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PAPER

Prior's ideal language

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Abstract

We present an axiom system for what we call Prior's Ideal Language and prove its completeness and pure completeness with respect to general models. With this is done, we explain, with examples, why this system provides a useful setting for exploring Arthur Prior's work.

Keywords: Arthur Prior; modal logic; hybrid logic; propositional quantifiers; universal modality; downarrow; Henkin semantics; general models; tense logic; description logic

1. Introduction

Arthur Prior is best known as the father of tense logic, but he also invented hybrid logic, arguably invented description logic as well, and made heavy use of propositional quantification throughout his logical career. In this paper, we introduce a language which combines the components we consider central to Prior's later work. We call the result *Prior's Ideal Language* (PIL). Most of this paper is devoted to proving a general completeness result for PIL; the final part of the paper explains in more detail some of the ways PIL is relevant to Prior's logical investigations.

PIL is easy to describe using contemporary terminology: it is a language of strong hybrid logic augmented with propositional quantifiers. The 'hybrid logic' part of this description means that we have a standard modal logic augmented with nominals and satisfaction operators, the 'strong' part means that we also have a universal modality and the downarrow binder, and the 'propositional quantifier' part means that we can quantify over the propositional variables of the underlying modal language. This is a highly expressive language – the inclusion of propositional quantification means we are working with a significant chunk of second-order logic – but we shall show that this system can be naturally 'tamed' by working with a Henkin-style semantics. In particular, we will show that there is an axiomatisation of this system that is complete – indeed, *purely complete* – with respect to what modal logicians call general models. Pure completeness tells us that not only is the basic logic complete, but that a wide range of axiomatic extensions are complete as well. We illustrate why this is useful with examples relevant to Prior's later work.

We proceed as follows. In Section 2, we define the syntax of PIL, drawing special attention to the soft-QDF formulas that play a key role in our axiomatisation. In Section 3, we introduce general frames and models and give a Henkin-style satisfaction definition for PIL. In Section 4, we present our axiomatisation and prove soundness, and then in Section 5, we prove basic and pure completeness. In Section 6, we explain in more detail why PIL is relevant to the study of Prior's work. Section 7 concludes.

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The present paper is a revised and extended version of the workshop paper Blackburn et al. (2023). In the workshop version, we extended basic hybrid logic with propositional quantification; in this paper, we extend strong hybrid logic with proportional quantification.

2. A Strong Hybrid Language with Propositional Quantifiers

We begin by defining PIL, the strong hybrid language with propositional quantifiers that we will work with for most of this paper. So, assume we have some language of basic (propositional) modal logic – that is, a propositional language built over a collection of propositional variables $PVAR = \{p, q, r, \dots\}$ using Boolean connectives and unary operators $\Box$ and $\Diamond$. In modal logic, propositional variables are standardly thought of as being true at some subset of possible worlds; this intuition lies at the heart of the Kripke semantics for modal logic, and nowadays it is common to identify 'propositions' with 'subsets of possible worlds'. We will not discuss basic modal logic further here, but turn straight to the task of enriching this simple language. It will be useful to think of the enrichment as having three main components; all three components play a prominent role in Prior's work, as we shall discuss in more detail in Section 6.

Component 1: Basic Hybrid Logic.

We turn our basic modal language into a basic hybrid language by adding *nominals* and *satisfaction operators*. Nominals are usually written i, j, k, and, crucially, each nominal is true at a *unique* possible world. Nominals play two syntactic roles in basic hybrid logic. First, they can be used as atomic formulas, in precisely the same way as ordinary propositional variables p, q, and r are. Because of the 'true at a unique world' restriction on their interpretation, in this first role, nominals can be thought of either as 0-place world-naming modalities or as special world-naming propositions; either way, nominals 'name' the unique worlds they are true at. Second, any nominal i can occur as a subscript to the symbol @. Any such $@_i$ is called a *satisfaction operator*, and for any formula φ, $@_i\varphi$ is true at a world w iff φ is true at the world that i names (i.e., at the unique world where i is true). Satisfaction operators can be thought of as 1-place *rigidifying modalities*. Note that any expression of the form $@_i\varphi$ is either true at all worlds or false at all worlds: it rigidly denotes the truth value that φ has at the world that i names. Any formula of the form $@_i\varphi$ is called a *satisfaction statement*.

Component 2: Strong Hybrid Logic.

We turn our basic hybrid language into a strong hybrid language by adding the *universal modality* and the *downarrow binder*. The box form of the universal modality is A, and $A\varphi$ means φ is true at all possible worlds in a model; its diamond form is E, and $E\varphi$ means φ is true at some world. A formulas like $\downarrow i\varphi$ allows us to assign a nominal to the possible world at which we are evaluating. More precisely, $\downarrow i\varphi$ assigns nominal i as a name for the world w at which we are evaluating, and φ is then evaluated with all (free) occurrences of i in φ interpreted as true at w ('name this world i and evaluate φ accordingly!'). As we shall see, downarrow is highly expressive, especially in combination with the universal modality.

Component 3: Propositional quantifiers

Lastly, we add propositional quantification. That is, we also allow ourselves to write expressions like $\forall p\varphi$, where p is a propositional variable of the underlying basic modal language.

Accordingly, we define the syntax of PIL as follows. Let $PVAR = \{p, q, r, \dots\}$ be a set of propositional variables and let $NOM = \{i, j, k \dots\}$ be a set of *nominals*. We assume that both sets are countable and disjoint, write $ATOM$ for $PVAR \cup NOM$ and call any element of $ATOM$ an *atomic symbol*. Then the language $\mathcal{L}$ of PIL built over $ATOM$ is defined by the following grammar:

$$\varphi ::= p \mid i \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid @_i\varphi \mid \forall p\varphi \mid A\varphi \mid \downarrow i\varphi,$$

where $p \in PVAR$ and $i \in NOM$. Other Boolean connectives are defined as expected, and $\Diamond$ and E are defined by $\neg\Box\neg \equiv \Diamond$ and $\neg A\neg \equiv E$. Similarly, $\exists p$ is defined by $\neg\forall\neg p \equiv \exists p$.

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It is clear from this definition that nominals in PIL can occur in formulas in three ways: either as an atomic formula (i.e., in *formula position*), as part of a satisfaction operator (in *operator position*), or right after a downarrow binder. Downarrow is a binder: occurrences of $\downarrow i$ bind all the free occurrences of i they have scope over. Note, however, that occurrences of $@_i$ do not bind occurrences of i; as we remarked above, satisfaction statements are modal operators. Unsurprisingly, occurrences of $\forall p$ bind all the free occurrences of the propositional variable p in their scope. The notions of free and bound nominals and propositional variables are defined standardly, and a *sentence* is a formula that contains no free nominals and no free propositional variables.

The result of substituting a formula ψ for a propositional variable q occurring in some formula φ, written $\varphi[\psi/q]$, is defined in the expected way. It is always possible to carry out such a substitution safely (i.e., without accidental binding) by relabelling the bound variables in φ. Similarly, $\varphi[j/i]$ is the result of substituting a nominal j for a nominal i; by relabelling the bound nominals in φ, such substitutions can always be carried out safely too.

In the Hilbert-style system presented in Section 4, the universal instantiation axiom has a side condition: only *quantifier and downarrow free formulas with no nominals in formula position* can be used to instantiate universal quantifications. We call such formulas soft-QDF formulas. The 'soft' part means: no nominals in formula position. The QDF part means: quantifier and downarrow free. That is, soft-QDF formulas are built using the grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid @_i\varphi \mid A\varphi,$$

where $p \in PVAR$ and $i \in NOM$. A substitution $\varphi[\psi/q]$ is called a *soft-QDF substitution* iff ψ is a soft-QDF formula.

Soft-QDF formulas behave well with respect to the Henkin-style semantics, that is, they are guaranteed to denote an admissible set in this semantics. On the other hand, nominals denote singleton sets, which need not be admissible, while downarrow can be thought of as introducing 'a hidden nominal in formula position' (we will see why in Section 6). Similarly, a quantified formula denotes an intersection of admissible sets, but admissible sets are only required to be closed under finite intersections, not arbitrary ones (see Definition 7 and Lemma 8).

3. A Henkin-Style Semantics

Our first goal in this paper is to prove completeness and pure completeness result for PIL. But we immediately face a problem. The basic propositional modal logic **K** (the set of basic modal formulas valid on all frames) is decidable in PSPACE. But when propositional quantifiers are added, and standardly interpreted on Kripke frames and models, the resulting set of validities is not even recursively enumerable! This was shown in Kit Fine's pioneering paper on modal languages with propositional quantifiers, namely Fine (1970). Such languages may look simple, but they are powerful second-order systems.

