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Two-Categorical Bundles and their Classifying Spaces

by

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Abstract

For a 2-category $2\mathcal{C}$ we associate a notion of a principal $2\mathcal{C}$-bundle. For the 2-category of 2-vector spaces, in the sense of M.M. Kapranov and V.A. Voevodsky, this gives the 2-vector bundles of N.A. Baas, B.I. Dundas and J. Rognes. Our main result says that the geometric nerve of a good 2-category is a classifying space for the associated principal 2-bundles. In the process of proving this we develop powerful machinery which may be useful in further studies of 2-categorical topology. As a corollary we get a new proof of the classification of principal bundles. Another 2-category of 2-vector spaces has been proposed by J.C. Baez and A.S. Crans. A calculation using our main theorem shows that in this case the theory of principal 2-bundles splits, up to concordance, as two copies of ordinary vector bundle theory. When $2\mathcal{C}$ is a cobordism type 2-category we get a new notion of cobordism-bundles which turns out to be classified by the Madsen–Weiss spaces.

Key Words: 2-bundles, classifying spaces, geometric nerve.

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1. Introduction and main result

The main purpose of this paper is to introduce a general notion of bundles associated to topological 2-categories, and classify these. This encompasses the notion of 2-vector bundles developed in [4].

What we refer to in this paper as a "2-category" is sometimes called a bicategory [7], or a weak 2-category, i.e. with relaxed associativity conditions for horizontal composition. Wherever strictness is essential, we will explicitly state that the 2-category under consideration is a strict 2-category.

The key idea behind principal $2\mathcal{C}$-bundles is to categorify the transition data description of a principal G-bundle by formally replacing the group G with a 2-category $2\mathcal{C}$. In the classical setting, given an ordered open cover $\{U_i\}$ of a space, one associates an element of G to points in each double intersection $U_i \cap U_j$, subject

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to a cocycle condition. In our case, we associate objects in $2\mathcal{C}$ to points in each U_i, 1-morphisms to points in each double intersection $U_i \cap U_j$ and 2-morphisms to points in each triple intersection. The classical cocycle condition is replaced by a higher cocycle condition on each quadruple intersection.

A 2-category has a geometric nerve, $\Delta 2\mathcal{C}$, which is a simplicial set, [11]. There are obvious generalizations of these nerves to topological 2-categories, which produce simplicial spaces. A principal 2-bundle is the same as a simplicial map from the ordered Čech complex, $U_\bullet$, of the ordered open cover to the geometric nerve of the 2-category. Alternatively, one may describe a principal 2-bundle as a 2-functor from a certain 2-category into $2\mathcal{C}$.

Two principal 2-bundles over X are concordant if they are the restriction to $X \times \{0,1\}$ of some principal 2-bundle over $X \times I$. This is an equivalence relation, and our aim is to determine the set of concordance classes. A space B is said to be a classifying space if there is a bijection between the set of concordance classes of principal 2-bundles over such X and the set of homotopy classes of maps from X into B. We restrict to base spaces X having the homotopy type of a CW complex. We need mild assumptions on the topology of the 2-categories. Definition 5.5 introduces the term "good", and this condition is satisfied for any $2\mathcal{C}$ whose total space of 2-morphisms is a CW complex and with all source-target maps being Hurewicz fibrations, see Theorem 5.6. Our main theorem, which is a 2-categorical answer to M. Weiss' question, [35], "What does the classifying space of a category classify?", says:

Theorem 1.1 *The realization of the geometric nerve of a good topological* 2*-category* $2\mathcal{C}$ *is a classifying space for principal* $2\mathcal{C}$*-bundles.*

We get the homotopy class in $[X, |\Delta 2\mathcal{C}|]$ corresponding to a principal $2\mathcal{C}$-bundle $U_\bullet \to \Delta 2\mathcal{C}$ by taking geometric realization.

In [4] Baas, Dundas and Rognes introduced 2-vector bundles and defined an associated second order K-theory. The representing spectrum of this cohomology theory is $K(ku)$, see [4, 2, 3], and it qualifies as a form of elliptic cohomology theory since the chromatic filtration is essentially 2, see [1]. We recover in Example 3.3 the notion of 2-vector bundles as the principal $2\mathcal{C}$-bundles for a suitable $2\mathcal{C}$ defined in the spirit of Kapranov and Voevodsky's 2-vector spaces.

A similar construction based on Baez and Crans's version of 2-vector spaces [5] leads to a cohomology theory which is two copies of ordinary K-theory, and hence not a form of elliptic cohomology, see Conclusion 3.12.

The last four sections are more technical. They deal with facts which are used in the proof of the main theorem. It is the method of our proof that naturally leads to the level of generality found here. In Section 4 we study the following

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problem. Suppose that $Z_\bullet$ is a simplicial space. By applying the simplicial functor degreewise, we get a bisimplicial set. Let us diagonalize this bisimplicial set. What are reasonable conditions on $Z_\bullet$ so that this diagonal simplicial set has the Kan property? The answer to this question, Theorem 4.4, is a topological Kan condition.

In Section 5 we provide criteria for what we think a good and sufficiently fibrant topological 2-category $2\mathcal{C}$ should be, i.e. we ask when the geometric nerve $\Delta 2\mathcal{C}$ is a good simplicial space and when it satisfies the topological Kan condition.

In Section 6 we generalize the concept of concordance classes by replacing the geometric nerve of a topological category with an arbitrary simplicial space, $Z_\bullet$. We give some general properties (homotopy invariance, exact sequence, gluing) for such generalized concordance classes. We also show how to replace arbitrary $Z_\bullet$ by another simplicial space $\tilde{Z}_\bullet$, satisfying the topological Kan condition, without changing the set of concordance classes. This relies on Quillen's small object argument.

In Section 7 we generalize the statement of Theorem 1.1, and prove Theorem 7.12; that concordance classes of $Z_\bullet$-bundles are classified by the geometric realization $|Z_\bullet|$ for good simplicial spaces $Z_\bullet$. As a bonus, we note that the generalized theorem also takes care of the cases where $Z_\bullet$ is an ∞-category in the sense of J. Lurie, see [20]. Such $Z_\bullet$ were called weak Kan complexes by J.M. Boardman and R.M. Vogt, and quasicategories by A. Joyal.

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2. Principal $2\mathcal{C}$-bundles

In this section we define principal $2\mathcal{C}$-bundles. In our definition we specify transition data. However, there are two other, equivalent, definitions; either as a simplicial map from the ordered Čech complex to the geometric nerve, or as a continuous functor from a certain 2-category into $2\mathcal{C}$. We end the section with the definition of concordance and the restatement of our main theorem.

Our structure 2-category $2\mathcal{C}$ will be *topological*. This may be understood in at least two different ways: either as a 2-category enriched in spaces or as a 2-category internal in spaces. We take the latter, which is the most general, point of view. Hence misunderstanding our usage of the term topological 2-category will not lead to any mistakes. Discrete (or ordinary) 2-categories are those cases where the spaces of objects, 1- and 2-morphisms have the discrete topology.

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We use the following notation: $2\mathcal{C}_0$, $2\mathcal{C}_1$, and $2\mathcal{C}_2$ denote the topological spaces of objects, 1-morphisms and 2-morphisms respectively. If $x, y \in 2\mathcal{C}_0$, then $2\mathcal{C}_1(x,y)$ stands for the space of 1-morphisms $x \to y$, while $2\mathcal{C}(x,y)$ for the topological category with 1-morphisms $x \to y$ as objects and 2-morphisms between these as morphisms. If $f,g \in 2\mathcal{C}_1(x,y)$, then $2\mathcal{C}_2(f,g)$ denotes the space of 2-morphisms $f \Rightarrow g$. Horizontal composition is denoted by $*$, whereas vertical composition (of 2-morphisms) is written by juxtaposition. The natural associativity, left and right unit coherence isomorphisms are denoted $\underline{\alpha}$, $\underline{\lambda}$, and $\underline{\rho}$ respectively.

As defined in [4], an *ordered open cover* $\mathcal{U}$ of a topological space X consists of a family $\{U_\alpha\}$ of open subsets $U_\alpha \subseteq X$ indexed over a partially ordered set $\mathcal{I}$ such that the family cover X, i.e. $\bigcup_{\alpha\in\mathcal{I}} U_\alpha = X$, and whenever a finite intersection $U_{\alpha_0\cdots\alpha_k} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}$ is nonempty then the partial ordering of $\mathcal{I}$ restricts to a total ordering on $\{\alpha_0,\ldots,\alpha_k\}$.

Definition 2.1 Let X be a topological space, and let $2\mathcal{C}$ be a topological 2-category. A *principal* $2\mathcal{C}$*-bundle* $\mathcal{E}$ over X consists of an ordered open cover $\mathcal{U}$, with indexing set $\mathcal{I}$, together with

(1) for each α in $\mathcal{I}$, a continuous map

$$V_\alpha \colon U_\alpha \to 2\mathcal{C}_0,$$

and

(2) for each $\alpha < \beta$, a continuous family of 1-morphisms

$$E_{\alpha\beta} \colon U_{\alpha\beta} \to 2\mathcal{C}_1$$

with source and target $V_\alpha \xrightarrow{E_{\alpha\beta}} V_\beta$, and

(3) for each $\alpha < \beta < \gamma$, a continuous family of 2-morphisms

$$\phi_{\alpha\beta\gamma} \colon U_{\alpha\beta\gamma} \to 2\mathcal{C}_2$$

with source and target $E_{\alpha\gamma} \overset{\phi_{\alpha\beta\gamma}}{\Rightarrow} E_{\beta\gamma} * E_{\alpha\beta}$, such that

(4) the diagram

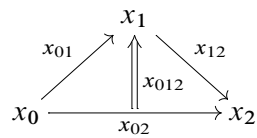
$$\begin{array}{ccc} E_{\gamma\delta} * (E_{\beta\gamma} * E_{\alpha\beta}) & \xRightarrow{\underline{\alpha}} & (E_{\gamma\delta} * E_{\beta\gamma}) * E_{\alpha\beta} \\ \Uparrow{\scriptstyle E_{\gamma\delta}*\phi_{\alpha\beta\gamma}} & & \Uparrow{\scriptstyle \phi_{\beta\gamma\delta}*E_{\alpha\beta}} \\ E_{\gamma\delta} * E_{\alpha\gamma} & \xLeftarrow[\phi_{\alpha\gamma\delta}]{} E_{\alpha\delta} & \xRightarrow[\phi_{\alpha\beta\delta}]{} E_{\beta\delta} * E_{\alpha\beta} \end{array}$$

commutes over $U_{\alpha\beta\gamma\delta}$ for each chain $\alpha < \beta < \gamma < \delta$ in $\mathcal{I}$. The 2-morphism $\underline{\alpha}$ is the natural associativity isomorphism of the 2-category $2\mathcal{C}$.

