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ASYMPTOTIC TRUTH-VALUE LAWS IN MANY-VALUED LOGICS

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Abstract. This paper studies which truth-values are most likely to be taken on finite models by arbitrary sentences of a many-valued predicate logic. The classical zero-one law (independently proved by Fagin and Glebskiĭ et al.) states that every sentence in a purely relational language is almost surely false or almost surely true, meaning that the probability that the formula is true in a randomly chosen finite structures of cardinal n is asymptotically 0 or 1 as n grows to infinity. We obtain generalizations of this result for any logic with values in a finite lattice-ordered algebra, and for some infinitely valued logics, including Łukasiewicz logic. The finitely valued case is reduced to the classical one through a uniform translation and Oberschelp's generalization of the zero-one law. Moreover, it is shown that the complexity of determining the almost sure value of a given sentence is PSPACE-complete (generalizing Grandjean's result for the classical case), and for some logics we describe completely the set of truth-values that can be taken by sentences almost surely.

§1. Introduction. Given a property P of finite structures, a natural question has been often considered (beginning with [16] in the context of graph theory): what is the probability that a structure satisfies P when randomly selected among finite structures with the same domain for a suitable probability measure? More interestingly, what are the asymptotic probabilities? Or, more precisely, what do these probabilities converge to (if anything) as the size of the domain of the structures grows to infinite? Sometimes, when the properties under consideration are expressible by formulas of a given logic, we may have a 'zero-one law': the probabilities converge to either 0 or 1 (and so we can say that the formula is either *almost surely false* or *almost surely true*). After early preliminary developments for monadic predicate logic [11], the topic of logical zero-one laws was properly started independently in the papers by Glebskiĭ et al [19] and Fagin [17] where they establish a zero-one law for first-order classical logic on finite purely relational vocabularies. The argument in [19] proceeds by a rather involved quantifier elimination, whereas the one in [17] makes use of the so-called 'extension axioms' that form a complete theory axiomatizing a countable random structure. Lynch in [33] attributes the discovery of this theory to Stanisław Jaśkowski; both the theory and the random structure appeared first in print in [18], where Gaifman acknowledged that the latter was an example suggested to him by Michael Rabin and Dana Scott. An important related result was obtained by Etienne Grandjean in [23] when he proved that the set

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of almost surely true formulas is PSPACE-complete. The reader may consult [15, 25, 32] for didactic introductions to zero-one laws and [14] for a rather comprehensive survey.

Zero-one laws have been only sporadically investigated in the the setting of many-valued logics, although they offer a number of intriguing issues that are invisible in the two-valued context. Indeed, in a many-valued setting we have a set of truth values A (with more than two elements) and formulas can take any element of A as a truth-value in a given structure. Given a formula φ, we can say that φ takes a truth-value $a \in A$ *almost surely* if the fraction of finite structures of cardinal n in which φ takes value a tends to 1 as n grows to infinity. Then, a natural question is: do we have such general asymptotic truth-value laws for any many-valued logic? That is, in the presence of a multiplicity of truth-values, are some of the values taken by sentences almost surely? Exactly which of them? And, whenever we do have such a law, what is the complexity of determining the truth-value that an arbitrary formula almost surely takes?

Zero-one laws have indeed been obtained for some finitely valued fuzzy logics in [5, 30], but, to the best of our knowledge, zero-one laws for infinitely valued logics remain largely unexplored. In the related context of semiring semantics, analogous issues have been recently put forward and successfully addressed by Grädel et al in [22] for sentences given in negation normal form (involving only the additional connectives of disjunction and conjunction) and letting interpretations assign truth-values directly to the literals (both atomic and negated atomic formulas) of the logic.

In this article, we consider asymptotic truth-value laws in many-valued logics with a general approach that allows for arbitrary probability measures and arbitrary lattice-based algebras, obtaining as main results:

- a general zero-one law for finitely valued predicate logics, and
- a zero-law for infinitely valued Łukasiewicz logic and other $[0, 1]$-valued predicate logics.

Moreover, for finitely values logics, we provide examples in which all values are almost sure and examples in which only a few are, and we generalize Grandjean's theorem on the complexity of asymptotic truth-values.

The article is arranged as follows. In Section 2, we recall some necessary notions from classical finite model theory and its zero-one laws. In Section 3, we present our general approach to these issues in the case of *finitely* valued logics, in Section 3.1 we obtain the general zero-one law for finitely valued logics, in Section 3.2 we discuss random models (which are behind the proof of the previous result) and their axiomatizations, and in Section 3.3 we show that the complexity of the problem of determining the asymptotic truth-value of an arbitrary formula is in PSPACE and, actually, it is PSPACE-complete in the presence of crisp identity, hence generalizing Grandjean's result for classical predicate logic. In Section 4, we provide natural examples of logics in which *all* truth-values are actually almost sure, and a rather wide class of logics in which the almost sure values are few. In Section 5, we prove the zero-one law for *infinitely* valued Łukasiewicz predicate logic. In Section 6, we consider again the problem of describing sets of almost sure values, now for *infinitely* valued logics, and we prove a zero-one law for a large family of $[0, 1]$-valued logics

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satisfying De Morgan laws. In particular, we obtain that all rational numbers in $[0, 1]$ are almost sure values of infinitely valued Łukasiewicz predicate logic. Finally, Section 7 contains some concluding remarks and open questions.

For readers less familiar with the literature on many-valued logic, a modern comprehensive reference in the general field of mathematical fuzzy logic is [12], while [26] remains even today a rather useful source that contains detailed proofs of many essential results; we recommend [35] as a nice article highlighting the importance of infinitely valued Łukasiewicz logic among other non-classical logics.

§2. Zero-one laws in two-valued predicate logic. In this section, we recall some facts from classical finite model theory that we aim to generalize later to the context of many-valued logics.

Consider first a purely relational vocabulary τ for the usual finitary first-order logic that we will denote by $\mathcal{L}_{\omega\omega}$, as it is customary in abstract model theory [6] and even in some works on many-valued logics (see, e.g., [4]), where the ω subscripts represent the finitary character of, respectively, conjunctions/disjunctions and quantifier strings. Sometimes, we will also refer to the stronger logics $\mathcal{L}_{\omega_1\omega}$ and $\mathcal{L}_{\infty\omega}$ which allow for conjunctions and disjunctions of, respectively, countably many or arbitarily many formulas, while keeping the finitary character of quantifier strings.

A τ-sentence is said to be *parametric* in the sense of [36, p. 277] (or, alternatively, it is an *Oberschelp condition*) if it is equivalent to a finite conjunction of first-order formulas of the form:

$$\forall^{\neq} x_1, \dots, x_k \, \phi(x_1, \dots, x_k),$$

which abbreviates $\forall x_1 \dots \forall x_k \, (\bigwedge_{i<j} \neg(x_i = x_j) \to \phi(x_1, \dots, x_k))$, where $\phi(x_1, \dots, x_k)$ is a quantifier-free formula such that all of its atomic subformulas $Rx_{i_1} \dots x_{i_k}$ distinct from identities have

$$\{x_{i_1} \dots x_{i_k}\} = \{x_1, \dots, x_k\}.$$

In the case where $k = 1$, any formula $\forall x \, \phi(x)$ where ϕ is quantifier-free, is parametric. For example, $\forall x \, \neg Rxx \wedge \forall^{\neq} x, y \, (Rxy \to Ryx)$ is parametric, whereas $\forall^{\neq} x, y, z \, (Rxy \wedge Ryz \to Rxz)$ is not. It is not difficult to check that any universal formula of the form $\forall x_1 \dots \forall x_k \, \phi(x_1, \dots, x_k)$ where each atomic subformula of ϕ distinct from identities contains all the variables $x_1, \dots, x_k$ is parametric. A class of structures is called an *Oberschelp class* if it can be axiomatized by parametric sentences.

Oberschelp's extension [36, Theorem 3] of the classical zero-one law says: *on finite models and finite purely relational vocabularies, for any class $\mathbb{K}$ definable by a finite set of parametric sentences, any first-order sentence ϕ is almost surely satisfied by finite members of $\mathbb{K}$ or almost surely not satisfied*. More precisely, if $\mathbb{K}_n$ is the set of models in $\mathbb{K}$ with domain $[n] = \{1, \dots, n\}$, then

$$\mu_n(\phi) = \frac{|\{\mathfrak{M} \in \mathbb{K}_n : \mathfrak{M} \models \phi\}|}{|\mathbb{K}_n|} \text{ converges to } 0 \text{ or } 1.$$

Oberschelp's result actually holds for sentences of the variable-bounded infinitary logic $\mathcal{L}^{\omega}_{\infty\omega}$, i.e., the fragment of the infinitary logic $\mathcal{L}_{\infty\omega}$ determined by formulas

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containing finitely many variables only. This is a generalization of a well-known result by Kolaitis and Vardi for the class of all finite models [29]. An accessible presentation of these results can be found in [15].

If there are no Oberschelp conditions, the counting probability measure μ_n in the above zero-one law corresponds to assigning probability $\frac{1}{2}$ to the probabilistically independent atomic events $R(i_1, \dots, i_k)$, $i_1, \dots, i_k \in [n]$, in the random model $\mathfrak{M} \in \mathbb{K}_n$. The zero-one law holds, with the same asymptotic probability for each ϕ, if the random model is obtained by assigning a fixed probability $p_R \in (0, 1)$ to the atomic event $R(x_{i_1}, \dots, x_{i_k})$ for each $R \in \tau$.[1] There is a wealth of positive and negative results pertaining to the case in which p_R varies with n (cf. [37]) or the class $\mathbb{K}$ is given by non-parametric conditions (cf. [24]).

§3. The case of finitely valued predicate logics. Let $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \dots \rangle$ be an algebra (which can be finite or infinite) with a lattice reduct $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}} \rangle$ and possibly with additional operations. We will call $\boldsymbol{A}$ a *lattice algebra*. The underlying lattice may be bounded, in which case 0 and 1 will denote, respectively, the bottom and top element (which may, but need not, be denoted by constants of the signature of $\boldsymbol{A}$).

For the associated $\boldsymbol{A}$-valued logic $\mathcal{L}^{\boldsymbol{A}}$ and a first-order relational vocabulary τ, the first-order logic $\mathcal{L}^{\boldsymbol{A}}(\tau)$ is built in the usual way from variables $x_1, x_2, \dots$, relation symbols in τ, connectives $\wedge, \vee, \dots$ (corresponding to the operations of $\boldsymbol{A}$), and quantifiers $\forall$ and $\exists$.

Models are $\boldsymbol{A}$-valued τ-structures $\mathfrak{M} = \langle M, R^{\mathfrak{M}} \rangle_{R \in \tau}$ with domain $M \neq \emptyset$, where $R^{\mathfrak{M}} \colon M^{\tau(R)} \longrightarrow A$, and $\tau(R)$ denotes the arity of $R \in \tau$; $\mathrm{Mod}_\tau^{\boldsymbol{A}}$ will denote the class of all $\boldsymbol{A}$-valued τ-structures.

Define inductively for each formula $\varphi(x_1, \dots, x_k)$ in $\mathcal{L}^{\boldsymbol{A}}(\tau)$ with k free variables, a mapping $||\varphi||^{\mathfrak{M}} \colon M^k \longrightarrow A$, interpreting the atomic formulas according to $\mathfrak{M}$, the connectives according to $\boldsymbol{A}$, and the quantifiers as suprema and infima in $\boldsymbol{A}$ ($\mathfrak{M}$ is supposed to be *safe*, that is, the required infima and suprema exist, which in the context of finite structures is always guaranteed).