Nearly 75 years ago, Leon Henkin introduced a way of 'taming' higher-order logics; see Henkin (1950).[1] His idea was to increase the number of models in a semantically natural way, thereby reducing the number of validities to the point where validity became recursively enumerable. An important paper by S. K. Thomason introduced a Henkin-style strategy into modal logic; see Thomason (1972). Thomason first showed an unexpected result: there are Priorean tense logics that are not complete with respect to any class of frames. The definition of frame validity requires us to quantify across *all* subsets of a Kripke frame, that is, across *all* propositions, and Thomason gave an elegant example showing that this form of second-order quantification was far from innocent. But Thomason also introduced a Henkin-style cure: he also showed that every Priorean tense logic is complete with respect to a class of what are now called *general frames*. Thomason's use of general frames to 'tame' standard Kripke semantics has become a fundamental semantical tool in modal logic, and general Kripke frames and models will provide the foundation of our semantics

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for PIL. The technical question that drives this paper is thus: *do general frames allow us to 'tame' the basic logic of PIL, and all its pure extensions?*

A general frame is a Kripke frame $\langle W, R\rangle$ together with a selection of propositions Π, that is, subsets of W. We don't insist that all subsets of W belong to Π; we simply insist that Π has enough structure to behave like a set of propositions. In particular, in any general frame, the set of admissible propositions should be closed under the operations corresponding to the Boolean operators and $\Box$. So there are a lot more general frames than frames – a single Kripke frame gives rise to multiple general frames, one for each choice of Π – and, it turns out that this expansion successfully 'tames' the set of validities. So, writing $\mathcal{P}(W)$ for the powerset of W, we define:

Definition 1 (Kripke frames, general frames, standard frames). *A Kripke frame is a pair $\mathcal{F} = \langle W, R\rangle$ where W is a non-empty set (of worlds) and R is a binary relation on W (the accessibility relation between worlds). A general frame is a pair $\mathcal{G} = \langle \mathcal{F}, \Pi\rangle$ where $\mathcal{F}$ is a Kripke frame and Π is a non-empty subset of $\mathcal{P}(W)$ that is closed under the following three conditions:*

- *relative complement: if $X \in \Pi$, then $W - X \in \Pi$*
- *intersection: if $X, Y \in \Pi$, then $X \cap Y \in \Pi$*
- *modal projection: if $X \in \Pi$ then $\{w \in W \mid \forall v(wRv \rightarrow v \in X)\} \in \Pi$.*

We call a subset Π of $\mathcal{P}(W)$ that satisfies these conditions a closed set of admissible propositions, and we call the elements of Π admissible propositions. A general frame is called a standard frame iff $\Pi = \mathcal{P}(W)$.

Next we must interpret PIL on general frames. The interpretation of the nominals will be taken care of by a naming function (or *nomination*) N assigning a world to each nominal, while the interpretation of the propositional variables will be given by a modal valuation function V.

Definition 2 (General models and standard models). *A general model $\mathcal{M}$ based on a general frame $\langle W, R, \Pi\rangle$ is a tuple $\langle W, R, \Pi, N, V\rangle$ where $N : NOM \rightarrow W$ and $V : PVAR \rightarrow \Pi$. A standard model is a model based on a standard frame.*

This definition builds in our central semantic design decision for PIL: *the interpretation of the nominals is independent of the choice of* Π. Nominals directly 'tag' arbitrary worlds via the nomination. This is important, because Π may not contain all – or even any – singleton subsets of W as admissible propositions. Our nominals ignore Π. They are hardwired to the underlying Kripke frame.

Definition 3 (Variants). *Let $\mathcal{M} = \langle \mathcal{G}, N, V\rangle$ and $\mathcal{M}^* = \langle \mathcal{G}, N', V'\rangle$ be general models. $N' \sim_i N$ means that N' agrees with N on all elements of NOM, save possibly on the nominal i. We say that N' is an i-variant of N. We apply analogous definitions for valuation functions V and V'.*

Now for a Henkin-style definition of satisfaction and truth:

Definition 4 (Satisfaction and truth). *Let $\mathcal{M} = \langle \mathcal{G}, N, V\rangle$ be a general model based on a general frame $\mathcal{G} = \langle W, R, \Pi\rangle$. We define what it means for $\mathcal{M}$ to satisfy a formula at a world w:*

- *$\mathcal{M}, w \models i$ iff $w = N(i)$, for any $i \in NOM$*
- *$\mathcal{M}, w \models p$ iff $w \in V(p)$, for any $p \in PVAR$*
- *$\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$*

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- $\mathcal{M}, w \models \varphi \wedge \psi$ *iff* $\mathcal{M}, w \models \varphi$ *and* $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \Box\varphi$ *iff for all* $v \in W$, *if* wRv *then* $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models @_i\varphi$ *iff* $\mathcal{M}, N(i) \models \varphi$
- $\mathcal{M}, w \models \forall p\varphi$ *iff for all* $\mathcal{M}' = \langle \mathcal{G}, N, V' \rangle$ *where* $V' \sim_p V$, *we have* $\mathcal{M}', w \models \varphi$
- $\mathcal{M}, w \models A\varphi$ *iff for all* $v \in W$, $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models {\downarrow} i\varphi$ *iff* $\mathcal{M}', w \models \varphi$ *where* $\mathcal{M}' = \langle \mathcal{G}, N', V \rangle$ *and* $N' \sim_i N$ *such that* $N'(i) = w$.

We say that a formula φ *is true at a world* w *in* $\mathcal{M}$ *iff* $\mathcal{M}, w \models \varphi$.

Definition 5 (Validity and consequence). *A formula* φ *is valid in a general model* $\mathcal{M}$ *iff it is true at all worlds in* $\mathcal{M}$; *we write this as* $\mathcal{M} \models \varphi$. *A formula* φ *is valid iff it is valid in all general models; and we write this as* $\models \varphi$.

A formula φ *is a consequence of a set of formulas* Γ, *written* $\Gamma \models \varphi$, *iff for all general models* $\mathcal{M}$ *and all worlds* w *in* $\mathcal{M}$, *if* $\mathcal{M}, w \models \Gamma$ *then* $\mathcal{M}, w \models \varphi$. *Here* $\mathcal{M}, w \models \Gamma$ *means: for all formulas* $\gamma \in \Gamma$, $\mathcal{M}, w \models \gamma$. *Note:* φ *is valid iff* $\emptyset \models \varphi$.

We could also have defined notions of *standard validity* and *standard consequence*; these are defined exactly as above but with 'standard model(s)' replacing 'general model(s)'. But, as discussed earlier, for the purposes of the present paper standard structures are of little interest. Completeness results are rare when working with standard structures, but by working with general models we will be able to prove Henkin-style completeness results that covers the basic logic and all its pure extensions.

The following semantic lemmas will be used in our soundness and completeness proofs. We start with the Agreement Lemma. This tells us that to ensure that nominations and valuations agree on whether φ is satisfied, it suffices that they agree on the atomic symbols occurring in φ.

Lemma 6 (Agreement Lemma). *Let* φ *be a formula, and let both* $\mathcal{M} = \langle W, R, \Pi, N, V \rangle$ *and* $\mathcal{M}^* = \langle W, R, \Pi, N^*, V^* \rangle$ *be general models based on* $\langle W, R, \Pi \rangle$ *such that:*

(i) $V(p) = V^*(p)$ *for all propositional variables* p *occurring in* φ, *and*
(ii) $N(i) = N^*(i)$ *for all nominals* i *occurring in* φ.

Then for all $w \in W$, *we have that* $\mathcal{M}, w \models \varphi$ *iff* $\mathcal{M}^*, w \models \varphi$.

Proof. By induction on the number of propositional connectives in φ. □

Definition 7. *Let* $\mathcal{M} = \langle W, R, \Pi, N, V \rangle$ *be a general model. Then for all formulas* φ, *we define*

$$[\mathcal{M}]_\varphi = \{w \in W \mid \mathcal{M}, w \models \varphi\}.$$

Next we see that all soft-QDF formulas pick out admissible propositions.

Lemma 8. *Let* $\mathcal{M} = \langle W, R, \Pi, N, V \rangle$ *be a general model. Then for any soft-QDF formula* φ, *we have* $[\mathcal{M}]_\varphi \in \Pi$.

Proof. By induction on the number of connectives in soft-QDF formulas. All propositional variables are assigned elements of Π by V, which establishes the base case. The inductive steps for $\neg\varphi$, $\varphi \wedge \psi$, and $\Box\varphi$, follow from the three closure conditions on Π. As for the $@_i\varphi$ and $A\varphi$ steps, note that any such formula is either true at all worlds, or false at all worlds, that is any such formula picks out either the proposition W or $\emptyset$. But these two propositions are always admissible: as Π

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is non-empty, it contains at least one proposition X. But then $\emptyset = X \cap (W - X)$ and $W = W - \emptyset$ are both in Π. □

Our next lemma tells us that the set of all propositions picked out by soft-QDF formulas is a subset of Π that is a closed collection of admissible propositions. Let us write $[\mathcal{M}]^{sqf}$ for $\{[\mathcal{M}]_\varphi \mid \varphi \textit{ is a soft-QDF formula}\}$. Then:

Lemma 9. *Given any general model* $\mathcal{M} = \langle W, R, \Pi, N, V\rangle$:
(1) $[\mathcal{M}]^{sqf} \subseteq \Pi$, *and*
(2) $[\mathcal{M}]^{sqf}$ *is a closed set of admissible propositions.*

Proof. Item 1 follows from Lemma 8. Item 2 holds because the three closure conditions correspond to the connectives $\neg$, $\wedge$ and $\Box$. Argue as follows:
Relative complement:
Consider $[\mathcal{M}]_\varphi$ for some soft-QDF formula φ. It is sufficient to show that $\Pi - [\mathcal{M}]_\varphi = [\mathcal{M}]_{\neg\varphi}$. But $w \in [\mathcal{M}]_{\neg\varphi}$ iff $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$ iff $w \in \Pi - [\mathcal{M}]_\varphi$.
Intersection:
Similar to the previous case.
Modal projection:
We show that $\{w \in W \mid \forall v(wRv \rightarrow v \in [\mathcal{M}]_\varphi)\} = [\mathcal{M}]_{\Box\varphi}$ for any soft-QDF formula φ. But $u \in [\mathcal{M}]_{\Box\varphi}$ iff $\mathcal{M}, u \models \Box\varphi$ iff $\forall v(uRv \rightarrow v \in [\mathcal{M}]_\varphi)$ iff $u \in \{w \in W \mid \forall v(wRv \rightarrow v \in [\mathcal{M}]_\varphi)\}$. □

Now for substitution lemmas. We need a lemma for when a formula is substituted for a propositional variable and another for when a nominal is substituted for a nominal. Note the restriction to soft-QDF formulas in the following.

