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The existence of global weak solutions for a multi-component plane flow

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We consider the dynamics of multi-component heat-conducting viscous incompressible flow in a plane domain when the viscosity and thermal conductivity of the medium depend on temperature. The dynamics of the flow is governed by an initial-boundary value problem for the Navier–Stokes system with heat conduction and heat transfer taken into account. The existence of a generalized global solution with velocity field and temperature of Hopf's class has been established in conjunction with the estimate of fractional smoothness of the order $1/2$ in the time variable.

1 Introduction

We consider the dynamics of a two-component fluid which occupies the domain $\Omega \in \mathbb{R}^2$, assumed to be bounded for the sake of simplicity. We observe that the same model, without any change, is valid as well for a multi-component fluid, but in the present paper we restrict ourselves to the two-component case. The domain Ω is divided into two parts $\Omega_1 = \Omega_1(t)$ and $\Omega_2 = \Omega_2(t)$ by an unknown interface which is denoted by $\Gamma(t)$, i.e. $\Omega = \Omega_1(t) \cup \Omega_2(t) \cup \Gamma(t)$. The outer boundary $\partial\Omega$ does not depend on time. The work is aimed at constructing the solution of the problem which describes the dynamics of a two-component heat-conducting viscous incompressible flow in a plane domain when the viscosity and the thermal conductivity of the medium depend on temperature. Both components are supposed to have the same constant density. The dynamics of the flow is described by the Navier–Stokes system of equations with heat conduction and heat transfer taken into account:

$$\begin{aligned} &\rho[v_t + (v \cdot \nabla)v] = f(x,t) + \mathrm{Div}\mathscr{T}, \\ &\Theta_t + (v \cdot \nabla)\Theta = \mathrm{div}(\varkappa \nabla \Theta) + g(x,t), \\ &\mathrm{div}\, v = 0,\ (x,t) \in Q_T = \Omega \times (0,T), \quad \Omega \subset \mathbb{R}^2, \quad T > 0. \end{aligned} \tag{1}$$

In system (1) and hereafter, variable t is time, $x = (x_1, x_2)$ denote spatial variables, $v(x,t) = (v_1, v_2)$ is the velocity field of the medium at point $x \in \Omega$ at time $t \in (0,T)$, ρ is the constant density of the medium, $\mathscr{T}$ is the stress tensor for the fluid, i.e. $\mathscr{T}_{ij} = \mu S_{ij} - \delta_{ij}P$, where δ_{ij} is Kroneker's symbol, P is the hydrodynamical pressure of the medium, $S_{ij} = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$ are the components of the strain tensor, $\mu(\vartheta)$ is the temperature dependent

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coefficient of viscosity. The vector field $f(x,t)=(f_1,f_2)$ is a given volume density of the external forces, $\vartheta(x,t)$ is the classical molecular temperature of the medium measurable by thermometer, $\varkappa(\vartheta)$ is the temperature dependent heat conductivity coefficient of the medium, $g(x,t)$ is the given density of the heat source.

Initial boundary value problems for the Navier–Stokes system with free boundaries are of great interest for hydrodynamics. Many authors have studied the problems of dynamics of multi-component viscous fluids with different parameters and with free boundaries separating the components. For example, the existence of global weak solutions in the case when the densities and viscosities are different but without heat conduction has been studied by Maslennikova & Bogovskii [4], as well as by Nouri & Poupaud [5]. The existence of solutions global in time, though for small initial values, with the consideration of heat conduction and compressibility has been established in the work of Zadrzyńska & Zajączkowski [7].

Antontsev *et al.* [1, pp. 109–156] considered the model of a nonhomogeneous viscous incompressible flow when the nonhomogenity of the incompressible flow was taken into account by the density's being piece-wise constant while the viscosity was assumed to be constant. This model was represented by the equations:

$$\begin{aligned} \rho[v_t+(v\cdot\nabla)v] &= \mu\Delta v-\nabla P+\rho f(x,t),\\ \rho_t+(v\cdot\nabla)\rho &= 0,\\ \operatorname{div} v &= 0. \end{aligned}$$

For the appropriate initial boundary value problem, in their work [1], the authors established the existence of a global generalized solution. A similar approach was employed by Nouri and Poupaud for taking into account the nonhomogenity of the fluid. More precisely, they couple the Navier–Stokes system with the transport equation for the piece-wise constant viscosity

$$\frac{\partial\mu}{\partial t}+(v\cdot\nabla)\mu=0$$

in conjunction with the initial condition

$$\mu|_{t=0}=\mu^0(x).$$

Maslennikova & Bogovskii [4] do not employ the transport equation neither for density nor for viscosity. Rather, they use a "Euler type" method involving the so-called pseudo-density function $\Phi(x,t)$, which is artificially introduced in order to describe the interface implicitly by the equation $\Phi(x,t)=0$. In so doing, the function $\Phi(x,t)$ satisfies the equation of free surface in conjunction with the appropriate initial condition. Such introduction of the pseudo-density function is the natural generalisation of the methods, on which the use of transport equation for piece-wise constant viscosity and density is based.