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In Section 3 we will give concrete examples of structure 2-categories giving rise to different types of principal 2-bundles. In particular, we recover the charted 2-vector bundles of [4] for a suitably chosen $2\mathcal{C}$. In the forthcoming examples, it is often possible to interpret V_α, $E_{\alpha\beta}$, and $\phi_{\alpha\beta\gamma}$ as bundles of objects, 1-morphisms, and 2-morphisms respectively.

Recall from [9, 11, 31], the notion of the geometric nerve $\Delta 2\mathcal{C}$ of a discrete 2-category $2\mathcal{C}$. The simplicial set $\Delta 2\mathcal{C}$ has 0-simplices the objects x_0 of $2\mathcal{C}$. The 1-simplices are the 1-morphisms $x_0 \xrightarrow{x_{01}} x_1$ of $2\mathcal{C}$, and the 2-simplices are triangles



where x_{012} is a 2-morphism $x_{02} \Rightarrow x_{12} * x_{01}$. For $n \geq 3$ the n-simplices are built from 2-simplices, such that for each subtetrahedron the obvious coherence condition for 2-morphisms is satisfied. The geometric nerve is 3-coskeletal, i.e. any simplex in the geometric nerve is uniquely determined by its 3-skeleton, see [6, Proposition 3.1]. The definition of the geometric nerve extends, in the obvious way, to the cases where $2\mathcal{C}$ is topological or simplicial, and then $\Delta 2\mathcal{C}$ becomes a simplicial space or a bisimplicial set respectively.

To an ordered cover $\mathcal{U}$ of X we associate the Čech complex $\check{C}_\bullet$ and the ordered Čech complex $U_\bullet$. Their definitions are as follows: $\check{C}_\bullet$ and $U_\bullet$ are both 1-coskeletal simplicial spaces (and thus determined by their 0- and 1-simplices). We define $\check{C}_0$ and $\check{C}_1$ to be

$$\coprod_\alpha U_\alpha \quad \text{and} \quad \coprod_{\alpha,\beta} U_{\alpha\beta}$$

respectively. The the ordered Čech complex $U_\bullet$ is defined as the sub-simplicial space of $\check{C}_\bullet$ having the same 0-simplices, but the 1-simplices being the disjoint union of all $U_{\alpha\beta}$'s with $\alpha \leq \beta$.

Let $|-|$ denote geometric realization. From [10, Theorem 2.1 and Proposition 2.6] we have that the spaces $|\check{C}_\bullet|$ and $|U_\bullet|$ are weakly equivalent to X via the natural maps.

With the geometric nerve, we get our first reformulation of the definition of a principal $2\mathcal{C}$-bundle:

Proposition 2.2 *There is a one-to-one correspondence between principal $2\mathcal{C}$-bundles subordinate to the ordered cover $\mathcal{U}$ and simplicial maps $U_\bullet \to \Delta 2\mathcal{C}$.*

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Proof: Let tr_3 denote the functor truncating a simplicial space at its 3-skeleton. Since the geometric nerve is 3-coskeletal, a simplicial map $U_\bullet \to \Delta 2\mathcal{C}$ corresponds to its truncation $\mathrm{tr}_3 U_\bullet \to \mathrm{tr}_3 \Delta 2\mathcal{C}$. The ordered Čech complex has free degeneracies, [10, Definition A.4], hence the truncated map is uniquely given as maps from the non-degenerate 0-, 1-, 2-, and 3-simplices of $U_\bullet$ satisfying the face relations. These data are precisely what is written out in Definition 2.1 above. □

Below we will give two additional reformulations of Definition 2.1. The idea is to look at $U_\bullet \to \Delta 2\mathcal{C}$ and think about the geometric nerve as a right adjoint. To illustrate this we consider the 1-categorical example $U_\bullet \to N_\bullet \mathcal{C}$, where $N_\bullet \mathcal{C}$ is the nerve of the category $\mathcal{C}$.

Define $X_{\mathcal{U}}^{\mathrm{ord}}$ to be the topological category whose spaces of objects and morphisms are the disjoint unions

$$\coprod_{\alpha} U_\alpha \quad \text{and} \quad \coprod_{\alpha \leq \beta} U_{\alpha\beta}$$

respectively. Whenever some underlying point $x \in X$ is contained in U_α we write x_α for the object of $X_{\mathcal{U}}^{\mathrm{ord}}$ corresponding to this copy of x. Similarly, we denote a typical morphism of $X_{\mathcal{U}}^{\mathrm{ord}}$ by $x_{\alpha\beta}$. The source and target of $x_{\alpha\beta}$ are x_α and x_β respectively, and composition is given by $x_{\beta\gamma} x_{\alpha\beta} = x_{\alpha\gamma}$.

Intuitively, $X_{\mathcal{U}}^{\mathrm{ord}}$ is the left adjoint of $N_\bullet$ applied to the ordered Čech complex $U_\bullet$. Hence there is a one-to-one correspondence between simplicial maps $U_\bullet \to N_\bullet \mathcal{C}$ and continuous functors $X_{\mathcal{U}}^{\mathrm{ord}} \to \mathcal{C}$. Also observe that $N_\bullet X_{\mathcal{U}}^{\mathrm{ord}}$ is isomorphic to $U_\bullet$, so it follows that $|N_\bullet X_{\mathcal{U}}^{\mathrm{ord}}|$ is weakly equivalent to X via the natural map.

Remark 2.3 The construction above is related to topological category $X_{\mathcal{U}}$ defined by Segal [28] as follows: To an open cover $\mathcal{U}$ indexed by an unordered set $\mathcal{I}$ we can associate a new open cover cover $\{U_S\}$, indexed over all finite non-empty subsets S of $\mathcal{I}$. There is a partial ordering of finite subsets given by inclusion $R \subseteq S$. Even though $\{U_S\}$ does not satisfy the conditions defining "an ordered open covering" the construction of $X_{\{U_S\}}^{\mathrm{ord}}$ is still well-defined. Moreover, the category $X_{\{U_S\}}^{\mathrm{ord}}$ is equal to $X_{\mathcal{U}}$. Explicitly; the space of objects of $X_{\mathcal{U}}$ is the disjoint union of all U_S and the space of morphisms is the disjoint union of U_S over all pairs $R \subseteq S$. The source map forgets R, and the target map is induced from the inclusions $U_S \subseteq U_R$. By [10, Proposition 2.7], $|N_\bullet X_{\mathcal{U}}|$ is weakly equivalent to X via the natural map.

Definition 2.4 Let $2X_{\mathcal{U}}^{\mathrm{norm}}$ be the topological 2-category constructed from $X_{\mathcal{U}}^{\mathrm{ord}}$ by adding identity 2-morphisms only.

We now get the following reformulation of the definition of a principal $2\mathcal{C}$-bundle subordinate to an ordered cover $\mathcal{U}$:

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Proposition 2.5 *There is a one-to-one correspondence between simplicial maps $U_\bullet \to \Delta 2\mathcal{C}$ and continuous normalized colax 2-functors $F\colon 2X_{\mathcal{U}}^{norm} \to 2\mathcal{C}$.*

Proof: Observe that $\Delta 2X_{\mathcal{U}}^{\mathrm{norm}} \cong N_\bullet X_{\mathcal{U}}^{\mathrm{ord}} \cong U_\bullet$. Hence, a continuous normalized colax 2-functor F yields a simplicial map $U_\bullet \to \Delta 2\mathcal{C}$ by functorality of the geometric nerve construction.

To reconstruct F from a simplicial map $U_\bullet \to \Delta 2\mathcal{C}$, notice that the restrictions to 0- and 1-morphisms, $U_0 \to (\Delta 2\mathcal{C})_0$ and $U_1 \to (\Delta 2\mathcal{C})_1$, determine F on objects and 1-morphisms respectively. Since F is normalized, it takes identities to identities, and there are no non-identity 2-morphisms in $2X_{\mathcal{U}}^{\mathrm{norm}}$. The remaining data of the colax 2-functor F are 2-cells

$$\phi\colon F(x_{\alpha\gamma}) \Rightarrow F(x_{\beta\gamma}) * F(x_{\alpha\beta}),$$

and we define these to be the value of $U_2 \to (\Delta 2\mathcal{C})_2$ on the 2-simplices $x_{\alpha\beta\gamma}$.

The verification of the cocycle condition on 3-cells is omitted. □

Considering Δ as a functor from the category of topological 2-categories and strict 2-functors into simplicial spaces, we could speculate about a left adjoint of Δ. Instead we explicitly construct 2-categories $2X_{\mathcal{U}}$ playing the role of the object adjoint to $U_\bullet$. Notice the presence of a non-associative version of Street's [31] oriental $\mathcal{O}_2$ in the construction.