Lemma 10 (Substitution Lemma for Propositional Variables). *Let* $\mathcal{M} = \langle W, R, \Pi, N, V\rangle$ *be a general model. Then for any safe substitution* $\varphi[\psi/p]$, *where* ψ *is a soft-QDF formula, we have that:*

$$\mathcal{M}, w \models \varphi[\psi/p] \textit{ iff } \mathcal{M}', w \models \varphi,$$

where $\mathcal{M}' = \langle W, R, \Pi, N, V'\rangle$ *such that* $V' \sim_p V$ *is defined by setting* $V'(p) = [\mathcal{M}]_\psi$.

Proof. First note that V' is well-defined since $[\mathcal{M}]_\psi \in \Pi$ by the previous lemma. The proof is by induction on the number of connectives in φ. The interesting case is where $\varphi = \forall q\theta$. We have three subcases:

(i) If $p = q$, then the result follows from the Agreement Lemma.
(ii) If $p \neq q$, but p does not occur free in θ, then the result again follows from the Agreement Lemma.
(iii) Finally, there is the case where $p \neq q$ and p occur free in θ. Then $\mathcal{M}, w \models (\forall q\theta)[\psi/p]$ iff $\mathcal{M}, w \models \forall q(\theta[\psi/p])$ iff for all $V'' \sim_q V$, we have $\mathcal{M}'', w \models \theta[\psi/p]$ where $\mathcal{M}'' = \langle W, R, \Pi, N, V''\rangle$. But by the induction hypothesis this is equivalent to for all $V'' \sim_q V$, we have $\mathcal{M}''', w \models \theta$ where $\mathcal{M}''' = \langle W, R, \Pi, N, V'''\rangle$ and $V''' \sim_p V''$ is defined by setting $V'''(p) = [\mathcal{M}'']_\psi$. But giving a V''' such that $V''' \sim_p V''$ where $V'''(p) = [\mathcal{M}'']_\psi$ and $V'' \sim_q V$ is equivalent to giving a V''' such that $V''' \sim_q V'$ where $V' \sim_p V$ is defined by setting $V'(p) = [\mathcal{M}'']_\psi$. But q cannot occur free in ψ, so $[\mathcal{M}'']_\psi = [\mathcal{M}]_\psi$ by the Agreement Lemma. The result follows from $\mathcal{M}', w \models \forall q\theta$ being equivalent to for all $V''' \sim_q V'$, it is the case that $\mathcal{M}''', w \models \theta$.
□

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Lemma 11 (Substitution Lemma for Nominals). *Let $\mathcal{M} = \langle W, R, \Pi, N, V\rangle$ be a general model. Then for any safe substitution $\varphi[j/i]$, we have that:*

$$\mathcal{M}, w \models \varphi[j/i] \text{ iff } \mathcal{M}', w \models \varphi,$$

where $\mathcal{M}' = \langle W, R, \Pi, N', V\rangle$ such that $N' \sim_i N$ is defined by setting $N'(i) = N(j)$.

Proof. A straightforward induction on the number of connectives in φ. □

4. The $\mathbf{K}_{PIL}$ Axiom System

We now present $\mathbf{K}_{PIL}$, our axiomatisation of PIL with respect to general frames. We first introduce $\mathbf{K}_h$+RULES, the axiomatisation of basic propositional hybrid logic presented in Chapter 7 Section 3 of Blackburn et al. (2001). We discuss both components of $\mathbf{K}_h$+RULES system (i.e., the $\mathbf{K}_h$ part and the RULES part) and then add what is required to handle the universal modality, the downarrow binder, and propositional quantification. $\mathbf{K}_{PIL}$ is the result.

Definition 12 (The $\mathbf{K}_h$ axiom system). *The $\mathbf{K}_h$ axiom system contains as axioms all propositional tautologies, and all instances of K for the modalities:*

$$K_{\Box} : \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$
$$K_{@} : @_i(\varphi \to \psi) \to (@_i\varphi \to @_i\psi).$$

It also contains all instances of the following interaction schemas:

$$Intro : i \wedge \varphi \to @_i\varphi$$
$$Agree : @_j@_i\varphi \leftrightarrow @_i\varphi$$
$$Back : \Diamond@_i\varphi \to @_i\varphi$$
$$Sdual : @_i\varphi \leftrightarrow \neg@_i\neg\varphi,$$

and in addition, all instances of the modal equality schemas:

$$Ref : @_i i$$
$$Sym : @_i j \leftrightarrow @_j i$$
$$Nom : @_i j \wedge @_j\varphi \to @_i\varphi.$$

The proof rules of $\mathbf{K}_h$ are:

$$MP : If \vdash \varphi \to \psi \text{ and } \vdash \varphi \text{ then } \vdash \psi$$
$$Gen_{\Box} : If \vdash \varphi \text{ then } \vdash \Box\varphi$$
$$Gen_{@} : If \vdash \varphi \text{ then } \vdash @_i\varphi$$

$\mathbf{K}_h$ proofs are Hilbert-style proofs and it is fairly straightforward to adapt the usual modal machinery of canonical models and prove that $\mathbf{K}_h$ is a (sound and) strongly complete axiom system for minimal propositional hybrid logic (see Chapter 7, Section 3 of Blackburn et al. 2001 for details). The interaction axioms (together with the $Gen_{@}$ rule) capture the logic of the satisfaction operators: self-dual normal modal operators that interact smoothly with the other connectives. The axioms *Ref*, *Sym*, and *Nom* capture the logic of atomic satisfaction statements like $@_i j$; such

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statements are 'modal equality assertions', modal equivalents of first-order atomic formulas of the form $i = j$. Clearly *Ref* and *Sym* express the reflexivity and symmetry of identity. The *Nom* axiom is more interesting. It can be read as a Leibniz-style identity axiom: 'if i and j are identical, and i has property φ, then j has property φ too'. But also note an important special case: if φ is k this becomes $@_i j \wedge @_j k \rightarrow @_i k$, which expresses the transitivity of identity.

Below are two schemas, both provable in $\mathbf{K}_h$, that are used in the completeness proof. Note that *Elim* is a contraposed form of the *Intro* axiom (using the *Sdual* axiom).

$$\textit{Elim}: \quad i \wedge @_i\varphi \rightarrow \varphi$$

$$\textit{Bridge}: \Diamond i \wedge @_i\varphi \rightarrow \Diamond\varphi$$

Nonetheless, despite the fact that $\mathbf{K}_h$ is complete with respect to the class of all Kripke models (i.e., it is the 'minimal hybrid logic'), it is more usual to work with more powerful proof systems such as $\mathbf{K}_h$+RULES.[2]

Definition 13 (The $\mathbf{K}_h$+RULES axiom system). *The $\mathbf{K}_h$+RULES axiom system contains all the axioms and rules of $\mathbf{K}_h$ plus the following two proof rules:*

$$\textit{NAME}: \frac{\vdash j \rightarrow \theta}{\vdash \theta} \qquad \textit{PASTE}: \frac{\vdash @_i\Diamond j \wedge @_j\varphi \rightarrow \theta}{\vdash @_i\Diamond\varphi \rightarrow \theta}$$

In both rules, j is a nominal distinct from i that does not occur in φ or θ.

As we shall see later, these two rules allow us to do some things that are *not* possible in $\mathbf{K}_h$, and these things will become important when we look at the pure extensions of $\mathbf{K}_h$. Anticipating this, we include the two rules already at this stage.

Having defined the axiom system $\mathbf{K}_h$+RULES for basic hybrid logic, we must next handle the strong hybrid component. Here are the axioms and rules for the universal modality and the downarrow binder.