For a finite (or complete) algebra $\boldsymbol{A}$, we may extend the semantics to the infinitary logic $\mathcal{L}^{\boldsymbol{A}}_{\omega_1\omega}$ by setting $|| \bigwedge_n \varphi_n ||^{\mathfrak{M}} := \inf_n ||\varphi_n||^{\mathfrak{M}}$ (all formulas φ_n having free variables in a common finite set) and similarly for $\bigvee_n \varphi_n$.

The classical logics $\mathcal{L}_{\omega\omega}$ and $\mathcal{L}_{\omega_1\omega}$ may be, respectively, identified with $\mathcal{L}^{\boldsymbol{B}_2}$ and $\mathcal{L}^{\boldsymbol{B}_2}_{\omega_1\omega}$, where $\boldsymbol{B}_2$ is the two-element Boolean algebra.

3.1. The zero-one law for finitely valued predicate logics. Let $\boldsymbol{A}$ be a finite lattice algebra, τ a finite relational vocabulary, and $\mathrm{Mod}^{\boldsymbol{A}}_{\tau,n}$ be the class of $\boldsymbol{A}$-valued τ-structures with domain $[n] = \{1, \dots, n\}$.[2] Given $\varphi(x_1, \dots, x_k) \in \mathcal{L}^{\boldsymbol{A}}(\tau)$, $i_1, \dots, i_k \leq n$, and $a \in A$, define the probability that a randomly chosen model $\mathfrak{M} \in \mathrm{Mod}^{\boldsymbol{A}}_{\tau,n}$ assigns the value a to $\varphi(i_1, \dots, i_k)$ as:

[1] In the presence of Oberschelp conditions, one must take conditional probabilities; thus, the atomic events are not necessarily independent and their probabilities may differ (but remain independent of the size of the universe).

[2] This may be identified with the product $\prod_{R \in \tau} A^{[n]^{\tau(R)}}$.

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$$\mu_n(\varphi(i_1,\dots,i_k)=a) := \frac{|\{\mathfrak{M}\in \mathrm{Mod}^{\boldsymbol{A}}_{\tau,n} : ||\varphi||^{\mathfrak{M}}(i_1,\dots,i_k)=a\}|}{|\mathrm{Mod}^{\boldsymbol{A}}_{\tau,n}|}.$$

This definition assigns the same weight to all models in $\mathrm{Mod}^{\boldsymbol{A}}_{\tau,n}$, and the same probability

$$\mu_n(R(i_1,\dots,i_{\tau(R)})=a) := \frac{1}{|A|},$$

to the event that an atomic statement $R(i_1,\dots,i_{\tau(R)})$ takes the value $a\in A$, for any a and numbers $i_1,\dots,i_{\tau(R)}\in[n]$.

More generally, and reciprocally, we may fix, for each $R\in\tau$ a probability distribution $p_R\colon A\longrightarrow[0,1]$ (that is, $\sum_{a\in A} p_R(a)=1$) such that $p_R(a)>0$ for any $a\in A$, and define a probability distribution $\mathrm{Pr}_n\colon \mathrm{Mod}^{\boldsymbol{A}}_{\tau,n}\longrightarrow[0,1]$ in the class of $\boldsymbol{A}$-valued models with domain $[n]$:

$$\mathrm{Pr}_n(\mathfrak{M}) := \prod\{p_R(R^{\mathfrak{M}}(i_1,\dots,i_{\tau(R)})) : R\in\tau,\ i_1,\dots,i_{\tau(R)}\in[n]\},$$

obtained by identifying $\mathfrak{M}$ with its atomic $\boldsymbol{A}$-valued diagram. Then, we may define for any formula $\varphi(x_1,\dots,x_k)\in\mathcal{L}^{\boldsymbol{A}}(\tau)$, $a\in A$, and $i_1,\dots,i_k\in[n]$,

$$\mu_n(\varphi(i_1,\dots,i_k)=a) := \sum\{Pr_n(\mathfrak{M}) : \mathfrak{M}\in\mathrm{Mod}^{\boldsymbol{A}}_{\tau,n}, ||\varphi||^{\mathfrak{M}}(i_1,\dots,i_k)=a\}.$$

Notice that, for the sake of clarity, we are using the simplified notation $\varphi(i_1,\dots,i_k)=a$ to denote the event $||\varphi||^{\mathfrak{M}}(i_1,\dots,i_k)=a$ for an arbitrary model $\mathfrak{M}\in\mathrm{Mod}^{\boldsymbol{A}}_{\tau,n}$.

It is not hard to check that, under this definition,

$$\mu_n(R(i_1,\dots,i_{\tau(R)})=a)=p_R(a),$$

and the atomic events $R(i_1,\dots,i_k)=a$, $R'(j_1,\dots,j_k)=b$ are (probabilistically) independent if $R\neq R'$ or $\langle i_1,\dots,i_k\rangle\neq\langle j_1,\dots,j_k\rangle$.

The next theorem generalizes the classical zero-one law to the finitely-valued context.

THEOREM 1. *For any sentence $\varphi\in\mathcal{L}^{\boldsymbol{A}}(\tau)$ and $a\in A$, $\lim_n\mu_n(\varphi=a)=1$ or $\lim_n\mu_n(\varphi=a)=0$. Equivalently, for any sentence $\varphi\in\mathcal{L}^{\boldsymbol{A}}(\tau)$, there is a (unique) $a\in A$ such that $\lim_n\mu_n(\varphi=a)=1$.*

PROOF. The equivalence of both statements follows from the fact that, for each φ, the events $\varphi=a$, with $a\in A$, are exhaustive and mutually incompatible. The first formulation follows from the classical zero-one laws via the following multi-translation $\theta\longmapsto\langle\theta^a\rangle_{a\in A}$, defined by simultaneous induction, from $\mathcal{L}^{\boldsymbol{A}}(\tau)$ into classical $\mathcal{L}_{\omega\omega}(\tau\times A)$, $\tau\times A=\{R^a : R\in\tau, a\in A\}$:

$$R(x_1,\dots,x_{\tau(R)})^a := R^a(x_1,\dots,x_{\tau(R)}),\ \text{for } R\in\tau$$

$$\circ(\varphi_1,\dots,\varphi_k)^a := \bigvee_{\circ^{\boldsymbol{A}}(b_1,\dots,b_k)=a}\varphi_1^{b_1}\wedge\dots\wedge\varphi_k^{b_k},\ \text{for any connective } \circ \text{ of arity } k \text{ in the signature of } \boldsymbol{A}$$

$$(\forall x\varphi)^a := \Big(\bigvee_{b_1\wedge\dots\wedge b_m=a,\ m\leq|A|}\exists x\,\varphi^{b_1}\wedge\dots\wedge\exists x\,\varphi^{b_m}\Big)\wedge\forall x\,\Big(\bigvee_{b\geq a}\varphi^b\Big)$$

$$(\exists x\,\varphi)^a := \Big(\bigvee_{b_1\vee\dots\vee b_m=a,\ m\leq|A|}\exists x\,\varphi^{b_1}\wedge\dots\wedge\exists x\,\varphi^{b_m}\Big)\wedge\forall x\,\Big(\bigvee_{b\leq a}\varphi^b\Big),$$

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and the corresponding model transformations from $\mathrm{Mod}_{\tau}^{\boldsymbol{A}}$ into $\mathrm{Mod}_{\tau\times A}$:

$$\mathfrak{M} \longmapsto \mathfrak{M}_{\tau\times A} := \langle M; (R^a)^{\mathfrak{M}_{\tau\times A}}\rangle_{R^a\in\tau\times A},$$

where $(R^a)^{\mathfrak{M}_{\tau\times A}} := \{\bar{i} \in M : R^{\mathfrak{M}}(\bar{i}) = a\}$, which are easily seen to satisfy:

$$||\theta||^{\mathfrak{M}}(i_1, \ldots, i_k) = a \;\Leftrightarrow\; \mathfrak{M}_{\tau\times A} \models \theta^a[i_1, \ldots, i_k].$$

The model transformations establish a domain-preserving bijection between $\mathrm{Mod}_{\tau}^{\boldsymbol{A}}$ and the class $\mathbb{K} \subseteq \mathrm{Mod}_{\tau\times A}$ of classical models of the parametric $(\tau \times A)$-sentences

$$\forall x_1 \ldots x_{\tau(R)} \bigvee\nolimits_{a\in A} R^a(x_1, \ldots, x_{\tau(R)}),$$
$$\forall x_1 \ldots x_{\tau(R)} \bigwedge\nolimits_{a\neq b} \neg(R^a(x_1, \ldots, x_{\tau(R)}) \wedge R^b(x_1, \ldots, x_{\tau(R)})).$$

Being an Oberschelp class, $\mathbb{K}$ satisfies the classical zero-one law if we assign probability $\mu_n^{\mathbb{K}}(R^a(x_1, \ldots, x_k)) := p_R(a)$ to the classical atomic events $R^a(x_1, \ldots, x_k)$. The latter assignment yields $\mathrm{Pr}_n(\mathfrak{M}_{\tau\times A}) = \mathrm{Pr}_n(\mathfrak{M})$ for $\mathfrak{M} \in \mathrm{Mod}_{\tau,n}^{\boldsymbol{A}}$ and the transformation is bijective from $\mathrm{Mod}_{\tau,n}^{\boldsymbol{A}}$ onto $\mathbb{K}_n$ (models in $\mathbb{K}$ with domain $[n]$). Then, $\mu_n(\varphi = a) =$
$\mu_n\{\mathfrak{M} \in \mathrm{Mod}_{\tau,n}^{\boldsymbol{A}} : ||\varphi||^{\mathfrak{M}} = a\} = \mu_n^{\mathbb{K}}\{\mathfrak{M}_{\tau\times A} \in \mathbb{K}_n : \mathfrak{M}_{\tau\times A} \models \varphi^a\} = \mu_n^{\mathbb{K}}(\varphi^a)$. Thus, $\mu_n(\varphi = a)$ converges to 0 or 1. ⊣

REMARK 2. Theorem 1 holds if the models are restricted to any class of $\boldsymbol{A}$-valued structures satisfying conditions that are parametrically expressible in $\mathcal{L}_{\omega\omega}(\tau \times A)$ via the translation. Just add these conditions to the definition of the class $\mathbb{K}$ in the proof. For example, $\boldsymbol{A}$-valued irreflexive symmetric graphs:

$$\forall x\, R^0xx, \forall^{\neq} xy \bigwedge\nolimits_{a\in A}(R^a xy \leftrightarrow R^a yx)\ (\textit{irreflexivity is expressed by using degree 0 on the relation}),$$

or structures with crisp identity:

$$\forall x\, (x \approx^1 x), \forall^{\neq} xy\, (x \approx^0 y).$$

REMARK 3. Theorem 1 holds also for sentences in finite vocabularies of the variable-bounded sublogic $(\mathcal{L}^{\boldsymbol{A}})^{\omega}_{\omega_1\omega}$ of $\mathcal{L}^{\boldsymbol{A}}_{\omega_1\omega}$, where each formula has finitely many free and bound variables, because of the obvious extension of the translation to the infinitary logic; for example,

$$(\bigwedge\nolimits_n \varphi_n)^a := \Big[\bigvee_{b_1\wedge\ldots\wedge b_k = a, k\leq|A|, n_1,\ldots,n_k} (\varphi_{n_1}^{b_1} \wedge \ldots \wedge \varphi_{n_k}^{b_m})\Big] \wedge \bigwedge\nolimits_n \bigvee_{b\geq a} \varphi_n^b,$$

sends it to classical $\mathcal{L}^{\omega}_{\omega_1\omega}$, known to satisfy the zero-one law under Oberschelp's conditions [15, 29]).

