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Sofic subshifts and piecewise isometric systems

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Abstract. We study the natural symbolic dynamics associated with piecewise continuous, non-invertible, dynamical systems. Our study is centered primarily on the relationship between the point-set topological properties of the partition of the system and the symbolic coding. We prove that for a class of maps locally preserving distances with regular partition, the associated symbolic dynamics cannot embed subshifts of finite type of positive entropy. Hence, in particular, almost sofic subshifts obtained from the symbolic dynamics have zero entropy. However, there are examples in Euclidean spaces of systems with non-regular partitions for which the coding maps can be surjective, particularly embedding all subshifts. For all such examples, the associated group of isometries is a subgroup of $O(\mathbb{R}, N)$.

0. *Introduction*

In this paper, we study natural generalizations of well-known interval exchange transformations to a class of piecewise continuous, not necessarily invertible systems. We address the question: which subshifts of finite type and which sofic subshifts can be embedded in the symbolic dynamics associated with our systems? Our study is centered primarily on the relationship between the point-set topological properties of the partition of the system and the symbolic coding. Our main result is the following.

THEOREM. (Theorem 2.1) *Let $T : X \to X$ be a generalized piecewise isometry with partition $\mathcal{P} = \{P_1, \ldots, P_r\}$. Suppose that one of the following holds:* (i) *the partition $\mathcal{P}$ is regular; or* (ii) *the coding of the exceptional set for T is countable. Then all subshifts of finite type embeddings in the symbolic coding of X have zero entropy.*

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The notion of a piecewise isometry in this article is quite general and it extends beyond Euclidean spaces. A partition $\mathcal{P}$ is regular if it satisfies some weak topological condition. Our result yields, in particular, that only almost sofic subshifts with zero entropy, and only coded systems with zero entropy can appear in the symbolic dynamics of T. (An *almost sofic* system is a subshift whose entropy is the supremum of the entropies of subshifts of finite type it contains. A *coded system* is a subshift whose dense subset can be presented by a countable irreducible labeled graph.) These classes contain the whole range of subshifts with Markovian behavior which are not of finite type [**22**].

In the second part of the article, we include an example which illustrates that either of the hypotheses (i) or (ii) is necessary for the theorem to hold. We construct (Theorem 4.1) a Euclidean piecewise rotation with irregular partition whose symbolic dynamics is a full shift on $r > 1$ symbols, in particular, the coding image embeds all subshifts. The main idea in the proof of the main result is the association of a certain compact subgroup of isometries giving rise to the construction of orbits visiting the partial boundaries of distinct atoms: $\partial P_{j_1} \cap P_{j_1}$ and sets $\partial P_{j_2} \cap P_{j_2}$ infinitely often. The main tools used in the proof are graph presentations of finite type systems, properties of compact transformation groups and properties of metric spaces.

Symbolic dynamics has been one of the most fundamental models for the study of hyperbolic dynamics of homeomorphisms. It has been investigated very extensively since the discovery of Markov partitions of the two-dimensional torus by Berg [**6**] and the fundamental paper by Adler and Weiss [**3**] establishing that toral automorphisms are measure theoretically isomorphic if and only if they have the same entropy. Systematic treatment of symbolic representations of dynamical systems by means of Markov partitions can be found in [**2**].

The goal of this article is different from the goal set up in [**6**] for two main reasons. The maps, which we study, lack hyperbolicity, and in our setup, the symbolic dynamics is naturally inherited from the generating transformation: $T : X \to X$. This can be illustrated in the following non-hyperbolic example (the toral map): $T : [0, 1] \times [0, 1]$, $Tx = Mx \bmod \mathbb{Z}^n$, where $M \in SO(\mathbb{R}, n)$ [**13**]. Since the entries of M are not integers, this map is not an automorphism; it is, however, a piecewise isometry. In dimension two, the domain is naturally partitioned into four maximal regions on which T is continuous (Figure 1). This partition gives rise to one-sided codings of orbits. The coding records the indices of atoms visited by the orbits. It is natural to ask: what subshifts can arise from this natural coding? Are there systems giving rise to subshifts of finite type with positive entropy? This question is of particular interest since the group of Euclidean isometries in dimensions three and higher contains free subgroups, and hence presumably may give rise to 'rich subshifts'.

Indirectly, these questions were addressed in papers investigating the symbolic dynamics of systems preserving distances, for example in [**1, 5, 9, 10, 14, 15, 16, 17, 21**]. In particular, in [**9**] Boshernitzan and Kornfeld showed that interval translations can give rise to subshifts called *substitution dynamical systems*. A result in [**15**], yields that two-dimensional polygon exchanges give rise only to subshifts with subexponential word complexity. In general, the question whether a non-invertible piecewise isometry with polyhedral partition can give rise to subshifts with positive entropy remains open. Also,

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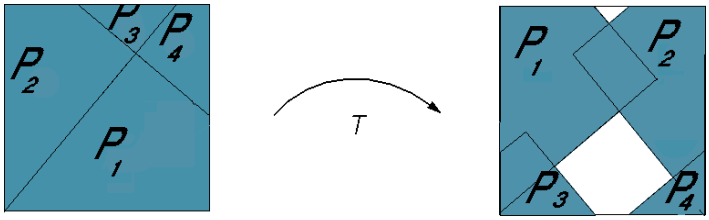


FIGURE 1. Piecewise rotation on the square: $T : [0, 1] \times [0, 1]$, $Tx = Mx \bmod Z^2$, where $M \in SO(\mathbb{R}, 2)$.

very recently in [**1**], Adler *et al* reported a study of two-dimensional piecewise affine maps on the torus which can be conjugated to an invertible piecewise rotation on the rhombus.

We structured the article as follows. The definition of the system, coding of piecewise isometries and important properties of sofic subshifts are included in §2. The main result is stated in §2. The proof, divided into fours steps, is included in §3. Section 4 illustrates an example of a piecewise isometry inducing full shift in the symbolic space. We conclude the article in §5 discussing the topological hypotheses imposed on the atoms and addressing some open questions.

1. *Preliminaries*

In this section, we define our systems, their natural symbolic dynamics and finally we list classes of subshifts for which our results hold.

1.1. *Definition of the system.*

Let Y be a metrizable topological space (Y need not be compact). Let $\mathcal{H}om(Y)$ denote the group of homeomorphisms of Y. Let $T_1, \ldots, T_r \in \mathcal{H}om(Y)$ ($r > 1$) be a finite collection of homeomorphisms of Y.

Definition 1.1. Let X be a compact subset of Y. A transformation $T : X \to X$ is called a *piecewise continuous map* with partition $\mathcal{P} = \{P_1, \ldots, P_r\}$ and induced homeomorphisms $\{T_1, \ldots, T_r\}$ if: (i) $T|_{P_i} = T_i|_{P_i}$ for every $i = 1, \ldots, r$; (ii) $P_i \cap P_j = \emptyset$ for $i \neq j$; and (iii) $\bigcup_{1 \leq j \leq r} P_j = X$. We do not require that T be surjective (Figure 1).

Definition 1.2. A piecewise continuous map $T : X \to X \subset Y$ with induced homeomorphisms $\{T_1, \ldots, T_r\}$ is called a *generalized piecewise isometry* if:

(i) there exists a metric $d_Y : Y \times Y \to \mathbb{R}^+ \cup \{0\}$ such that every T_i preserves d_Y;

(ii) every bounded in d_Y subset of Y has a compact closure in Y.