Definition 14 (Strong hybrid axioms and rules). *For the universal modality we add:*

(i) all axioms and rules of $\mathbf{K}_h$ containing the $\Box$ modality, but with $\Box$ replaced by A,

(ii) the following version of the PASTE rule for the universal modality, where j is a nominal distinct from i that does not occur in φ or θ:

$$\frac{\vdash @_j\varphi \rightarrow \theta}{\vdash @_i E\varphi \rightarrow \theta}$$

(iii) one more axiom:

$$\textit{Incl}_i : Ei$$

As for the downarrow binder, we add a single axiom:

$$\textit{DA}: @_i(\downarrow j\varphi \leftrightarrow \varphi[i/j]).$$

That is, the axiomatisation for A exactly mirrors the axiomatisation for $\Box$, but with one extra axiom which says: i must be true somewhere in any model (we return to this point in Section 6). As for downarrow, several different options are explored in Blackburn and ten Cate (2006). We chose this one as it reflects the semantics of the downarrow binder in a particularly simple and

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direct way.[3] So we are now ready to assemble the system we are really interested in: $\mathbf{K}_{PIL}$, our axiomatisation of PIL.

Definition 15 (The $\mathbf{K}_{PIL}$ axiom system). *The $\boldsymbol{K}_{PIL}$ axiom system contains all the axioms and rules of $\boldsymbol{K}_h$+RULES plus the above axioms for the universal modality and the downarrow binder. Moreover, it contains the following axioms:*

$$Q1 : \forall p(\varphi \to \psi) \to (\varphi \to \forall p\psi), \textit{ where } \varphi \textit{ contains no free occurrences of } p$$

$$Q2\text{-}sqf : \forall p\varphi \to \varphi[\psi/p], \textit{ where } \varphi[\psi/p] \textit{is a soft-QDF substitution}$$

$$Barcan_{@} : \forall p@_i\varphi \leftrightarrow @_i\forall p\varphi,$$

and one more proof rule:

$$Gen_{\forall} : \textit{If} \vdash \varphi \textit{ then} \vdash \forall p\varphi.$$

While *Q1* is familiar from modal and classical logic, and $Barcan_{@}$ from hybrid logic, the side condition on *Q2-sqf* deserves comment. We cannot substitute nominals (as not all admissible sets of propositions contain singleton subsets) nor can we substitute quantified formulas either (our Substitution Lemma 10 does not cover such substitutions, since the admissible sets are only required to be closed under finite intersections, not arbitrary intersections). Allowing only soft-QDF substitutions ensures soundness.[4]

Definition 16 (Provability and consistency in $\mathbf{K}_{PIL}$). *A formula φ is $\mathbf{K}_{PIL}$-provable iff there is a $\mathbf{K}_{PIL}$ Hilbert-style proof of φ; we write $\vdash \varphi$ for provability and $\nvdash \varphi$ for unprovability. A formula φ is $\mathbf{K}_{PIL}$-provable from a set of formulas Σ iff for some conjunction σ of formulas from Σ we have $\vdash \sigma \to \varphi$. A formula φ is $\mathbf{K}_{PIL}$-consistent iff $\nvdash \neg\varphi$. A set of formulas Σ is $\mathbf{K}_{PIL}$-consistent iff there is no conjunction σ of formulas from Σ such that $\vdash \neg\sigma$.*

Theorem. *$\boldsymbol{K}_{PIL}$ is sound with respect to general frames.*

Proof. To prove this we need to show that (a) all the axioms are valid on all general frames, and (b) that the proof rules preserve validity. This is known for all the axioms and rules in K_h+RULES, and for the axioms for the universal modality and the downarrow binder as well, so we only need to check that the *Q1* and *Q2-sqf* are valid and that $Gen_{\forall}$ preserves validity. That $Gen_{\forall}$ preserves validity is more-or-less immediate. The axioms are also easy to handle; we present the argument for *Q2-sqf* and leave *Q1* and $Barcan_{@}$ for the reader.

So: choose an arbitrary general model $\mathcal{M} = \langle W, R, \Pi, N, V\rangle$ and let $w \in W$. Then to show that $\forall p\varphi \to \varphi[\psi/p]$ is valid, where $\varphi[\psi/p]$ is a soft-QDF substitution, suppose that $\mathcal{M}, w \models \forall p\varphi$. This is equivalent to: for all V' such that $V' \sim_p V$, we have $\mathcal{M}', w \models \varphi$, where $\mathcal{M}' = \langle W, R, \Pi, N, V'\rangle$. Define $V'' \sim_p V$ by setting $V''(p) = [\mathcal{M}]_\psi$; Lemma 8 tells us that V'' is a well-defined variant of V because ψ is a soft-QDF formula. Hence $\mathcal{M}'', w \models \varphi$ and so, using the Substitution Lemma 10, $\mathcal{M}, w \models \varphi[\psi/p]$. □

Lemma 17. *If $\vdash \sigma \to \theta[c/p]$ and c does not occur in θ or σ, then $\vdash \sigma \to \forall q\theta[q/p]$, where q is any variable not occurring in θ or σ.*

Proof. Left to the reader. □

Lemma 18. *Suppose that q can be safely substituted for p in φ and that φ has no free occurrences of q. Then $\vdash \forall p\varphi \leftrightarrow \forall q\varphi[q/p]$.*

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Proof. Left to the reader. □

One final remark. We formulated $Barcan_@$ in its ∀-form, that is, in the form $\forall p @_i \varphi \leftrightarrow @_i \forall p \varphi$. Its ∃-form is $@_i \exists p \varphi \leftrightarrow \exists p @_i \varphi$. Strictly speaking, the left-to-right arrows of both forms are Barcan formulas, while the right-to-left directions of both are converse Barcan formulas.

5. Pure Completeness

We will now extend a standard model-building strategy used to prove the strong completeness of $\mathbf{K}_h + RULES$ (Chapter 7, Section 3 of Blackburn et al. 2001) to show that every $\mathbf{K}_{PIL}$-consistent set of formulas has a general model. The general model we shall construct has a number of special properties (described below) which will enable us to prove strong completeness not only for $\mathbf{K}_{PIL}$ itself, but for all of its pure extensions as well.

Definition 19 ($\mathbf{K}_{PIL}$ Maximal Consistent Sets). *Fix a language of $\mathcal{L}$. A set of formulas Σ in this language is a $\mathbf{K}_{PIL}$-MCS iff it is $\mathbf{K}_{PIL}$-consistent, and any proper extension (in the same language) is inconsistent.*

Lemma 20 (Named sets yielded by an MCS). *Let Γ be a $\mathbf{K}_{PIL}$-MCS. For every nominal i, let $\Delta_i = \{\varphi \mid @_i \varphi \in \Gamma\}$. Then:*

(1) *For every nominal i, Δ_i is a $\mathbf{K}_{PIL}$-MCS that contains i.*
(2) *For all nominals i and j, if $i \in \Delta_j$, then $\Delta_j = \Delta_i$.*
(3) *For all nominals i and j, $@_i \varphi \in \Delta_j$ iff $@_i \varphi \in \Gamma$.*
(4) *If a nominal $k \in \Gamma$, then $\Gamma = \Delta_k$.*

Proof. This is Lemma 7.24 of Blackburn et al. (2001), and proof details can be found there. In fact, Lemma 7.24 shows that only $\mathbf{K}_h$ reasoning is needed to prove this lemma. □

The Δ_i in this lemma are called the *named sets yielded by* Γ. Clause 1 of the previous lemma tells us that each of these is a MCS containing at least one nominal; all of these MCSs are 'hidden inside' the original $\mathbf{K}_h$-MCS. Clause 2 tells us that each nominal picks out a unique such MCS, and Clause 3 tells us that any satisfaction statement is either in all the Δ_j or in none of them. So every $\mathbf{K}_h$-MCS contains *almost* all that is required to build structures in which *every world is named by some nominal*. But not quite. Note that Clause 4 is only a conditional – we have no guarantee that Γ itself is one of the MCSs 'hidden inside' Γ.

Indeed, this weakness in Clause 4 is the key reason for using $\mathbf{K}_h$+RULES instead of $\mathbf{K}_h$. The addition of the NAME and PASTE rules *does* allow us to guarantee that Γ itself is one of the MCSs 'hidden inside' Γ. This will let us create a 'named and pasted' MCS, which contains all the information required to build a frame in which each world is named by a nominal, which will prove crucial for the pure extensions completeness result. Furthermore, the axioms and rules for propositional quantification in $\mathbf{K}_{PIL}$ also ensure that this MCS is 'witnessed', which allows us to define a suitable set of admissible propositions over its frame, thereby creating a general model. We first define what we mean by 'named', 'pasted' and 'witnessed' and then prove the Lindenbaum-style lemma which will lead us to these goals.

Definition 21 (Named, pasted and witnessed MCSs). *Let Σ be a $\mathbf{K}_{PIL}$-MCS. Then we say:*

- *Σ is named iff for some nominal i, $i \in \Sigma$.*
- *Σ is pasted iff for every formula of the form $@_i \Diamond \varphi \in \Sigma$, there is some nominal j such that $@_i \Diamond j \in \Sigma$ and $@_j \varphi \in \Sigma$, and similarly, for every formula of the form $@_i E \varphi \in \Sigma$, there is some nominal j such that $@_j \varphi \in \Sigma$.*

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- Σ *is witnessed iff for every formula of the form* $@_i\exists p\varphi$, *there is some propositional variable* q *such that* $@_i\varphi[q/p] \in \Sigma$.