REMARK 4. Zero-one laws have been studied for classical modal logics (e.g., [21, 27, 31, 39]). In [27], the authors prove zero-one laws for model validity for various systems of modal logic and state a similar result for frame validity. Unfortunately, the latter is refuted in [31] using ideas from [21]. In the finitely valued case, any fully modal sentence of the $\boldsymbol{A}$-valued version $\mathrm{S5}_c^{\boldsymbol{A}}$ of propositional bimodal logic S5 over finite universal crisp frames (cf. [9, 10]) satisfies a zero-one law for model validity,

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because this logic may be interpreted as the one-variable fragment of $\mathcal{L}^{\boldsymbol{A}}$ on unary predicates via the translation $\varphi \mapsto \widehat{\varphi}$:

$$\begin{aligned}
&\widehat{p_i} := P_i(w)\\
&\widehat{\Box\varphi} := \forall w \widehat{\varphi}(w)\\
&\widehat{\Diamond\varphi} := \exists w \widehat{\varphi}(w)\\
&\widehat{t(\varphi_1, \dots, \varphi_k)} := t(\widehat{\varphi}_1(w), \dots, \widehat{\varphi}_k(w)),
\end{aligned}$$

which sends any fully modal formula to a sentence.

3.2. Finitely valued random models and extension axioms. In the above proof, the finitely valued zero-one law lurks a countable $\boldsymbol{A}$-valued random model, corresponding to the classical random model that one obtains for the language of the translation.

THEOREM 5. *There is a countable $\boldsymbol{A}$-valued τ-model $\mathfrak{R}_\tau^{\boldsymbol{A}}$ such that, for any sentence $\varphi \in \mathcal{L}^{\boldsymbol{A}}(\tau)$, we have $\lim_n \mu_n(\varphi = a) = 1$ iff $||\varphi||^{\mathfrak{R}_\tau^{\boldsymbol{A}}} = a$.*

PROOF. Take the $\boldsymbol{A}$-model $\mathfrak{R}_\tau^{\boldsymbol{A}}$ corresponding to the countable ω-categorical random model $\mathfrak{R}_{\tau\times A}$ of the class $\mathbb{K}$ introduced in the proof of Theorem 1. Then, $\lim_n \mu_n(\varphi = a) = 1$ iff $\lim_n \mu_n^{\mathbb{K}}(\varphi^a) = 1$ iff $\mathfrak{R}_{\tau\times A} \models \varphi^a$ iff $||\varphi||^{\mathfrak{R}_\tau^{\boldsymbol{A}}} = a$. ⊣

The classical extension axioms characterizing $\mathfrak{R}_{\tau\times A}$, and indirectly $\mathfrak{R}_\tau^{\boldsymbol{A}}$, are rarely expressible in $\mathcal{L}^{\boldsymbol{A}}$, but they are expressible in $(\mathcal{L}_{\omega\omega})_{\tau\times A}$ as:

$$Ext_A(k, f) : \forall^{\neq} x_1, \dots, x_k \, \exists x_{k+1} \neq x_1, \dots, x_k \bigwedge\nolimits_{\varphi\in F_{k+1}} \varphi^{f(\varphi)}(x_1, \dots, x_{k+1}),$$

where F_{k+1} is the set of atomic τ-formulas with variables in $x_1, \dots, x_k, x_{k+1}$, truly containing the variable x_{k+1}, and $f : F_{k+1} \longrightarrow A$. These axioms are automatically consistent with the set of parametric sentences defining the class $\mathbb{K}$ in the proof of Theorem 1. If there are additional parametric conditions, only the extension axioms consistent with them are considered.

EXAMPLE 6. According to Remark 2, the random $\boldsymbol{A}$-valued irreflexive symmetric graph may be described as the unique countable $\boldsymbol{A}$-weighted irreflexive symmetric graph $\mathfrak{R} = \langle G, R\rangle$ such that, for any distinct $x_1, \dots, x_k \in G$ and $a_1, \dots, a_k \in A$, there is an $x_{k+1} \neq x_1, \dots, x_k$ in G such that $R(x_i, x_{k+1}) = a_i$. This structure was studied as a Fraïssé limit in [3].

Since the extension axioms and thus the random model are independent of the probabilities $\{p_R\}_{R\in\tau}$, we can use Theorem 5 to obtain the following corollary.

COROLLARY 7. *For any sentence $\varphi \in \mathcal{L}^{\boldsymbol{A}}$, the value $a \in A$ such that $\lim_n \mu_n(\varphi = a) = 1$ depends only on φ and not on the probability distributions p_R (provided that $p_R(a) > 0$ for all $a \in A$).*

We call the value a in Corollary 7 the *almost sure* or *asymptotic* truth-value of φ. This corollary entails that assigning the same probability $\frac{1}{|A|}$ to all truth-values has the same effect in the limit as assigning a positive probability to each one.

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REMARK 8. If p_R is allowed to take value 0 at some $a \in A$, this corresponds to asking the Oberschelp conditions $\forall x_1 \ldots \forall x_n \neg R^a(x_1, \ldots, x_n)$. Thus, the asympotic value of φ still exists and depends on the support of p_R only.

3.3. Complexity of almost sure values. Grandjean [23] showed that the problem of determining for the classical zero-one law if a sentence is almost surely true or not is in PSPACE, and it has been observed [14] that this holds in the presence of Oberschelp conditions. Moreover, it is PSPACE-complete, even for the vocabulary of pure identity.

Since our translation is exponential in the length of the original formula, it yields only an exponential space upper bound for the computation of the almost sure value of an $\boldsymbol{A}$-valued sentence. However, the shape of the translation will allow us to give a simple direct proof of an upper polynomial space bound. PSPACE-completeness will be achieved for logics with crisp identity.

THEOREM 9. *For any sentence $\varphi \in \mathcal{L}^{\boldsymbol{A}}(\tau)$, the almost sure value of φ may be computed in polynomial space.*

We denote by Ext_A the set of all extension axioms. It is enough to show that for any $a \in A$, it may be determined in polynomial space whether $Ext_A \vdash \varphi^a$. Then, one may check this for each $a \in A$ utilizing the same space until finding the (unique) a for which it holds. Using Grandjean's strategy [23], we can do this without writing φ^a explicitly.

Given a sequence of variables $\mathbf{x} = x_1, \ldots, x_k$, a *complete description* of $\mathbf{x}$ is a $(\tau \times A)$-formula $\Delta(\mathbf{x})$ of the form $\bigwedge_{\psi \in At_k} \psi^{f(\psi)}$ where At_k is the set of atomic τ-formulas with variables in $\mathbf{x}$ and $f : At_k \to A$. It is easy to see that any such $\Delta(\mathbf{x})$ is realizable in the classical random model $\mathfrak{R}_{\tau \times A}$, because existential sentences are always almost surely true.

Notice that, due to the strong homogeneity of $\mathfrak{R}_{\tau \times A}$ (any isomorphism between finite substructures may be extended to an automorphism, see [28]), all tuples of $\mathfrak{R}_{\tau \times A}$ satisfying the same complete description $\Delta(\mathbf{x})$ have the same type in $\mathfrak{R}_{\tau \times A}$; therefore, the theory $Ext_A + \Delta(\mathbf{x})$ is complete for all formulas with their free variables in $\mathbf{x}$.

In what follows, $|\alpha|$ will denote the length of a formula or a sequence of variables, or the cardinality of a finite set. For simplicity we assume that the symbols in τ and $\tau \times A$ as well as the variables have length 1. Particularly, $|\theta^a| = |\theta|$ for any atomic $\theta \in \mathcal{L}^{\boldsymbol{A}}_\tau$ and $a \in A$.

Set $M_\tau = \max\{\tau(R) : R \in \tau\}$. Then, it is easy to check that $|\Delta(\mathbf{x})| = \sum_{R \in \tau}(\tau(R) + c')|\mathbf{x}|^{\tau(R)} \leq |\tau|(M_\tau + c')|\mathbf{x}|^{M_\tau} = c|\mathbf{x}|^{M_\tau}$ for a suitable constant c.

LEMMA 10. *Given a complete description $\Delta(\mathbf{x})$, any formula $\theta \in \mathcal{L}^{\boldsymbol{A}}_\tau$ with its free variables contained in $\mathbf{x}$ and $a \in A$, it may be decided whether $Ext_A + \Delta(\mathbf{x}) \vdash \theta^a$ or not in space $c(|\mathbf{x}| + |\theta|)^{M_\tau}$.*

PROOF. Induction on the complexity of θ.

(1) If θ is atomic, then $Ext_A + \Delta(\mathbf{x}) \vdash \theta^a$ iff θ^a is a conjunct of $\Delta(\mathbf{x})$, and this is clearly verifiable in space $|\Delta(\mathbf{x})| + |\theta^a| = |\Delta(\mathbf{x})| + |\theta| \leq c|\mathbf{x}|^{M_\tau} + |\theta| \leq c(|\mathbf{x}| + |\theta|)^{M_\tau}$.

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(2) If θ is $\circ(\varphi_1, \dots, \varphi_k)$ where $\circ$ is a connective, then by definition of the translation and completeness of $Ext_A + \Delta(\mathbf{x})$: $Ext_A + \Delta(\mathbf{x}) \vdash \theta^a$ iff there are $b_1, \dots, b_k \in A$ such that $\circ^{\mathcal{A}}(b_1, \dots, b_k) = a$ and

$$Ext_A + \Delta(\mathbf{x}) \vdash \varphi_i^{b_i}, \text{ for } i = 1, \dots, k.$$

But the k statements in the display may be verified each one in space $c(|\mathbf{x}| + |\theta|)^{M_\tau} \geq c(|\mathbf{x}| + |\varphi_i|)^{M_\tau}$ by induction hypothesis. And we may do this sequentially in the same space for each b_i of each relevant tuple $b_1, \dots, b_k$, until finding a tuple for which the displayed statement holds.

(3) If θ is $\forall x\, \varphi(x)$, then by definition of the translation and completeness: $Ext_A + \Delta(\mathbf{x}) \vdash (\forall x\, \theta(x))^a$ iff there are $b_1, \dots, b_m \in A$ such that $b_1 \wedge \dots \wedge b_m = a$, and

$$Ext_A + \Delta(\mathbf{x}) \vdash \exists x\, \varphi(x)^{b_1}, \dots, \exists x\, \varphi(x)^{b_m}, \forall x\, (\bigvee_{b \geq a} \varphi(x)^b),$$

equivalently,

$$\begin{aligned} Ext_A + \Delta(\mathbf{x}) \quad &\vdash \quad \exists x\, \varphi(x)^{b_i}, \text{ for } i = 1, \dots, m \\ Ext_A + \Delta(\mathbf{x}) \quad &\nvdash \quad \exists x\, \varphi(x)^{b}, \text{ for all } b \not\geq a \end{aligned}$$

because $\forall x\, (\bigvee_{b \geq a} \varphi(x)^b)$ is equivalent in Ext_A to $\bigwedge_{b \not\geq a} \neg \exists x\, \varphi(x)^b$, and $Ext_A + \Delta(\mathbf{x}) \vdash \neg \exists x\, \varphi(x)^b$ iff $Ext_A + \Delta(\mathbf{x}) \nvdash \exists x\, \varphi(x)^b$ due to completeness. Now, $Ext_A + \Delta(\mathbf{x}) \vdash \exists x\, \varphi(x)^{b_1}$ if and only if there is an expansion $\Delta'(x, \mathbf{x})$ of $\Delta(\mathbf{x})$ such that

$$Ext_A + \Delta'(x, \mathbf{x}) \vdash \varphi(x)^{b_1},$$

(if $\mathbf{a}$ is a tuple in $\mathfrak{R}_{\tau \times A}$ satisfying $\Delta(\mathbf{x})$ and b is a witness of $\exists x\, \varphi(x, \mathbf{a})^{b_1}$, then the complete description $\Delta'(x, \mathbf{x})$ of $\mathfrak{R}_{\tau \times A} \restriction \{b, \mathbf{a}\}$ does the job). By induction hypothesis, we may check this for each possible expansion of $\Delta(\mathbf{x})$ in space $c(|\mathbf{x}| + 1 + |\varphi(x)|)^{M_\tau} \leq c(|\mathbf{x}| + |\theta|)^{M_\tau}$. Doing this sequentially in the same space for each $b_1, \dots, b_m, b$ such that $b_1 \wedge \dots \wedge b_m = a$ and $b \not\geq a$, we may check whether $Ext_A + \Delta(\mathbf{x}) \vdash (\forall x\, \theta(x))^a$.