Even though Y is not assumed to be always compact, it is worthwhile noting that for compact spaces Y, generalized piecewise isometries can also be characterized by the restriction on the group of the induced homeomorphisms. For if $\overline{\langle T_1, \ldots, T_r \rangle}$ is compact with respect to the C^0 topology on $\mathcal{H}om(Y)$, then there exists a probability Haar measure μ_Γ invariant under the action of Γ. The metric $d_Y : Y \times Y \to \mathbb{R}^+ \cup \{0\}$ defined by

$$d_Y(y_1, y_2) = \int_\Gamma d_Y(Ty_1, Ty_2)\, d\mu_\Gamma \tag{1}$$

is then preserved under the action of elements in Γ. Hence, if Y and $\overline{\langle T_1, \ldots, T_r \rangle}$ are compact, then T is a generalized piecewise isometry.

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1.2. *Examples of Euclidean piecewise isometries.* Piecewise isometries in dimension one, such as interval exchanges [**4, 8, 12, 18, 19, 24, 26**] and interval translations [**9**], can be easily defined in our settings. For example, an *interval translation transformation* is a piecewise continuous map acting on a left open interval $X \subset \mathbb{R} = Y$. The partition $\mathcal{P}$ consists of a finite number of smaller left open intervals and the homeomorphisms $\{T_1, \ldots, T_r\}$ are translations of $\mathbb{R}$. An *interval exchange transformation* is an invertible interval translation.

In this section, we briefly define two new examples.

1.2.1. *Piecewise rotation on the lattice mesh.* One of the very beautiful and intriguing new examples of Euclidean piecewise isometries is the *piecewise rotation on the unit lattice mesh*, $T : [0, 1]^n \to [0, 1]^n$, $Tx = Mx \bmod \mathbb{Z}^n$, where $M \in SO(\mathbb{R}, n)$. In two dimensions (Figure 1), the map T rotates the unit square about the origin, and then shifts back (modulo integer lattice) the pieces which fall outside of the original square. For rotations by an angle other than an integer multiple of $\pi/4$, X is partitioned by four convex atoms. The piecewise rotation of the unit lattice mesh was introduced in [**13**]. It results in very beautiful and intriguing dynamics.

1.2.2. *Example on a sphere.* Let $Y = X = S^n$ be the unit sphere centered at the origin in $\mathbb{R}^n$ ($n \geq 3$). Let $\{P_1, P_2\}$ be a partition of X into two sets homeomorphic to hemispheres, P_1 open and P_2. Let $T_1 : Y \to Y$ and $T_2 : Y \to Y$ be two rotations about the origin generating a free subgroup of $O(\mathbb{R}, n)$. The rotations $\{T_1, T_2\}$ and the domains $\{P_1, P_2\}$ define a non-invertible piecewise isometry T on the sphere X.

1.3. *Coding of orbits.* The partition $\mathcal{P} = \{P_1, \ldots, P_r\}$ associated to a piecewise continuous map T of X gives rise to a natural one-sided coding map $\phi : X \to \Omega_r = \{1, \ldots, r\}^{\mathbb{N}}$ for T. The map ϕ encodes the forward orbit of a point by recording the indices of atoms visited by the orbit, that is $\phi(x) = w_0 w_1 \ldots$, where $T^k x \in P_{w_k}$.

The coding map ϕ conjugates T with the (one-sided) *shift map* $S : \Omega_r \to \Omega_r$, $S(w_0 w_1 w_2 \ldots) = w_1 w_2 \ldots$:

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ \Omega_r & \xrightarrow{S} & \Omega_r \end{array} \tag{2}$$

The shift map $S : \Omega_r \to \Omega_r$ is continuous with respect to the product topology of $\mathbb{N}$ copies of the finite set $\{1, \ldots, r\}$ each with discrete topology. This product topology can also be generated by the metric $d_{\Omega_r} : \Omega_r \times \Omega_r \to \mathbb{R}^+ \cup 0$, $d_{\Omega_r}(\phi_1, \phi_2) = \sum_{i=0}^{\infty} |\phi_1(i) - \phi_2(i)|/r^i$.

Even though $\phi : X \to \Omega_r$ is uniquely defined for all points in X, the coding ϕ has some disadvantages as often neither the coding image $\phi(X)$ nor the subsets of points in X with the same code are compact. Sometimes, in order to have a reasonable relation between topologies in X and the sequence space, it is desirable to modify the definition of the coding by assigning more than one code to orbits which meet the boundaries of the

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atoms. A *generalized orbit* of x is a sequence $\{x_k\}$ such that $x_0 = x$ and for all positive k there is some $w_k \in \{1, \ldots, r\}$ for which $x_{w_k} \in \overline{P_{w_k}}$ and $T_{w_k} : x_k \mapsto x_{k+1}$. The *coding of the generalized orbit* $\{x_k\}$ is the sequence $w_0 w_1 \ldots$. Finally, the *coding* $\phi^*(x)$ is defined to be the union of all the codings of generalized orbits of x. (Naturally, for all points x whose orbit under T never meets $\partial\mathcal{P} = \partial P_1 \cup \cdots \cup \partial P_r$, the codings $\phi(x)$ and $\phi^*(x)$ coincide.) Then, the set of points x such that $\phi^*(x)$ contains the coding beginning with a given word is closed. Thus $\phi^*(X)$ is also always closed, and $\overline{\phi(X)} \subset \phi^*(X)$. Both codings ϕ and ϕ^* are included in the main result.

1.3.1. *Example of the Baker map.* Encoding locally hyperbolic systems frequently give rises to full subshifts of finite type. For instance, at the most basic level, the well-known example of the *Baker map* gives rise to full shifts on two symbols. Let $X = [0, 1]\times[0, 1] \subset \mathbb{R}^2$ be partitioned by sets: $P_1 = [0, \frac{1}{2}) \times [0, 1]$ and $P_2 = [\frac{1}{2}, 1] \times [0, 1]$. Define the homeomorphisms $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$, $T_1(x, y) = (2x, 0.5y)$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$, $T_2(x, y) = (2x - 1, 0.5y + 0.5)$. The Baker map is defined by $T(x) = T_i(x)$ for $i = 1, 2$. The coding of orbits in the Baker map depends only on the first coordinate of points, and since the projection of T onto the first coordinate is a doubling map, it is not difficult to see that $\overline{\phi(X)} = \phi^*(X) = \{1, 2\}^{\mathbb{N}}$.

1.4. *The graph construction.* In this section, we argue that the symbolic coding defined previously is of fundamental importance as it gives rise to a space $\hat{X}$ on which T has a continuous extension $\hat{T}$, and hence it allows us to apply many standard results in dynamical systems.

The idea of the construction is to 'separate' sets of points encoded by the same sequences by placing them at different levels in the set $X \times \Omega$. Let $G = \{(x, \phi(x)),\ x \in X\}$ be the graph of $\phi : X \to \Omega$ topologized by the product metric $d_G : G \times G \to \mathbb{R}^+ \cup 0$, $d_G((x_1, \phi(x_1)), (x_2, \phi(x_2))) = \max\{d(x_1, x_2), d_{\Omega_r}(\phi(x_1), \phi(x_2))\}$, where d is a metric on X.