To prove a Lindenbaum-style lemma for $\mathbf{K}_{PIL}$, we must extend the language. Suppose we start with language $\mathcal{L}$. We will add a countably infinite set of nominals (called *NewN*) and a countably infinite set of new propositional variables (called *NewL*), and call the extended language $\mathcal{L}'$. We will use *NewN* for naming and pasting and *NewL* for witnessing.

Lemma 22 (Lindenbaum). *Every* $\mathbf{K}_{PIL}$*-consistent set of formulas in language* $\mathcal{L}$ *can be extended to a named, pasted and witnessed* $\mathbf{K}_{PIL}$*-MCS in language* $\mathcal{L}'$.

Proof. Given a consistent set of $\mathcal{L}$-formulas Σ, add *NewN* and *NewP* as just described to form $\mathcal{L}'$, and enumerate all three sets. Define Σ_k to be $\Sigma \cup \{k\}$, where k is the first nominal in *NewN*. Σ_k is consistent. For suppose not. Then for some conjunction of formulas θ from Σ, we have $\models k \rightarrow \neg\theta$. But k is a new nominal, so it does not occur in θ; hence, by the NAME rule we have $\models \neg\theta$. This contradicts the consistency of Σ, so Σ_k must be consistent.

Define Σ^0 to be Σ_k, and suppose we have defined Σ^m, where $m \geq 0$. Let φ_{m+1} be the $(m+1)$-th formula in our enumeration of $\mathcal{L}'$. We define Σ^{m+1} as follows:

If $\Sigma^{m+1} \cup \{\varphi_{m+1}\}$ is inconsistent, then $\Sigma^{m+1} = \Sigma^m$. Otherwise:

(1) $\Sigma^{m+1} = \Sigma^m \cup \{\varphi_{m+1}\}$, if φ_{m+1} is not of the form $@_i\Diamond\varphi$, $@_iE\varphi$ or $@_i\exists p\varphi$.

(2) $\Sigma^{m+1} = \Sigma^m \cup \{\varphi_{m+1}\} \cup \{@_i\Diamond j \wedge @_j\varphi\}$, if φ_{m+1} is of the form $@_i\Diamond\varphi$. (Here j is the first nominal in the enumeration of *NewN* that does not occur in Σ^m or $@_i\Diamond\varphi$).

(3) $\Sigma^{m+1} = \Sigma^m \cup \{\varphi_{m+1}\} \cup \{@_j\varphi\}$, if φ_{m+1} is of the form $@_iE\varphi$. (Analogous to the previous case, j is the first nominal in the enumeration of *NewN* that does not occur in Σ^m or $@_iE\varphi$).

(4) $\Sigma^{m+1} = \Sigma^m \cup \{\varphi_{m+1}\} \cup \{@_i\varphi[r/p]\}$, if φ_{m+1} is of the form $@_i\exists p\varphi$. (Here r is the first propositional variable in the enumeration of *NewL* that does not occur in Σ^m or $@_i\exists p\varphi$).

Let $\Sigma^* = \bigcup_{n\geq 0} \Sigma^n$. Clearly this set is named (as we added k in the first step), maximal (by construction), pasted and witnessed. Furthermore, it is consistent, for the only aspects of the expansion that require checking are those given in by the second, third and fourth steps.

Preservation of consistency by the second step is precisely what the PASTE rule guarantees, and similarly, consistency by the third step is guaranteed by the version of PASTE for the universal modality.

As for the fourth step, we argue as follows. Assume for the sake of contradiction that $\Sigma^m \cup \{@_i\exists p\varphi\} \cup \{@_i\varphi[r/p]\}$ is inconsistent. Then there are formulas $\sigma_1, \ldots, \sigma_n$ in Σ^m such that $\vdash \neg(\sigma_1 \wedge \ldots \wedge \sigma_n \wedge @_i\exists p\varphi \wedge @_i\varphi[r/p])$. Writing $\sigma_1 \wedge \ldots \wedge \sigma_n$ as σ, propositional logic yields $\vdash (\sigma \wedge @_i\exists p\varphi) \rightarrow \neg@_i\varphi[r/p]$. The conditions of Lemma 17 are fulfilled, so we have $\vdash (\sigma \wedge @_i\exists p\varphi) \rightarrow \forall q\neg@_i\varphi[q/p]$ where q is a new propositional variable. This is equivalent to $\vdash (\sigma \wedge @_i\exists p\varphi) \rightarrow \neg\exists q@_i\varphi[q/p]$, so using (the $\exists$-form of) $Barcan_@$ we have $\vdash (\sigma \wedge @_i\exists p\varphi) \rightarrow \neg@_i\exists q\varphi[q/p]$. Moreover, the conditions of Lemma 18 are fulfilled as well, so $\vdash @_i\exists p\varphi \leftrightarrow @_i\exists q\varphi[q/p]$, and we have that $\vdash (\sigma \wedge @_i\exists p\varphi) \rightarrow \neg@_i\exists p\varphi$. This contradicts the consistency of $\Sigma^m \cup \{@_i\exists p\varphi\}$, thus the fourth step must preserve consistency after all. □

Lemma 23 (Generated admissible sets). *Let* $\mathcal{F} = \langle W, R\rangle$ *be a Kripke frame, let* $\mathcal{P}(W)$ *be the powerset of* W, *let* $B \subseteq \mathcal{P}(W)$ *and define* $\Pi(B)$, *the admissible set generated by* B, *to be the smallest subset of* $\mathcal{P}(W)$ *containing* B *that is closed under relative complement, intersection, and modal projection. Then* $\langle W, R, \Pi(B)\rangle$ *is a general frame.*

Proof. Immediate by definition of $\Pi(B)$. □

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Lemma 24. *Let $\mathcal{F} = \langle W, R\rangle$ be a Kripke frame, and let N be any nomination on $\mathcal{F}$. Let V be any mapping such that $V : PVAR \to \mathcal{P}(W)$. Then $\mathcal{M} = \langle W, R, \Pi(im(V)), N, V\rangle$ is a general model (here $im(V)$ is the image of V).*

Proof. As $\langle W, R\rangle$ is a Kripke frame and $im(V) \subseteq \mathcal{P}(W)$, by the previous lemma $\langle W, R, \Pi(im(V))\rangle$ is a general frame. Hence V is a mapping into $\Pi(im(V))$, thus V is a valuation on a general frame. Thus $\mathcal{M}$ is a general model. □

Definition 25 (Canonical named structures). *Let Γ be a named, pasted and witnessed $\mathbf{K}_{PIL}$-MCS. Let $\mathcal{M}^\Gamma$ be $\langle W^\Gamma, R^\Gamma, \Pi^\Gamma, N^\Gamma, V^\Gamma\rangle$ where:*

- *W^Γ is the set of all named sets yielded by Γ.*
- *R^Γ is the standard modal canonical relation between MCSs. That is, for any $u, v \in W^\Gamma$ we define $uR^\Gamma v$ iff for all formulas φ, $\varphi \in v$ implies $\Diamond\varphi \in u$. Or equivalently: $uR^\Gamma v$ iff for all formulas φ, $\Box\varphi \in u$ implies $\varphi \in v$.*
- *$N^\Gamma : NOM \to W^\Gamma$ is defined as follows. For any nominal i, $N(i)$ is the unique $w \in W^\Gamma$ such that $i \in w$; that is, $N(i) = \Delta_i$.*
- *$V^\Gamma : PVAR \to \mathcal{P}(W^\Gamma)$ is the standard modal canonical valuation (for proposition variables). That is, $V^\Gamma(p) = \{w \in W^\Gamma \mid p \in w\}$, for any proposition variable p.*
- *Π^Γ is $\Pi(im(V^\Gamma))$.*

We now check that this definition does indeed give rise to Kripke frames and general models where every world is named by some nominal.

Lemma 26. *Let Γ be a named, pasted and witnessed $\mathbf{K}_{PIL}$-MCS, and let $\mathcal{M}^\Gamma$ be the canonical named general model yielded by Γ. Then $\langle W^\Gamma, R^\Gamma\rangle$ is a Kripke frame and $\mathcal{M}^\Gamma$ is a named general model.*

Proof. To see that $\langle W^\Gamma, R^\Gamma\rangle$ is a Kripke frame, first note that by Lemma 20 (1), for every nominal i, Δ_i is a $\mathbf{K}_{PIL}$-MCS containing i. As W^Γ is a non-empty set of MCSs, the standard modal canonical relation R^Γ can be defined over it, thus $\langle W^\Gamma, R^\Gamma\rangle$ is a Kripke frame. Moreover N^Γ is a well-defined nomination, for Lemma 20 (2) guarantees that Δ_i is the unique element of W^Γ such that $i \in w$, and it clearly 'names' every world in W^Γ. Finally, V^Γ is well-defined, so we have all we need to apply Lemma 24 and conclude that $\mathcal{M}^\Gamma$ is a named general model. □

So from now on we will call $\langle W^\Gamma, R^\Gamma\rangle$ the *canonical named Kripke frame* yielded by Γ, and $\mathcal{M}^\Gamma$ the *canonical named general model* yielded by Γ. We now examine them more closely. Our first lemma tells us that R^Γ works exactly as it does in ordinary propositional modal logic.