(4) If θ is $\exists x\, \varphi(x)$, then, similarly to the previous case, $Ext_A + \Delta(\mathbf{x}) \vdash (\exists x\, \varphi(x))^a$ iff there are $b_1, \dots, b_m \in A$ such that $b_1 \vee \dots \vee b_m = a$, and

$$\begin{aligned} Ext_A + \Delta(\mathbf{x}) \quad &\vdash \quad \exists x\, \varphi(x)^{b_i}, \text{ for } i = 1, \dots, m \\ Ext_A + \Delta(\mathbf{x}) \quad &\nvdash \quad \exists x\, \varphi(x)^{b}, \text{ for all } b \not\leq a \end{aligned}$$

and this claim may be verified in space $c(|\mathbf{x}| + |\theta|)^{M_\tau}$.

⊣

COROLLARY 11. *The almost sure value of any sentence $\varphi \in \mathcal{L}^{\mathcal{A}}(\tau)$ may be computed in space $c(1 + |\varphi|)^{M_\tau}$.*

PROOF. Let $\Delta(x)$ be any complete description in one variable. Then, $Ext_A \vdash \varphi^a$ iff $Ext_A + \exists x \Delta(x) \vdash \varphi^a$ (because $Ext_A \vdash \exists x\, \Delta(x)$) iff $Ext_A + \Delta(x) \vdash \varphi^a$. Thus, $Ext_A \vdash \varphi^a$ may be decided in space $c(1 + |\varphi|)^{M_\tau}$. Check this for any $a \in A$ until finding the only a for which it holds. ⊣

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If the logic satisfies additional Oberschelp conditions in the translation (besides the necessary ones), the previous result is obtained by considering only complete descriptions compatible with those conditions.

$\boldsymbol{A}$-valued logic with crisp identity, in symbols $\mathcal{L}^{\boldsymbol{A}}_{\approx}$, will be $\mathcal{L}^{\boldsymbol{A}}$ enriched with a distinguished binary relation symbol $\approx$ interpreted in each model as the characteristic function of the diagonal.

This corresponds to having the Oberschelp conditions $\forall x\,(x \approx^1 x)$, $\forall^{\neq} xy\,(x \approx^0 y)$ in the translated logic. Hence, this logic has a zero-one law and satisfies the above results.

THEOREM 12. *If $\boldsymbol{A}$ has a term u such that $u^{\boldsymbol{A}}(1) = 0$ and $u^{\boldsymbol{A}}(0) = 1$, then computing the almost sure values of sentences of $\mathcal{L}^{\boldsymbol{A}}_{\approx}$ is PSPACE-complete.*

PROOF. Consider the translation $\varphi \longmapsto \widehat{\varphi}$ from classical pure identity logic into $\mathcal{L}^{\boldsymbol{A}}_{\approx}$ that sends $=$ to $\approx$ and $\neg$ to u. A straightforward induction on the complexity of φ shows that, for any $\mathbf{m}$ in $\mathfrak{R}^{\boldsymbol{A}}_{\tau}$, $||\widehat{\varphi}||^{\mathfrak{R}^{\boldsymbol{A}}_{\tau}}(\mathbf{m}) \in \{0,1\}$ and $||\widehat{\varphi}||^{\mathfrak{R}^{\boldsymbol{A}}_{\tau}}(\mathbf{m}) = 1$ iff $\mathfrak{R}_{\tau\times A} \models \varphi[\mathbf{m}]$. Notice that for the classical random model of the extension axioms of this logic: $\mathfrak{R}_{\tau\times A} \models m \approx^1 n$ iff $m = n$, $\mathfrak{R}_{\tau\times A} \models m \approx^0 n$ iff $m \neq n$. Therefore, for any classical identity sentence φ, $\lim_n \mu_n(\widehat{\varphi} = 1) = 1$ iff $||\widehat{\varphi}||^{\mathfrak{R}^{\boldsymbol{A}}_{\tau}} = 1$ iff $Ext_A \vdash \varphi$ iff $I_\infty \vdash \varphi$, where I_∞ is the theory of identity in infinite sets. But deducibility in the latter theory is PSPACE-hard by results of Stockmeyer [38]. ⊣

§4. The set of almost sure values in finitely valued predicate logics. Call $a \in A$ an *almost sure* truth-value of $\mathcal{L}^{\boldsymbol{A}}$ (of $\boldsymbol{A}$) if there is a sentence $\varphi \in \mathcal{L}^{\boldsymbol{A}}$ such that $\lim_n \mu_n(\varphi = a) = 1$. Do all the elements of A arise as almost sure values? If not, which are those values? We will see that for many familiar classes of finite algebras the answer to the first question is positive (for example MV-chains for Łukasiewicz logic, or G-chains for Gödel logic), for others have a small subset of almost sure values (0 or 1 for Boolean algebras, at most four values for De Morgan algebras, at most three for the lattice-negation reducts of G-chains, etc.).

The first assertion follows from the following general lemma.

LEMMA 13. *Let $t(v_1,\dots,v_k)$ be a term in the signature of a finite lattice algebra $\boldsymbol{A}$, then $m = \inf_{a_1,\dots,a_k\in A} t^{\boldsymbol{A}}(a_1,\dots,a_k)$ and $s = \sup_{a_1,\dots,a_k\in A} t^{\boldsymbol{A}}(a_1,\dots,a_k)$ are almost sure truth-values of $\mathcal{L}^{\boldsymbol{A}}$.*

PROOF. Let $R(x)$ be an atomic formula with a single variable x. We show that

$$\mu_n(\forall x_1 \dots \forall x_k\, t(R(x_1),\dots,R(x_k)) = m) \text{ converges to 1.}$$

Fix tuples $\langle a^1_j\rangle_j,\dots,\langle a^\ell_j\rangle_j \in |A|^k$, such that $t^{\boldsymbol{A}}(a^1_1,\dots,a^1_k) \wedge^{\boldsymbol{A}} \dots \wedge^{\boldsymbol{A}} t^{\boldsymbol{A}}(a^\ell_1,\dots,a^\ell_k) = m$, and for each $n \geq k\ell$ find $K \subseteq [n]^k$ of cardinal $|K| = \lfloor n/k\ell \rfloor$ such that all tuples in K have all their components distinct, and two different tuples in K are distinct at each coordinate. Then,

$$\begin{aligned}
&\mu_n(\forall x_1 \dots \forall x_k\, t(R(x_1),\dots,R(x_k)) = m)\\
&= \Pr_n\Big(\inf_{i_1,\dots,i_k\in[n]} t^{\boldsymbol{A}}(R(i_1),\dots,R(i_k)) = m\Big)\\
&\geq \Pr_n\,(\text{for some } \langle i^1_j\rangle_j,\dots,\langle i^\ell_j\rangle_j \text{ in } [n]^k : R(i^i_j) = a^i_j, \text{ for } i = 1,\dots,\ell, j = 1,\dots,k)
\end{aligned}$$

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$$\geq \Pr_n(\text{for some } \langle i_j^1\rangle_j, \dots, \langle i_j^\ell\rangle_j \text{ in } K : R(i_j^i) = a_j^i, \text{ for } i = 1, \dots, \ell, j = 1, \dots, k)$$
$$= 1 - \Pr_n(\text{for all } \langle i_j^1\rangle_j, \dots, \langle i_j^\ell\rangle_j \text{ in } K \text{ is false: } R(i_j^i) = a_j^i, \text{ for } i = 1, \dots, \ell, j = 1, \dots, k)$$
$$= 1 - \prod_{\langle i_1^1,\dots,i_k^1\rangle,\dots,\langle i_1^\ell,\dots,i_k^\ell\rangle \in K} [1 - \Pr_n(R(i_j^i) = a_j^i \text{ for } i = 1, \dots, \ell, j = 1, \dots, k))]$$
$$= 1 - (1 - p_R(a_1^1) \dots p_R(a_k^\ell))^{\lfloor n/k\ell \rfloor}.$$

For the last two identities, we use that the $k\ell$ events $R(i_j^i) = a_j^i$ form a mutually independent set for distinct $i_1^1, \dots, i_k^1, \dots, i_1^\ell, \dots, i_k^\ell \in K$. Since $p_R(a_i^j) > 0$, the last quantity converges to 1. A similar proof shows that

$$\mu_n(\exists x_1 \dots \exists x_k\, (t(R(x_1), \dots, R(x_k)) = s) \text{ converges to 1.}$$

⊣

EXAMPLE 14. It follows that 0 and 1 are always almost sure values, even if they do not appear in the signature of A, because they are inf and sup of the term $t(v) := v$.

EXAMPLE 15. Let $\mathrm{L}_{N+1} = \langle \{0, \frac{1}{N}, \dots, \frac{N-1}{N}, 1\}, \wedge, \vee, \neg, \to \rangle$ be the MV-chain of $N+1$ elements, where $\wedge, \vee$, are min and max, and $\neg, \to$ are Łukasiewicz negation and implication. Then all values of L_{N+1} are almost sure values of the corresponding predicate logic. To see this, notice that using (Chang) connectives:

$a \oplus b := \neg a \to b$, $a \odot b := \neg(a \to \neg b)$, $na := a \oplus \dots \oplus a$, $a^n := a \odot \dots \odot a$ $(n \geq 1)$, the term $t(v) = v^N \oplus \neg v$ has minimum value $\frac{1}{N}$ in L_{N+1} (taken at $a = \frac{N-1}{N}$), and thus $t_{k/N}(v) = k(v^N \oplus \neg v) = v^k \to kv^N$ has minimum value $\frac{k}{N}$.

EXAMPLE 16. Let $\boldsymbol{G}_{N+1} = \langle \{0 = g_0 < \dots < g_N = 1\}, \wedge, \vee, \neg, \to \rangle$ be the G-chain of $N+1$ elements, where $\neg, \to$ are Gödel implication and negation, then all values of $\boldsymbol{G}_{N+1}$ are almost sure values of the corresponding predicate logic. Define:

$$t_1(v_1) := v_1 \vee \neg v_1, t_{k+1}(v_1, \dots, v_{k+1}) := v_{k+1} \vee (v_{k+1} \to t_k(v_1, \dots, v_k))$$
$$= v_{k+1} \vee (v_{k+1} \to (v_k \vee (v_k \to \dots \to (v_2 \vee (v_2 \to (v_1 \vee \neg v_1)) \dots)))$$

then the minimum possible value of $t_1(v_1)$ in $\boldsymbol{G}_{N+1}$ is $g_1 \vee \neg g_1 = g_1$, since $g_0 \vee \neg g_0 = 1$ and $g_j \vee \neg g_j = g_j$ for $g_j > g_1$. For $2 \leq k \leq N$:

$$\min_{v_1,\dots,v_k \in G_{N+1}} t_k(v_1, \dots, v_k) = g_k,$$

obtained at $g_1, \dots, g_k$, because

$$t_k(x_1, \dots, x_k) = \begin{cases} 1 & \text{if } x_{i+1} \leq x_i \text{ for some } i \\ x_k & \text{if } x_i < x_{i+1} \text{ for all } i. \end{cases}$$

REMARK 17. Notice that in the proof of Lemma 13, we have used a single predicate symbol and k variables to write the sentences

$$\forall x\, t(P_1(x), .., P_k(x))$$
$$\exists x\, t(P_1(x), .., P_k(x))$$

having the desired almost sure values m and s. We may use instead k (unary) predicates and a single variable with a similar proof. Therefore, via the translation

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explained in Remark 4, the formulas $\Box t(p_1, \dots, p_k)$ and $\Diamond t(p_1, \dots, p_k)$ of the $\mathrm{S5}_c^A$ have almost sure values m and s, respectively. It follows then from Example 15 that all values of L_{N+1} are almost sure values of $\mathrm{S5}_c^{\mathrm{L}_{N+1}}$, since the formula $\Box(p^k \to kp^N)$ has almost sure value $\frac{k}{N}$, and it follows, from Example 16, that all values of G_{N+1} are almost sure values of $\mathrm{S5}_c^{G_{N+1}}$, since the modal formula $\Box(p_k \vee (p_k \to (p_{k-1} \vee (p_{k-1} \to (\dots (p_1 \vee \neg p_1) \dots))$ takes the almost sure value g_k.

