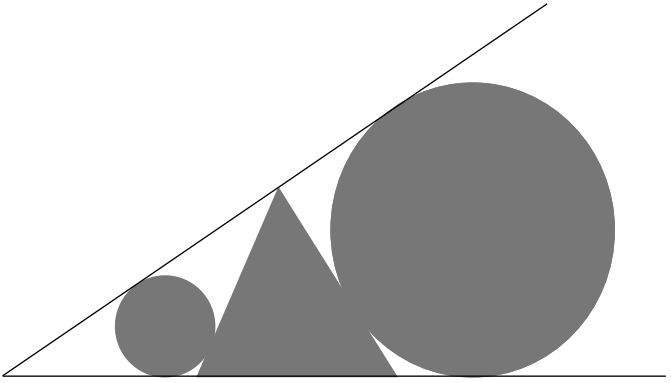
Student Problems

Students up to the age of 19 are invited to send solutions to either or both of the following problems to Stan Dolan, 126A Harpenden Road, St Albans, Herts. AL3 6BZ.

Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive solutions for either problem. It is, therefore, not necessary to submit solutions to both. Solutions should arrive by January 20th 2018. Please give your School year, the name and address of your School or College, and the name of a teacher through whom the award will be made. Please print your own name clearly! The names of all successful solvers will be published in the March 2018 edition.

Problem 2017.5 (Paul Stephenson)

The diagram shows three shaded shapes fitting snugly between two lines.



The perimeters of the small circle, triangle and large circle are 1, 2 and 3 units, respectively.

How far apart are the centres of the two circles?

Problem 2017.6 (Geoff Strickland)

If a and b are positive real numbers with $a + b = 1$, then prove that

$$\left(a + \frac{1}{b}\right)^2 + \left(b + \frac{1}{a}\right)^2 \geqslant \frac{25}{2}.$$

https://doi.org/10.1017/mag.2017.154 Published online by Cambridge University Press

Solutions to 2017.3 and 2017.4

Problem 2017.3 was solved by Sae Koyama (The King's College London Maths School).

Problem 2017.4 was solved by Andy Li (Dulwich College) and Andrew Ng (Tonbridge School).

Problem 2017.3 (Gordon Screaton)

(a) Find all prime numbers which (in binary form) are $10101\ldots0101_2$.

(b) Express each composite number $10101\ldots0101_2$ as a non-trivial product of two binary numbers.

Solution

(a) Sae noted that the only prime is $101_2 = 5$.

(b) Sae then split the proof into two cases. For $n > 1$, a number of the given form with $2n + 1$ digits factorises as shown below.

For n even

$$\frac{2^{2(n+1)} - 1}{2^2 - 1} = \left(2^n + 2^{n-1} + \ldots + 1\right)\left(2^n - 2^{n-1} + \ldots + 1\right).$$

This is the $n + 1$ bit number, $11\ldots11_2$, multiplied by the n bit number, $10101\ldots11_2$.

For n odd

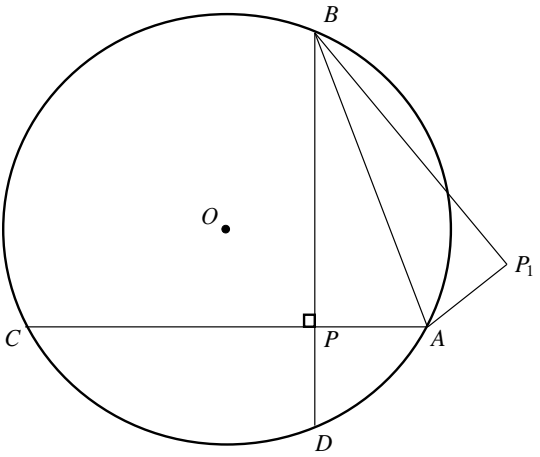
$$\frac{2^{2(n+1)} - 1}{2^2 - 1} = \left(2^{n+1} + 1\right)\left(2^{n-1} + 2^{n-3} + \ldots + 1\right).$$

This is the $n + 2$ bit number, $100\ldots01_2$, multiplied by the n bit number, $1010\ldots101_2$.

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Problem 2017.4 (Paul Stephenson)

A cyclic quadrilateral $ABCD$ has circumcentre O and perpendicular diagonals which intersect at P. The reflections of P in each of the sides of $ABCD$ are P_1, P_2, P_3 and P_4. The point P_1 is shown in the diagram.



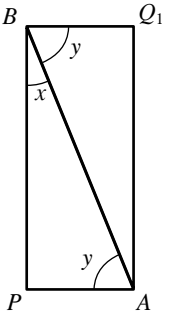
Prove that P_1, P_2, P_3 and P_4 lie on a circle centre O.

Solution

Andy's and Andrew's solutions were quite similar. What follows is closer in spirit to Andy's succinct solution. Let Q_1 be the reflection of P_1 in the perpendicular bisector of AB. Since O lies on the perpendicular bisector, $OP_1 = OQ_1$.

As shown in the diagram, $x + y = 90°$ and therefore $BPAQ_1$ is a rectangle.

Then $|OP_1|^2 = \dfrac{|AC|^2}{4} + \dfrac{|BD|^2}{4}$.



Identical results for P_2, P_3 and P_4 prove that P_1, P_2, P_3 and P_4 lie on a circle centre O.

The first prize of £25 is awarded to Sae Koyama. The second prize of £20 is awarded to Andy Li .

STAN DOLAN

https://doi.org/10.1017/mag.2017.154 Published online by Cambridge University Press