Definition 2.6 Define the topological 2-category $2X_{\mathcal{U}}$ as follows: Let the space of objects, $(2X_{\mathcal{U}})_0$, be the disjoint union $\coprod U_\alpha$. A typical object is denoted x_α. Let the space of 1-morphisms be non-associatively freely generated by the disjoint union $\coprod_{\alpha<\beta} U_{\alpha\beta}$. A typical 1-morphism is thus a parenthesized sequence of $x_{\alpha_{i-1}\alpha_i}$'s. For example if $\alpha < \beta < \gamma < \delta < \epsilon$ and $x \in U_{\alpha\beta\gamma\delta\epsilon}$, then all the following expressions are different 1-morphisms:

$$x_{\alpha\beta}, \quad x_{\alpha\gamma}, \quad x_{\beta\gamma} * x_{\alpha\beta}, \quad (x_{\gamma\delta} * x_{\beta\gamma}) * x_{\alpha\beta},$$
$$x_{\gamma\delta} * (x_{\beta\gamma} * x_{\alpha\beta}), \quad (x_{\delta\epsilon} * x_{\gamma\delta}) * (x_{\beta\gamma} * x_{\alpha\beta}).$$

Observe that any 1-morphism $f\colon x_{\alpha_0} \to x_{\alpha_k}$ has an associated chain $\alpha_0 < \alpha_1 < \cdots < \alpha_k$ of indices, thus f is some parenthesizing of $x_{\alpha_{k-1}\alpha_k} * \cdots * x_{\alpha_1\alpha_2} * x_{\alpha_0\alpha_1}$.

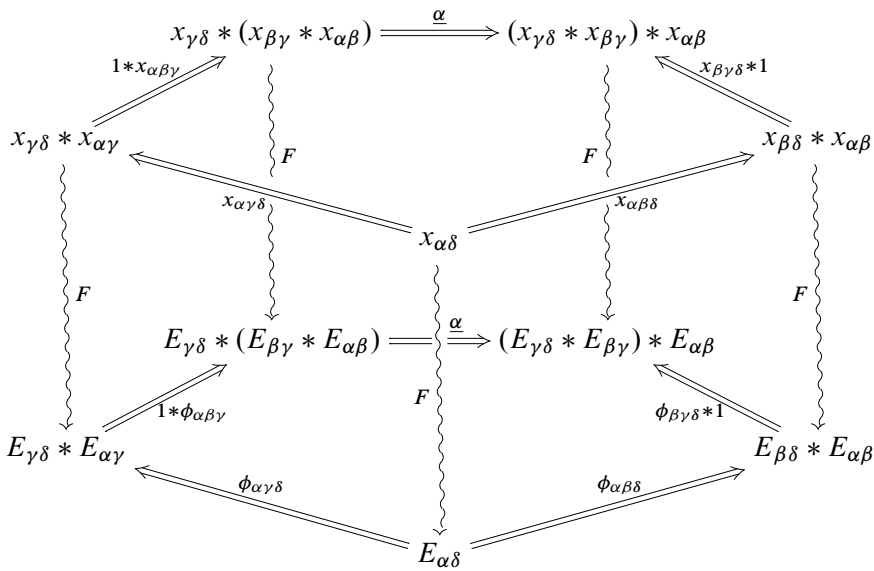
The space of 2-morphisms $(2X_{\mathcal{U}})_2$ is defined as follows: for arbitrary 1-morphisms $f\colon x_{\alpha_0} \to x_{\alpha_k}$ and $g\colon y_{\beta_0} \to y_{\beta_l}$ there are either one or zero 2-morphisms $f \Rightarrow g$. If $x = y$, $\alpha_0 = \beta_0$ and $\alpha_k = \beta_l$, and the chain $\alpha_0 < \alpha_1 < \cdots < \alpha_k$ is a refinement of $\beta_0 < \beta_1 < \cdots < \beta_l$, then there exists a unique 2-morphism $f \Rightarrow g$. Otherwise, there is no 2-morphism $f \Rightarrow g$. Let $x_{\alpha\beta\gamma}$ denote the unique 2-morphism $x_{\alpha\gamma} \Rightarrow x_{\beta\gamma} * x_{\alpha\beta}$.

This leads to the following reformulation of Definition 2.1:

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Proposition 2.7 *There is a one-to-one correspondence between simplicial maps $U_\bullet \to \Delta 2\mathcal{C}$ and continuous strict 2-functors $F\colon 2X_{\mathcal{U}} \to 2\mathcal{C}$.*

Proof: The proof is to inspect the definitions. Suppose first that we are given such a strict 2-functor F. Since $\Delta 2\mathcal{C}$ is 3-coskeletal, a map $U_\bullet \to \Delta 2\mathcal{C}$ is uniquely determined by the maps of 0-, 1-, 2-, and 3-simplices. Restricting F to the objects, 1-morphisms, and 2-morphisms of the form x_α, $x_{\alpha\beta}$, and $x_{\alpha\beta\gamma}$ we recover the maps $U_0 \to (\Delta 2\mathcal{C})_0$, $U_1 \to (\Delta 2\mathcal{C})_1$, and $U_2 \to (\Delta 2\mathcal{C})_2$ respectively. To get the map on 3-simplices, we need to verify the tetrahedron coherence condition of $\Delta 2\mathcal{C}$. Take a 3-simplex of $U_\bullet$. It is represented by a point x in $U_{\alpha\beta\gamma\delta}$. Now inspect how F, in the following diagram, transforms the top, consisting of 2-morphisms in $2X_{\mathcal{U}}$, into the bottom, consisting of 2-morphisms in $2\mathcal{C}$:



This verifies the tetrahedron coherence condition.

Next, we consider the problem of reconstructing a strict 2-functor $F\colon 2X_{\mathcal{U}} \to 2\mathcal{C}$ from a simplicial map $U_\bullet \to \Delta 2\mathcal{C}$. The map on 0-, 1-, and 2-simplices determines F on objects, 1-morphisms, and 2-morphisms of the form x_α, $x_{\alpha\beta}$, and $x_{\alpha\beta\gamma}$. Now observe that $2X_{\mathcal{U}}$ is generated by such 1- and 2-morphisms with the associators. Thus the strict functor F is uniquely determined by the data above. We omit further details. □

Proposition 2.8 *Let $\mathcal{U}$ be an ordered open cover of X. The corresponding ordered Čech complex $U_\bullet$ embeds as a simplicial deformation retract of $\Delta 2X_{\mathcal{U}}$, and consequently we get a homotopy equivalence $|\Delta 2X_{\mathcal{U}}| \simeq X$.*

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We know that $U_\bullet \cong \Delta 2X_{\mathcal{U}}^{\mathrm{norm}}$. If $2X_{\mathcal{U}}^{\mathrm{norm}}$ and $2X_{\mathcal{U}}$ are biequivalent as 2-categories, then the result would follow from [32, Section 10]. However, a direct argument is not difficult:

Proof: Observe that $\Delta 2X_{\mathcal{U}}$ actually is 2-coskeletal, and we can write any n-simplex as a flag

$$\begin{pmatrix} x_{\alpha_0} & f_{01} & f_{02} & f_{03} & \cdots & f_{0n} \\ & x_{\alpha_1} & f_{12} & f_{13} & \cdots & f_{1n} \\ & & x_{\alpha_2} & f_{23} & \cdots & f_{2n} \\ & & & x_{\alpha_3} & \cdots & f_{3n} \\ & & & & \ddots & \vdots \\ & & & & & x_{\alpha_n} \end{pmatrix},$$

where $\alpha_0 < \alpha_1 < \ldots < \alpha_n$, $x \in U_{\alpha_0 \cdots \alpha_n}$ and $f_{ij} : x_{\alpha_i} \to x_{\alpha_j}$ are 1-morphisms in $2X_{\mathcal{U}}$ such that for every $i < j < k$ there exists a 2-morphism $f_{ik} \Rightarrow f_{jk} * f_{ij}$. Define $U_\bullet \to \Delta 2X_{\mathcal{U}}$ by sending the n-simplex $x \in U_{\alpha_0 \alpha_1 \cdots \alpha_n}$ to the n-simplex in $\Delta 2X_{\mathcal{U}}$ given by $f_{ij} = x_{\alpha_i \alpha_j}$. Observe that this gives an inclusion, and we view $U_\bullet$ as a sub-simplicial space of $\Delta 2X_{\mathcal{U}}$.

Now we define a simplicial deformation retraction h from $\Delta 2X_{\mathcal{U}}$ down to $U_\bullet$ by

$$h_k(f_{ij}) = \begin{pmatrix} x_{\alpha_0} & \cdots & x_{\alpha_0 \alpha_{k-1}} & x_{\alpha_0 \alpha_k} & x_{\alpha_0 \alpha_k} & x_{\alpha_0 \alpha_{k+1}} & \cdots & x_{\alpha_0 \alpha_n} \\ & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ & & x_{\alpha_{k-1}} & x_{\alpha_{k-1} \alpha_k} & x_{\alpha_{k-1} \alpha_k} & x_{\alpha_{k-1} \alpha_{k+1}} & \cdots & x_{\alpha_{k-1} \alpha_n} \\ & & & x_{\alpha_k} & 1_{x_{\alpha_k}} & x_{\alpha_k \alpha_{k+1}} & \cdots & x_{\alpha_k \alpha_n} \\ & & & & x_{\alpha_k} & f_{k\,k+1} & \cdots & f_{kn} \\ & & & & & x_{\alpha_{k+1}} & \cdots & f_{k+1\,n} \\ & & & & & & \ddots & \vdots \\ & & & & & & & x_{\alpha_n} \end{pmatrix}.$$

These maps $h_k : (\Delta 2X_{\mathcal{U}})_n \to (\Delta 2X_{\mathcal{U}})_{n+1}$, for $0 \leq k \leq n$, verify the usual identities for a simplicial deformation retraction. □

Given a principal $2\mathcal{C}$-bundle $\mathcal{E}$ over X and a subspace A of X it is clear that we may restrict $\mathcal{E}$ to A. We denote the restriction by $\mathcal{E}|_A$. More generally, there is a pullback $f^*\mathcal{E}$ over Y, for every continuous map $f : Y \to X$.

We define equivalence of principal $2\mathcal{C}$-bundles by concordance:

Definition 2.9 Two principal $2\mathcal{C}$-bundles $\mathcal{E}_0$ and $\mathcal{E}_1$ over X are *concordant* if there exists a principal $2\mathcal{C}$-bundle $\mathcal{E}$ over $X \times I$ such that the restrictions $\mathcal{E}|_{X \times \{0\}}$ and $\mathcal{E}|_{X \times \{1\}}$ are $\mathcal{E}_0$ and $\mathcal{E}_1$ respectively.