PROPOSITION 1.1. *The extension map $T_G : G \to G$, $(x, \phi(x)) \to (Tx, \phi(Tx))$ is continuous.*

Proof of Proposition 1.1. Let $x_0 \in P_i$. For all $x \notin P_i$, $d_G((x_0, \phi(x_0)), (x, \phi(x))) \geq d_{\Omega_r}(\phi(x_0), \phi(x)) \geq 1$. Therefore, in the limit statement below, we may also assume that $x \in P_i$.

$$
\begin{aligned}
&\lim_{(xx,\phi(x))\to(x_0,\phi(x_0))} T_G(x, \phi(x)) \\
&\qquad = \lim_{(x,\phi(x))\to(x_0,\phi(x_0))} (Tx, \phi(Tx)) && \text{by definition of } T_G \\
&\qquad = \lim_{(x,\phi(x))\to(x_0,\phi(x_0))} (T_i x, S(\phi(x))) && \text{since } x \in P_i \text{ and by (2)} \\
&\qquad = (T_i x_0, S(\phi(x_0))) && \text{by continuity of } T_i \text{ and } S \\
&\qquad = (T_i x_0, \phi(T x_0)) = T_G(x_0, \phi(x_0)) && \text{by (2) and definition of } T_G. \qquad \square
\end{aligned}
$$

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Let $\Pi_X : X \times \Omega_r \to X$ and $\Pi_{\Omega_r} : X \times \Omega_r \to \Omega_r$ be natural projections. We now have the following diagrams:

$$\begin{array}{ccc} G & \xrightarrow{T_G} & G \\ {\scriptstyle \Pi_X}\downarrow & & \downarrow{\scriptstyle \Pi_X} \\ X & \xrightarrow{T} & X \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{T_G} & G \\ {\scriptstyle \Pi_{\Omega_r}}\downarrow & & \downarrow{\scriptstyle \Pi_{\Omega_r}} \\ \phi(X) & \xrightarrow{S} & \phi(X) \end{array}$$

COROLLARY 1.1. *Let $\overline{G} = \hat{X}$. Let $\hat{T} : \hat{X} \to \hat{X}$ be the continuous extension of $T_G : G \to G$. Then $\Pi_{\Omega_r}|_{\hat{X}} : \hat{X} \to \overline{\phi(X)}$ is a continuous, surjective mapping and $S|_{\overline{\phi(X)}} : \overline{\phi(X)} \to \overline{\phi(X)}$ is a topological factor map of $\hat{T}$. In particular, the following diagram commutes:*

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\hat{T}} & \hat{X} \\ {\scriptstyle \Pi_{\Omega_r}}\downarrow & & \downarrow{\scriptstyle \Pi_{\Omega_r}} \\ \overline{\phi(X)} & \xrightarrow{S} & \overline{\phi(X)} \end{array}$$

1.5. *Subshifts.* In this section, we point out that our result can be applied to a broad class of subshifts which are not necessarily of finite type.

A subset $Y \subset \Omega_r$ which is closed and invariant ($\phi(Y) = Y$) under the shift map S is called a *subshift* (sometimes, when it is clear from the context, the shift map S restricted to a closed invariant set is also called a subshift.) The most basic types of subshifts are *subshifts of finite type* described by a finite set of forbidden words†.

Our main result can be applied to a broad class $\mathcal{Y}$ of subshifts $Y \subset \mathcal{Y}$ with the property that if the entropy of Y is positive, then Y contains a subshift of finite type of positive entropy. In particular, this class includes sofic, almost sofic and coded systems.

Sofic subshifts are subshifts presented by a finite labeled graph (multiple edges may carry the same label). The entropy of a sofic subshift Y is equal to the supremum of the entropies of subshifts of finite type contained in Y [**23**]. A simple example of a subshift which is sofic but not of finite type is the *even shift*, the set $F = \{11222112, 211211112, \ldots\}$ of all codings in Ω_r for which the digit one must appear in blocks of even length. Sofic subshifts were introduced by Weiss [**27**] as the smallest collection of subshifts consisting of all subshifts of finite type together with their topological factors. Sofic shifts are important for a number of reasons. They are analogous to regular languages in automata theory. They are also natural models for information storage and transmission.

A subshift Y is *almost sofic* if its entropy is the supremum of the entropies of subshifts of finite type contained in Y. A class of almost sofic subshifts contains sofic subshifts.

† In our setting, we consider one-sided subshifts. Beyond subshifts of finite type, other more general subshifts, i.e. sofic, almost sofic, and coded, are frequently studied as the double-sided subshifts. However, there is a natural one-to-one correspondence between double-sided subshifts and one-sided subshifts which preserves the entropy. Given a double-sided subshift Y, one can associate a one-sided subshift by considering the right-infinite blocks in elements of Y. One can then define the corresponding one-sided subshifts.

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Another generalization of sofic subshifts is the class of coded systems. A subshift Y is *coded* if its dense subset can be presented by a countable irreducible labeled graph (such a graph can be always chosen to be right-resolving [**7, 11**]). A coded system Y can also be characterized as the closure of the increasing union of irreducible subshifts of finite type contained in Y. Our result can also be applied to coded systems because a coded subshift of positive entropy contains a sofic subshift (and hence a subshift of finite type) of positive entropy†. (This follows, for example, from the fact that a right-resolving graph presentation of a non-periodic coded subshift must contain two distinct loops based at the same vertex, such as in diagram (3), giving rise to a sofic subshift of positive entropy.)

The class of coded systems contains the whole range of subshifts with Markovian behavior (e.g. synchronized systems) which are not necessarily of finite type and which have been studied in the literature (see [**20, 22**] where a number of further references are given).

2. *Theorem*

The main result of this section states that for typical generalized piecewise isometries, all subshift of finite type embeddings have zero entropy (Theorem 2.1). An exceptional case of a piecewise isometry giving rise to a subshift with positive entropy is illustrated in §5 (actually, in §5, we show how a full shift can be realized). The partition $\mathcal{P} = \{P_1, \ldots, P_r\}$ constructed there has the property that even though $P_i \cap P_j = \emptyset$ for $i \neq j$, $\overline{P_i} = \overline{P_j}$. Moreover, all are contained in $\partial P_1 \cup \cdots \cup \partial P_r$. These are precisely the kinds of partitions that we would like to avoid in our main result. We, therefore, introduce the following definitions.

Definition 2.1 (Regularity). A partition $\mathcal{P} = \{P_1, \ldots, P_r\}$ is *regular* if for all non-empty subsets $A_i \subset P_i$ and $A_j \subset P_j$, $\overline{A_i} \neq \overline{A_j}$ for all $i \neq j$.

Definition 2.2 (Exceptional set). The *exceptional set* E_T of a piecewise continuous map $T : X \to X$ with partition $\mathcal{P} = \{P_1, \ldots, P_r\}$ is the set of all points in X whose orbits under T intersect the set $\partial\mathcal{P} = \partial P_1 \cup \cdots \cup \partial P_r$ infinitely many times. Similarly, define the *exceptional set* E_T^* to be the set of points in X whose generalized orbits intersect the set $\partial\mathcal{P}$ infinitely many times.

Regularity is a very weak condition on the boundaries of the atoms—we shall propose sufficient conditions implicating regularity later in §5. Here, we just mention that, for example, in Euclidean spaces, partitions consisting of polygons or convex sets are regular.

THEOREM 2.1. *Let $X \subset Y$ be a compact set and let $T : X \to X$ be a generalized piecewise isometry with partition $\mathcal{P} = \{P_1, \ldots, P_r\}$ and induced homeomorphisms $\{T_1, \ldots, T_r\} \subset \mathcal{H}om(Y)$. Suppose that one of the following conditions holds:*

(i) *the partition $\mathcal{P}$ consists of regular sets; or*

(ii) *the coding $\phi(E)$ of the exceptional set E is countable.*

Then every subshift of finite type embedding of $\phi(X)$ has zero entropy. Moreover, if

(ii′) *$\phi^*(E^*)$ is countable (recall that always $\phi(E) \subset \phi^*(E^*)$),*

then all subshifts of finite type embeddings in $\phi^*(X)$ have zero entropy ($\phi(X) \subset \phi^*(X)$).