Lemma 27 (Existence Lemma). *Let Γ be a named, pasted, and witnessed $\mathbf{K}_{PIL}$-MCS, and let $\mathcal{M}^\Gamma$ be the canonical named general model yielded by Γ. If $u \in W^\Gamma$ and $\Diamond\varphi \in u$, then there is a $v \in W^\Gamma$ such that $uR^\Gamma v$ and $\varphi \in v$. Analogously, if $u \in W^\Gamma$ and $E\varphi \in u$, then there is a $v \in W^\Gamma$ such that $\varphi \in v$.*

Proof. The case with $\Diamond$ is essentially Lemma 7.27 from Blackburn et al. (2001). A similar (in fact simpler) argument works for the case with E: As $u \in W^\Gamma$, for some nominal i we have that $u = \Delta_i$. Hence as $E\varphi \in u$, $@_i E\varphi \in \Gamma$. But Γ is pasted, so for some nominal j, $@_j\varphi \in \Gamma$, and so $\varphi \in \Delta_j$. Now, Δ_j is a suitable choice of v. □

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Lemma 28. *Let Γ be a named, pasted and witnessed $\mathbf{K}_{PIL}$-MCS, let $\mathcal{M}^\Gamma$ be the canonical named general model yielded by Γ, and let $u \in W^\Gamma$. Then for all quantifier-free formulas φ, we have that:*
(1) $\mathcal{M}^\Gamma, u \models \varphi$ *iff* $\varphi \in u$
(2) $\mathcal{M}^\Gamma, \Delta_i \models \varphi$ *iff* $@_i\varphi \in \Gamma$.

Proof. Item 1 follows by induction on the number of connectives. It is essentially an extended version of Lemma 7.28 from Blackburn et al. (2001). The base case, for the atomic formulas, is clear by the definitions of V^Γ and N^Γ. As our inductive hypothesis (IH) we assume that for any quantifier-free formula φ with at most n connectives we have $\mathcal{M}^\Gamma, u \models \varphi$ iff $\varphi \in u$. So suppose that θ is a formula with $n+1$ connectives; we want to show that this property also holds for θ.

For θ a Boolean formula, the argument is straightforward, and for θ of the form $\Box\psi$, the standard modal argument using the Existence Lemma 27 establishes it.

To highlight the role of the axiom $Incl_i$, we cover the case with the universal modality. Without loss of generality, we deal with E instead of A. So suppose θ has the form $E\psi$. We have: $\mathcal{M}^\Gamma, u \models E\psi$ iff (by the truth-condition for E) for some v, $\mathcal{M}^\Gamma, v \models \psi$ iff (by the IH) for some v, $\psi \in v$ iff $E\psi \in u$. In the last equivalence, we use the Existence Lemma 27 in the right-to-left direction. In the left-to-right direction, we note that by the definition of $\mathcal{M}^\Gamma$, we can assume that for some nominal j, $v = \Delta_j$, and hence, by Lemma 20 item (1) and *Intro*, $@_j\psi \in \Delta_j$, but by Lemma 20 item (3), $@_j\psi \in \Gamma$, and again Lemma 20 item (3), $@_j\psi \in \Delta_k$ where k is such that $u = \Delta_k$. Finally, by the axiom $Incl_j$ and a version of *Bridge* for the universal modality, we get $E\psi \in \Delta_k$.

Suppose θ has the form $@_i\psi$. We have: $\mathcal{M}^\Gamma, u \models @_i\psi$ iff $\mathcal{M}^\Gamma, N^\Gamma(i) \models \psi$ iff $\mathcal{M}^\Gamma, \Delta_i \models \psi$ iff $\psi \in \Delta_i$ (by the IH) iff $@_i\psi \in \Delta_i$ (using *Intro* for one direction and *Elim* for the other) iff $@_i\psi \in \Gamma$ (Lemma 20 item (3)), iff $@_i\psi \in u$ (Lemma 20 item (3)).

Suppose θ has the form $\downarrow i\psi$. By the definition of $\mathcal{M}^\Gamma$, we can assume that for some nominal j, $u = \Delta_j$, that is, $u = N^\Gamma(j)$. We have: $\mathcal{M}^\Gamma, u \models \downarrow i\psi$ iff $\mathcal{M}, u \models \psi$ where $\mathcal{M} = \langle W^\Gamma, R^\Gamma, \Pi^\Gamma, N', V^\Gamma\rangle$ and $N' \sim_i N^\Gamma$ such that $N'(i) = u$, that is, $N'(i) = N^\Gamma(j)$ (by the truth-condition for $\downarrow$) iff $\mathcal{M}^\Gamma, u \models \psi[j/i]$ (by the Substitution Lemma 11) iff $\psi[j/i] \in u$ (by the IH) iff $@_j(\psi[j/i]) \in u$ (using *Intro* and *Elim*) iff $@_j \downarrow i\psi \in u$ (using the *DA* and $K_@$ axioms) iff $\downarrow i\psi \in u$ (using *Intro* and *Elim*).

This completes the inductive proof of Item 1. Item 2 then follows by the definition of Δ_i. □

Next for a simple but important lemma:

Lemma 29. *Let Γ be a named, pasted and witnessed $\mathbf{K}_{PIL}$-MCS, let $\mathcal{M}^\Gamma$ be the canonical named general model yielded by Γ. Then $[\mathcal{M}^\Gamma]^{sqf} = \Pi^\Gamma$.*

Proof. Item 1 of Lemma 9 tells us that $[\mathcal{M}^\Gamma]^{sqf} \subseteq \Pi^\Gamma$. Item 2 of same lemma tells us that $[\mathcal{M}^\Gamma]^{sqf}$ is a closed set of admissible propositions. Now, $im(V^\Gamma) \subseteq [\mathcal{M}]^{sqf}$, as these are the atomic propositions picked out by the propositional variables. But Π^Γ is $\Pi(im(V^\Gamma))$, the *smallest* closed set of admissible propositions containing $im(V^\Gamma)$. So $\Pi^\Gamma \subseteq [\mathcal{M}^\Gamma]^{sqf}$. □

The previous lemma is important because it gives us a *syntactic* handle on the elements of Π^Γ: every proposition in Π^Γ is 'picked out' by some soft-QDF formula; this syntactic characterisation enables us to prove the final lemma leading to completeness.

Lemma 30 (Truth Lemma). *Let Γ be a named, pasted, and witnessed $\mathbf{K}_{PIL}$-MCS, and let $\mathcal{M}^\Gamma$ be the canonical named general model yielded by Γ, and let $u \in W^\Gamma$. Then, for all formulas φ, we have that $\mathcal{M}^\Gamma, u \models \varphi$ iff $\varphi \in u$.*

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Proof. For any formula φ, let $con(\varphi)$ be the number of connectives in φ. Moreover, let $quan(\varphi)$ be the maximal depth of quantifier nesting in φ, that is:

$$quan(\varphi) = \begin{cases} 0 & \text{if } \varphi \text{ is atomic} \\ sup\{quan(\psi), quan(\theta)\} & \text{if } \varphi = \psi \wedge \theta \\ quan(\psi) & \text{if } \varphi \in \{\neg\psi, @_i\psi, \Box\psi, A\psi, \downarrow i\psi\} \\ quan(\psi) + 1 & \text{if } \varphi = \forall p\psi \end{cases}$$

We prove the result by induction on pairs $(quan(\varphi), con(\varphi))$ ordered lexicographically, that is, $(b', c') < (b, c)$ iff (1) $b' < b$ or (2) $b' = b$ and $c' < c$.

Note that Lemma 28 has established this for all formulas φ associated with the pair $(0, 0)$ (i.e., atomic formulas) and indeed for all formulas associated with pairs $(0, n)$ for any natural number n (i.e., all quantifier-free formulas). So the base cases are established. As our inductive hypothesis (IH) we assume that for natural numbers q and c, we have that $\mathcal{M}^\Gamma, u \models \varphi$ iff $\varphi \in u$ for all formulas φ associated with pairs $(b', c') < (b, c)$.

We proceed case by case. If θ is a Boolean formula, or of the form $\Box\psi$, $A\psi$, $@_i\psi$ or $\downarrow i\psi$, we can argue as in Lemma 28, where we use our new IH just as before to formulas associated with pairs (b, c') such that $c' < c$. The critical case is where θ is of the form $\forall p\psi$.