Concordance is an equivalence relation. The main theorem, Theorem 1.1, classifies the set of concordance classes of principal $2\mathcal{C}$-bundles over X: under

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mild assumptions on X and $2\mathcal{C}$, there is a bijection between concordance classes and homotopy classes of maps $X \to |\Delta 2\mathcal{C}|$.

Remark 2.10 There are other ways to define an equivalence relation between principal $2\mathcal{C}$-bundles. In all the following cases one should allow elementary refinements of the cover, compare [4, Definition 2.4]. In addition one declares two principal $2\mathcal{C}$-bundles $\mathcal{E}_0$ and $\mathcal{E}_1$, subordinate to the same ordered open cover $\mathcal{U}$, to be equivalent if either

i) there exists a natural transformation between the 2-functors $2X_{\mathcal{U}} \to 2\mathcal{C}$ corresponding to $\mathcal{E}_0$ and $\mathcal{E}_1$,

ii) there is an adjoint equivalence relating $\mathcal{E}_0$ and $\mathcal{E}_1$ in the 2-category of 2-functors, natural transformations and modifications, or

iii) there is a simplicial homotopy $U_\bullet \times \Delta^1_\bullet \to \Delta 2\mathcal{C}$ between the simplicial maps $U_\bullet \to \Delta 2\mathcal{C}$ representing $\mathcal{E}_0$ and $\mathcal{E}_1$ respectively.

This list of possible equivalence relations is not exhaustive. Under what circumstances will this actually lead to a set of equivalence classes different from the concordance classes? Discussing this in detail would be beyond the scope of this article, but we believe that for compact X and those $2\mathcal{C}$ that occur most commonly, the equivalences of type i)—iii) should all coincide with concordance.

3. Examples

In this section we give several examples of structure 2-categories, their principal 2-bundles, and the corresponding classifying space. Let us start off by discussing a simple example which illuminates the fact that different categories can have the same concordance classes of principal bundles:

Example 3.1 (Complex line bundles) In this example we consider the more elementary setting of topological categories. In this case the nerve, $N_\bullet$, and the geometric nerve, Δ, coincide. To do calculations we need a topological version of Quillen's Theorem A, [26], and refer to Moerdijk [24, page 64].

Let $U(1)$ be the topological group of complex numbers with modulus 1. A principal bundle with structure group $U(1)$ corresponds to a complex line bundle. When considering $U(1)$ as a topological category, it has only one object, $*$, and $U(1)$ is the space of morphisms. We may also think about the space $\mathbb{CP}^\infty$ as a category: The space of objects is $\mathbb{CP}^\infty$, and we add identity morphisms only. It is classical that maps $X \to \mathbb{CP}^\infty$ correspond to complex line bundles by pulling back the canonical line bundle.

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There is an intermediate topological category which we will denote by $\mathcal{CU}$. The space of objects is $\mathbb{CP}^\infty$ and the total space of morphisms can be identified with $S^\infty \times_{U(1)} S^\infty$. In more detail the morphisms $\mathcal{CU}(L',L)$ between two complex lines in $\mathbb{C}^\infty$ are the unitary linear transformations $f: L' \to L$. To topologize the total space of all morphisms we consider basis vectors b' and b of length 1 for L' and L respectively, together with a point $c \in U(1)$. To the triple (b',b,c) we associate the linear transformation $f: zb' \mapsto zcb$. Therefore $\mathcal{CU}_1$ is defined as the quotient of all such triples, where we identify (b',b,c) with $(\theta' b', \theta b, \theta^{-1}\theta' c)$ for all $\theta,\theta' \in U(1)$.

Fix a complex line L_0 in $\mathcal{CU}_0$. This gives an inclusion functor $i: U(1) \to \mathcal{CU}$. We form the comma category $\mathbb{CP}^\infty \backslash i$, and observe that the canonical map $B(\mathbb{CP}^\infty \backslash i) \to \mathbb{CP}^\infty$ is a fiber bundle with contractible fiber $EU(1)$. The topological version of Quillen's Theorem A implies that i induces a weak equivalence

$$BU(1) = |N_\bullet U(1)| \xrightarrow{\simeq} |N_\bullet \mathcal{CU}| = B\mathcal{CU}.$$

There is also an inclusion $j: \mathbb{CP}^\infty \to \mathcal{CU}$ given as the identity map on objects. Observe that all morphisms of the comma category $\mathbb{CP}^\infty \backslash j$ are identity morphisms. We identify the map $B(\mathbb{CP}^\infty \backslash j) \to \mathbb{CP}$ with the source map $\mathcal{CU}_1 \to \mathbb{CP}$, which is a fiber bundle whose fiber is the unit sphere in $\mathbb{C}^\infty$. Consequently, by the topological version of Quillen's Theorem A, it follows that j induces a weak equivalence

$$\mathbb{CP}^\infty = |N_\bullet \mathbb{CP}^\infty| \xrightarrow{\simeq} |N_\bullet \mathcal{CU}| = B\mathcal{CU}.$$

Example 3.2 (Gerbes) Recall from [25, Section 3.1] the notion of a *local bundle gerbe*. This fits into our framework as follows. Let $2\mathcal{G}$ be the topological 2-category with $2\mathcal{G}_0 = *$, $2\mathcal{G}_1 = \mathbb{CP}^\infty$, and, interpreting $L, L' \in \mathbb{CP}^\infty$ as complex lines in $\mathbb{C}^\infty$, a 2-morphism $f: L \Rightarrow L'$ is the same as a unitary linear transformation between these lines. Represent the lines as $L = [z_1, z_2, z_3, \ldots]$ and $L' = [z'_1, z'_2, z'_3, \ldots]$. Then we may write their tensor product as line represented by the collection of all products $z_i z'_j$, i.e. $L \otimes L' = [z_1 z'_1, z_2 z'_1, z_1 z'_2, z_1 z'_3, z_2 z'_2, z_3 z'_1, \ldots]$. This gives a map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$, and which we take as the definition of horizontal composition of 1-morphisms.

Now consider a principal $2\mathcal{G}$-bundle. Since $\mathbb{CP}^\infty$ classifies complex line bundles, we interpret $E_{\alpha\beta}$ as a complex line bundle over $U_{\alpha\beta}$. From this point of view, $\phi_{\alpha\beta\gamma}$ is a bundle isomorphism from $E_{\alpha\gamma}$ to $E_{\beta\gamma} \otimes E_{\alpha\beta}$, and it corresponds to the trivialization $\theta_{\alpha\beta\gamma}$ considered by [25]. There is a one-to-one correspondence between stable isomorphism classes of gerbes over X and elements of the cohomology group $H^3(X;\mathbb{Z})$, see [25, Theorem 3.3]. Thus we write

$$\mathrm{Gerbe}(X) = [X, K(\mathbb{Z},3)].$$

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Example 3.3 (BDR 2-vector bundles) In order to classify the concordance classes of 2-vector bundles it is sufficient to describe the corresponding structure 2-category. By Theorem 1.1 the geometric nerve of the structure 2-category classifies the concordance classes. Since this result depends only on transition data, the description of the fiber plays no role here.

Based on the 2-vector spaces of [19], we define a topological 2-category $2\mathcal{KV}(n)$ as follows: Take a single point $*$ as the space of objects, and let the 1-morphisms be the space of $n \times n$-matrices of complex Grassmannians

$$\coprod \left(\mathrm{Gr}(a_{ij})\right)_{i,j=1}^{n},$$

where the disjoint union runs over all $n \times n$-matrices (a_{ij}) which are invertible in $GL_n(\mathbb{Z})$ and have non-negative integers as entries.. Thus a map $E_{\alpha\beta} \colon U_{\alpha\beta} \to (2\mathcal{KV}(n))_1$ defines an $n \times n$-matrix $(E_{ij})_{\alpha\beta}$ of complex vector bundles such that $\det(\dim(E_{ij})_{\alpha\beta}) = \pm 1$. The 2-morphisms are defined as matrices of linear isomorphisms, and the composition, $*$, of 1-morphisms is defined as reverse matrix multiplication using the exterior $\otimes$ of Grassmannians as product between matrix entries, and $\oplus$ as sum. This recovers the definition of *charted 2-vector bundles* given in [4]. Their composition $\cdot$ is dual to our $*$, and our $\phi_{\alpha\beta\gamma}$ is the inverse of their $\phi^{\alpha\beta\gamma}$. We write

$$2\mathrm{Vect}_n(X) = [X, |\Delta 2\mathcal{KV}(n)|].$$

This leads to a geometrically defined form of elliptic cohomology: Baas, Dundas and Rognes show in [4] how to define a kind of second order K-theory, namely the $2K$-theory of 2-vector bundles. They conjectured that the representing spectrum was $K(ku)$, and this has been proved by Baas, Dundas, Richter and Rognes in [2, 3]. Calculations by Ausoni and Rognes [1] show that this spectrum has telescopic complexity 2, and hence it qualifies as a form of elliptic cohomology.

Remark 3.4 Observe that $2\mathcal{KV}(n)$ for $n > 1$ is not group-complete, for example when $n = 2$ the 1-morphism $\left(\begin{smallmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} \end{smallmatrix}\right)$ has no inverse. We do not see the lack of inverses as an obstacle to taking a geometric point of view, and by allowing non-group-complete $2\mathcal{C}$ in our definition of principal $2\mathcal{C}$-bundles, Definition 2.1, we try to encourage geometric thinking. To get a *theory*, some kind of group completion machine is needed. For the case above we recommend ring completion of rig categories as constructed in [3].

Example 3.5 (String bundles) Let $2\mathcal{S}$ denote the surface 2-category defined by U. Tillmann in [34, Section 2]. Its objects, 1-morphisms, and 2-morphisms are closed strings, oriented cobordisms, and their diffeomorphisms respectively. Define *string bundles* as principal $2\mathcal{S}$-bundles. The classifying space $|\Delta 2\mathcal{S}|$ is an infinite

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loop space, and it is related to the stabilized mapping class group of oriented surfaces, Γ_∞, by the formula:

$$\Omega|N_\bullet 2\mathscr{S}| \cong \mathbb{Z} \times B\Gamma_\infty^+,$$

see [34, Theorem 3.1].