† It is not true, however, that a coded subshift must be almost sofic; see [**25**].

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COROLLARY 2.1. *Suppose that the hypothesis in Theorem 2.1 is satisfied. Then almost all sofic subshift embeddings of $\phi(X)$ have zero entropy.*

Since all mixing (non-constant) subshifts of finite type are irreducible and non-periodic (Proposition 4.5.10 in [**22**]), Theorem 2.1 yields the following corollary.

COROLLARY 2.2. *Suppose that the hypothesis in Theorem 2.1 is satisfied. Then all non-constant subshifts of finite type which are embedded in $\phi(X)$ are non-mixing.*

Remark 2.2. (Piecewise isometry for which $\phi(X)$ is a reducible subshift of finite type.) Theorem 2.1 yields that if a system of finite type is embedded in the symbolic dynamics of T, then it must be periodic or reducible (since irreducible, non-periodic subshifts of finite type have positive entropy). Both cases actually occur. Any piecewise isometry for which there exists a periodic point embeds a periodic subshift. The following is a simple example that the second case also occurs.

Let $T_0, T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotations by angles $\alpha_0, \alpha_1 \in (0, 2\pi)$, respectively, non-commensurable with π about the centers $S_0 = (-r_1, 0)$ and $S_1 = (r_2, 0)$, respectively ($r_1, r_2 \in \mathbb{R}^+$). Let $X = B(S_1, R_1) \cup B(S_1, R_2)$ be the union of the two closed discs such that $R_1 > r_1$, $R_2 > r_2$ and $R_2^2 - r_2^2 = R_1^2 - r_1^2$. Let $P_1 = \{(x, y) \in X : x < 0\}$ and $P_2 = \{(x, y) \in X : x \geq 0\}$. Then $T_1 P_1 \subset X$ and $T_2 P_2 \subset X$, hence the piecewise isometry $T : X \to X$ with induced isometries $\{T_1, T_2\}$ and the partition $\{P_1, P_2\}$ of X is well defined. Observe that the coding $\phi(X)$ contains the two constant codings $1 \ldots 1 \ldots$ and $2 \ldots 2 \ldots$ (the centers of rotations are fixed by T) and all codings $1 \ldots 122 \ldots$ consisting of a finite number of ones followed by an infinite number of twos. In other words, a reducible subshift of finite type given by the transition matrix $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is contained in $\phi(X)$.

Theorem 2.1 can be applied to all examples from §1.2 in order to obtain conclusions about $\sigma(X)$. In particular, in the case of the piecewise rotation on the lattice mesh, since the partition of X is regular (for example because all atoms are convex, see Proposition 5.1) Theorem 2.1 yields the following.

COROLLARY 2.3. *Let $T : X \mapsto X$, $Tx = Mx \bmod \mathbb{Z}^n$ be a piecewise rotation on the unit lattice mesh X. Then $\phi(X)$ does not embed sofic subshifts with positive entropy.*

3. *Proof of Theorem 2.1*

The proof of Theorem 2.1 is based on the following key lemma.

KEY LEMMA. *Suppose that an irreducible non-periodic subshift of finite type is embedded in $\phi(X)$; then the partition $\mathcal{P}$ is not regular. Moreover, there exists an uncountable set of codings $C \subset \phi(X)$ such that the orbit $\{T^n x, n \geq 0\}$ of a point $x \in X$ with code $\phi(x) \in C$ visits the sets ∂P_{j_1} and ∂P_{j_2} infinitely often, where $j_1, j_2 \in \{1, \ldots, r\}$ are distinct indexes.*

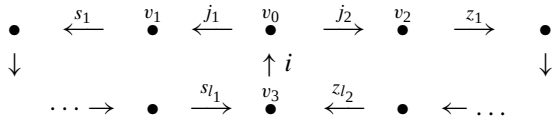
Proof of the Key Lemma. The proof consists of four steps. First, we include a well-known fact that the existence of an irreducible non-periodic subshift of finite type implies that there is a free semigroup of words $\Omega_r(w_1, w_2) \subset \phi(X)$ (step 1). This will imply that the corresponding subgroup of isometries K can be viewed as a compact group (step 2).

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The compactness of K will allow us to construct certain dense semigroup orbits (step 3). Finally, we will show that these dense semigroup orbits imply the irregularity of the partition and also that they correspond to codings in $\phi(X)$ of orbits visiting the boundaries of two distinct atoms infinitely often.

3.1. *Step 1.* First, we show that the graph of a non-periodic irreducible subshift of finite type contains two different loops based at the same point (Lemma 3.1, diagram (3.1)).

LEMMA 3.1. *Let $H \subset \Omega_r$ and suppose that an irreducible non-periodic subshift of finite type is embedded in H. Then there exist two finite sequences $w_1 = j_1 s_1 \dots s_{l_1} i$ and $w_2 = j_2 z_1 \dots z_{l_2} i$ of digits in $\{1, \dots, r\}$ beginning with distinct digits $j_1 \neq j_2$ such that $\Omega_r(w_1, w_2) \subset H$, where $\Omega_r(w_1, w_2)$ denotes the semigroup of all infinite codings generated by w_1 and w_2.*



(3)

Proof of Lemma 3.1. Let F be a non-periodic irreducible subshift and let $f : F \to H \subset F$ be a continuous injective map. Then the set $f(F)$ is conjugate to H. Since the conjugacy preserves finite type, irreducibility and non-periodicity, $f(F) \subset \phi(X)$ is also an irreducible non-periodic subshift. Let G be an irreducible labeled right-resolving graph presentation of $f(F)$. Since G is non-periodic, G contains a vertex v_0 connected by an edge with two distinct vertices v_1 and v_2. Since G is right-resolving the labels of the edges connecting v_0 with v_1 and v_2 are distinct: $j_1, j_2 \in \{1, \dots, r\}$, $j_1 \neq j_2$. Let $i \in \{1, \dots, r\}$ be the label of an edge connecting some vertex v_3 with v_1 (there always exists an edge terminating at any given vertex). Since the graph G is irreducible, there are allowable finite paths joining v_1 with v_3 and v_2 with v_3. Let $w_1 = j_1 s_1 \dots s_{l_1} i$ and $w_2 = j_2 z_1 \dots z_{l_2} i$ be the labels of such paths; see diagram (3). Hence, the semigroup $\Omega(w_1, w_2)$ of codings generated by composing w_1 and w_2 are allowed in $f(F) \subset H$. □

3.2. *Step 2.* We now show (Lemma 3.2) that a certain restriction of the group $\langle T_{w_1}, T_{w_2} \rangle$ generated by $T_{w_1} = T_i T_{s_{l_1}} \dots T_{s_1} T_{j_1}$ and $T_{w_2} = T_i T_{z_{l_2}} \dots T_{z_1} T_{j_2}$ has a compact closure.