We turn now to algebras with few almost sure values and obtain the following result inspired by the techniques in [22]. Notice that the proof of this theorem actually gives a new proof of the zero-one law in the cases considered.

THEOREM 18. *Let $\boldsymbol{A} = \langle A, \wedge, \vee, \neg \rangle$ be a finite distributive lattice with a negation satisfying*:

(1) $\neg(v \wedge w) = \neg v \vee \neg w$, $\neg(v \vee w) = \neg v \wedge \neg w$
(2) $v \leq \neg\neg v$
(3) $\neg 1 = 0$.

Then, the almost sure values of $\mathcal{L}^{\boldsymbol{A}}$ are $0, \varepsilon, \varepsilon', \delta, \delta'$, and 1, where

$$\varepsilon = \sup_{x \in A}(x \wedge \neg x), \varepsilon' = \sup_{x \in A}(\neg x \wedge \neg\neg x),$$
$$\delta = \inf_{x \in A}(x \vee \neg x), \delta' = \inf_{x \in A}(\neg x \vee \neg\neg x).$$

PROOF. We already know these values are almost sure by Lemma 13. In order to see that they are the only ones, notice first the following derived laws of $\boldsymbol{A}$:

(4) $v \leq w$ implies $\neg w \leq \neg v$
(5) $\neg\neg\neg v = \neg v$
(6) $\neg 0 = \neg\neg 1 = 1$, $\neg\neg 0 = 0$.
(7) $0 \leq \varepsilon \leq \varepsilon' \leq \delta \leq \delta' \leq 1$, $\neg\varepsilon = \neg\varepsilon' = \delta'$, $\neg\delta = \neg\delta' = \varepsilon'$, thus $E = \{0, \varepsilon, \varepsilon', \delta, \delta', 1\}$ is a subalgebra of $\boldsymbol{A}$.

Expand $\boldsymbol{A}$ to the algebra $\boldsymbol{A}^+ = \langle A, \wedge, \vee, \neg, 0, \varepsilon, \varepsilon', \delta, \delta', 1 \rangle$ and use the same symbols in the signature to denote the constants in E. By distributivity, (1), (5), (6), and (7), any term $t(v_1, \dots, v_k)$ in the signature of $\boldsymbol{A}^+$ may be put in disjunctive normal form: $\bigvee_s C_s$ with $C_s = e \wedge \bigwedge_r \neg^{n_r} v_{i_r}$, where $e \in E$, $v_{i_r} \in \{v_1, \dots, v_k\}$ and $n_r \leq 2$. Moreover, by (2), we may assume no pair $\{v_i, \neg\neg v_i\}$ appears in C_s, since $v_i \wedge \neg\neg v_i = v_i$. By idempotency and commutativity of $\wedge$, we may assume that each variable appears at most once in C_s as: v_i, $\neg v_i$, or $\neg\neg v_i$, or at most twice as: $v_i \wedge \neg v_i$, or $\neg v_i \wedge \neg\neg v_i$. Similarly, $t(v_1, \dots, v_k)$ may be put in conjunctive normal form $\bigwedge_s D_s$ with $D_s = e \vee \bigvee_r \neg^{n_r} v_{i_r}$ and no occurrence of both $v_i, \neg\neg v_i$. We prove now a result on elimination of quantifiers. Let $Ext(\ell)$ denote the conjunction of the extension axioms $Ext_A(k, f)$ for $k \leq \ell$, and define for formulas $\varphi, \psi \in \mathcal{L}_\tau^{\boldsymbol{A}^+}$: $\varphi \equiv_{Ext(\ell)} \psi$ iff $||\varphi(m_1, \dots, m_k)||^{\mathfrak{M}} = ||\psi(m_1, \dots, m_k)||^{\mathfrak{M}}$ for any model $\mathfrak{M}$ of $\mathcal{L}_\tau^{\boldsymbol{A}^+}$ satisfying $Ext(\ell)$, and any $m_1, \dots, m_k \in M$.

CLAIM. *Any formula $\varphi \in (\mathcal{L}_\tau^{\boldsymbol{A}^+})^{(\ell)}$ is $Ext(\ell)$-equivalent to a quantifier-free formula in the same free variables as φ.*

We proceed by induction on the complexity of φ. Assume by induction hypothesis that $\varphi(x_1, \dots, x_k, x_{k+1}) \equiv_{Ext(\ell)} \psi(x_1, \dots, x_k, x_{k+1})$ quantifier-free, and $k + 1 \leq \ell$, we must show the same for $\exists x_{k+1} \varphi(x_1, \dots, x_k, x_{k+1})$ and $\forall x_{k+1} \varphi(x_1, \dots, x_k, x_{k+1})$.

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By the previous algebraic remarks, we may assume ψ is in disjunctive normal form, i.e., $\bigvee_s C_s(x_1, \dots, x_k, x_{k+1})$ where each C_s is a conjunction:

$$e \wedge C'_s(x_1, \dots, x_k) \wedge \bigwedge_r \neg^{n_r} R_r(x_1, \dots, x_k, x_{k+1}),\ n_r \in \{0, 1, 2\},$$

$C'_s(x_1, \dots, x_k)$ being the part not containing the variable x_{k+1}, such that each atomic $R_r(x_1, \dots, x_k, x_{k+1})$ appearing in C_s does it in one and only one of the following forms:

$$R_r \text{ alone}, \neg R_r \text{ alone}, \neg\neg R_r \text{ alone}, R_r \wedge \neg R_r, \neg R_r \wedge \neg\neg R_r.$$

Substitute the first three types of formulas with (the symbol) 1, the fourth with ε, and the fifth with ε'; and let c be the minimum of e and the substituted constants to obtain:

$$C^+_s(x_1, \dots, x_k) := c \wedge C'_s(x_1, \dots, x_k).$$

Since we have substituted the highest possible values in each case, we have for any model $\mathfrak{M}$ and $x_1, \dots, x_k, i \in M$:

$$||C_s(x_1, \dots, x_k, i)||^{\mathfrak{M}} \leq ||C^+_s(x_1, \dots, x_k)||^{\mathfrak{M}}.$$

On the other hand, if $\mathfrak{M}$ satisfies $Ext(\ell)$ and $k + 1 \leq \ell$, there is, for any $x_1, \dots, x_k$ in M, an $i_0 \in M$ such that for each R_r appearing in $C_s(x_1, \dots, x_k, x_{k+1})$, the atomic formula $R_r(x_1, \dots, x_k, i_0)$ takes a value making the configuration in which it appears to take the highest possible value; that is,

$$||C_s(x_1, \dots, x_k, i_0)||^{\mathfrak{M}} = ||C^+_s(x_1, \dots, x_k)||^{\mathfrak{M}}.$$

Therefore, for any $x_1, \dots, x_k \in M$

$$\begin{aligned} ||\exists x_{k+1}\, C_s(x_1, \dots, x_k, x_{k+1})||^{\mathfrak{M}} &= \sup_{i \in M} ||C_s(x_1, \dots, x_k, i)||^{\mathfrak{M}} \\ &= ||C^+_s(x_1, \dots, x_k)||^{\mathfrak{M}} \end{aligned}$$

and, finally,

$$\begin{aligned} ||\exists x_{k+1}\, \varphi(x_1, \dots, x_k, x_{k+1})||^{\mathfrak{M}} &= ||\bigvee_s \exists x_{k+1}\, C_s(x_1, \dots, x_k, i)||^{\mathfrak{M}} \\ &= ||\bigvee_s C^+_s(x_1, \dots, x_k)||^{\mathfrak{M}}, \end{aligned}$$

showing that $\exists x_{k+1}\, \varphi(x_1, \dots, x_k, x_{k+1}) \equiv_{Ext(\ell)} \bigvee_s C^+_s(x_1, \dots, x_k)$.

Symmetrically, one eliminates the quantifier in $\forall x_{k+1}\, \varphi(x_1, \dots, x_k, x_{k+1})$ using a conjunctive normal form of ψ and substituting in the elementary disjunction each configuration by its lowest possible value:

$$\begin{aligned} &R_r \text{ alone}, \neg R_r \text{ alone}, \neg\neg R_r \text{ alone go to } 0, \\ &R_r \vee \neg R_r \text{ goes to } \delta, \\ &\neg R_r \vee \neg\neg R_r \text{ goes to } \delta'. \end{aligned}$$

Finally, any sentence $\varphi \in (\mathcal{L}^{A^+}_\tau)^{(\ell)}$ is $Ext(\ell)$-equivalent to a quantifier-free sentence ψ in $\mathcal{L}^{A^+}_\tau$, necessarily a $\wedge, \vee, \neg$ combination of elements of E and thus an element e of E. Hence, $\lim_n \mu_n(\varphi = e) \geq \lim_n \mu_n(Ext(\ell)) = 1$.

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Clearly, the conditions of the theorem are satisfied by De Morgan algebras, which include Boolean algebras and the $\{\wedge, \vee, \neg\}$-reducts of MV-algebras. They are satisfied also by the $\{\wedge, \vee, \neg\}$-reduct of any G-algebra since the law $\neg(v \wedge w) = \neg v \vee \neg w$ fails in arbitrary Heyting algebras but holds in prelinear ones.

EXAMPLE 19. The almost sure values for any finite Boolean algebra are just 0 and 1, since $\varepsilon = \varepsilon' = 0$ and $\delta = \delta' = 1$.

EXAMPLE 20. The almost sure values for the $\{\wedge, \vee, \neg\}$-fragment of $\mathcal{L}^{\mathbf{L}_{N+1}}$ are $\{0, \frac{1}{2}, 1\}$ for even N, and $\{0, \frac{N-1}{2N}, \frac{N+1}{2N}, 1\}$ for odd N. Indeed, $\varepsilon = \varepsilon' = \delta = \delta' = \frac{N/2}{N}$ in the first case, and $\varepsilon = \varepsilon' = \max(v \wedge (1 - v)) = \frac{(N-1)/2}{N}$, $\delta = \delta' = \min(v \vee (1 - v)) = \frac{(N+1)/2}{N}$ in the second.

EXAMPLE 21. For any finite G-algebra $\boldsymbol{G}$, the almost sure values of the $\{\wedge, \vee, \neg\}$-fragment of $\mathcal{L}^{\boldsymbol{G}}$ are $0, \delta, 1$, where $\delta = \inf(v \vee \neg v)$, since $\varepsilon = \varepsilon' = 0$ and $\delta' = \inf(\neg v \vee \neg\neg v) = 0$. Notice that δ is the minimum dense element of $\boldsymbol{G}$ and the smallest positive element if $\boldsymbol{G}$ is a chain.