Let $\mathbb{CP}^\infty_{-1}$ denote the spectrum whose $2k$th space is the Thom space of the complex bundle complementary to the canonical complex line bundle over $\mathbb{CP}^{k-1}$. By I. Madsen and M. Weiss' proof of the generalized Mumford conjecture [23], the classifying space of string bundles is $\Omega^{\infty-1}\mathbb{CP}^\infty_{-1}$. In [15, Definition 5.1] S. Galatius, I. Madsen, U. Tillmann, and M. Weiss introduce a topological category $\mathcal{C}_2^+$ of embedded oriented surfaces. By their new proof of the generalized Mumford conjecture, the notion of principal $\mathcal{C}_2^+$-bundles coincides with the notion of string bundles, up to concordance. They also introduce the notation $MTSO(2)$ for the spectrum $\mathbb{CP}^\infty_{-1}$. To summarize, the concordance classes of string bundles are classified by $\Omega^{\infty-1}MTSO(2) = \Omega^{\infty-1}\mathbb{CP}^\infty_{-1}$, i.e.

$$\mathrm{String}(X) = [X, \Omega^{\infty-1}MTSO(2)] = [X, \Omega^{\infty-1}\mathbb{CP}^\infty_{-1}].$$

Example 3.6 (Cobordism bundles) Extending the previous example, we now define *d-cobordism bundles* as principal $\mathcal{C}_d$-bundles, where $\mathcal{C}_d$ is the topological category of embedded $(d-1)$-manifolds and embedded cobordisms introduced in [15, Section 2.1]. Furthermore, let $MTO(d)$ be the spectrum whose $(n+d)$th space is the Thom spectrum of the bundle $U^\perp_{d,n}$ complementary to the canonical d-plane bundle over the Grassmannian of d-planes in $\mathbb{R}^{n+d}$. The main theorem of Galatius, Madsen, Tillmann, and Weiss shows that the classifying space of d-cobordism bundles is $\Omega^{\infty-1}MTO(d)$. There are variations of these constructions by placing various types of tangential structure on the cobordism category, e.g. orientation, etc. We write

$$\mathrm{Cob}^d(X) = [X, \Omega^{\infty-1}MTO(d)].$$

Example 3.7 (Graph bundles) Let $\mathcal{C}_\infty$ denote the graph cobordism category defined by Galatius in [14, Section 4]. We define *graph bundles* as principal $\mathcal{C}_\infty$-bundles. On one hand, the classifying space $B\mathcal{C}_\infty$ is weakly equivalent to $\Omega^{\infty-1}S$, where S is the sphere spectrum. On the other hand, there is a weak equivalence $\mathbb{Z}\times B\mathrm{Aut}^+_\infty \simeq \Omega B\mathcal{C}_\infty$, where Aut_∞ is the automorphism group of the free group on n generators as n tends to infinity. For the concordance classes of graph bundles over X we write

$$\mathrm{Graph}(X) = [X, \Omega^{\infty-1}S].$$

Example 3.8 (BC 2-vector bundles) Let us now discuss choices of $2\mathcal{C}$ based on Baez and Crans's 2-vector spaces. There are numerous variations depending on

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how strictly the 1-morphisms should be equivalences, how much structure the 2-morphisms should contain, and how to choose the objects from 2-vector spaces within the same weak equivalence class. Let b_1 and b_0 be non-negative integers and let C_* be a 2-term chain complex of vector spaces.

i) Let $2\mathcal{B}_{\text{strict}}(C_*)$ be the 2-category with a single object C_*, the 1-morphisms are the chain isomorphisms $f\colon C_* \to C_*$, and the 2-morphisms $f \Rightarrow g$ are chain homotopies ϕ from f to g.

ii) Let $2\mathcal{B}_{\text{weak}}(C_*)$ be the 2-category with a single object C_*, the 1-morphisms are the chain equivalences $f\colon C_* \to C_*$, and the 2-morphisms $f \Rightarrow g$ are chain homotopies ϕ from f to g.

iii) Let $2\mathcal{B}_{\text{eq}}(C_*)$ be the 2-category with a single object C_*, the 1-morphisms are tuples $(f,\bar{f},\iota_f,\epsilon_f)$, where f and $\bar{f}$ are chain maps $C_* \to C_*$, and $\iota_f\colon 1 \Rightarrow f\bar{f}$ and $\epsilon_f\colon \bar{f}f \Rightarrow 1$ are chain homotopies, and the 2-morphisms $(f,\bar{f},\iota_f,\epsilon_f) \Rightarrow (g,\bar{g},\iota_g,\epsilon_g)$ are pairs of chain homotopies $\phi\colon f \Rightarrow g$ and $\bar{\phi}\colon \bar{f} \Rightarrow \bar{g}$ such that the following identities hold:

$$\iota_g - \iota_f = f\bar{\phi} + \phi\bar{g} \quad \text{and} \quad \epsilon_f - \epsilon_g = f\bar{\phi} + \phi\bar{g}.$$

iv) Let $2\mathcal{B}_{\text{ad}}(C_*)$ be the 2-category with a single object C_*, the 1-morphisms are tuples $(f,\bar{f},\iota_f,\epsilon_f)$, where f and $\bar{f}$ are chain maps $C_* \to C_*$, and $\iota_f\colon 1 \Rightarrow f\bar{f}$ and $\epsilon_f\colon \bar{f}f \Rightarrow 1$ are chain homotopies such that the zigzag identities

$$f\epsilon_f + \iota_f f = 0 \quad \text{and} \quad \epsilon_f\bar{f} + \bar{f}\epsilon_f = 0$$

hold, and the 2-morphisms $(f,\bar{f},\iota_f,\epsilon_f) \Rightarrow (g,\bar{g},\iota_g,\epsilon_g)$ are pairs of chain homotopies $\phi\colon f \Rightarrow g$ and $\bar{\phi}\colon \bar{f} \Rightarrow \bar{g}$ such that the following identities hold:

$$\iota_g - \iota_f = f\bar{\phi} + \phi\bar{g} \quad \text{and} \quad \epsilon_f - \epsilon_g = f\bar{\phi} + \phi\bar{g}.$$

v) Let $2\mathcal{B}_{\text{weak}}(b_1,b_0)$ be the 2-category with objects all 2-term chain complexes C_* with Betti-numbers b_0,b_1, the 1-morphisms are the chain equivalences $f\colon C_* \to C'_*$, and the 2-morphisms $f \Rightarrow g$ are chain homotopies ϕ from f to g.

vi) Let $2\mathcal{B}_{\text{eq}}(b_1,b_0)$ be the 2-category with objects all 2-term chain complexes C_* with Betti-numbers b_0, b_1, the 1-morphisms are tuples $(f,\bar{f},\iota_f,\epsilon_f)$, where $f\colon C_* \to C'_*$ and $\bar{f}\colon C'_* \to C_*$ are chain maps, and $\iota_f\colon 1 \Rightarrow f\bar{f}$ and $\epsilon_f\colon \bar{f}f \Rightarrow 1$ are chain homotopies, and the 2-morphisms $(f,\bar{f},\iota_f,\epsilon_f) \Rightarrow$

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$(g,\bar{g},\iota_g,\epsilon_g)$ are pairs of chain homotopies $\phi\colon f \Rightarrow g$ and $\bar{\phi}\colon \bar{f} \Rightarrow \bar{g}$ such that the following identities hold:

$$\iota_g - \iota_f = f\bar{\phi} + \phi\bar{g} \quad \text{and} \quad \epsilon_f - \epsilon_g = f\bar{\phi} + \phi\bar{g}.$$

vii) Let $2\mathcal{B}_{\mathrm{ad}}(b_1,b_0)$ be the 2-category with objects all 2-term chain complexes C_* with Betti-numbers b_0,b_1, the 1-morphisms are tuples $(f,\bar{f},\iota_f,\epsilon_f)$, where $f\colon C_* \to C'_*$ and $\bar{f}\colon C'_* \to C_*$ are chain maps, and $\iota_f\colon 1 \Rightarrow f\bar{f}$ and $\epsilon_f\colon \bar{f}f \Rightarrow 1$ are chain homotopies such that the zigzag identities

$$f\epsilon_f + \iota_f f = 0 \quad \text{and} \quad \epsilon_f \bar{f} + \bar{f}\epsilon_f = 0$$

hold, and the 2-morphisms $(f,\bar{f},\iota_f,\epsilon_f) \Rightarrow (g,\bar{g},\iota_g,\epsilon_g)$ are pairs of chain homotopies $\phi\colon f \Rightarrow g$ and $\bar{\phi}\colon \bar{f} \Rightarrow \bar{g}$ such that the following identities hold:

$$\iota_g - \iota_f = f\bar{\phi} + \phi\bar{g} \quad \text{and} \quad \epsilon_f - \epsilon_g = f\bar{\phi} + \phi\bar{g}.$$

We will now calculate the classifying space in the case **v)** of the example above. Set $2\mathcal{B} = 2\mathcal{B}_{\mathrm{weak}}(b_1,b_0)$. In order to study $2\mathcal{B}$ in more detail, we choose for each object C_* isomorphisms

$$C_1 \cong H_1 \oplus B \quad \text{and} \quad C_0 \cong B \oplus H_0,$$

where H_1 and H_0 are complex vector spaces of complex dimension b_1 and b_0 respectively, and B is an arbitrary complex vector space. Moreover, we can choose these isomorphisms such that the differential $d\colon C_1 \to C_0$ is identified with the matrix $\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$. A 1-morphism $f\colon C_* \to C'_*$ is then identified with a pair of matrices,

$$\begin{pmatrix} f_{H'_1}^{H_1} & f_{H'_1}^{B} \\ 0 & f_{B'}^{B} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} f_{B'}^{B} & f_{B'}^{H_0} \\ 0 & f_{H'_0}^{H_0} \end{pmatrix}$$

representing f_1 and f_0 respectively. Here the notation $f_{B'}^{H_1}$ indicates a linear map from H_1 to B', etc. Observe that f is a chain equivalence if and only if both $f_{H'_1}^{H_1}$ and $f_{H'_0}^{H_0}$ are invertible. A chain homotopy ϕ from f to g is a linear map $C_0 = B \oplus H_0 \to H'_1 \oplus B' = C'_1$, and thus represented by a matrix

$$\begin{pmatrix} \phi_{H'_1}^{B} & \phi_{H'_1}^{H_0} \\ \phi_{B'}^{B} & \phi_{B'}^{H_0} \end{pmatrix},$$

such that

$$\phi_{H'_1}^{B} = f_{H'_1}^{B} - g_{H'_1}^{B}, \quad \phi_{B'}^{B} = f_{B'}^{B} - g_{B'}^{B}, \quad \text{and} \quad \phi_{B'}^{H_0} = f_{B'}^{H_0} - g_{B'}^{H_0}.$$

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Horizontal composition corresponds to matrix multiplication and vertical composition to addition of matrices.

