\|$ on M, and constants $0 < \lambda_1 < \lambda_2$ with $0 < \lambda_1 < 1$, such that for every $x \in X$,

$$\|Df_x v\| \le \lambda_1 \|v\| \quad \text{for } v \in E^s_x,$$
$$\|Df_x v\| \ge \lambda_2 \|v\| \quad \text{for } v \in E^{cu}_x.$$

We have the following analogue of Lemma 2.4 in the case of partially hyperbolic diffeomorphisms with a stable, centre-unstable splitting.

THEOREM 5.2. (Shub [14]) *For any $x \in X$ and for $\delta > 0$ sufficiently small, there are two C^1 embedded discs, $W^s_\delta(x)$ and $W^{cu}_\delta(x)$, tangent to E^s_x and E^{cu}_x respectively, and satisfying the following properties.*

(1) *Choose $\rho > 0$ such that $\lambda_1 < \rho < \lambda_2$. Then the δ-local stable manifold is given by*

$$W^s_\delta(x) = \{y \in M : d(f^n x, f^n y) \le \delta, \text{ for all } n \ge 0 \text{ and } d(f^n x, f^n y)/\rho^n \to 0 \text{ as } n \to \infty\}.$$

(2) *$f(W^s_\delta(x)) \subset W^s_\delta(f(x))$ and f contracts distances by a constant close to λ_1.*

(3) *$f(W^{cu}_\delta(x)) \cap B(x, \delta) \subset W^{cu}_\delta(f(x))$.*

For all results in this section we will also assume that E^{cu} is integrable. By this we mean that there exists a foliation of M by immersed $\dim(E^{cu})$-manifolds whose leaves are everywhere tangent to E^{cu} (see [4]). The key here is that by assuming integrability of E^{cu} we have an analogue of the local product structure, Theorem 2.5.

Our results require an additional condition which is less restrictive than Lyapunov stability. Heuristically, we assume that the contraction of distances in the centre-unstable manifold needs to be subexponential.

Definition 5.3. (Subexponential contraction in the centre-unstable manifold) A partially hyperbolic diffeomorphism with stable, centre-unstable splitting $f : X \to X$ satisfies

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subexponential contraction in the centre-unstable manifold if there exists an increasing $g : \mathbb{N} \to \mathbb{R}_{\geq 1}$ such that for $x \in X$, $y, z \in W^{cu}_{\delta}(x)$ and $n \in \mathbb{N}$,

$$d_{cu}(f^n y, f^n z) > \frac{1}{g(n)} d_{cu}(y, z),$$

with

$$\limsup_{n\to\infty} \frac{1}{n} \log g(n) = 0.$$

In [7, §4], Climenhaga, Fisher and Thompson describe a class of derived from Anosov systems with a constraint on the derivative in the centre-stable direction. The inverse of such systems are examples of partially hyperbolic diffeomorphisms with integrable E^{cu} satisfying Definition 5.3. For another class of systems satisfying Definition 5.3 consider the following variant of the Smale–Williams solenoid found in [6]. In the classical setting the unstable direction arises from the doubling map on the circle. Example 2.6 in [6] replaces the doubling map with a family of differentiable Manneville–Pomeau maps on the circle to form partially hyperbolic solenoids satisfying Definition 5.3.

5.1. *Growth of centre-unstable manifolds.* As in Proposition 3.2, we need the following characterization of the pressure in terms of growth rates of appropriately weighted centre-unstable manifolds.

PROPOSITION 5.4. *Let $f : X \to X$ be a mixing partially hyperbolic attracting diffeomorphism with subexponential contraction in centre-unstable manifolds. For any continuous function $G : X \to \mathbb{R}$, $x \in X$ and $\delta > 0$ sufficiently small, we have*

$$P(G) = \limsup_{n\to+\infty} \frac{1}{n} \log \int_{f^n W^{cu}_{\delta}(x)} \exp\{G^n(f^{-n}y)\}\, d\lambda_{f^n W^{cu}_{\delta}(x)}(y). \tag{5.1}$$

In Proposition 3.2 we require Lyapunov stability to restrict growth in the centre-stable direction so that the centre direction does not contribute to the pressure. In the present case we require that the centre-unstable direction does not contract exponentially, otherwise this contributes to the right-hand side of (5.1).