Unlike other approaches to the problems of the dynamics of the multi-component incompressible fluid with unknown interfaces, the method developed by Maslennikova and Bogovskii remains valid even in the case when the viscosity and temperature conductivity coefficients are variable and not necessarily piece-wise constant, depending for instance

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on temperature ϑ. In the present work, as in Maslennikova & Bogovskii [4], the method of introducing the pseudo-density function is used.

We note that prior to the present work the existence of global weak solutions in the dynamics of the multi-component heat-conducting viscous incompressible flow has been an open question in case the coefficients of viscosity and temperature-conductivity depend on temperature. Similar problems but without heat conduction had been studied by different authors. For instance, the existence of global generalized solution in case of different density and viscosity but without heat-conduction has been established in Maslennikova & Bogovskii [4], as well as in Nouri & Poupaud [5]. The solution, global in time but for small initial values, with consideration of heat conduction and compressibility, has been established in Zadrzyńska & Zajaçzkowski [7].

In the present work we use an approach similar to Maslennikova & Bogovskii [4] based on the "Euler type" method involving the pseudo-density function $\Phi(x,t)$ which satisfies the equation of free surface

$$\frac{\partial \Phi}{\partial t} + v \cdot \nabla \Phi = 0. \tag{2}$$

For the system (1) on the fixed outer surface $\partial\Omega$ are given the following boundary conditions

$$v|_{\partial\Omega} = 0, \tag{3}$$

$$\vartheta|_{\partial\Omega} = 0. \tag{4}$$

The temperature dependent kinematic viscosity and thermal conductivity of the medium on the subsets $Q_T^j = \{(x,t) \in Q_T : x \in \Omega_j(t), t \in (0,T)\}$, $j = 1,2$, coincide with the given functions $\mu_j(\vartheta)$ and $\varkappa_j(\vartheta)$, $j = 1,2$. Precisely,

$$\mu = \begin{cases} \mu_1(\vartheta) & \text{in} \quad Q_T^1, \\ \mu_2(\vartheta) & \text{in} \quad Q_T^2, \end{cases} \qquad \varkappa = \begin{cases} \varkappa_1(\vartheta) & \text{in} \quad Q_T^1, \\ \varkappa_2(\vartheta) & \text{in} \quad Q_T^2. \end{cases}$$

We denote the part of the surface $\Gamma(t)$ from the side $\Omega_j(t)$, $j = 1,2$, by $\Gamma_j = \Gamma_j(t)$. For the velocity vector field on the unknown interface are given the two vector conjugate conditions, namely

$$v|_{\Gamma_1(t)} = v|_{\Gamma_2(t)}, \tag{5}$$

which is continuity of the velocity field, and

$$\mathscr{T} v|_{\Gamma_1(t)} = \mathscr{T} v|_{\Gamma_2(t)} \tag{6}$$

which is continuity of stress, where ν is the unit normal to the surface $\Gamma(t)$. Here the coefficient of viscosity μ in the stress tensor is a function of temperature $\mu_1(\vartheta)$ or $\mu_2(\vartheta)$ on $\Gamma_1(t)$ or $\Gamma_2(t)$, respectively. For the temperature on the interface are given two scalar conjugate conditions, namely,

$$\vartheta|_{\Gamma_1(t)} = \vartheta|_{\Gamma_2(t)} \tag{7}$$

which is continuity of temperature of the medium, and

$$\left[\varkappa_1(\vartheta)\frac{\partial \vartheta}{\partial \nu} - (v \cdot \nu)\vartheta\right]\Big|_{\Gamma_1(t)} = \left[\varkappa_2(\vartheta)\frac{\partial \vartheta}{\partial \nu} - (v \cdot \nu)\vartheta\right]\Big|_{\Gamma_2(t)} \tag{8}$$

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which is continuity of the heat flux in the medium, where $(v \cdot v)$ indicates the scalar product in $\mathbb{R}^2$.