Because $\phi^{H_0}_{H_1'}$ is arbitrary, the space of chain homotopies between two fixed chain maps f and g admits an affine structure. This indicates that there are too many 2-morphisms in $2\mathcal{B}$. Therefore, define $2\mathcal{B}^{\mathrm{Ho}}$ as the quotient of $2\mathcal{B}$, where the objects and 1-morphisms are the same, and two 2-morphisms from $2\mathcal{B}$ are identified in the quotient if their source and target 1-morphisms are pairwise equal. Explicitly, the class of ϕ in $2\mathcal{B}^{\mathrm{Ho}}$ is represented by a matrix

$$\begin{pmatrix} \phi^{B}_{H_1'} & * \\ \phi^{B}_{B'} & \phi^{H_0}_{B'} \end{pmatrix},$$

where we do not care about the stuff in the upper right corner. Not unexpectedly, we get:

Lemma 3.9 *The geometric nerves $\Delta 2\mathcal{B}$ and $\Delta 2\mathcal{B}^{\mathrm{Ho}}$ are weakly equivalent.*

Proof: Here is one way to show this: both 2-categories satisfies the assumptions **G1'**, **G2'**, **G3**, and **G4** of Section 5, and the canonical map $2\mathcal{B} \to 2\mathcal{B}^{Ho}$ is a continuous strict 2-functor. In order to apply Lemma 5.8, we fix chain complexes C_* and C'_*, and compare nerves of morphism categories:

$$N_\bullet 2\mathcal{B}(C_*, C'_*) \to N_\bullet 2\mathcal{B}^{\mathrm{Ho}}(C_*, C'_*).$$

At each simplicial degree k, this map is a fiber bundle with fiber an affine space, namely a $(k-1)$-tuple of $\phi^{H_0}_{H_1}$'s. Since the fiber is contractible, the map is a weak equivalence. □

We finish the calculation by comparing $2\mathcal{B}^{\mathrm{Ho}}$ to the product of the complex general linear groups $GL(b_1)$ and $GL(b_0)$. We have 2-functors

$$GL(b_1) \times GL(b_0) \xrightarrow{i} 2\mathcal{B}^{\mathrm{Ho}} \xrightarrow{H} GL(b_1) \times GL(b_0),$$

where i includes $GL(b_1) \times GL(b_0)$ as the automorphisms of the chain complex $\mathbb{C}^{b_1} \xrightarrow{0} \mathbb{C}^{b_0}$, and H is homology of the chain complex C_*, i.e. we choose bases for H_1 and H_0 for each object of $2\mathcal{B}^{\mathrm{Ho}}$. For $\mathbb{C}^{b_1}$ and $\mathbb{C}^{b_0}$ we choose the standard bases. Observe that the composition Hi is the identity on $GL(b_1) \times GL(b_0)$. We will now construct a normalized colax transformation σ from the identity on $2\mathcal{B}^{\mathrm{Ho}}$ to the composition iH.

The choice of bases for H_1 and H_0 specifies projection maps $C_1 \cong H_1 \oplus B \to \mathbb{C}^{b_1}$ and $C_0 \cong B \oplus H_0 \to \mathbb{C}^{b_0}$. This yields for each chain complex C_* a chain map

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σ_{C_*} into $\mathbb{C}^{b_1} \xrightarrow{0} \mathbb{C}^{b_0}$. For a chain equivalence $f : C_* \to C'_*$ we observe that

$$\begin{array}{ccc} H_1 \oplus B & \xrightarrow{\begin{pmatrix} f^{H_1}_{H'_1} & f^{B}_{H'_1} \\ 0 & f^{B}_{B'} \end{pmatrix}} & H'_1 \oplus B' \\ \downarrow{\scriptstyle \sigma_{C_*}} & & \downarrow{\scriptstyle \sigma_{C'_*}} \\ \mathbb{C}^{b_1} & \xrightarrow{[f^{H_1}_{H'_1}]} & \mathbb{C}^{b_1} \end{array}$$

does not commute. Therefore we define σ_f as the class of chain homotopies represented by the matrix $\begin{pmatrix} f^{B}_{H'_1} & * \end{pmatrix}$. We leave it to the reader to verify that σ is a normalized colax transformation.

Remark 3.10 To see why we need the intermediate 2-category $2\mathcal{B}^{\mathrm{Ho}}$, try to replace $2\mathcal{B}^{\mathrm{Ho}}$ with $2\mathcal{B}$, and then define σ as above. In that case take $\sigma_f = \begin{pmatrix} f^{B}_{H'_1} & 0 \end{pmatrix}$. Given a pair of composable chain maps $C_* \xrightarrow{f} C'_* \xrightarrow{g} C''_*$, observe that, in general, we have $\sigma_{gf} \neq g \circ \sigma_f + \sigma_g \circ f$ in $2\mathcal{B}$. Hence σ is not well-defined on $2\mathcal{B}$.

We now apply the following standard technique:

Lemma 3.11 *Let F and G be normalized colax 2-functors $2\mathcal{C} \to 2\mathcal{D}$. A normalized colax transformation $\phi\colon F \to G$ induces a homotopy between the induced maps $|\Delta F|$ and $|\Delta G|$.*

Proof: Let $\mathcal{I}$ be the category with two objects, 0 and 1, and one non-identity morphisms, $0 \to 1$. Our normalized colax transformation ϕ defines a 2-functor $2\mathcal{C} \times \mathcal{I} \to 2\mathcal{D}$, which restricts to F and G over 0 and 1 respectively. Since the geometric nerve of $\mathcal{I}$ is an interval, we get our homotopy. □

This lemma immediately implies that the geometric nerve of $2\mathcal{B}^{\mathrm{Ho}}$ is weakly equivalent to the geometric nerve of $GL(b_1) \times GL(b_0)$. We arrive at:

Conclusion 3.12 The geometric nerve of $2\mathcal{B} = 2\mathcal{B}_{\mathrm{weak}}(b_1, b_0)$ is weakly equivalent to the nerve $N_\bullet GL(b_1) \times N_\bullet GL(b_0)$. Hence, the corresponding Baez–Crans 2-vector bundles splits, up to concordance, as two copies of ordinary vector bundles. With little or no modification, similar calculations can be carried out for the other 2-categories of Example 3.8. In all cases the geometric nerve splits, up to weak equivalence, as $N_\bullet GL(b_1) \times N_\bullet GL(b_0)$. We can therefore conclude that $2K$-theories based on Baez–Crans 2-vector bundles are given as two copies of ordinary K-theory.

4. The topological Kan condition

We consider a simplicial space $Z_\bullet$ with a topological Kan property. We apply the following notation: let Δ^n be the topological n-simplex, i.e. the subspace of $\mathbb{R}^{n+1}$

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consisting of tuples $(t_0,t_1,\ldots,t_n)$ such that each $t_i \geq 0$ and $\sum t_i = 1$. Let $\Delta^n_\bullet$ denote the simplicial n-simplex, i.e. the simplicial set with q-simplices all order preserving maps $[q] \to [n]$. We define the *space of k-horns of dimension n* in the simplicial space $Z_\bullet$ to be

$$\Lambda^n_k(Z_\bullet) = \{(z_0,\ldots,z_{k-1},-,z_{k+1},\ldots,z_n) \mid d_i(z_j) = d_{j-1}(z_i), i < j, i \neq k \neq j\} \subset \underbrace{Z_{n-1} \times Z_{n-1} \times \cdots \times Z_{n-1}}_{n \text{ factors}}$$

We give $\Lambda^n_k(Z_\bullet)$ the subspace topology. For our purposes, it seems more natural to associate this horn to the complementary set $I(k) = \{0,1,\ldots,k-1,k+1,\ldots n\}$. So we think of a k-horn as an $I(k)$-cohorn. If $I \subset [n] = \{0,1,\ldots,n\}$, we define the *space of I-cohorns* to be

$$\Lambda^n_I(Z_\bullet) = \{(z_i)_{i\in I} \mid d_i(z_j) = d_{j-1}(z_i), i < j, i \in I \ni j\} \subset \underbrace{Z_{n-1} \times Z_{n-1} \times \cdots \times Z_{n-1}}_{|I| \text{ factors}}$$

Note that $\Lambda^n_k = \Lambda^n_{I(k)}$. The face maps $d_i \colon Z_n \to Z_{n-1}$, $i \neq k$ induce a canonical map $c^n_k \colon Z_n \to \Lambda^n_k(Z_\bullet)$, and more generally, for any $I \subset [n]$ we have a canonical map $c^n_I \colon Z_n \to \Lambda^n_I(Z_\bullet)$.

Definition 4.1 We say that $Z_\bullet$ satisfies the *discrete Kan condition* if the maps c^n_k are surjective for each n with $0 \leq k \leq n$. We say that $Z_\bullet$ satisfies the *topological Kan condition* if the maps c^n_k are surjective Serre fibrations for each n with $0 \leq k \leq n$.

Remark 4.2 E.H. Brown and R.H. Szczarba consider in [8, Definition 2.1] a related notion of fibrancy.