We want to show that $\mathcal{M}^\Gamma, u \models \forall p\psi$ iff $\forall p\psi \in u$. For the left-to-right direction, we show the contrapositive. So suppose that $\forall p\psi \notin u$. As u is an MCS, $\neg\forall p\psi \in u$, that is, $\exists p\neg\psi \in u$. The MSC Γ is witnessed, so for some propositional variable q, $@_j\neg\psi[q/p] \in \Gamma$. But then $\neg\psi[q/p] \in \Delta_j$. But $\neg\psi[q/p] \in \Delta_j$ is associated with $(b - 1, c)$, so the IH applies and we have $\mathcal{M}^\Gamma, u \models \neg\psi[q/p]$. It follows from the Substitution Lemma that $\mathcal{M}, u \models \neg\psi$ where $\mathcal{M} = \langle W^\Gamma, R^\Gamma, \Pi^\Gamma, N^\Gamma, V\rangle$ such that $V \sim_p V^\Gamma$ is defined by setting $V(p) = [\mathcal{M}^\Gamma]_q$. Hence, $\mathcal{M}, u \not\models \psi$. It follows that $\mathcal{M}^\Gamma, u \not\models \forall p\psi$, and we have proved the contrapositive.

For the right-to-left direction, suppose that $\forall p\psi \in u$, and further suppose for the sake of contradiction that $\mathcal{M}^\Gamma, u \not\models \forall p\psi$. Then for some $V \sim_p V^\Gamma$ we have $\mathcal{M}, u \models \neg\psi$ where $\mathcal{M} = \langle W^\Gamma, R^\Gamma, \Pi^\Gamma, N^\Gamma, V\rangle$. Now $V(p) \in \Pi^\Gamma$, but by Lemma 29 we know $\Pi^\Gamma = [\mathcal{M}^\Gamma]^{sqf}$, hence $V(p) = [\mathcal{M}^\Gamma]_\theta$ for some soft-QDF formula θ. The Substitution Lemma tells us that $\mathcal{M}^\Gamma, u \models \neg\psi[\theta/p]$ *iff* $\mathcal{M}, u \models \neg\psi$, and hence we have $\mathcal{M}^\Gamma, u \models \neg\psi[\theta/p]$. As $\neg\psi[\theta/p]$ is associated with the pair $(b - 1, c')$ for some c', we can apply the IH to conclude $\neg\psi[\theta/p] \in u$. But this leads to a contradiction. As u is a MCS, for all soft-QDF formulas ξ, we have $\psi[\xi/p] \in u$ by the *Q2-sqf* axiom. In particular, $\psi[\theta/p] \in u$. We conclude that $\mathcal{M}^\Gamma, u \models \forall p\psi$ after all. □

Theorem (Strong basic completeness). *Every* $\mathbf{K}_{PIL}$*-consistent set of sentences has a named model.*

Proof. Follows from the previous lemma in the familiar way. □

So we have proved the *basic* strong completeness result. But the general model we have built is named, so we can immediately extend this to cover all pure extensions of $\mathbf{K}_{PIL}$. In hybrid logic, a formula is called 'pure' if all its atomic subformulas are nominals. Here are three well-known examples: $i \to \Diamond i$ (the *Irreflexivity* axiom), $i \to \Box(\Diamond i \to i)$ (the *Antisymmetry* axiom), and $@_i\Diamond j \vee @_i j \vee @_j\Diamond i$ (the *Trichotomy* axiom). A pure formula φ *defines* a class of frames F iff: $(W, R) \in \mathsf{F} \Leftrightarrow (W, R) \models \varphi$. It is easy to check that our three examples define the class of irreflexive, antisymmetric, and trichotomous frames respectively. More importantly: adding *any* (combination of) pure formula(s) as extra axiom(s) to $\mathbf{K}_h$ + *RULES* is a proof system that is complete with respect to the class of frames defined. For a more detailed statement and discussion of this result, see Blackburn and ten Cate (2006). Here we shall simply record that:

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Theorem (Strong pure completeness). *Let Λ be a set of pure formulas, and let $\mathbf{K}_{PIL} + \Lambda$ be the Hilbert system obtained by using the pure formulas in Λ as extra axioms. Then every $\mathbf{K}_{PIL} + \Lambda$-consistent set of sentences has a named model built over a Kripke frame belonging to the frame-class defined by the pure formulas in Λ.*

Proof. This is essentially the same as the proof of Theorem 7.29 from Blackburn et al. (2001). Because nominals directly 'tag' worlds in the underlying Kripke frames, the standard completeness result for pure formulas carry over unchanged. □

Let us look at an example using $\downarrow$. As a first step, let $\Lambda = \{i \rightarrow \neg\Diamond i, \Diamond\Diamond i \rightarrow \Diamond i\}$. These two formulas jointly define the class of strictly partially ordered frames: $i \rightarrow \neg\Diamond i$ defines irreflexivity, and $\Diamond\Diamond i \rightarrow \Diamond i$ defines transitivity, so adding them as axioms to $\mathbf{K}_{PIL}$ gives us the complete logic of this frame class. But suppose we want to work with *confluent* strictly partially ordered frames. Confluence means that 'any two successors of a given world share a common successor world'. This property is not definable using a *basic* pure formula (see Proposition 4.2.11 in Ten Cate 2005) but it can be defined using the following strong pure formula:

$$@_i \wedge @_j \rightarrow \Diamond(i \wedge \Diamond \downarrow k @_j \Diamond k)).$$

This says: 'if the world at which we are evaluating has successors i and j, then there is a successor state of i – call it k – such that k is also a successor to j'. So adding this as a third axiom yields the logic of confluent strictly partially ordered frames.

6. Prior's Ideal Language

Having proved Theorem 5, pure completeness for $\mathbf{K}_{PIL} + \Lambda$, we are in a better position to explain why we called this system 'Prior's Ideal Language'. First some historical background.

Arthur Prior was a philosophical logician, best known as the inventor of tense logic (see Copeland 2017; Øhrstrøm and Hasle 2007). However he also invented hybrid logic (see Blackburn 2006; Braüner 2021), and routinely used propositional quantification throughout his career. Shortly before his death in 1969, Prior began to explore such ideas as 'quasi-modalities' and developed a system he called 'egocentric logic'. With the benefit of hindsight, it is now clear that egocentric logic was an early example of the description logics that would become popular in computer science in the 1980s. Such investigations lead him to start rethinking the boundaries between logics of time, worlds, and individuals, but this last phase of his work remains little known and under-discussed.[5] We believe that PIL provides an excellent setting for exploring Prior's work, and in particular the work from this period. We now explain why.

First, PIL is a *general* tool. In this paper, our basic hybrid language contained only one box-diamond pair, $\Box$ and $\Diamond$. But the results of this paper, and in particular Theorem 5, generalise straightforwardly to basic hybrid languages containing multiple modalities. Actually, our treatment of the universal modality pretty much shows how to do this. Recall from Definition 14 that we axiomatised A by insisting that it had the same basic axioms and rules as $\Box$ and then adding the pure axiom

$$Ei$$

which tells us what is special about the universal modality (namely that it sees every world/time/individual). We can extend this strategy to as many box-diamond pairs $\Box_m$ and $\Diamond_m$ as we like. We simply add them to the basic hybrid language, interpret each by its own relation R_m on W, let each of these modalities mirror the same logic as $\Box$, and then add extra pure axioms to shape them to our needs.[6]

For example, if we want a Prior-style tense logic, we could start with a basic hybrid language containing the pairs G and F, and H and P, interpret these by relations R_G and R_H. To axiomatise

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this, we would give each of these the logic of $\Box$, exactly as in Definition 14, and also add the following two pure axioms:

$$i \to HFi \qquad i \to GPi.$$

These guarantee that R_G and R_H are converse relations, thus we have a *tense* logic: paired operators that look forward and backward along the same relation.

It was sometime during the late 1960s that Prior had the insight that led him to rethink the boundaries between logics of time, worlds, and individuals. An unexpected result of his successful investigation of strong hybrid logic in the setting of tense logic, was the realisation – which anticipated a major theme of the Amsterdam-style modal logic by more than 20 years – that modalities could be used to talk about absolutely *anything*. Modal logic was not just a tool for reasoning about abstract entities like possible worlds and times, it could be applied to everyday objects as well. This shift in Prior's thinking was clearly and dramatically marked in a remarkable pair of papers, namely 'Tense logic and the logic of earlier and later', and 'Quasi-propositions and quasi-individuals', both of which can be found in the first edition of *Papers on Time and Tense*; Prior (1968).[7] Egocentric logic was introduced in 'Quasi-propositions and quasi-individuals', and is an early example of what is computer scientists now call description logic (see Baader et al. 2003): it is a modal logic, indeed a hybrid logic, for reasoning about people.[8] In Prior's egocentric logic, the 'worlds' are people and their properties and relationships (e.g., their relative heights) are described using what Prior called 'quasi modalities' such as [*taller*], $\langle$*taller*$\rangle$, [*shorter*] and $\langle$*shorter*$\rangle$.[9] Prior talks about 'people propositions' (nominals), and makes use of the universal modality, quantification over nominals, and propositional quantification. It is an important paper, and set the agenda for the final phase of his work.

Egocentric logic is something that PIL can deal with; indeed, egocentric logic was a major motivation for PIL. As we have already noted, PIL can handle multiple modalities such Prior's [*taller*] and [*shorter*]; a pure axiomatisation that captures Prior's egocentric logic of relative heights can be found in Kofod (2022). More generally, we are interested in reviving Prior's project of rethinking the boundaries between logics of time, worlds, and individuals. There are interesting connections to be made with description logic here, and perhaps with other logics for knowledge representation too; for example Belardinelli et al. (2018) recently suggested that epistemic logic augmented with propositional quantifiers was useful for reasoning about knowledge in AI.