EXAMPLE 22. The almost sure values for a finite De Morgan algebra are at most four since $\varepsilon = \varepsilon'$ and $\delta = \delta'$. Examples 19 and 20 show they may be exactly two, exactly three or exactly four.

EXAMPLE 23. As the conditions of the theorem are equational, they are preserved by finite products; in particular, the reader may verify that the logic associated with the algebra $\mathrm{G}_3 \times \mathrm{Ł}_4$ has 5 distinct almost sure values: $0, \varepsilon = \varepsilon', \delta, \delta', 1$. We let the reader find an example where six distinct values are achieved.

Notice that no new almost sure values are added by $(\mathcal{L}^{\boldsymbol{A}})^{(\omega)}_{\omega_1\omega}$ because any sentence of this logic is asymptotically equivalent to a finitary one, as we prove next using our translation. Let $(\mathcal{L}^{\boldsymbol{A}})^{(k)}_{\omega_1\omega}$ denote the formulas in $(\mathcal{L}^{\boldsymbol{A}})^{(\omega)}_{\omega_1\omega}$ utilizing at most the variables $\mathbf{x} = x_1, \ldots, x_k$, free or bound. Define $(\mathcal{L}^{\boldsymbol{A}})^{(k)}$ similarly, and denote by θ_k the conjunction of the classical extension axioms $Ext_A(i, f)$ for $i \leq k$.

LEMMA 24. *Given a finite lattice algebra $\boldsymbol{A}$, for any formula $\varphi \in (\mathcal{L}^{\boldsymbol{A}})^{(k)}_{\omega_1\omega}$, there is a formula $\varphi' \in (\mathcal{L}^{\boldsymbol{A}})^{(k)}$ such that $\theta_k \models \forall\mathbf{x}[\varphi^a \leftrightarrow \varphi'^a]$ for any $a \in A$. Hence, $||\varphi||^{\mathfrak{M}} = ||\varphi'||^{\mathfrak{M}}$ in almost all finite models.*

PROOF. We do it by induction on the complexity of φ. We prove the inductive step for infinite conjunctions; the other inductive steps are similar or trivial. Assume $\varphi_n \in (\mathcal{L}^{\boldsymbol{A}}_\tau)^{(k)}_{\omega_1\omega}$, τ finite, and $\theta_k \models \forall\mathbf{x}[\varphi_n^a \leftrightarrow \varphi_n'^a]$ with $\varphi_n' \in (\mathcal{L}^{\boldsymbol{A}}_\tau)^{(k)}$ for each $a \in A$ and $n \in \omega$. Then the classical random τ-model $\mathfrak{R}_{\tau\times A}$ (cf. Theorem 5) satisfies for each $a \in A$:

$$(\bigwedge_{n\in\omega} \varphi_n)^a \leftrightarrow [\bigvee_{b1\wedge\ldots\wedge b_k = a, k\leq|A|, n_1,\ldots,n_k\in\omega} \varphi_{n_1}'^{\,b_1} \wedge \ldots \wedge \varphi_{n_k}'^{\,b_k}] \wedge \bigwedge_{b\geq a} \bigvee_{n\in\omega} \varphi_n'^{\,b}.$$

As $Th(R)$ is ω-categorical, by the Ryll–Nardzewski theorem, there is a finite subset $S \subseteq \omega$ such that any formula $\varphi_n'(\mathbf{x})^b$ (with $n \in \omega$, $b \in A$) is equivalent to one in the set $\{\varphi_n'(\mathbf{x})^b : n \in S, b \in A\}$. Thus, $(\bigwedge_{n\in\omega} \varphi_n)^a$ is equivalent in $\mathfrak{R}_{\tau\times A}$ to

$$[\bigvee_{b1\wedge\ldots\wedge b_k, k\leq|A|, n_1,\ldots,n_k\in S} \varphi_{n_1}'^{\,b_1} \wedge \ldots \wedge \varphi_{n_k}'^{\,b_k}] \wedge \bigwedge_{b\geq a} \bigvee_{n\in S} \varphi_n'^{\,b},$$

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clearly equivalent to $(\bigwedge_{n\in S}\varphi'_n)^a$. Then $\theta_k \models \forall \mathbf{x}[(\bigwedge_{n\in\omega}\varphi_n)^a \leftrightarrow (\bigwedge_{n\in S}\varphi_n)^a]$; otherwise, by [15, Claim (4), in Section 4.2], we obtain $\theta_k \models \neg\forall\mathbf{x}[(\bigwedge_{n\in\omega}\varphi_n)^a \leftrightarrow (\bigwedge_{n\in S}\varphi_n)^a]$, contradicting that the equivalence holds in $\mathfrak{R}_{\tau\times A}$. ⊣

§5. A zero-one law for infinitely valued Łukasiewicz predicate logic. Proving zero-one laws for logics given by infinite lattice algebras seems challenging. The translation yields in this case infinitary formulas with essentially infinite vocabularies, and it is known that in the case of vocabularies with an infinite supply of unary predicate symbols, the zero-one law for classical $\mathcal{L}^{\omega}_{\omega_1\omega}$ fails [15, Exercise 4.1.9.]. Moreover, we cannot expect positive probabilities $p_R(a)$ for all the truth-values (except for countable A).

We will deal mainly with infinitely valued Łukasiewicz logic, determined by the standard MV-algebra $[0,1]_{\mathrm{L}} = \langle [0,1], \wedge, \vee, \to, \neg\rangle$ with crip identity (cf. [26]).

Consider σ-additive Borel probability measures $\{p_R\}_{R\in\tau}$ over $[0,1]$ which assign positive measure to all non-trivial intervals of $[0,1]$, say, $p_R =$ Lebesgue measure. These induce a product measure ρ_n in the infinite space of models with domain $[n]$: $\mathrm{Mod}^{[0,1]}_{\tau,n} = \prod_{R\in\tau}[0,1]^{[n]^{\tau(R)}}$, which permits to define for any measurable set $S\subseteq[0,1]$ the probability that a sentence φ takes a value in S :

$$\mu_n(\varphi\in S) = \rho_n\{\mathfrak{M}\in \mathrm{Mod}^{[0,1]}_{\tau,n} : \varphi^{\mathfrak{M}}\in S\},$$

(the map $\mathfrak{M}\longmapsto\varphi^{\mathfrak{M}}$ is measurable because it may be shown to be continuous).

It is important to stress that for each pair of predicates R and R', and $i_1,\dots,i_k,i'_1,\dots,i'_k\in[n]$, we have that $R(i_1,\dots,i_k)\in S$ and $R'(i'_1,\dots,i'_k)\in S'$ become independent events provided that the tuples $\langle R,i_1,\dots,i_k\rangle$ and $\langle R',i'_1,\dots,i'_k\rangle$ are distinct.

For simplicity, we will use the notation $\mathfrak{M}\models\varphi$ as an equivalent of $||\varphi||^{\mathfrak{M}} = 1$.

THEOREM 25. *For any sentence $\varphi\in\mathcal{L}^{[0,1]_{\mathrm{L}}}$, there is a unique $a\in[0,1]$ such that*

$$\lim_n \mu_n(\varphi\in V) = 1 \text{ for any open interval } V \text{ containing } a.$$

Equivalently,

$$\lim_n \mu_n(|\varphi - a| < \varepsilon) = 1 \text{ for any } \varepsilon > 0.$$

The two formulations are clearly equivalent. They say that φ takes approximately value a with probability 1, for any degree of approximation. To prove the theorem, we will need the following observations on the expressivity of the logic (see [7, 13, 34]). For each fraction $r = \frac{n}{m}\in[0,1]$, there are connectives of Łukasiewicz logic $()_{\geq r}$, $()_{\leq r}$ such that

$$\mathfrak{M}\models\varphi_{\geq r} \text{ iff } ||\varphi||^{\mathfrak{M}}\geq r,$$
$$\mathfrak{M}\models\varphi_{\geq r} \text{ iff } ||\varphi||^{\mathfrak{M}}\leq r.$$

More generally and precisely (see [8]),

$$||\varphi_{\geq r}||^{\mathfrak{M}}\geq 1-\varepsilon \text{ iff } ||\varphi||^{\mathfrak{M}}\geq r-\frac{\varepsilon}{m}$$
$$||\varphi_{\leq r}||^{\mathfrak{M}}\geq 1-\varepsilon \text{ iff } ||\varphi||^{\mathfrak{M}}\leq r+\frac{\varepsilon}{m}$$

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We will assume also that the logic $\mathcal{L}^{[0,1]_{\mathrm{L}}}$ has a 'crisp identity' $\approx$ taking only the values 0 or 1.[3]

Given $k, N \in \omega$, let F_{k+1} consist of all the atomic formulas distinct from identities truly containing the variable x_{k+1}, and $A_N = \{[\frac{j}{N}, \frac{j+1}{N}], j = 0, \dots, N-1\}$. For each choice of intervals $g : F_{k+1} \longrightarrow A_N$, define the extension axiom $Ext_A(k, N, g)$:

$$\forall x_1 \dots x_k \Big(\bigvee\nolimits_{i \neq j} x_i \approx x_j \vee \exists x_{k+1} \bigwedge\nolimits_{i \leq k} \neg x_{k+1} \approx x_j \wedge \bigwedge\nolimits_{\varphi \in F_{k+1}} \varphi_{g(\varphi)}(x_1, \dots, x_k, x_{k+1})\Big)$$

Then define the sentence $Ext_A(k, N)$:

$$\bigwedge\nolimits_{g:\, F_{k+1} \longrightarrow A_N} Ext_A(k, N, g)$$

and the theory

$$Ext := \{Ext_A(k, N)\}_{k,N}.$$

Due to the semantical interpretation of quantifiers as infima and suprema, we have that $Ext_A(k, N, g)^M \geq 1 - \varepsilon$ if and only if for any distinct $i_1, \dots, i_k$ in M there is an i in $M \setminus \{i_1, \dots, i_k\}$ such that $||\varphi_{g(\varphi)}||^{\mathfrak{M}}(i_1, \dots, i_k, i) \geq 1 - \varepsilon$ for each $\varphi \in F_{k+1}$. More precisely, if $g(\varphi) = [\frac{j}{N}, \frac{j+1}{N}]$,

$$||\varphi||^{\mathfrak{M}}(i_1, \dots, i_k, i) \in \left[\frac{j}{N} - \frac{\varepsilon}{m}, \frac{j+1}{N} + \frac{\varepsilon}{m}\right],$$

for each $\varphi \in F_{k+1}$, which is a relaxation of the intended condition

$$||\varphi||^{\mathfrak{M}}(i_1, \dots, i_k, i_{k+1}) \in \left[\frac{j}{N}, \frac{j+1}{N}\right].$$

In the following, we will denote by $g(\varphi)_{\pm\varepsilon}$ the displayed relaxed interval.

LEMMA 26. *For any $\varepsilon > 0$, the sequence $\mu_n(Ext_A(k, N, g) \geq 1 - \varepsilon)$ converges to* 1.

PROOF. Fix to natural numbers k and N, a mapping $g : F_{k+1} \longrightarrow A_N$ and recall that according to the previous observation $Ext_A(k, g, N)^M < 1 - \varepsilon$ means that there are distinct $i_1, \dots, i_k$ in $[n]$ such that:

for all $i \in [n] \setminus \{i_1, \dots, i_k\}$, it is false that $\bigwedge_{\varphi \in F_{k+1}} ||\varphi||^{\mathfrak{M}}(i_1, \dots, i_k, i) \in g(\varphi)_{\pm\varepsilon}$.