For the velocity vector field and temperature are given the initial conditions

$$v|_{t=0} = v^0(x), \tag{9}$$

$$\vartheta|_{t=0} = \vartheta^0(x). \tag{10}$$

The initial form $\Gamma(0)$ of the interface is given implicitly by the equation $\Phi_0(x) = 0$, which corresponds to the initial condition

$$\Phi|_{t=0} = \Phi_0(x). \tag{11}$$

It is assumed that $\Gamma(0) \in C^1$, precisely, $\Phi_0(x) \in C^1(\overline{\Omega})$ and $|\nabla \Phi_0(x)| \neq 0$ in the closure $\overline{\Omega}$. Besides, the introduction of the pseudo-density function method requires that at the initial moment the following conditions are fulfilled

$$\begin{aligned} &\Phi_0(x) > 0, \quad x \in \Omega_1(0), \\ &\Phi_0(x) = 0, \quad x \in \Gamma(0), \\ &\Phi_0(x) < 0, \quad x \in \Omega_2(0), \end{aligned} \tag{12}$$

due to which, for sufficiently smooth vector field $v(x,t)$ and for all $t \in [0,T]$, the pseudo-density function $\Phi(x,t)$ will satisfy the conditions

$$\begin{aligned} &\Phi(x,t) > 0, \quad x \in \Omega_1(t), \\ &\Phi(x,t) = 0, \quad x \in \Gamma(t), \\ &\Phi(x,t) < 0, \quad x \in \Omega_2(t), \end{aligned} \tag{13}$$

which follows from the corollary of Theorem 1, stated later.

The use of the pseudo-density function $\Phi(x,t)$ is convenient for defining the coefficients of viscosity and temperature conductivity of the two media. We will consider that $\mu = \mu(\vartheta, \Phi)$ and $\varkappa = \varkappa(\vartheta, \Phi)$, where

$$\mu(\vartheta,\lambda) = \begin{cases} \mu_1(\vartheta), & \lambda > 0, \\ \mu_2(\vartheta), & \lambda < 0, \end{cases} \qquad \varkappa(\vartheta,\lambda) = \begin{cases} \varkappa_1(\vartheta), & \lambda > 0, \\ \varkappa_2(\vartheta), & \lambda < 0. \end{cases}$$

In the present case the original nonlinear problem (1)–(11) is neither simplified nor replaced by any approximation and the theorem on global generalised solution of the problem (1)–(11) is proved without any smallness supposition of whatsoever.

2 Weak formulation of the problem

For defining the generalized solution, we introduce the following functional spaces. By the symbol $W^{1,0}_{2,x,t}(Q_T)$ we denote, as usual, the non-isotropic Sobolev space with the norm

$$\|u\|_{W^{1,0}_{2,x,t}(Q_T)} = \|u\|_{L_2(Q_T)} + \|\nabla_x u\|_{L_2(Q_T;\mathbb{R}^2)}.$$

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By $L^{x,t}_{2,\infty}(Q_T)$ we denote the non-isotropic Lebesgue space with the norm

$$\|u\|_{L^{x,t}_{2,\infty}(Q_T)} = \operatorname*{ess\,sup}_{t\in(0,T)} \|u\|_{L_2(\Omega)}.$$

The space for the solenoidal vector field is denoted by the standard notation

$$\overset{\circ}{J}{}^{\infty}(\Omega) = \{u \in \overset{\circ}{C}{}^{\infty}(\Omega;\mathbb{R}^2) : \operatorname{div} v = 0\}.$$

The closure in $W^1_2(\Omega;\mathbb{R}^2)$ of its subspace $\overset{\circ}{J}{}^{\infty}(\Omega)$ is denoted by $\overset{\circ}{J}{}^1_2(\Omega)$. By $\overset{\circ}{J}{}_2(\Omega)$ is denoted the closure of its subspace $\overset{\circ}{J}{}^{\infty}(\Omega)$ in $L_2(\Omega;\mathbb{R}^2)$. The velocity field is sought for in the Hopf class

$$H_T = \left\{u(x,t) \in W^{1,0}_{2,x,t}(Q_T;\mathbb{R}^2) \cap L^{x,t}_{2,\infty}(Q_T;\mathbb{R}^2) : \operatorname{div} u = 0, u|_{\partial\Omega} = 0\right\}$$

with the corresponding induced norm. In the definition of the generalized solution the test vector-functions are taken from the class

$$H^{\infty}_T = \{u(x,t) \in C^{\infty}(\overline{Q}_T;\mathbb{R}^2) : u|_{t=T} = 0,\ u(x,t) \in \overset{\circ}{J}{}^{\infty}(\Omega)\,\forall t \in [0,T]\}.$$