Lemma 4.3 *If the simplicial space $Z_\bullet$ satisfies the topological Kan condition, then for each n and every non-empty proper subset I of $[n]$ the map c^n_I is a surjective Serre fibration.*

Proof: The proof is by double induction. First by induction on n: assume that all $c^{n'}_{I'}$ are surjective Serre fibrations for $n' < n$ and every non-empty proper subset I' of $[n']$. Next by downwards induction on $|I| = m$: assume that all c^n_J are surjective Serre fibrations for all non-empty proper subsets J of $[n]$ with $|J| > m$. This is true for $m = n-2$ by assumption, since the J's in this case have the form $I(k)$ and $c^n_J = c^n_{I(k)} = c^n_k$. For the inductive step assume that $k \notin I$ and $J = I \cup \{k\}$,

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so that by the induction hypothesis c^n_J is a surjective Serre fibration. Consider the canonical map which forgets z_k

$$r^J_I : \Lambda^n_J(Z_\bullet) \to \Lambda^n_I(Z_\bullet).$$

Then $c^n_I = r^J_I \circ c^n_J$. So it is enough to show that the map r^J_I is a surjective Serre fibration.

We denote the injective map which omits k by $\delta_k : [n-1] \to [n]$. Thus the face map $d_k : Z_n \to Z_{n-1}$ is the map induced by δ_k. Let the sequence $(z_i)_{i\in I}$ be a point in $\Lambda^n_I(Z_\bullet)$. Let $I' \subset [n-1]$ be the subset determined by $I = \delta_k(I')$. There is such a subset since $k \notin I$. Moreover, $|I'| = |I|$ and the map $\delta_k : I' \to I$ is a bijection. For $i \in I'$ we define elements

$$y_i = \begin{cases} d_{k-1}(z_{\delta_k(i)}) = d_{k-1}(z_i) & \text{if } i \in I', \text{and } i < k, \\ d_k(z_{\delta_k(i)}) = d_k(z_{i+1}) & \text{if } i \in I', \text{and } k \le i. \end{cases}$$

We claim that the sequence $(y_i)_{i\in I'}$ forms an I'-cohorn of dimension $(n-1)$ in $Z_\bullet$. This can be seen either by meditating on the geometry of the simplex, or more computationally in the following way: for $i, j \in I'$ and $i < j$, we have

$$d_i y_j = \begin{cases} d_i d_{k-1}(z_j) = d_{k-2} d_i(z_j) & i < j < k \\ d_i d_k(z_{j+1}) = d_{k-1} d_i(z_{j+1}) & i < k \le j \\ d_i d_k(z_{j+1}) = d_k d_{i+1}(z_{j+1}) & k \le i < j \end{cases}$$

and also, using that $(z_i)_{i\in I}$ is in the space of I-cohorns,

$$d_{j-1} y_i = \begin{cases} d_{j-1} d_{k-1}(z_i) = d_{k-2} d_{j-1}(z_i) = d_{k-2} d_i(z_j) & i < j \le k \\ d_{j-1} d_{k-1}(z_i) = d_{k-1} d_j(z_i) = d_{k-1} d_i(z_{j+1}) & i < k < j \\ d_{j-1} d_k(z_{i+1}) = d_k d_j(z_{i+1}) = d_{i+1}(z_{j+1}) & k \le i < j. \end{cases}$$

The association $(z_i)_{i\in I} \mapsto (y_i)_{i\in I'}$ defines a continuous map $P^I_k : \Lambda^n_I(Z_\bullet) \to \Lambda^{n-1}_{I'}(Z_\bullet)$. Since $k \in J$ there is also a continuous map $p_k : \Lambda^n_J(Z_\bullet) \to Z_{n-1}$ defined by picking out the appropriate component, i.e. we have $p_k((z_i)_{i\in J}) = z_k$. These maps fit into a diagram of spaces

$$\begin{array}{ccc} \Lambda^n_J(Z_\bullet) & \xrightarrow{p_k} & Z_{n-1} \\ \downarrow{\scriptstyle r^J_I} & & \downarrow{\scriptstyle c^{n-1}_{I'}} \\ \Lambda^n_I(Z_\bullet) & \xrightarrow{P^I_k} & \Lambda^{n-1}_{I'}(Z_\bullet) \end{array}$$

This is actually a pullback diagram. The right map is a surjective Serre fibration by the induction hypothesis of the first induction, so it follows that r^J_I is also a surjective Serre fibration. □

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We can now prove the main theorem of this section:

Theorem 4.4 *Let $Z_\bullet$ be a simplicial space. Assume that $Z_\bullet$ satisfies the topological Kan condition. Then the diagonal simplicial set $D_\bullet = \mathrm{diag}(\mathrm{Sing}_\bullet Z_\bullet)$ is a Kan complex, i.e. it satisfies the discrete Kan condition.*

Proof: We need to show that any horn in $D_\bullet$ can be filled. The Kan condition is always satisfied for $n = 1$, since a horn of dimension 1 is filled by the degeneracy map $s_0 \colon D_0 \to D_1$. So we assume without restriction that $n \geq 2$. A k-horn of dimension n in $D_\bullet$ consists of a sequence of continuous maps

$$z_i : \Delta^{n-1} \to Z_{n-1} \quad \text{defined for } 0 \leq i \leq n-1,\ i \neq k,$$

with the property that if z_i and z_j are defined and $i < j$, then the following two continuous maps agree:

$$\Delta^{n-2} \xrightarrow{\delta_i} \Delta^{n-1} \xrightarrow{z_j} Z_{n-1} \xrightarrow{d_i} Z_{n-2},$$
$$\Delta^{n-2} \xrightarrow{\delta_{j-1}} \Delta^{n-1} \xrightarrow{z_i} Z_{n-1} \xrightarrow{d_{j-1}} Z_{n-2}.$$

Here, and in the rest of the proof, we abuse notation by not distinguishing between a map $[m] \to [n]$ and its affine extension to a continuous map $\Delta^m \to \Delta^n$ between the associated topological standard simplices. The way to visualize these z_i's is to consider the topological horn

$$\Lambda^n_k = \cup_{i \neq k} \delta_i(\Delta^{n-1}) \subset \Delta^n.$$

The domain of z_i can be identified with the subset $\delta_i(\Delta^{n-1})$ of Λ^n_k. The maps corresponding to z_i and z_j do not necessarily agree on the intersection $\delta_i(\Delta^{n-1}) \cap \delta_j(\Delta^{n-1})$ of these subsets, but satisfy a more complicated compatibility condition.

Main step. We define a continuous map

$$y : \Lambda^n_k \to Z_n$$

such that for every $i \neq k$ we have that $z_i = d_i \circ y \circ \delta_i$. We define this map inductively. The induction is done over a sequence of subspaces of the skeleton of Λ^n_k, which we define now. By $v_i \in \Delta^n$ we denote the i'th vertex. Let W_r be the union of all r-dimensional subsimplices of Δ^n which contain v_k. Then

$$\{v_k\} = W_0 \subset W_1 \subset \ldots W_{n-1} = \Lambda^n_k.$$

Start of induction. W_0 is included in every subspace $\delta_i(\Delta^n)$, $i \neq k$, so there exists a sequence of vertices $\phi^i \in \Delta^{n-1}$, $i \neq k$, such that $\delta_i \phi^i = v_k$. Let $(b_i)_{i \neq k}$ be

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the sequence in Z_{n-1} given by $b_i = z_i\phi^i$. Take $i < j$, $i \neq k \neq j$, then there exists a vertex $\psi \in \Delta^{n-2}$ such that $\delta_j\delta_i(\psi) = v_k$, hence $\phi^i = \delta_{j-1}(\psi)$, and $\phi^j = \delta_i(\psi)$. Now we can check that b_i and b_j satisfy

$$d_i b_j = d_i z_j \phi^j = d_i z_j \delta_i(\psi) = d_{j-1} z_i \delta_{j-1}(\psi) = d_{j-1} z_i \phi^i = d_{j-1} b_i.$$

Thus, $(b_i)_{i\neq k}$ lies in $\Lambda^n_k(Z_\bullet)$. By assumption, the map c^n_k is surjective so we can find an element $a \in Z_n$ such that $d_i(a) = b_i$ for all $i \neq k$. We define y on $W_0 = \{v_k\}$ by $y(v_k) = a \in Z_n$.

Induction step. Assume that we have defined $y\colon W_{r-1} \to Z_{n+1}$ such that for each $i \neq k$ we have a commutative diagram:

$$\begin{array}{ccccc} W_{r-1} \cap \delta_i(\Delta^{n-1}) & \hookrightarrow & W_{r-1} & \xrightarrow{\ y\ } & Z_n \\ \downarrow & & & & \downarrow{\scriptstyle d_i} \\ \Delta^{n-1} & & \xrightarrow{\qquad z_i \qquad} & & Z_{n-1}. \end{array}$$

We want to extend y to a map from W_r with the same property. To define this extension, it is sufficient to define it on each r-simplex in W_r. Such a simplex is determined by an injective order preserving map $\tau\colon [r] \hookrightarrow [n]$ with the property that $\tau(k') = k$ for some k' in $[r]$. Let $I \subset [n]$ be the complement of the image of τ. Let us now translate the problem of extending the map y to the simplex $\tau \subset W_r$ into a problem of constructing a map $y^\tau\colon \Delta_r \to Z_n$ with certain properties. Since τ is a general r-simplex of W_r the collection of all these maps y^τ defines the extension of y to W_r. We have to be careful about what the transformed problem is.

First, note that $\tau(\Delta^r) \cap W_{r-1} = \tau(\Lambda^r_{k'})$. Thus y^τ is already defined on $\Lambda^r_{k'}$ by induction.

There are two cases for the maps z_i. If $i \notin I$, then $\tau(\Delta^r) \cap \delta_i(\Delta^{n-1})$ is contained in W_{r-1}. So the z_i's corresponding to these i's do not give extra conditions on y^τ, beyond that it has to be an extension of the already defined map $y^\tau\colon \Lambda^r_{k'} \to Z_n$. On the other hand, if $i \in I$, we do get an extra condition.

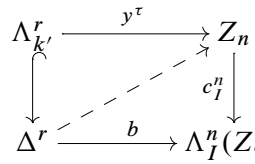
The image $\tau(\Delta^r)$ is included in every subspace $\delta_i(\Delta^n)$ for $i \in I$, so there exists a sequence of injective order preserving maps $\phi^i\colon [r] \hookrightarrow [n-1]$ where $i \in I$, such that $\delta_i\phi^i = \tau$. Let $(b_i)_{i\in I}$ be the sequence of maps $\Delta^r \to Z_{n-1}$ given by $b_i = z_i\phi^i$.