LEMMA 3.2. *Define $T_{w_1} = T_i T_{s_{l_1}} \dots T_{s_1} T_{j_1}$, $T_{w_2} = T_i T_{z_{l_2}} \dots T_{z_1} T_{j_2}$, and let $U = \overline{\langle T_{w_1}, T_{w_2} \rangle^+ X}$ be the closure of the semigroup orbit of X. Then U is compact and $T_{w_1}(U) = U$, $T_{w_2}(U) = U$. Moreover, $\overline{\langle T_{w_1}|_U, T_{w_2}|_U \rangle}$ is a compact subgroup of $\mathcal{H}om(U)$ (topologized by the C^0 metric).*

Proof. We first use the following *shadowing* idea (estimates (4) and (5)) to show that the semigroup orbit of X

$$U^+ = \langle T_{w_1}, T_{w_2} \rangle^+ X$$

is bounded. Fix an arbitrary (finite) word w generated by w_1 and w_2 and a point $p \in X$. Let $w = w_{j_0} \dots w_{j_k}$ and $T_w = T_{w_{j_k}} \dots T_{w_{j_0}}$, where $j_k \in \{1, 2\}$. Since by Lemma 3.1,

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$\Omega_r(w_1, w_2) \subset \phi(X)$, there exists a point $x_w \in X$ whose coding begins with the word w. Therefore, $d_Y(T_w x_w, x_w) \leq \operatorname{diam}(X)$ ($\operatorname{diam}(X)$ denotes the diameter of X with respect to the metric d_Y), and the following key estimate holds for every $x_0 \in X$:

$$\begin{aligned} d_Y(T_w x_0, x_0) &\leq d_Y(T_w x_0, T_w x_w) + d_Y(T_w x_w, x_w) + d_Y(x_w, x_0) \\ &= d_Y(x_0, x_w) + d_Y(T_w x_w, x_w) + d_Y(x_w, x_0) \leq 3 \operatorname{diam}(X). \end{aligned} \tag{4}$$

Then,

$$\begin{aligned} d_Y(p, T_w X) &= \sup_{x_0 \in X} d_Y(p, T_w x_0) \leq \sup_{x_0 \in X} (d_Y(T_w x_0, x_0) + d_Y(x_0, p)) \\ &\leq \sup_{x_0 \in X} (3 \operatorname{diam}(X) + \operatorname{diam}(X)) = 4 \operatorname{diam}(X). \end{aligned} \tag{5}$$

Since the word w was chosen arbitrarily, from (5) it follows that U^+ is bounded in the metric d_Y. Hence, by (ii) in Definition 1.2, the set $U = \overline{U^+}$ is compact.

Next, we show that every element of the group $\langle T_{w_1}, T_{w_2} \rangle$ fixes U. Since for every $G \in \langle T_{w_1}, T_{w_2} \rangle^+$, $G(U^+) \subset U^+$ and hence $G(U) \subset U$. Moreover, observe that $G^{-1}(U) = U$ for every $G \in \langle T_{w_1}, T_{w_2} \rangle^+$. Suppose otherwise, let $u_0 \in U - G(U)$. Since U is compact, $d_Y(u_0, G(U)) = \eta > 0$. For every integer $m > 0$, $\operatorname{dist}(u_0, G^m u_0) \geq \eta$. Since G is an isometry, for every positive $k \neq m$, $d_Y(G^k u_0, G^m u_0) \geq \eta$, a contradiction with the compactness of U. Therefore, every element of the group $\langle T_{w_1}, T_{w_2} \rangle$ fixes U.

Finally, $\langle T_{w_1}|_U, T_{w_2}|_U \rangle$, is a subgroup of the group of isometries $\mathcal{J}(U)$ of the compact space U. Since the group of isometries of any compact space is always compact with respect to the C_0 topology, it follows that $\langle T_{w_1}|_U, T_{w_2}|_U \rangle$ has a compact closure. □

3.3. *Step 3.* We now prove a general lemma (Lemma 3.3) stating the existence of certain orbits of compact groups. We will then apply this lemma to the group $\Gamma = \overline{\langle T_{w_1}|_U, T_{w_2}|_U \rangle}$ and homeomorphisms $T_{w_1}|_U$ and $T_{w_2}|_U$.

Recall that a *right invariant metric* d_Γ of a topological group Γ is a metric generating the given topology and such that for all $V, W, Z \in \Gamma$, $d_\Gamma(VZ, WZ) = d_\Gamma(V, W)$.

LEMMA 3.3. *Let Γ be a compact group equipped with a right invariant metric d_Γ and let $g_1, g_2 \in \Gamma$. For a code $c = c_0 c_1 \cdots \in \Omega_2 = \{1, 2\}^{\mathbb{N}}$, we define the following partial orbits $S_1(c)$ and $S_2(c)$ contained in the semigroup $\langle g_1, g_2 \rangle^+$. Let $S_1, S_2 : \Omega_2 \to 2^{\langle g_1, g_2 \rangle^+}$ such that*

$$S_1(c_0 c_1 \dots) = \{g_{c_n} \dots g_{c_0} : c_{n+1} = 1\}$$

and

$$S_2(c_0 c_1 \dots) = \{g_{c_n} \dots g_{c_0} : c_{n+1} = 2\}.$$

Then there exists an uncountable set of codings $C \subset \Omega_2$ such that for every $c \in C$,

$$\overline{S_1(c)} = \overline{S_2(c)} = \overline{\langle g_1, g_2 \rangle^+} = \overline{\langle g_1, g_2 \rangle}.$$

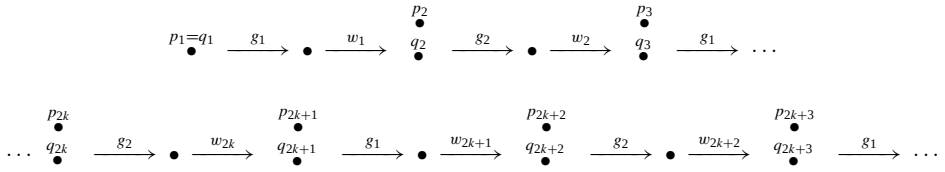
Proof. We first show that there exists at least one code $c \in \Omega_2$ such that $\overline{S_1(c)} = \overline{S_2(c)} = \overline{\langle g_1, g_2 \rangle^+} = \overline{\langle g_1, g_2 \rangle}$. The goal of the proof is to construct a certain sequence of increasing words $q_n \in \langle g_1, g_2 \rangle^+$ (q_n is a right subword of q_{n+1}) meeting all neighborhoods of all

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elements of $\langle g_1, g_2\rangle^+$. (The sequence $\{q_n\}$ will then define a code $c \in \Omega_2$ satisfying the conclusion of the lemma.)

The idea of the construction is to approximate words in $\langle g_1, g_2\rangle$ by words in $\langle g_1, g_2\rangle^+$, thus we need to observe that $\langle g_1, g_2\rangle^+$ is dense in $\langle g_1, g_2\rangle$. For $i = 1, 2$, the semigroup $\langle g_i\rangle^+$ has an accumulation point in Γ (here we use the compactness of Γ). Thus for every $\epsilon > 0$, there are some n, m, $n > m > 0$ such that $d_\Gamma(g_i^m, g_i^n) < \epsilon$, and by the right invariance of the metric d_Γ, $d_\Gamma(g_i^{-1}, g_i^{n'}) < \epsilon$ for $n' = n - m - 1 \geq 0$. In other words, every negative power of g_i can be approximated by positive powers of g_i. Since the group operation in Γ is continuous, for every $g \in \langle g_1, g_2\rangle^+$ and every $\epsilon > 0$, there is some $g' \in \langle g_1, g_2\rangle^+$ such that $d_\Gamma(g, g') < \epsilon$.

Let $\{p_n\}$ be the list of all elements in $\langle g_1, g_2\rangle^+$ such that for an arbitrarily large K, $\langle g_1, g_2\rangle^+ = \{p_{2k} : k > K\} = \{p_{2k+1} : k > K\}$. Using induction, we now construct two sequences $\{q_n\}$ and $\{w_n\}$ of words in $\langle g_1, g_2\rangle^+$. Let $q_1 = p_1$. By the density of $\langle g_1, g_2\rangle^+$ in $\langle g_1, g_2\rangle$, and the right invariance of the metric d_Γ, choose $w_1 \in \langle g_1, g_2\rangle^+$ so that $d_\Gamma(p_2, w_1 g_1 q_1) < 1$. Let $q_2 = w_1 g_1 q_1$. Choose $w_2 \in \langle g_1, g_2\rangle^+$ so that $d_\Gamma(p_3, w_2 g_2 q_3) < \frac{1}{2}$.