A second reason we think that PIL is a useful setting for exploring Prior's work because it offers a powerful – yet nuanced – form of expressivity. How powerful? Well, we said that the strong component of PIL was A and $\downarrow$. But this is not exactly the way Prior did it: he allowed direct quantification over nominal. That is, Prior used expressions like

$$\forall i(i \to \neg\Diamond i).$$

Note that this is universal closure of the pure axiom for irreflexivity. So, if we place the universal A in front of it we get

$$A\forall i(i \to \neg\Diamond i)$$

a statement which is true only in models where every point is irreflexive. So it is fairly easy to say how powerful Prior's version of strong hybrid logic is: it allows us to capture 'full first-order classical quantification' over the elements of W. Prior himself knew this in the late 1960s: he showed that his strong language captured the UT-calculus, which is essentially what modal logicians would nowadays call the first-order correspondence language; see Blackburn (2006) for further discussion and Blackburn and Seligman (1995) for relevant technical results.

But the PIL approach is as powerful as Prior's. First, given Prior's language, we can define $\downarrow$ as follows:

$$\downarrow i\varphi := \exists i(i \wedge \varphi).$$

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Note that this translation reveals the hidden nominal in formula position that motivates the 'D' in 'soft-QDF'. On the other hand, using PIL we can define Priors direct quantification over nominals as follows:

$$\exists i\varphi := {\downarrow}jE{\downarrow}iE(j \wedge \varphi).$$

So the approaches are expressively equivalent. But we like the locality of $\downarrow$. Prior himself emphasised the locality of modal logic, and although he did not isolate downarrow, he came close: for example, he drew attention to the strong validity $\exists ii$, 'some proposition is true right here and now'. For example, this formula is axiom T9 in the paper 'Tense logic and the logic of earlier and later'. Furthermore, expressing $\exists i\varphi$ as ${\downarrow}jE{\downarrow}iE(j \wedge \varphi)$ breaks down the classical quantifier in a suggestive way: it decomposes it into global search for truth (using E) combined with local binding (using $\downarrow$). Finally, the fragment of strong hybrid logic containing $\downarrow$ and @ (but without A) has turned out to be important: it is equivalent to the bounded fragment of first-order logic, and gives rise to general interpolation results for many forms of hybrid logic (see Areces et al. 2001). Indeed, this fragment even turns out to have an unexpected link with propositional quantification. Every formula of modal logic with propositional quantifiers (interpreted over standard models) expresses a second-order property. Some of these properties don't have first-order equivalents, but many important ones do. Balder ten Cate showed that every such elementary property is equivalent to a sentence of $\downarrow$ and @; see (see Theorem 12.0.3, p. 179, in Ten Cate 2005). Note that Ten Cate's result shows that Theorem 5, the pure completeness theorem for $\mathbf{K}_{PIL} + \Lambda$, covers all elementary frame classes definable in modal logic with propositional quantifiers under the *standard* interpretation. In short, both conceptually and technically, PIL embodies ideas that seem relevant to a deeper and more nuanced technical understanding of Prior's later work.

Our third and final point concerns propositional quantification. As we have mentioned, Arthur Prior was an 'early adopter' of propositional quantifiers in modal logic (see, in particular, Prior 1958, 1961), and his students Robert Bull and Kit Fine both published important technical papers of the subject (see Bull 1969 and Fine 1970). But Arthur Prior was also the inventor of hybrid logic, and these two interests became intertwined. For much of his career, Prior had oscillated between two views of nominals: we call these his 'tag' view and his 'telescope' view. The 'tag' view is standard in contemporary hybrid logic, and that is the way we presented nominals in this paper: nominals are 'labels' or 'indices' or '0-place world identifying modalities'. However, when he adopts the 'telescope' view, Prior speaks of nominals identifying worlds, times or individuals by means of *content*. As he puts it in his paper 'Tense Logic and the Logic of Earlier and Later', nominals combine 'all those propositions which would ordinarily be said to be true' at that world, time or individual. He seemed to view this combination as being something like the infinite conjunction of all the relevant propositions describing that world, time or individual.

And here propositional quantifiers enter the picture in an interesting way. Sometimes Arthur Prior introduced nominals not as a second sort of primitive propositional symbol, but as an ordinary propositional variables preceded by the Q operator, and he spelt out what Qp meant using propositional quantifiers:

$$Qp = Ep \wedge \forall q(A(p \rightarrow q) \vee A(p \rightarrow \neg q))$$

(Prior uses the symbols $\Box$ and $\Diamond$, rather than A and E, in this definition. But it is clear that he is using what is nowadays called the universal modality.) In particular, he defines nominals using the Q-operator in both 'Quasi-propositions and quasi-individuals' and the later and more detailed 'Egocentric Logic' in both papers it seems pretty clear that he views Q as giving rise to the tag interpretation. And this is correct: under the *standard* interpretation of propositional quantification, this definition of Q makes a 'nominal' defined by the formula Qp true at a unique element of W, that is, Qp says that p is true *somewhere* and that p strictly implies *every* proposition q or its negation (read $A(p \rightarrow q)$ as 'p is included in q'). More interestingly, however, is that under the *general* interpretation of propositional quantification, Qp does not say that p is true at a unique element

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of W, but rather that p is true at exactly the elements of some atom, that is, a non-empty minimal admissible subset. Thus, p is true at the elements of an atom belonging to 'all those propositions which would ordinarily be said to be true'. So Qp read in this way can be thought of identifying worlds, times and individuals via descriptions, via content. It is a tool for making telescope nominals.

Thus, PIL offers access to *both* tag and telescope nominals. The tag nominals are built into the basic syntax of PIL, and together with the satisfaction operators and downarrow binder which manipulate them, they give rise to many strong 'logics of indices', as Theorem 5 testifies. But, in parallel to this, we can also define 'telescope nominals' of the form Qp, at least when working with the general frame semantics of this paper.

7. Conclusion

We introduced a language called PIL (strong hybrid logic enriched with propositional quantifiers), interpreted this language over general frames, and proved a pure completeness theorem for it. To prove completeness, we adapted and extended standard methods from the hybrid logical literature; in particular, we used a Henkin-style model construction in which the required general model was built out of a single MCS. We then explained why we call this language 'Prior's Ideal Language'. First, we pointed out that the methods we used here generalise straightforwardly: it is easy to adapt to multimodal languages, and (when suitable pure axioms can be found) completeness results for these richer systems are immediate. This generality opens the door to Prior's final project: rethinking the boundaries between logics of time, worlds, and individuals. Second, we pointed out the strong hybrid component – particularly when viewed in terms of $\downarrow$ and A – gives us an interesting perspective on quantification over worlds, times, and individuals, a perspective which decomposes classical quantification into a blend of local binding coupled with the use of special modalities like the universal modality. We also noted that the strong language is extremely expressive when it comes to elementary properties, which means that we have many potentially useful pure axioms at our disposal. Third, we pointed out that PIL is a setting where Prior's 'tag' and 'telescope' nominals live together in one system. Prior's later papers are under-explored (we suspect that very few people have read them in detail). Nonetheless, we believe they contain ideas worth exploring, and that PIL provides contemporary tools and results relevant to the task.

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Notes

1 Various Henkin style 'taming' strategies have been previously used in hybrid logic: see for example Areces et al. (2014) and Manzano et al. (2023). Both papers examine ways of combining sample type theory with hybrid logical machinery, but neither allows propositional quantification.

2 Several such systems have been explored; see Areces and ten Cate (2007) and Blackburn and ten Cate (2006) for more. Here we follow Blackburn et al. (2001).

3 That said, a slightly more complex setup for $\downarrow$ would allow us to jettison the *Name* and *Paste* rules. The $\downarrow$ operator is powerful enough to capture their force using three axioms (one of them is DA) and the rule $Gen_{\downarrow}$: If $\vdash \varphi$ then $\vdash \downarrow i\varphi$; see Blackburn and ten Cate (2006).

4 Note that on *standard* models we could drop the restriction prohibiting substitution of quantified formulas as standard models admit all subsets of the frame as propositions. That is, the rule which permits any soft substitution is sound on

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standard models. However this does not lead to a completeness result for standard models. The results of Fine (1970) show that the set of all standard validities is not recursively enumerable.

5 See, in particular, the technical papers in the new edition of his book *Papers on Time and Tense* Prior (2003), and the posthumous volume *Worlds, Times and Selves* Prior and Fine (1977).

6 This is *almost* what we did for the universal modality – except there we were able to use a slightly simpler version of the PASTE rule. However, in the general case, the PASTE rule for each $\Diamond_m$ should exactly mirror the PASTE rule for $\Diamond$.

7 The new edition Prior (2003) contains both papers and also several more papers that build on them, including 'Egocentric logic'. Julie Kofod's PhD thesis (see Kofod 2022) discusses these developments in detail.

8 For a detailed account of the links between hybrid logic and description logic, see Areces (2000).

9 See Kofod (2022) for more on this first version of egocentric logic. We have used contemporary notation, rather than Prior's, for the egocentric modalities.

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