The temperature is taken from the Hopf class

$$\Theta_T = \left\{\vartheta(x,t) \in W^{1,0}_{2,x,t}(Q_T) \cap L^{x,t}_{2,\infty}(Q_T) : \vartheta|_{\partial\Omega} = 0\right\}$$

with the corresponding induced norm. In the definition of the generalized solution the test functions are taken from the class

$$\Theta^{\infty}_T = \{\psi(x,t) \in C^{\infty}(\overline{Q}_T) : \psi|_{t=T} = 0,\ \psi(x,t) \in \overset{\circ}{C}{}^{\infty}(\Omega)\,\forall t \in [0,T]\}.$$

Definition 1 *The triplet* $\{v,\vartheta,\Phi\}$*, where* $v \in H_T$, $\vartheta \in \Theta_T$, $\Phi \in L_\infty(Q_T)$ *is said to be a generalized solution of the problem* (1)–(11)*, if* v,ϑ,Φ *satisfy the following three integral identities:*

$$-\int_{Q_T}(v\cdot u_t)\,dx\,dt - \int_{Q_T}(v\cdot(v\cdot\nabla)u)\,dx\,dt + \sum_{k=1}^{2}\int_{Q_T}\mu(S_k\cdot\nabla u_k)\,dx\,dt = \int_{Q_T}(f\cdot u)\,dx\,dt + \int_{\Omega}(v^0(x)\cdot u(0,x))\,dx \quad \forall u \in H^{\infty}_T,$$

$$-\int_{Q_T}\vartheta\psi_t\,dx\,dt - \int_{Q_T}\vartheta(v\cdot\nabla\psi)\,dx\,dt + \int_{Q_T}\varkappa(\nabla\vartheta\cdot\nabla\psi)\,dx\,dt = \int_{Q_T}g\psi\,dx\,dt + \int_{\Omega}\vartheta^0(x)\psi(0,x)\,dx \quad \forall\psi \in \Theta^{\infty}_T,$$

$$\int_{Q_T}\Phi(\gamma_t + (v\cdot\nabla\gamma))\,dx\,dt = -\int_{\Omega}\Phi_0(x)\gamma(x,0)\,dx \quad \forall\gamma \in C^{\infty}(\overline{Q}_T) : \gamma(x,T) = 0,$$

where the vector-function $S_k = (S_{k1}, S_{k2})$ is the kth row of the tensor of deformation S, $k = 1,2$.

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We prove two theorems on correctness of the, above stated, definition 1 of the generalized solution for (1)–(11).

Theorem 1 *Let the generalized solution in the sense of definition* 1 *be such that* $v(x,t)$ *satisfies a Lipschitz condition in* Q_T. *Then* $\Phi(x,t)$ *also satisfies a Lipschitz condition in* Q_T *and there exists a Lipschitz surface* $\Gamma(t)$, *which satisfies the equation* $\Phi(x,t)=0$ *for all* $t \in [0,T]$.

Proof We consider the equation of the free surface

$$\frac{\partial \Phi}{\partial t} + (v \cdot \nabla \Phi) = 0,\ (x,t) \in Q_T,$$

$$\Phi|_{t=0} = \Phi_0(x),\ x \in \Omega.$$

There exists an unique solution of this problem (follows, for example, from DiPerna & Lions [2]). This solution can be written explicitly in the form

$$\Phi(x,t) = \Phi_0(h(x,t)), \quad x \in \Omega, \quad t \in [0,T],$$

where the mapping $y = h(x,t)$ is inverse to the mapping $x = g(y,t) : \Omega \to \Omega$. The mapping $x = g(y,t)$ is defined as the solution of the Cauchy problem

$$\begin{cases} \frac{dx}{dt} = v(x,t), & t \in (0,T) \\ x|_{t=0} = y, & y \in \Omega. \end{cases} \tag{14}$$

If the velocity field $v(x,t)$ satisfies a Lipschitz condition in Q_T, then owing to the uniqueness of the solution of the Cauchy problem (14) the mapping $x = g(y,t)$ will be a homeomorphism. In addition, according to Sukhinin [6], the mapping $x = g(y,t)$ will satisfy the Lipschitz condition in Q_T. Then the inverse mapping $y = h(x,t)$, being the solution of the Cauchy problem, inverse to (14), will as well satisfy in Q_T the Lipschitz condition.

It is obvious that, for the mapping $x = g(y,t)$, the set $\Gamma(t) \subset \Omega$ is the image of the set $\Gamma(0)$, i.e. $\Gamma(t) = g(\Gamma(0),t)$ for all $t \in [0,T]$. At the initial moment $\Gamma(0) \in C^1$, therefore the image $\Gamma(t)$ will be a Lipschitz curve at every moment $t \in (0,T]$. □

Corollary From Theorem 1, in particular, it follows that the fulfillment of conditions (12) implies the fulfillment of (13).