Let $S(M, \varepsilon, i_1, \dots, i_k)$ be the above classical statement for fixed ε and $i_1, \dots, i_k$, and $\mathfrak{M}$ chosen randomly in $\mathrm{Mod}^{[0,1]}_{\tau,n}$. Then, since the set of events

$$\{||\varphi||^{\mathfrak{M}}(i_1, \dots, i_k, i) \in g(\varphi)_{\pm\varepsilon} : \varphi \in F_{k+1},\ i \in [n] \setminus \{i_1, \dots, i_k\}\},$$

is independent,

$$\mu_n(S(M, \varepsilon, i_1, \dots, i_k)) = \prod\nolimits_{i \in [n] \setminus \{i_1, \dots, i_k\}} (1 - \prod\nolimits_{\varphi \in F_{k+1}} \mu_n(\varphi(i_1, \dots, i_k, i) \in g(\varphi)_{\pm\varepsilon}))$$

But $\mu_n(\varphi(i_1, \dots, i_k, i) \in g(\varphi)_{\pm\varepsilon}) = p_R(g(\varphi)_{\pm\varepsilon}) \geq p_R([\frac{j}{N}, \frac{j+1}{N}]) > 0$, for some relation symbol R and some j. Then, taking $\delta_N = \min\{p_R([\frac{j}{N}, \frac{j+1}{N}]) : R \in \tau,$

[3] This does not imply loss of generality because any model of Ł may be equipped with a crisp identity, without altering the model theoretic properties of the logic.

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$0 \leq j \leq N-1\}$, we have

$$\mu_n(S(M,\varepsilon,i_1,\dots,i_k)) \leq (1-\delta_N)^{n-k},$$

Therefore,

$$\mu_n(Ext_A(k,g,N) < 1-\varepsilon) \leq \sum\nolimits_{i_1,\dots,i_k,} \mu_n(S(M,\varepsilon,i_1,\dots,i_k)) \leq n^k(1-\delta_N)^{n-k}$$

which clearly converges to 0, since $1-\delta_N < 1$. Therefore, $\mu_n(Ext_A(k,g,N) \geq 1-\varepsilon)$ converges to 1. ⊣

Recall that $Ext_A(k,N,g)^{\mathfrak{M}} \geq 1-\varepsilon$ iff $M \models Ext_A(k,N,g)_{\geq 1-\varepsilon}$. Therefore, the previous lemma shows that any finite subset of the theory $Ext^* = \{\psi_{\geq 1-\varepsilon} : \psi \in Ext, \varepsilon > 0)$ is satisfiable. Hence, by satisfiability compactness of Łukasiewicz logic it has a model, which will be necessarily a model of Ext. As Ext is countable, this model may be chosen to be countable by the downward Löwenheim–Skolem theorem.

DEFINITION 1. Let $\overline{a},\overline{b}$ be tuples of the same length k of distinct elements of $\mathfrak{A}$ and $\mathfrak{B}$, respectively, then $\overline{a} \sim_\varepsilon \overline{b}$ if and only if $|\varphi^{\mathfrak{A}}(\overline{a}) - \varphi^{\mathfrak{B}}(\overline{b})| \leq \varepsilon$ for any atomic φ with variables in $\{x_1,\dots,x_k\}$. We say then that the pair $\overline{a},\overline{b}$ (or the assignment $a_i \mapsto b_i$) is an *ε-partial isomorphism*.

Considering tuples enumerating $\mathfrak{A}$ and $\mathfrak{B}$ respectively, we may define, similarly, the notion of ε-isomorphism. Partial ε-isomorphisms between models of Ext have the extension property, more precisely.

LEMMA 27. *If $\mathfrak{A},\mathfrak{B} \models Ext(k,N)$, $N \geq \frac{2}{\varepsilon}$, then $\overline{a} \sim_\varepsilon \overline{b}$ (of length $\leq k$) implies for any $a \in A \setminus \{\overline{a}\}$ there is $b \in B \setminus \{\overline{b}\}$ such that $\overline{a}a \sim_\varepsilon \overline{bb}$ and viceversa. Particular case: for any $a \in A$, there is $b \in B$ such that $a \sim_\varepsilon b$.*

PROOF. Assume $\overline{a} \sim_\varepsilon \overline{b}$, and consider $g\colon F_{k+1} \longrightarrow A_N$ such that $\varphi^{\mathfrak{A}}(\overline{a},a) \in g(\varphi)$ for each $\varphi \in F_{k+1}$, then $Ext(k,N,g)$ is an extension axiom which must be satisfied by $\mathfrak{B}$; hence, there is b such that $||\varphi_{g(\varphi)}||^{\mathfrak{B}}(\overline{b},b) \geq 1-\varepsilon/2$. From this, we obtain that $\varphi^{\mathfrak{B}}(\overline{b},b) \in [\frac{j}{N} - \varepsilon/2, \frac{j+1}{N} + \varepsilon/2]$, thus $|\varphi^{\mathfrak{A}}(\overline{a},a) - \varphi^{\mathfrak{B}}(\overline{b},b)| \leq \frac{1}{N} + \frac{\varepsilon}{2} \leq \varepsilon|$. ⊣

We will utilize the following observation: $|(r \oplus s) - (r' \oplus s')| \leq 2\max\{|r-r'|, |s-s'|\}$. For any formula $\#\varphi$, let (φ) be the number of occurrences of $\oplus$ in φ.

LEMMA 28. *If $\mathfrak{A},\mathfrak{B} \models Ext(k,N)$, $N \geq \frac{2}{\varepsilon}$, then for any formula φ with free variables in $\{x_1,\dots,x_k\}$, $\overline{a} \sim_\varepsilon \overline{b}$ implies $|\varphi^{\mathfrak{A}}(\overline{a}) - \varphi^{\mathfrak{B}}(\overline{b})| \leq 2^{\#(\varphi)}\varepsilon$.*

PROOF. By induction on the complexity of $\varphi(\overline{x}) \in \text{Ł}\forall(\tau)$, based in $\oplus, \neg, \exists$.

- If φ is atomic, then the result is immediate by definition of partial ε-isomorphism.
- If φ is $\neg\theta$, then it is trivial because $|(\neg\theta)^{\mathfrak{A}}[\overline{a}] - (\neg\theta)^{\mathfrak{B}}[\overline{b}]| = |\theta^{\mathfrak{A}}[\overline{a}] - \theta^{\mathfrak{B}}[\overline{b}]|$.
- If φ is $\theta \oplus \theta'$, then, by induction hypothesis, $|\theta^{\mathfrak{A}}[\overline{a}] - \theta^{\mathfrak{B}}[\overline{b}]| \leq 2^{\#(\theta)}\varepsilon$, $|\theta'^{\mathfrak{A}}[\overline{a}] - \theta'^{\mathfrak{B}}[\overline{b}]| \leq 2^{\#(\theta')}\varepsilon$ and $|(\theta \oplus \theta')^{\mathfrak{A}}[\overline{a}] - (\theta \oplus \theta')^{\mathfrak{B}}[\overline{b}]| \leq 2 \cdot 2^{\max(\#(\theta),\#(\theta'))}\varepsilon \leq 2^{\#(\theta)+\#(\theta')+1}\varepsilon = 2^{\#(\varphi)}\varepsilon$.
- If φ is $\exists v\,\theta$ and $\overline{a} \sim_\varepsilon \overline{b}$, then, by the previous lemma, for each $a \in A$ there is $b_a \in B$ such that $\overline{a}a \sim_\varepsilon \overline{b}b_a$, and the inductive hypothesis yields $|\theta^{\mathfrak{A}}[\overline{a}a] -$

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$\theta^{\mathfrak{B}}[\bar{b}b_a]| \leq 2^{\#(\theta)}\varepsilon$. Hence, $\theta^{\mathfrak{A}}[\bar{a}a] \leq \theta^{\mathfrak{B}}[\bar{b}b_a] + 2^{\#(\theta)}\varepsilon$. Taking suprema over a, we obtain $(\exists v\,\theta)^{\mathfrak{A}}[\bar{a}] \leq \sup_{b_a} \theta^{\mathfrak{B}}[\bar{b}b_a] + 2^{\#(\theta)}\varepsilon \leq (\exists v\,\theta)^{\mathfrak{B}} + 2^{\#(\theta)}\varepsilon[\bar{b}]$, and thus $(\exists v\,\theta)^{\mathfrak{A}}[\bar{a}] - (\exists v\,\theta)^{\mathfrak{B}}[\bar{b}] \leq 2^{\#(\theta)}\varepsilon$. Symmetrically, we get the same for the negation and thus $|(\exists v\,\theta)^{\mathfrak{A}}[\bar{b}] - (\exists v\theta)^{B}[\bar{a}]| \leq 2^{\#(\theta)}\varepsilon = 2^{\#(\varphi)}\varepsilon$.

⊣

We will write $\mathfrak{A} \equiv_{\mathcal{L}^{[0,1]_{\mathrm{L}}}} \mathfrak{B}$ whenever $\varphi^{\mathfrak{A}} = \varphi^{\mathfrak{B}}$ for each sentence φ.

COROLLARY 29 (Completeness). *If* $\mathfrak{A}, \mathfrak{B} \models Ext$, *then* $\mathfrak{A} \equiv_{\mathcal{L}^{[0,1]_{\mathrm{L}}}} \mathfrak{B}$. *Therefore*, Ext *is complete.*

PROOF. Given a sentence φ and $\varepsilon > 0$, let $x_1, \ldots, x_k$ be its variables and $N \geq 2 \cdot 2^{\#(\varphi)}/\varepsilon$. Then, by Lemma 27, there are $a \in A$, $b \in B$ such that $a \sim_{\varepsilon/2^{\#(\varphi)}} b$ and, by Lemma 28, $|\varphi^{\mathfrak{A}} - \varphi^{\mathfrak{B}}| = |\varphi^{\mathfrak{A}}[a] - \varphi^{\mathfrak{B}}[b]| \leq 2^{\#(\varphi)}\varepsilon/2^{\#(\varphi)} = \varepsilon$. Since this is true for all $\varepsilon > 0$, we have $\varphi^{\mathfrak{A}} = \varphi^{\mathfrak{B}}$. ⊣

It is easy to see that a theory T is complete iff for any sentence φ and rational r, we have $T \models \varphi_{\geq r}$ or $T \models \varphi_{\leq r}$.

PROOF OF THEOREM 25. Fix a sentence φ. By completeness, one has $Ext \models \varphi_{\geq r}$ or $Ext \models \varphi_{\leq r}$ for each rational $r \in [0,1]$. Then, the sets $L = \{r \in \mathbb{Q} : Ext \models \varphi_{\geq r}\}$, $U = \{s \in \mathbb{Q} : Ext \models \varphi_{\leq s}\}$ are easily seen to determine a real cut $a \in [0,1]$. Assume first that $a \in (0,1)$ and let $a \in (r,s)$ and $r < r' < a < s' < s$. Then, $Ext \models \varphi_{\geq r'} \wedge \varphi_{\leq s'}$ by construction. Take a positive $\varepsilon < s - s', rs' - r$. By the approximate consequence compactness of Ł, there is a subset $Ext^* \subseteq_{fin} Ext$ such that $Ext^* \models (\varphi_{\geq r'} \wedge \varphi_{\leq s'})_{\geq 1-\varepsilon}$, thus $Ext^* \models \varphi_{[r'-\varepsilon,\, s'+\varepsilon]}$. From the fact that $\lim_n \mu_n(Ext^*) = 1$, we obtain $\lim_n \mu_n(\varphi \in [r' - \varepsilon, s' + \varepsilon]) = 1$ and, a fortiori, $\lim_n \mu_n(\varphi \in (r,s)) = 1$. This shows the existence of a. Uniqueness follows easily since $b \neq a$ implies the existence of rationals u, v, r such that $u < b < r < a < s$. As $\varphi \in (r,s)$ has asymptotic probability 1, the incompatible event $\varphi \in (u,r)$ must have asymptotic probability 0.