Suppose that we have defined the words $q_1, \ldots, q_{2k-1}, q_{2k}$ and $w_1, \ldots, w_{2k-1}, w_{2k}$. Then let

$$q_{2k+1} = w_{2k} g_2 q_{2k}. \tag{6}$$

We choose $w_{2k+1} \in \langle g_1, g_2\rangle^+$ so that

$$d_\Gamma(p_{2k+2}, w_{2k+1} g_1 q_{2k+1}) < \frac{1}{2k+1}. \tag{7}$$

Define

$$q_{2k+2} = w_{2k+1} g_1 q_{2k+1} \tag{8}$$

and choose $w_{2k+2} \in \langle g_1, g_2\rangle^+$ so that

$$d_\Gamma(p_{2k+3}, w_{2k+2} g_2 q_{2k+2}) < \frac{1}{2k+2}. \tag{9}$$

From (6) and (8), it follows that for every $n > 0$, $q_n \in \langle g_1, g_2\rangle^+$, and q_n is a right subword of q_{n+1}. Also, it follows that there is a code $c = c_0 \ldots c_l \ldots \in \Omega_2$ such that for every $k > 0$, $q_{2k} = g_{c_l} \ldots g_{c_0}$ for some $l > 0$ and $q_{2k+1} = g_{c_{l'}} \ldots g_{c_0}$ for some $l' > l$, and $\{q_{2k}\} \subset S_1(c)$. Also, it follows that $\{q_{2k+1}\} \subset S_2(c)$ for every $k \geq 0$.

Finally, observe that $\{q_{2k}\}$ and $\{q_{2k+1}\}$ are both dense in $\langle g_1, g_2\rangle^+$. Indeed, by (6)–(9), we obtain

$$d_\Gamma(p_{2k+1}, q_{2k+1}) = d_\Gamma(p_{2k+1}, w_{2k} g_2 q_{2k}) < \frac{1}{2k}$$

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and

$$d_\Gamma(p_{2k+2}, q_{2k+2}) = d_\Gamma(p_{2k+2}, w_{2k+1}g_1q_{2k+1}) < \frac{1}{2k+1}.$$

Since for an arbitrarily large K, $\{p_{2k} : k > K\} = \{p_{2k+1} : k > K\} = \langle g_1, g_2\rangle^+$, from the above inequalities, it follows that for an arbitrary $\epsilon > 0$, $\{q_{2k}\}$ and $\{q_{2k+1}\}$ are both ϵ-dense in $\langle g_1, g_2\rangle^+$. This means that $\{q_{2k}\}$ and $\{q_{2k+1}\}$ are dense in $\langle g_1, g_2\rangle^+$. Hence, since $\{q_{2k}\} \subset S_1(c) \subset \langle g_1, g_2\rangle^+$ and $\{q_{2k+1}\} \subset S_2(c) \subset \langle g_1, g_2\rangle^+$, we finally conclude that $\overline{S_1(c)} = \overline{S_2(c)} = \overline{\langle g_1, g_2\rangle^+} = \overline{\langle g_1, g_2\rangle}$.

We have shown the existence of at least one code $c \in \Omega_2$ satisfying the lemma. In order to conclude the lemma, we will construct an injective function: $f : \Omega_2 \to \Omega_2$ such that for every $\omega \in \Omega_2$, the code $f(\omega)$ will satisfy the conclusion of Lemma 3.3. First, extend the definition of $S_1, S_2 : \mathcal{B}_2 \to 2^{\langle g_1, g_2\rangle^+}$ on the set of finite sequences of finite words $\mathcal{B}_2$ of digits in $\{1, 2\}$. Let

$$S_1(c_0c_1 \dots c_m) = \{g_{c_n} \dots g_{c_0} : c_{n+1} = 1, n < m\}$$

and

$$S_2(c_0c_1 \dots c_m) = \{g_{c_n} \dots g_{c_0} : c_{n+1} = 2, n < m\}.$$

Let $\{c^n\}$ be the sequence of left subwords of the code c of length increasing to infinity such that the sets $S_1(c^n)$ and $S_2(c^n)$ are both $(1/n)$-dense in $\overline{\langle g_1, g_2\rangle}$. Observe that by the right metric invariance, given a set $A \in \overline{\langle g_1, g_2\rangle}$, if A is ϵ-dense in $\overline{\langle g_1, g_2\rangle}$ so must its right translation Ag for every $g \in \overline{\langle g_1, g_2\rangle}$. Hence, $S_1(\omega c^n)$ and $S_2(\omega c^n)$ are ϵ-dense in $\overline{\langle g_1, g_2\rangle}$ for every finite word ω. In particular, for all $\omega_i \in \{1, 2\}$, the sets $S_1(\omega_0c^0\omega_1c^1\omega_2 \dots c^n)$ and $S_2(\omega_0c^0\omega_1c^1\omega_2 \dots c^n)$ are also both $(1/n)$-dense in $\overline{\langle g_1, g_2\rangle}$. We can thus define

$$f : \Omega_2 \to \Omega_2, \ f(\omega_0\omega_1 \dots) = \omega_0c^0\omega_1c^1\omega_2 \dots \omega_nc^n \dots.$$

Hence, $f(\omega)$ satisfies $\overline{S_1(f(\omega))} = \overline{S_2(f(\omega))} = \overline{\langle g_1, g_2\rangle}$. The set $f(\Omega_2)$ is uncountable for f is injective and Ω_2 is uncountable. Consequently, the uncountable set of codings $C = f(\Omega_2)$ satisfies the lemma. □

3.4. *Step 4.* We are ready to undertake the last step in proving the Key Lemma. Apply Lemma 3.3 to the group $\Gamma = \overline{\langle T_{w_1}|_U, T_{w_2}|_U\rangle}$ (defined in Lemma 3.2), homeomorphisms $T_{w_1}|_U$ and $T_{w_2}|_U$ (Lemma 3.2), and the metric $d_\Gamma = d_{C_0}|_{\Gamma\times\Gamma}$. (The C_0 metric on $\mathcal{H}om(U)$ is easily checked to be a right-invariant metric on Γ.) There are uncountably many codings $c = c_0c_1 \dots \in \Omega_2$ such that both sets

$$A_1 = \{T_{w_{c_n}} \dots T_{w_{c_0}}|_U : c_{n+1} = 1\}$$

and

$$A_2 = \{T_{w_{c_n}} \dots T_{w_{c_0}}|_U : c_{n+1} = 2\}$$

are dense in the semigroup $\langle T_{w_1}|_U, T_{w_2}|_U\rangle^+$.