3 Main results

Theorem 2 *In the conditions of Theorem* 1, *let the generalized solution further possess the one-sided smoothness* $v \in W^{2,1}_{2,x,t}(Q_T^j;\mathbb{R}^2)$, $\vartheta \in W^{2,1}_{2,x,t}(Q_T^j)$, $j=1,2$. *Then the generalized solution will be a solution in the ordinary sense, i.e. there exists* $\nabla P \in L_2(Q_T^j;\mathbb{R}^2)$, $j=1,2$, *and a.e. in* Q_T *equations* (1), (2) *are satisfied, in addition the initial conditions* (9)–(11) *are satisfied a.e. in* Ω, *and for every* $t \in [0,T]$ *the conjugate conditions* (5)–(8) *are satisfied a.e. on the free surface* $\Gamma(t)$.

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Theorem 2 is proved in the conventional way. In Definition 1 we first integrate by parts the identities for the velocity field, temperature and Φ. Then by choosing the appropriate test functions we show the validity of Theorem 2.

We now formulate the main theorem of this present work.

Theorem 3 *Let the exterior boundary $\partial\Omega$ be Lipschitz, $T>0$, the functions $\mu_j(\vartheta)$, $\varkappa_j(\vartheta)$ satisfy Lipschitz conditions on ϑ, $0<M_1\leqslant\mu_j(\vartheta)\leqslant M_2<\infty$, $0<K_1\leqslant\varkappa_j(\vartheta)\leqslant K_2<\infty$, $j=1,2$; and let $f\in L_2(Q_T;\mathbb{R}^2)$, $g\in L_2(Q_T)$, $v^0\in \overset{\circ}{J}_2(\Omega)$, $\vartheta^0(x)\in L_2(\Omega)$. Then there exists a global generalized solution $v\in H_T\cap W^{1,1/2}_{2,x,t}(Q_T;\mathbb{R}^2)$, $\vartheta\in\Theta_T\cap W^{1,1/2}_{2,x,t}(Q_T)$.*

We recall here the norm in the non-isotropic Sobolev–Slobodetskii space $W^{1,s}_{2,x,t}$ with $s=1/2$, which is defined as

$$\|u\|^2_{W^{1,1/2}_{2,x,t}(Q_T)}=\|u\|^2_{L_2(Q_T)}+\|\nabla_x u\|^2_{L_2(Q_T;\mathbb{R}^2)}+\int_\Omega\int_0^T\int_0^T\frac{|u(x,t)-u(x,\tau)|^2}{|t-\tau|^2}\,dt\,d\tau\,dx.$$

If the function is defined on the whole space then this norm is equivalent to

$$\|v\|_{W^{1,1/2}_{2,x,t}(Q_T)}=\int_{-\infty}^{\infty}\int_\Omega\left|(1+\xi^2)^{\frac{1}{4}}\widehat{v}(x,\xi)\right|^2dx\,d\xi+\int_{-\infty}^{\infty}\int_\Omega|\nabla_x v(x,t)|^2\,dx\,dt$$

The proof of Theorem 3 is based on a Galerkin approximation, which for the velocity vector field is sought for in the following way:

$$v^N(x,t)=\sum_{k=1}^N C_k^N(t)w^k(x)$$

with the unknown coefficients $C_k^N(t)$, $k=1,\dots,N$, where $\{w^k(x)\}_{k=1}^\infty$ are known basis vector-functions $w^k(x)\in\overset{\circ}{J}{}^\infty(\Omega)$. Here $\{w^k(x)\}_{k=1}^\infty$ is a complete linearly independent system in the space $J_2^1(\Omega)$. The Galerkin approximation for temperature is sought in the following way:

$$\vartheta^N(x,t)=\sum_{k=1}^N B_k^N(t)\xi^k(x)$$

with unknown coefficients $B_k^N(t)$, $k=1,\dots,N$, where $\{\xi^k(x)\}_{k=1}^\infty$ are known basis functions $\xi^k(x)\in\overset{\circ}{C}{}^\infty(\Omega)$. Here $\{\xi^k(x)\}_{k=1}^\infty$ is the complete linear-independent system in the Sobolev space $W_2^1(\Omega)$.