If $a \in [0,s)$, choose $a < s' < s$. Then, $Ext \models \varphi_{\leq s'}$ and $Ext^* \models (\varphi_{\leq s'})_{1-\varepsilon}$ for an $\varepsilon < s - s'$, and hence $Ext^* \models \varphi_{[0,s'+\varepsilon]}$. Since $\lim_n \mu_n(Ext^*) = 1$, we have $\lim_n \mu_n(\varphi \in [0, s' + \varepsilon]) = 1$, and, a fortiori, $\lim_n \mu_n(\varphi \in [0,s)) = 1$. The case $a = 1$ is similar. ⊣

REMARK 30. The inductive step in Lemma 28 works without extra effort for any continuous connective $\circ$ under the rank definition: $\#(\circ(\theta_1, \ldots, \theta_k)) = \max_i \#(\theta_i) + 1$. So Ext is complete for any expansion of Łukasiewicz logic by continuous connectives, and thus for full continuous logic with crisp identity, and the same is true of the zero-one law.

REMARK 31. From lemmas 27 and 28, one may show that, for any countable $\mathfrak{A}, \mathfrak{B} \models Ext$ and $\varepsilon > 0$, there are enumerations $a_0, a_1, \ldots$ of A and $b_0, b_1, \ldots$ of B such that $a_i \mapsto b_i$ is an ε-isomorphism. These enumerations depend on ε and it is not possible to obtain ω-categoricity in this way. However, it follows from the above proof that any sentence ϕ takes in all models of Ext the same value, precisely its almost sure truth-value.

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§6. Further results for infinitely valued predicate logics. Although we cannot expect a complete description of the infinitely valued case, in this section we still obtain some results generalizing those in Section 4. In particular, we obtain that all rational numbers in $[0, 1]$ are almost sure values of infinitely valued Łukasiewicz predicate logic, and we prove a zero-one law for a large family of logics satisfying De Morgan laws with values in closed sublattices of $[0, 1]$.

Observe that the analogue of Lemma 13 holds for infinitely valued Łukasiewicz logic with a similar proof, due to the continuity of McNaughton functions.

THEOREM 32. *Let $t(v_1, \dots, v_k)$ be a term in the signature of* MV*-algebras. Then, the values $m = \inf_{a_1,\dots,a_k \in A} t^{\boldsymbol{A}}(a_1, \dots, a_k)$ and $s = \sup_{a_1,\dots,a_k \in A} t^{\boldsymbol{A}}(a_1, \dots, a_k)$ are almost sure for $\boldsymbol{A} = [0, 1]_{\mathrm{L}}$.*

PROOF. Notice first that m and s exist by compactness of $[0, 1]$ and the continuity of $t^{\boldsymbol{A}}$. Let $R(x)$ be an atomic formula in one variable. We show that for any open interval V of $[0, 1]$ containing m:

$$\mu_n(\forall x_1 \dots \forall x_k\, (t(R(x_1), \dots, R(x_k))) \in V) \text{ converges to 1.}$$

By definition of m, there is a tuple $\langle a_j \rangle_j \in |A|^k$ such that $t^{\boldsymbol{A}}(a_1, \dots, a_k) \in V$ and by continuity of $t^{\boldsymbol{A}}$, there are open intervals V_j such that $a_j \in V_j$ and $v_j \in V_j$ implies $t^{\boldsymbol{A}}(v_1, \dots, v_k) \in V$.

For each $n \geq k$, find $K \subseteq [n]^k$ of cardinal $|K| = \lfloor n/k \rfloor$ such that all tuples in K have all their components distinct, and two different tuples in K are distinct at each coordinate. Then,

$$\begin{aligned}
&\mu_n(\forall x_1 \dots \forall x_k\, (t(R(x_1), \dots, R(x_k)) \in V)\\
&= \Pr(\inf_{i_1,\dots,i_k \in [n]} t^{\boldsymbol{A}}(R(i_1), \dots, R(i_k)) \in V)\\
&= \Pr(\text{for some } (i_j)_j \text{ in } [n]^k : t^{\boldsymbol{A}}(R(i_1), \dots, R(i_k)) \in V) (\text{since } V \text{ is an open}\\
&\qquad \text{interval containing } m)\\
&\geq \Pr(\text{for some } (i_j)_j \text{ in } [n]^k : R(i_j) \in V_j, \text{ for } j = 1, \dots, k).\\
&\geq \Pr(\text{for some } (i_j)_j \text{ in } K : R(i_j) \in V_j, \text{ for } j = 1, \dots, k)\\
&= 1 - \Pr(\text{for all } (i_j)_j \text{ in } K \text{ is false: } R(i_j) \in V_j, \text{ for } j = 1, \dots, k)\\
&= 1 - \prod_{(i_1 \dots i_k) \in K} [1 - \Pr(R(i_j) \in V_j \text{ for } j = 1, \dots, k)]\\
&= 1 - (1 - p_R(V_1) \dots p_R(V_k))^{\lfloor n/k \rfloor}.
\end{aligned}$$

For the last two identities, we use that the events $R(i_j) \in V_j$ form a mutually independent set for the coordinates of the k-tupes in K. Since $p_R(V_i) > 0$, the last quantity converges to 1 when n goes to ∞.

A similar argument shows that, for any open set V containing s,

$$\mu_n(\exists x_1 \dots \exists x_k\, (t(R(x_1), \dots, R(x_k)) \in V)$$

converges to 1. ⊣

COROLLARY 33. *All rational numbers are almost sure values of* $[0, 1]$*-valued Łukasiewicz logic* ($\mathcal{L}^{[0,1]_{\mathrm{L}}}$).

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PROOF. Use the same terms of Example 15. ⊣

It is easy to see that Theorem 32 generalizes to any bounded linearly ordered lattice algebra for which the additional operations are continuous with respect to the order topology because its proof only uses the continuity of the functions interpreting the terms.

THEOREM 34. *Let $t(v_1, \dots, v_k)$ be a term for a bounded linearly ordered lattice algebra $\boldsymbol{A}$. If $t^{\boldsymbol{A}}$ is continuous w.r.t. the order topology, then $m = \inf_{a_1,\dots,a_k \in A} t^{\boldsymbol{A}}(a_1, \dots, a_k)$ and $s = \sup_{a_1,\dots,a_k \in A} t^{\boldsymbol{A}}(a_1, \dots, a_k)$ are almost sure for $\mathcal{L}^{\boldsymbol{A}}$ whenever they exist.*

EXAMPLE 35. Consider Łukasiewicz logic with product. Then observe that $\frac{1}{2}(\sqrt{5} - 1) = \inf(\neg v \vee (v \cdot v))$ is an almost-sure value.

EXAMPLE 36. Let $\boldsymbol{G}_{\uparrow}$ be the G-chain consisting of an ascending sequence from 0 to 1. Then, it may be checked that Gödel implication is continuous in this case. Therefore, all elements of $\boldsymbol{G}_{\uparrow}$ are almost sure (compare with Example 39 below).

Finally, notice that the analogue of Lemma 18 holds for infinite closed sublattices of $[0, 1]$ with a continuous negation and actually gives a proof of the zero-one law in this case.

THEOREM 37. *Let $\boldsymbol{A} = \langle A, \wedge, \vee, \neg \rangle$ be an infinite closed sublattice of $[0, 1]$ containing 0 and 1 with a continuous negation satisfying*:

(1) $\neg(v \wedge w) = \neg v \vee \neg w$, $\neg(v \vee w) = \neg v \wedge \neg w$
(2) $v \leq \neg\neg v$
(3) $\neg 1 = 0$

Then, $\mathcal{L}^{\boldsymbol{A}}$ satisfies the zero-one law and its only almost sure values are 0, 1, and

$$\varepsilon = \sup_{x \in A}(x \wedge \neg x), \varepsilon' = \sup_{x \in A}(\neg x \wedge \neg\neg x),$$
$$\delta = \inf_{x \in A}(x \vee \neg x), \delta' = \inf_{x \in A}(\neg x \vee \neg\neg x).$$

PROOF. This is essentially the same argument as in Theorem 18. By compactness of $\boldsymbol{A}$ and continuity of the operations, the displayed infima and suprema are attained, for example $\varepsilon = (a \wedge \neg a)$ for some $a \in A$, and they are almost sure by Theorem 34. Thus, only finitely many extension axioms are needed to ensure that the suprema of the various configurations R, $\neg R$, $\neg\neg R$, $R \wedge \neg R$, $\neg R \wedge \neg\neg R$ appearing in the proof of Lemma 18 are realized (the same holds for the infima), and the quantifier elimination may be achieved with asymptotic probability 1. ⊣

EXAMPLE 38. The logic given by the $\{\wedge, \vee, \neg\}$-reduct of the Łukasiewicz algebra $[0, 1]_{\mathrm{L}}$ has almost sure values $\{0, \frac{1}{2}, 1\}$.

EXAMPLE 39. The logic given by the $\{\wedge, \vee, \neg\}$-reduct of any closed subalgebra of the standard G-chain over $[0, 1]$ in which 0 is isolated has a zero-one law, and its almost sure values are $\{0, \delta, 1\}$ where δ is the minimum positive element since $\wedge, \vee$ are continuous and, under isolation of 0, Gödel negation becomes continuous.

§7. Final remarks. The translation we have used in the case of finite algebras has appeared in [1, 2] and permits to transfer straightforwardly from classical to

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finitely valued logics most relevant model-theoretic facts and concepts: compactness, ultraproducts, elementary embeddings, type omission, etc.

Grädel et al. [22] consider certain semirings $\boldsymbol{A} = \langle A, +, \cdot, 0, 1\rangle$ as algebras of truth values in which $\vee$ and $\wedge$ are interpreted as $+$ and $\cdot$, and the quantifiers $\exists$ and $\forall$ are interpreted as iterated $+$ and $\cdot$, respectively. Negation is allowed at the atomic level only, and no further operations (connectives) are allowed. In this setting, they obtain zero-one laws for finite distributive lattices and "absorptive" semirings such as $\langle \mathrm{L}_{N+1}, \vee, \odot, 0, 1\rangle$. Our results in the present paper generalize those on lattice semirings because we do not require distributivity, and we allow arbitrary additional operations. Moreover, the treatment of negation in [22] can be expressed in terms of Oberschelp conditions.

Notice that the zero-one law fails for the simplest non-lattice semiring, namely the two-element field $\mathbb{F}_2 = \langle \{0, 1\}, +, \cdot, 0, 1\rangle$, as witnessed by the sentence $\exists x(Px \vee \neg Px)$ whose value in universes of cardinal n is $\Sigma_{x\in[n]}1$, which oscillates between 0 and 1 with the parity of n (negation is interpreted by the function $v + 1$ so that $Pi \vee \neg Pi$ takes the value $v + v + 1 = 1$ for any $i \in [n]$, and $\exists$ is interpreted by Σ).

It is worth noting that, in the context of continuous model theory, a zero-one law is proved in [20] for the theory of metric spaces, which yields a zero-one law for Łukasiewicz of pure identity under certain restrictions.

We conclude with some questions that merit further investigation:

(1) Does $\mathcal{L}^{[0,1]_{\mathrm{L}}}$ have almost sure irrational values?
(2) Does full Łukasiewicz logic over [0, 1] with product and its (discontinuous) residuated implication have a zero-one law?
(3) Does full Gödel logic over [0, 1] have a zero-one law?
(4) The set A_{as} of almost sure values of a logic $\mathcal{L}^{\boldsymbol{A}}$ is an invariant of $\boldsymbol{A}$, in fact, a subalgebra. Is there a characterization of A_{as} in purely algebraic terms?

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