a finite subcomplex of X, hence in $\prod_{d<m<k} R_m$ for some k; call this Z. Since $fg \sim 1$, and $\Pi_r(Z) = 0$ for $r \geqslant k$, we see that $\Pi_r(S) = 0$ for $r \geqslant k$, in contradiction with a well-known result of J.-P. Serre.

Therefore X, Y are of the same n-type for all n, but not of the same homotopy type.

I wish to acknowledge my indebtedness to Prof. J. H. C. Whitehead, both on general grounds and for encouraging me to publish this answer to his query.

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ANTITHESIS OR REGRESSION?*

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In two recent papers (3, 4), Hammersley and his collaborators have introduced the idea of 'antithetic variates', relating and distinguishing their use from the 'control variate method' and the use of correlated samples. The applications which they have made indicate that their approach will prove very fruitful, but the relations of various methods deserve somewhat more attention.

At p. 470 of (3), Hammersley and Morton give what they consider 'a fairly general specification of the antithetic variate problem'. Here each antithetic variate depends on a corresponding α 'which may vary as the sampling procedure varies, but which are subject to given constraints...' and for which we have (rearranging their formula)

$$\text{ave}\left\{\frac{t_i}{m_i}\right\} = \theta,$$

where the m_i are known, and we seek to choose b_i's so that

$$t = \Sigma b_i t_i = \Sigma(b_i m_i)\left(\frac{t_i}{m_i}\right)$$

is an unbiased minimum-variance estimator of θ.

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Clearly the m_i, which are known, play no essential role, and may as well be taken as unity (redefining the t_i as may be necessary). Then, very much as Hammersley and Morton have hinted in their footnote on p. 449, we may write

$$t = \sum_1^n b_i t_i = t_1 + \sum_2^n b_i(t_i - t_1),$$

where we have used $\Sigma b_i = 1$—a direct consequence of unbiasedness when $m_i = 1$. In this form we can readily recognize t as a type of estimate long familiar to statisticians, and already introduced into Monte Carlo (1, 6, 7), namely, a regression estimate—regression on $t_2 - t_1, t_3 - t_1, \ldots, t_n - t_1$ being used to reduce the variance of the unbiased estimate t_1.

The desirable features claimed for the 'antithetic variate method' in comparison with the control variate method are just those well known to sampling statisticians (2, 5, 9) as arising in the comparison of a regression estimate with a ratio estimate.

The distinction stressed by Hammersley and Morton in their footnote (p. 449)—that to know that ave $\{t_1 - t_2\} = 0$ is qualitatively different from knowing a non-zero average value—is valid and important. But it falls into the place it should occupy when we regard the 'antithetic variate method' as an important special case of regression estimation.

In passing, it is interesting to note that a technique for the Buffon needle problem equivalent to, and possibly simpler than, Hammersley and Morton's crossed needles was suggested and examined by Mantel (8) in 1954.

Attention has been called to these aspects of the 'antithetic variate method' for the sole purpose of discouraging a separation of Monte Carlo theory from the classical theory of sampling. Both theories can learn from each other's progress—can, and should!

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