The coefficients of the Galerkin approximation with the unknown interface $\Phi^N(x,t)=0$ are defined as the solution of the Cauchy problem

$$\begin{aligned}&\int_\Omega\left(v_t^N(x,t)\cdot w^m(x)\right)dx+\int_\Omega((v^N(x,t)\cdot\nabla)v^N(x,t)\cdot w^m(x))\,dx\\&\quad+\sum_{k=1}^2\int_\Omega\mu(\vartheta^N,\Phi^N)\left(S_k^N\cdot\nabla w_k^m(x)\right)dx=\int_\Omega(f(x,t)\cdot w^m(x))\,dx,\end{aligned}\tag{15}$$

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$$\int_\Omega \vartheta_t^N(x,t)\xi^m(x)\,dx - \int_\Omega \vartheta^N(x,t)(v^N(x,t)\cdot\nabla\xi^m(x))\,dx + \int_\Omega \varkappa(\vartheta^N,\Phi^N)(\nabla\vartheta^N(x,t)\cdot\nabla\xi^m(x))\,dx = \int_\Omega g(x,t)\xi^m(x)\,dx, \tag{16}$$

$$\Phi_t^N(x,t) + (v^N(x,t)\cdot\nabla\Phi^N(x,t)) = 0,\ (x,t)\in Q_T, \tag{17}$$

where m takes the values 1 to N, with initial conditions

$$\int_\Omega (v^N(x,0)\cdot w^m(x))\,dx = \int_\Omega (v^0(x)\cdot w^m(x))\,dx, \quad 1\leqslant m\leqslant N, \tag{18}$$

$$\int_\Omega \vartheta^N(x,0)\xi^m(x)\,dx = \int_\Omega \vartheta^0(x)\xi^m(x)\,dx, \quad 1\leqslant m\leqslant N, \tag{19}$$

$$\Phi^N(x,0) = \Phi_0(x), \quad x\in\Omega. \tag{20}$$

For the given function $\Phi(x,t)\in C(\overline{Q}_T)$ the existence of Galerkin approximations $v^N(x,t)$ and $\vartheta^N(x,t)$ is derived, as usual, from the a priori estimate of these approximations. The existence of the Galerkin approximation with unknown interface $\Phi^N(x,t)=0$ is proved in the following way. First the Galerkin approximations $v^N(x,t)$ and $\vartheta^N(x,t)$ are found for any given function $\Phi(x,t)\in C(\overline{Q}_T)$ with initial condition $\Phi(x,0)=\Phi_0(x)$. Then the Cauchy problem

$$\Psi_t(x,t) + (v^N(x,t)\cdot\nabla\Psi(x,t)) = 0,\ (x,t)\in Q_T,$$
$$\Psi(x,0) = \Phi_0(x),\ x\in\Omega,$$

is solved, with the help of which the mapping $\Psi=\mathscr{F}_N(\Phi)$ is defined with respect to the given Φ. The existence of the solution of the problem (15)–(20) with unknown interface $\Phi^N(x,t)=0$ follows from the existence of the fixed point $\Phi=\Phi^N(x,t)$ of the mapping $\mathscr{F}_N$: $C(\overline{Q}_T)\to C(\overline{Q}_T)$.

For the given function $\Phi_0(x)$ we denote by M_0 the set

$$M_0 = \left\{\Phi(x,t)\in C(\overline{Q}_T)\ :\ \Phi(x,0)=\Phi_0(x),\ \min_{x\in\Omega}\Phi_0(x)\leqslant\Phi(x,t)\leqslant\max_{x\in\Omega}\Phi_0(x)\ \forall(x,t)\in Q_T\right\}.$$

The following lemma holds.

Lemma 1 *The mapping $\mathscr{F}_N(\Phi)$ transforms the set M_0 into itself and there exists a fixed point $\Phi=\Phi^N\in C^1(\overline{Q}_T)$ in M_0.*

The lemma is proved using Schauder's fixed point theorem. The lemma on estimates of the type of maximum principle from Maslennikova & Bogovskii [4] is used to verify the fulfillment of conditions of the Schauder theorem.

Then we obtain the usual *a priori* estimate for the Galerkin approximation : $v^N(x,t)\in H_T$, $\vartheta^N(x,t)\in\Theta_T$ in the respective induced norms. Later we obtain fractional smoothness in time variable t for $v^N(x,t)$ and $\vartheta^N(x,t)$, namely the following lemma is proved.