In particular, for every $x \in X \subset U$ where U is compact,

$$\overline{A_1x} = \overline{A_2x}. \tag{10}$$

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Given a code $c \in C$, choose $x = x_c$ such that $\phi(x_c) = w_{c_0} \dots w_{c_n} \dots$. Since $\Omega_r(w_1, w_2) \subset \phi(X)$ (Lemma 3.1), for every $c \in C$, such a choice is possible. Since the words $w_1 = j_1 \dots i$ and $w_2 = j_2 \dots i$ begin with the distinct digits j_1 and j_2 respectively, by the definition of A_1, A_2 and by the definition of the coding, we obtain

$$A_1 x_c \subset P_{j_1} \quad \text{and} \quad A_2 x_c \subset P_{j_2}. \tag{11}$$

The statements (10) and (11) imply that the atoms P_{j_0} and P_{j_1} are the elements of an irregular partition. Also, $A_1 x_c \subset \partial P_{j_1}$ and $A_2 x_c \subset \partial P_{j_2}$. Moreover, by the definition of A_1 and A_2, for $i = 1, 2$,

$$T^n x_c \in A_i x_c \text{ for infinitely many } n. \tag{12}$$

Finally, the function $C \to \Omega_r(w_1, w_2)$, $c_0 c_1 \dots \to w_{c_0} w_{c_1} \dots$ is injective, since w_0 and w_1 begin with distinct digits, therefore, the set of points $x_c \in X$ satisfying (12) has an uncountable number of codings. This concludes the proof of the Key Lemma. □

Theorem 2.1 immediately follows from the Key Lemma. If the partition $\mathcal{P}$ is regular or the coding of the exceptional set E_T is countable, then an irreducible non-periodic subshift of finite type cannot be embedded in $\phi(X)$. Since the entropy of a subshift of finite type is realized by the entropy of one of its irreducible embeddings, and the entropy of a periodic subshift is zero, it follows that only subshifts of finite type of zero entropy can be embedded in $\phi(X)$.

The proof of the last part of Theorem 2.1 related to generalized orbits is identical to the previous part. □

4. *Examples with irregular partition inducing full shift*

In this section, we show that without the hypothesis (i) or (ii), Theorem 2.1 does not hold. We construct a piecewise isometry in the Euclidean space whose coding map is surjective, in particular the coding image is a full shift on r symbols (hence embeds all subshifts).

In Euclidean spaces there are no examples of piecewise isometries on the interval with surjective coding map. (An iteration of a non-trivial translation eventually takes a point out of an interval.) The 'smallest' compact space on which piecewise isometries inducing surjective codings may exist is a circle [**13**].

THEOREM 4.1. *Let $X = \{e^{2\pi i x} : x \in [0, 1)\}$ be the unit circle. There exists a piecewise isometry $T : X \to X$ with partition $\mathcal{P}$ such that the associated coding map $\phi : X \to \Omega$ is surjective.*

If X is the unit disk, then there exists a piecewise isometry $T : X \mapsto X$ with a Lebesgue measurable partition $\mathcal{P}$ such that the associated coding map $\phi : X \mapsto \Omega$ is surjective; see Remark 4.3.

Proof of Theorem 4.1. Let T_i, for $i = 1, \dots, r$, be the rotation of X by the angle $2\pi\theta_i$, where $\theta_i \in [0, 1)$ and the numbers $\{1, \theta_1, \dots, \theta_r\}$ are rationally independent. Let $V(1, \theta_1, \dots, \theta_r)$ denote the integer lattice spanned by vectors $\{1, \theta_1, \dots, \theta_r\}$. We partition $[0, 1)$ into disjoint classes by the following equivalence relation: $\sim$, $x \sim y$ if and only if $x - y \in V(1, \theta_1, \dots, \theta_r)$. Since the number of different classes is countable, a set

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A consisting of exactly one element from each class (here we use the axiom of choice) has cardinality α_0. Since Ω_r also has cardinality α_0, it follows that there is a bijection $b : \Omega_r \to A$. The goal is to construct a partition $\{P_1, \dots, P_r\}$ of X such that given $w \in \Omega_r$, the point $e^{2\pi i b(w)} \in X$ has precisely the code w under the piecewise rotation $T : X \to X$ generated by rotations T_i and the partition $\{P_1, \dots, P_r\}$.

First, for $k = \{1, \dots, r\}$, define the following sets Q_k (we will later put $P_k = Q_k$ for $k = 2, \dots, r$ and $P_1 = X - \bigcup_{2 \le i \le r} P_k$):

$$Q_k = \left\{ e^{2\pi i x} : x = \sum_{j=0}^{n-1} \theta_{c_j} + b(c_0 \dots c_{n-1} k \dots) \text{ for some } n > 0 \text{ and } c_j \in \{1, \dots, r\} \right\}. \tag{13}$$

We show that the sets $\{Q_1, \dots, Q_r\}$ are pairwise disjoint. Suppose that for some $k, k' \in \{1, \dots, r\}$, $k \neq k'$, and $Q_k \cap Q_{k'} \neq \emptyset$. Then, for some $n, n' \in \mathbb{N}$, $n \le n'$, $c_j \in \{1, \dots, r\}$ and $c'_j \in \{1, \dots, r\}$,

$$\sum_{j=0}^{n-1} \theta_{c_j} + b(c_0 \dots c_{n-1} k \dots) = \sum_{j=0}^{n'-1} \theta_{c'_j} + b(c'_0 \dots c'_{n'-1} k' \dots) \pmod 1,$$

or,

$$\sum_{j=0}^{n-1} \theta_{c_j} - \sum_{j=0}^{n'-1} \theta_{c'_j} = b(c_0 \dots c_{n-1} k \dots) - b(c'_0 \dots c'_{n'-1} k' \dots) \pmod 1. \tag{14}$$

The right-hand-side of equation (14) is the difference of two numbers in A belonging to the same equivalence class $\sim$. Hence, $b(c_0 \dots c_{n-1} k \dots) = b(c'_0 \dots c'_{n'-1} k' \dots)$ and since b is a bijection,

$$c_0 \dots c_{n-1} k \dots = c'_0 \dots c'_{n'-1} k' \dots . \tag{15}$$

From (14), it also follows that

$$\sum_{j=n}^{n'-1} \theta_{c_j} = 0 \pmod 1. \tag{16}$$

From (16) and the rational independence of $\{1, \theta_1, \dots, \theta_r\}$, we conclude that $n = n'$. From (15), it follows that $k = k'$, a contradiction. Hence, the sets $\{Q_1, \dots, Q_r\}$ are pairwise disjoint.

Define the partition $\{P_1, \dots, P_r\}$ of X. Let $P_k = Q_k$ for $k = 2, \dots r$ and $P_1 = X - \bigcup_{2 \le k \le r} P_k$. Let $T : X \to X$ be the piecewise rotation generated by the rotations $\{T_1, \dots, T_r\}$ and the partition $\{P_1, \dots, P_r\}$. Finally, we show that given $w = c_0 \dots c_n \dots \in \Omega_r$, the point $x_w = e^{2\pi i b(w)} \in X$ has code w. We use induction on n. If $n = 0$, then by (13), $e^{2\pi i b(w)} \in Q_{c_0} \subset P_{c_0}$. Suppose now that the coding of x_w begins with the word $w_{c_0} \dots w_{c_{n-1}}$. Then

$$\begin{aligned} T^n x_w &= T_{c_{n-1}} \dots T_{c_0}(e^{2\pi i b(w)}) = (e^{2\pi i \sum_{j=0}^{n-1} \theta_{c_j}})(e^{2\pi i b(w)}) \\ &= e^{2\pi i [\sum_{j=0}^{n-1} \theta_{c_j} + b(w_{c_0} \dots w_{c_n} \dots)]} \in Q_{w_{c_n}} \subset P_{w_{c_n}}. \end{aligned}$$

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Since the nth iterate of x_w lies in $P_{w_{c_n}}$, w_{c_n} is the nth digit in the coding of x_w. We have shown that given a code w, there exits a point $x_w \in X$ realizing w, and hence that the coding map of the piecewise rotation T is surjective. □

Remark 4.2. The induced isometries in the example above have a common fixed point. Using the shadowing idea from Lemma 3.2, it is not difficult to show that the induced isometries $T_1, \ldots, T_r$ of all Euclidean piecewise isometries acting on spaces with compact closures and whose codings are surjective must have a common fixed point—they are elements of $O(\mathbb{R}, n)$. The shadowing idea implies that the action of the isometries $T_1, \ldots, T_r$ can be reduced to a compact space and in Euclidean spaces this implies the existence of a fixed point. Hence, $T_1, \ldots, T_r \subset O(\mathbb{R}, n)$.