The proof of Proposition 5.4 is similar to the proof of Proposition 3.2. The overall approach is the same, but we have to be more careful constructing the spanning set in the first half of the proof. In particular, we no longer cover $f^{n+m} W^{cu}_{\delta}(x)$ with balls of a fixed radius.

Proof. We start with the following analogue of Lemma 3.6.

LEMMA 5.5. *For any $\epsilon > 0$ there exists an $m > 0$ such that $f^m W^{cu}_{\delta}(x)$ is ϵ-dense in X. In particular, we can assume that $X = \bigcup_{y \in f^m W^{cu}_{\delta}(x)} W^s_{\epsilon}(y)$.*

This is a direct consequence of mixing and the local product structure.

To get a lower bound on the growth rate in Proposition 5.4, given $\epsilon > 0$ and $n \geq 1$, we want to construct an $(n, 2\epsilon)$-spanning set. The way we construct the spanning set is

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similar to Proposition 3.2, although here the cover of $f^{n+m}W_\delta^{cu}(x)$ uses balls whose radius depends on n.

We begin by choosing a covering of $f^{n+m}W_\delta^{cu}(x)$ by balls of radius $\epsilon_n = g(n)^{-1}\epsilon < \epsilon$,

$$B_{d_{cu}}(x_i, \epsilon_n) : i = 1, \dots, N := N(n+m, \epsilon_n),$$

contained within the centre-unstable manifold with respect to the induced metric d_{cu}, and let $A_{\epsilon_n} := f^{n+m}W_\delta^{cu}(x) \setminus \bigcup_{y \in \partial f^{n+m}W_\delta^{cu}(x)} B_{d_{cu}}(y, \epsilon_n/2)$, where $\partial f^{n+m}W_\delta^{cu}(x)$ is the boundary of $f^{n+m}W_\delta^{cu}(x)$. We can choose a maximal set $S = \{x_1, \dots, x_{N(n+m,\epsilon_n)}\}$ with the additional properties that $d_{cu}(x_i, x_j) > \epsilon_n/2$ for $i \neq j$ and $x_i \in A_{\epsilon_n}$. By our choice of S we have that

$$A_{\epsilon_n} \subset \bigcup_{i=1}^{N(n+m,\epsilon_n)} B_{d_{cu}}(x_i, \epsilon_n/2).$$

By the triangle inequality we have that

$$f^{n+m}W_\delta^{cu}(x) \subset \bigcup_{i=1}^{N(n+m,\epsilon_n)} B_{d_{cu}}(x_i, \epsilon_n).$$

Since $B_{d_{cu}}(x_i, \epsilon_n/4) \cap B_{d_{cu}}(x_j, \epsilon_n/4) = \emptyset$ for $i \neq j$, we have that the disjoint union satisfies

$$\bigcup_{i=1}^{N(n+m,\epsilon_n)} B_{d_{cu}}(x_i, \epsilon_n/4) \subset f^{n+m}W_\delta^{cu}(x).$$

By Lemma 5.5, for any point $z \in X$ we can choose a point $y \in f^m(W_\delta^u(x))$ with $z \in W_\epsilon^s(y)$ and $d(y, z) < \epsilon$. Then $d(f^j z, f^j y) < \epsilon$ for $0 \leq j \leq n$. By construction, we can then choose an x_i such that $f^n y \in W_\delta^{cu}(x_i)$ and $d_{cu}(f^n y, x_i) < \epsilon_n = (1/g(n))\epsilon$. Additionally, for every $0 \leq j \leq n$, we have $d_{cu}(f^{n-j}y, f^{-j}x_i) < \epsilon$. Otherwise, let $j \in \{0, \dots, n-1\}$ be the smallest such that $d_{cu}(f^{n-j}y, f^{-j}x_i) > \epsilon$. Then $d_{cu}(f^n y, x_i) < (1/g(n))\epsilon < (1/g(n))d_{cu}(f^{n-j}y, f^{-j}x_i) < (1/g(j))d_{cu}(f^{n-j}y, f^{-j}x_i)$ (as g is increasing). Since $f^{n-j}y$ and $f^{-j}x_i$ are in the same piece of centre-unstable manifold, this would contradict the assumption of Definition 5.3. In particular, letting $y_i = f^{-n}x_i$, by the triangle inequality,

$$d(f^j z, f^j y_i) \leq d(f^j z, f^j y) + d(f^j y, f^j y_i) \leq 2\epsilon$$

for $0 \leq j \leq n$. Therefore, $\{y_1, \dots, y_{N(n+m,\epsilon_n)}\}$ is an $(n, 2\epsilon)$-spanning set.