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Lemma 2 *For the Galerkin approximation, we have the estimates*

$$\sup_{N\geqslant 1}\|v^N\|^2_{W^{0,1/2}_{2,x,t}(Q_T)}\leqslant M_1(v^0,f), \tag{21}$$

$$\sup_{N\geqslant 1}\|\vartheta^N\|^2_{W^{0,1/2}_{2,x,t}(Q_T)}\leqslant M_2(v^0,\vartheta^0,f,g), \tag{22}$$

$$\min_{x\in\overline{\Omega}}\Phi_0(x)\leqslant\Phi^N(x,t)\leqslant\max_{x\in\overline{\Omega}}\Phi_0(x)\quad\forall(x,t)\in Q_T, \tag{23}$$

where M_1, M_2 are some polynomials of third order with respect to the corresponding norms v^0, ϑ^0, f, g.

The method proposed in Maslennikova & Bogovskii [4] is used for obtaining the estimates (21), which consists in the following. The coefficients of the Galerkin approximation in (15) are further redefined on the whole axis t and are multiplied by the suitable smooth cut function $\eta(t)$. Modified thereby (15) is multiplied by $H[\eta(t)C_k^N(t)]$ and summed up for all k from 1 to N, where H denotes the Hilbert transformation, i.e. the integral transformation of the type

$$H[\varphi(t)]=\frac{1}{\pi}PV\int_{-\infty}^{\infty}\frac{\varphi(s)}{t-s}\,ds,$$

which is an isometry of $L_2(\mathbb{R}^1)$ onto itself. The identity obtained thereby is integrated with respect to t from $-\infty$ to $+\infty$. Finally, using the Plancherel theorem for Fourier transforms, the properties of the Hilbert transform, multiplicative inequality and the already achieved usual estimates of the Galerkin approximation we get the estimate (21). For the Galerkin approximations for temperature ϑ^N the estimate (22) is deduced in the same manner. Estimate (23) follows from the maximum principle [3].

Owing to the estimates obtained, there exists subsequence of the Galerkin approximation $\{v^{N_k},\vartheta^{N_k},\Phi^{N_k}\}$ such that v^{N_k} converges in $W^{1,1/2}_{2,x,t}(Q_T;\mathbb{R}^2)$ weakly to some $v(x,t)\in H_T\cap W^{1,1/2}_{2,x,t}(Q_T;\mathbb{R}^2)$, ϑ^{N_k} converges weakly in $W^{1,1/2}_{2,x,t}(Q_T)$ to some $\vartheta(x,t)\in\Theta_T\cap W^{1,1/2}_{2,x,t}(Q_T)$. The embedding of $W^{1,1/2}_{2,x,t}(Q_T)$ in $L_p(Q_T)$ is compact for $p<4$. Therefore for $p<4$ the subsequences v^{N_k} and ϑ^{N_k} converge strongly in $L_p(Q_T;\mathbb{R}^2)$ and $L_p(Q_T)$ to v and ϑ respectively. Then from Theorem II.4 of DiPerna & Lions [2], it follows that for any $p\geqslant 1$ the subsequence $\Phi^{N_k}(x,t)$ converges strongly in $L_p(Q_T)$ to some function $\Phi(x,t)\in L_\infty(Q_T)$, as well due to (23) a.e. in Q_T we have the estimates

$$\min_{x\in\overline{\Omega}}\Phi_0(x)\leqslant\Phi(x,t)\leqslant\max_{x\in\overline{\Omega}}\Phi_0(x).$$

Without loss of generality, we shall consider that the subsequence $\Phi^{N_k}(x,t)$ converges to the function $\Phi(x,t)$ a.e. in Q_T.

The proof of Theorem 3 is completed by referring to the following auxiliry theorem.

Theorem 4 *The triplet $\{v,\vartheta,\Phi\}$ is a generalized solution of the initial-boundary value problem* (1)–(11) *in the sense of Definition* 1.

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The proof of Theorem 4 is based essentially on the fact that for the obtained $\Phi(x,t)$ the set $\sigma = \{(x,t) \in Q_T : \Phi(x,t) = 0\}$ has null measure. Precisely the following lemma, which specifies the structure of the global generalized solution of the problem (1)–(11), holds.

Lemma 3 *The Lebesgue measure* $\operatorname{mes}\sigma = 0$. *Besides,* $\operatorname{mes}\Omega_j(t) = \operatorname{mes}\Omega_j(0)$ *for a.e.* $t \in (0,T)$, $j = 1,2$.

The equality $\operatorname{mes}\sigma = 0$ is shown from the contrary. We suppose the Lebesgue measure of the set σ is equal to some $\delta > 0$. In addition, for every fixed number $N \geqslant 1$ the function $\Phi^N(x,t)$ is the solution of the Cauchy problem (17), (20).