Remark 4.3. Let X be the unit disk. One can simplify the proof above, not use the axiom of choice, and construct a measurable partition $\mathcal{P}$ of X for which there exists a piecewise isometry with surjective coding map. Let f be a bijection $\Omega_r \mapsto (0, 1]$. Given a code $w \in \Omega$, we partition the concentric circle C_w of radius $f(w)$ so that it contains an orbit of a point realizing the code w. For $k = \{1, \ldots, r\}$, define the following sets Q_k:

$$Q_k = \left\{ te^{2\pi ix} : x = \sum_{j=0}^{n-1} \theta_{c_j}, , t = f(c_0 \ldots c_k \ldots) \text{ for some } n > 0 \text{ and } c_j \in \{1, \ldots, r\} \right\}. \tag{17}$$

Let $P_k = Q_k$ for $k = 2, \ldots, r$ and $P_1 = X - \bigcup_{2 \le i \le r} P_k$. The rational independence of $1, \theta_1, \ldots, \theta_r$ guarantees that the sets $Q_1, \ldots, Q_r$ (and hence the sets $P_1, \ldots, P_r$) are pairwise disjoint. As in the proof above, one checks that given a coding $w \in \Omega_r$, the point $f(w) \in X$ has the code w. The constructed partition is Lebesgue measurable, because each $Q_k \subset \langle T_1, \ldots, T_r \rangle^+(0, 1]$ and $\langle T_1, \ldots, T_r \rangle^+(0, 1]$ is a countable union of segments of zero (two-dimensional Lebesgue) measure, hence each Q_k has Lebesgue measure zero.

Remark 4.4. Let $M = \bigcap_{n \ge 0} T^n X$. Observe that, by the definition of Q_k, $Q_k \cap M = \emptyset$. Hence, $M \subset P_1$. Since any T-invariant Borel measure on X must be concentrated on M, and since $T|_M = T_1|_M$ is a rotation, it follows that the measure-theoretic entropy (with respect to any Borel T-invariant measure) of T is zero.

We do not know if there exists a piecewise isometry $T : X \mapsto X$ with a T-invariant Borel measure on X of positive entropy. (A lift of a Bernoulli measure from the shift space does not result in a Borel measure.)

Remark 4.5. The existence of a piecewise isometry on the unit circle with surjective coding map was suggested to the author by Michael Boshernitzan.

5. *Concluding remarks*

5.1. *Sufficient conditions implying regularity.* In this section, we make a few topological remarks about the regularity hypothesis on the atoms.

PROPOSITION 5.1. *In $\mathbb{R}^n$, a partition $\mathcal{P}$ consisting of convex sets and any partition homeomorphic to $\mathcal{P}$ is regular.*

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Proof. Since regularity is a topological invariance in the space of all partitions, it is enough to prove Lemma 5.1 for partitions consisting of convex sets. Let $\mathcal{P} = \{P_1, \ldots, P_r\}$ be a partition consisting of convex sets. Let $A_i \in P_i, A_j \in P_j$. We show that $\overline{A_i} \neq \overline{A_j}$ by induction on $d = \max(\dim P_i, \dim P_j)$. If $d = 0$ then P_i and P_j are points and the conclusion is clear. Suppose that $d > 0$. There exists a hyperplane H (of dimension $d-1$) separating the convex sets P_i and P_j (P_i and P_j are contained in the opposite sides of the closed half-spaces determined by H). Suppose that for some $A_i \subset P_i$, $A_j \subset P_j$, $\overline{A_i} = \overline{A_j}$. Then $A_i \subset P_i \cap H$ and $A_j \subset P_j \cap H$. By the inductive hypothesis applied to the convex sets $P_i \cap H$ and $P_j \cap H$ whose dimensions do not exceed $d-1$, we conclude $\overline{A_i} \neq \overline{A_j}$, a contradiction. □

More generally, we note that in some cases regularity can be checked effectively with the aid of the following proposition.

PROPOSITION 5.2. *Define the boundary operator $\beta(U) = U \cap \overline{\partial U \cap U^c}$. Suppose that for every atom P_i of the partition $\mathcal{P} = \{P_1, \ldots, P_r\}$ there is some positive integer k such that $\beta^k(P_i) = \emptyset$; then $\mathcal{P}$ is regular.*

In Euclidean spaces, the boundary operator β can reduce the dimension of sets. In particular, Proposition 5.2 can be used to show that generalized polytopes are regular. (A *generalized polytope* is a subset of $\mathbb{R}^n$ consisting of a finite number of disjoint simplexes: $\{\alpha_1 v_1 + \cdots + \alpha_k v_k, 0 < \alpha_i < 1, 1 \leq k \leq n\}$. It can be shown that for a generalized polytope $U \subset \mathbb{R}^n$, $\dim(\beta(U)) \leq \dim(U) - 2$, where $\dim(\emptyset) = -1$ and the dimension of a polytope is defined to be the maximum dimension of its simplexes.)

Proof of Proposition 5.2. Let a set U be called *regular* if for all non-empty subsets $A \subset U$ and $B \subset U^c$, $\overline{A} \neq \overline{B}$. We first show that a set U is regular if $\beta(U)$ is regular. Suppose otherwise, let $A \subset U$, $B \subset U^c$ and $\overline{A} = \overline{B}$. Then $B \subset \partial U \cap U^c$ and $A \subset \overline{B} \subset \overline{\partial U \cap U^c}$. Thus $A \subset U \cap \overline{\partial U \cap U^c} = \beta(U)$ and $B \subset U^c \subset \beta(U)^c$. However, the regularity of $\beta(U)$ implies that $\overline{A} \neq \overline{B}$, a contradiction. Since the empty set is regular, the set $\beta^k(P_i) = \emptyset$ for some k is regular, and thus P_i is regular. In particular, the partition $\mathcal{P}$ is also regular. □

5.2. *Questions.* (1) Keith Burns asked whether and how far the assumption of the existence of an invariant metric preserved by induced homeomorphisms can be weakened or changed and still restrict possible subshifts of finite type to reducible or periodic only. For instance, the hypothesis that there exists a measure μ on Y invariant under the action of $\langle T_1, \ldots, T_r \rangle$ is too weak. (In the Baker map illustrated in §1, both induced homeomorphisms T_1 and T_2 preserve Lebesgue measure but $\overline{\phi(X)} = \phi^*(X) = \{1, 2\}^{\mathbb{N}}$ embeds all sofic subshifts.)

(2) In this article, we showed that sofic systems embedded in the symbolic dynamics of a generalized piecewise isometry with a regular partition have zero entropy. In general metric spaces, if the partition is regular, the question whether the restriction of the shift map to $\overline{\phi(X)}$ has zero entropy remains open. Corollary 2.1 suggests that constructions of systems with positive entropy on the symbolic space, if at all exist, should not be based on sofic subshifts.

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In the very particular case of two-dimensional invertible piecewise isometries, Gutkin and Haydn showed that this entropy is 0 [**15**]. In the Euclidean higher-dimensional cases, the group of isometries contains free subgroups, and this contributes to the complexity of the dynamics and might distinguish higher-dimensional cases from $\mathbb{R}^2$.

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