It remains to relate $Z_{0,G}(n, 2\epsilon)$ to an integral over $f^{n+m}W_\delta^{cu}(x)$. By the properties of our choice of ϵ_n-cover for $f^{n+m}W_\delta^{cu}(x)$ and using Lemma 3.7, we have that for all $n \geq 1$,

$$\begin{aligned} Z_{0,G}(n, 2\epsilon) &\leq \sum_{i=1}^{N} \frac{1}{M(n)} \int_{B_{d_{cu}}(x_i, \epsilon_n/4)} \exp\{G^n(f^{-n}x_i)\}\, d\lambda_{f^{n+m}W_\delta^{cu}(x)}(z) \\ &\leq \frac{1}{M(n)} e^{n\tau} \int_{f^{n+m}W_\delta^{cu}(x)} \exp\{G^n(f^{-n}z)\}\, d\lambda_{f^{n+m}W_\delta^{cu}(x)}(z) \end{aligned} \tag{5.2}$$

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where $M(n) = M(n, \epsilon_n) = \inf_z \lambda_{f^{n+m}W^{cu}_\delta(x)}(B_{d_{cu}}(z, \epsilon_n/4)) > 0$. For ϵ small, there is a constant K_1 such that $\lambda(B_{d_{cu}}(z, \epsilon_n/4)) \geq K_1 g(n)^{-\dim(E^{cu})}\lambda(B_{d_{cu}}(z, \epsilon/4))$. Therefore, by the defining property of g, $\limsup_{n\to\infty}(-\log M(n)/n) = 0$. Finally, we can bound

$$\int_{f^{n+m}W^{cu}_\delta(x)} \exp\{G^n(f^{-n}z)\}\, d\lambda(z) \leq e^{m\|G\|_\infty} \int_{f^{n+m}W^{cu}_\delta(x)} \exp\{G^{n+m}(f^{-(n+m)}z)\}\, d\lambda(z). \tag{5.3}$$

Comparing equations (5.2) and (5.3), we see that

$$\lim_{\epsilon\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log Z_{0,G}(n, 2\epsilon) \leq \limsup_{n\to+\infty} \frac{1}{n} \log \int_{f^n W^{cu}_\delta(x)} \exp\{G^n(f^{-n}z)\}\, d\lambda(z) + \tau.$$

Since $\tau > 0$ can be chosen arbitrarily small the lower bound follows.

To get an upper bound on the growth rate in Proposition 5.4, we proceed the same way as in the second half of the proof of Proposition 3.2, so we shall omit it. □

5.2. *Constructing equilibrium states.* We conclude with a statement about the construction of equilibrium states for partially hyperbolic systems with subexponential contraction in the centre-unstable direction.

THEOREM 5.6. *Let $f : X \to X$ be a topologically mixing partially hyperbolic attracting diffeomorphism satisfying Definition 5.3, and let $G : X \to \mathbb{R}$ be a continuous function. Given $x \in X$ and $\delta > 0$ small, consider the sequence of probability measures $(\lambda_n)_{n=1}^\infty$ supported on $W^u_\delta(x)$ and absolutely continuous with respect to the induced volume $\lambda_{W^u_\delta(x)}$ given by*

$$\lambda_n(A) = \frac{\int_{f^n(A)} \exp\{\sum_{i=1}^n G(f^{-i}y)\}\, d\lambda_{f^n W^u_\delta}(y)}{\int_{f^n W^u_\delta(x)} \exp\{\sum_{i=1}^n G(f^{-i}z)\}\, d\lambda_{f^n W^u_\delta}(z)} \quad \text{for Borel } A \subset W^u_\delta(x).$$

Then the weak limit points of the averages*

$$\mu_n := \frac{1}{n} \sum_{k=0}^{n-1} f^k_* \lambda_n, \quad n \geq 1,$$

are equilibrium states for G.

The proof of Theorem 5.6 is the same as that of Theorem 1.1, so we will not repeat it.

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