The cut-off function $\eta(\xi) \in \overset{\circ}{C}{}^{\infty}(\mathbb{R}^1)$ is so selected that $\eta(\xi) = 1$ for $|\xi| \leqslant 1$ and $\eta(\xi) = 0$ for $|\xi| \geqslant 2$. Multiplying equation (17) by $\frac{1}{\varepsilon}\eta'(\frac{\Phi^N}{\varepsilon})$ and integrating over $x \in \Omega$, we have

$$\int_\Omega \eta\left(\frac{\Phi^N}{\varepsilon}\right) dx = \int_\Omega \eta\left(\frac{\Phi_0}{\varepsilon}\right) dx \quad \forall t \in [0,T],\ \forall N \geqslant 1,\ \forall \varepsilon > 0.$$

From here, by integrating over t from 0 to T and using the Lebesgue theorem we get

$$\lim_{N_k \to \infty} \int_{Q_T} \eta\left(\frac{\Phi^{N_k}}{\varepsilon}\right) dx\, dt = \int_{Q_T} \eta\left(\frac{\Phi}{\varepsilon}\right) dx\, dt = T \int_\Omega \eta\left(\frac{\Phi_0}{\varepsilon}\right) dx \tag{24}$$

for all $\varepsilon > 0$. But, on the other hand

$$\int_{Q_T} \eta\left(\frac{\Phi}{\varepsilon}\right) dx\, dt \geqslant \int_\sigma \eta\left(\frac{\Phi}{\varepsilon}\right) dx\, dt = \delta \quad \forall \varepsilon > 0. \tag{25}$$

Then from (24) and (25) we find the lower bound

$$\int_\Omega \eta\left(\frac{\Phi_0}{\varepsilon}\right) dx \geqslant \frac{\delta}{T} \quad \forall \varepsilon > 0,$$

which implies that for all $\varepsilon > 0$ holds the inequality

$$\frac{\delta}{T} \leqslant \int_\Omega \eta\left(\frac{\Phi_0}{\varepsilon}\right) dx \leqslant \operatorname{mes}\{x \in \Omega : \Phi_0(x) < 2\varepsilon\}. \tag{26}$$

It is evident that the right-hand side of (26) tends to zero as $\varepsilon \to 0$, whereas the left hand side of (26) is strictly positive and independent of ε. Hence from the contradiction it follows that $\operatorname{mes}\sigma = 0$. The equalities $\operatorname{mes}\Omega_j(t) = \operatorname{mes}\Omega_j(0)$ for a.e $t \in (0,T)$, $j = 1,2$, are established similarly.

The main difficulty in Theorem 4 consists in taking limit in the nonlinear terms when $N_k \to \infty$

$$\int_{Q_T} \left(\mu(\vartheta^{N_k}, \Phi^{N_k}) S_j^{N_k} \cdot \nabla w_j^m\right) \varphi(t)\, dx\, dt \to \int_{Q_T} \left(\mu(\vartheta, \Phi) S_j \cdot \nabla w_j^m\right) \varphi(t)\, dx\, dt, \tag{27}$$

$j = 1,2$. The fact that the function $\mu(\vartheta, \Phi)$ is bounded with respect to all the variables and Lipschitz on ϑ, allows to establish the strong convergence of $\mu(\vartheta^{N_k}, \Phi^{N_k})$ to $\mu(\vartheta^N, \Phi^N)$ in $L_2(Q_T)$. From this strong convergence and the weak convergence of v^{N_k} in $W_{2,x,t}^{1,1/2}(Q_T; \mathbb{R}^2)$,

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according to the Lebesgue theorem follows (27) for every $m \geqslant 1$ and for all $\varphi(t) \in C^{\infty}[0, T]$. In the same way the limit for nonlinear terms for the equation of heat conduction is established.

4 Conclusion

We have discussed the problem of the dynamics of two-component heat-conducting viscous incompressible flow. For the initial-boundary value problem for the Navier–Stokes system with the consideration of the heat conduction and heat transfer in a two-component plane flow with the unknown interface, the existence of the global generalised solution has been proved, and the structure of the solution has been studied. The coefficients of viscosity and heat conductivity are taken temperature dependent. Fractional smoothness of the order $1/2$ in time variable t is established, namely, the velocity vector field and temperature are estimated in the norm of the nonisotropic Sobolev–Slobodetskii space $W^{1,s}_{2,x,t}$ with $s = 1/2$.

The results of the present work could be used for modelling the dynamics of multi-component heatconducting viscous incompressible flow with unknown interfaces. The method used in the present work can be applied to a wide range of problems involving the multi-component flow with interfaces. The method used here would be as well of interest for the nonlinear PDE theory. The results of the present work are also of practical interest, since the method proposed in the present work can as well underlie the effective algorithm for numerical solution of the problem under consideration.

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