Take $i < j$, $i \notin I \ni j$, then there exists an injective order preserving map $\psi\colon [r] \hookrightarrow [n-2]$ such that $\delta_j\delta_i\psi = \tau$, hence $\phi^i = \delta_{j-1}\psi$, and $\phi^j = \delta_i\psi$. As before b_i and b_j satisfy

$$d_i b_j = d_i z_j \phi^j = d_i z_j \delta_i \psi = d_{j-1} z_i \delta_{j-1} \psi = d_{j-1} z_i \phi^i = d_{j-1} b_i.$$

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Thus, the collection $(b_i)_{i\in I}$ gives a map $b\colon \Delta^r \to \Lambda^n_k(Z_\bullet)$. The extension problem we have to solve is the following:



This is solvable since the map c^n_I is a Serre fibration by Lemma 4.3. This completes the induction step.

Final step. We have constructed a continuous map $y\colon \Lambda^n_k \to Z_n$ such that $z_i = d_i \circ y \circ \delta_i$ for every $i \neq k$. The composition of y with a retraction $\Delta^n \to \Lambda^n_k$ fills the k-horn in $D_\bullet$ that we considered initially. □

Remark 4.5 A k-horn, $0 \le k \le 2$, in dimension 2 in $D_\bullet = \mathrm{diag}(\mathrm{Sing}_\bullet Z_\bullet)$ can be filled if $Z_\bullet$ satisfies the weaker condition that the three boundary maps $d_i : Z_2 \to Z_1$ are Serre fibrations, and that $Z_\bullet$ satisfies the discrete Kan extension condition in dimension 2.

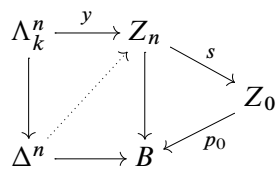
To see this use the Kan extension property at the vertex v_k to obtain a point in Z_2. Then use that $d_i : Z_2 \to Z_1$ is a Serre fibration to obtain a continuous map $y\colon \Lambda^k_2 \to Z_2$. Finally extend this to the topological 2-simplex.

In this case, we do not need Lemma 4.3.

We end this section by relativizing Theorem 4.4 in a special case:

Corollary 4.6 *Let $Z_\bullet$ be a simplicial space, and $p_0\colon Z_0 \to B$ a map of topological spaces. Then the induced map* $\mathrm{diag}(\mathrm{Sing}_\bullet Z_\bullet) \to \mathrm{Sing}_\bullet B$, *extending p_0, is a Kan fibration provided that p_0 is a Serre fibration and $Z_\bullet$ satisfies the topological Kan condition.*

Proof: Consider the problem of filling a simplicial n-dimensional k-horn in $\mathrm{diag}(\mathrm{Sing}_\bullet Z_\bullet)$ over a matching n-simplex in $\mathrm{Sing}_\bullet B$. Forgetting the n-simplex in B, we may use the theorem to fill the horn, thus producing a map $z\colon \Delta^n \to Z_n$. Projecting into B we get a new map $\Delta^n \to B$, which can be compared with the given n-simplex in B. Restricting to the topological k-horn $\Lambda^n_k \subseteq \Delta^n$, these simplices in B coincide. Thus we have reduced the original problem to a lifting problem of spaces, namely



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Here, y is the restriction of z to the topological k-horn Λ^n_k. Observe that the degeneracy map $s\colon Z_n \to Z_0$ is a surjective Serre fibration by Lemma 4.3, and p_0 is a Serre fibration by assumption. Hence, the lifting exists. □

5. Good and sufficiently fibrant 2-groupoids

In this section we will give criteria on a topological 2-category $2\mathcal{C}$ to ensure that its geometric nerve $\Delta 2\mathcal{C}$ is good, and in the case where $2\mathcal{C}$ is a topological 2-groupoid, we will give criteria that ensure that $\Delta 2\mathcal{C}$ satisfies the topological Kan condition.

We will use the notion of a *locally equiconnected* (abbr. LEC) space X, i.e. the diagonal map $X \to X \times X$ is a closed cofibration. All CW-complexes are LEC spaces by E. Dyer and S. Eilenberg's adjunction theorem, see [13]. On the other hand, as a corollary of their Theorem II.7, we get:

Theorem 5.1 (Dyer, Eilenberg) *Assume that X is a LEC space, and $i\colon A \hookrightarrow X$ admits a retraction $r\colon X \to A$. Then i is a closed cofibration, and A is a LEC space.*

Proof: View A as a subset of X. A *haloing function* for A is a function $q\colon X \to I$ such that $q^{-1}(0) = A$. A *halo retract* A of X is a retract which admits a haloing function. By [13, Theorem II.7] the result above follows if we can find a haloing function for our retract A.

By Dyer and Eilenberg's characterization of LEC spaces X, their Theorem II.1, there exists a haloing function $k\colon X \times X \to I$ for the diagonal in $X \times X$. Now define $q\colon X \to I$ by the formula

$$q(x) = k(x, ir(x)).$$

Since $q(x) = 0$ if and only if $x \in A$, we have constructed our haloing function for A. □

Corollary 5.2 *A simplicial space $Z_\bullet$, where Z_q is a LEC space for infinitely many q, is a good simplicial space.*

Proof: By downward induction on q we assume that Z_q is LEC. Any degeneracy $s_i\colon Z_{q-1} \to Z_q$ has a retract, namely d_i. Hence, s_i is a closed cofibration and Z_{q-1} is LEC. □

One basic tool to prove that a space is LEC is:

Theorem 5.3 (Heath, [16]) *Assume that X, B and E are LEC spaces. Then the pullback of $X \xrightarrow{f} B \xleftarrow{p} E$ is a LEC space provided that p is a Hurewicz fibration.*

For nerves of topological categories we now get the following:

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Corollary 5.4 *Let $\mathcal{C}$ be a topological category. If the total space of morphisms $\mathcal{C}_1$ is LEC and the source map $s\colon \mathcal{C}_1 \to \mathcal{C}_0$ is a Hurewicz fibration, then $N_\bullet\mathcal{C}$ is a good simplicial space.*

Proof: The space of objects, $\mathcal{C}_0$, is LEC, since it is a retract of $\mathcal{C}_1$. We will prove by induction that $N_q\mathcal{C}$ is LEC, and the result then follows by Corollary 5.2. Observe that $N_{q+1}\mathcal{C}$ is given as the pullback of $N_q\mathcal{C} \xrightarrow{v_q} \mathcal{C}_0 \xleftarrow{s} \mathcal{C}_1$, where v_q is the last vertex map, $N_q\mathcal{C}$ is LEC by induction, and s is a Hurewicz fibration by assumption. By Heath's theorem $N_{q+1}\mathcal{C}$ is also LEC. □

Definition 5.5 We shall call a topological 2-category $2\mathcal{C}$ *good* if its geometric nerve $\Delta 2\mathcal{C}$ is a good simplicial space.

Let us now list the criteria on a topological 2-category $2\mathcal{C}$ that together imply that $2\mathcal{C}$ is good:

G1 The source map $s_1\colon 2\mathcal{C}_1 \to 2\mathcal{C}_0$ from 1-morphisms to objects is a Hurewicz fibration.

G2 The target map $t_2\colon 2\mathcal{C}_2 \to 2\mathcal{C}_1$ from 2-morphisms to 1-morphisms is a Hurewicz fibration.

G3 The total space of 2-morphisms, $2\mathcal{C}_2$, is LEC.

G4 All 2-morphisms are isomorphisms vertically and the map $(-)^{-1}\colon 2\mathcal{C}_2 \to 2\mathcal{C}_2$ sending a 2-morphism to its vertical inverse is continuous.

The assumptions **G1**–**G3** exclude pathological 2-categories. All examples in Section 3 satisfy assumptions **G1**–**G4**.

Theorem 5.6 *A topological 2-category $2\mathcal{C}$ is good if it satisfies the assumptions **G1**, **G2**, **G3**, and **G4**.*

Before proving the theorem, we consider the following lemma which, under assumption **G4**, characterizes n-simplices of $\Delta 2\mathcal{C}$:

Lemma 5.7 *Let $2\mathcal{C}$ be a topological 2-category which satisfies **G4**. Then the space*

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of n-simplices $(\Delta 2\mathcal{C})_n$ is homeomorphic to the space of flags

$$\begin{pmatrix} x_0 & f_{01} & \phi_{012} & \phi_{013} & \phi_{014} & \phi_{015} & \cdots & \phi_{01n} \\ & x_1 & f_{12} & \phi_{123} & \phi_{124} & \phi_{125} & \cdots & \phi_{12n} \\ & & x_2 & f_{23} & \phi_{234} & \phi_{235} & \cdots & \phi_{23n} \\ & & & x_3 & f_{34} & \phi_{345} & \cdots & \phi_{34n} \\ & & & & x_4 & f_{45} & \cdots & \phi_{45n} \\ & & & & & x_5 & \cdots & \phi_{56n} \\ & & & & & & \ddots & \vdots \\ & & & & & & & x_n \end{pmatrix},$$

where $x_0, x_1, \ldots, x_n$ are objects of $2\mathcal{C}$, the $f_{i\,i+1} \colon x_i \to x_{i+1}$ are 1-morphisms, and $\phi_{i\,i+1\,j} \colon f_{ij} \Rightarrow f_{i+1\,j} * f_{i\,i+1}$, $i+1<j$, are 2-morphisms.

Proof: The 1-morphisms f_{ij} with $i+1<j$ are implicitly defined as the source of the 2-morphism $\phi_{i\,i+1\,j}$.

Given a flag of the form above, the only data of an n-simplex in $\Delta 2\mathcal{C}$ missing are the ϕ_{ijk} with $0 \le i < j < k \le n$ and $j \ne i+1$. We claim that these may be reconstructed continuously from the flag and the coherence condition of the geometric nerve.

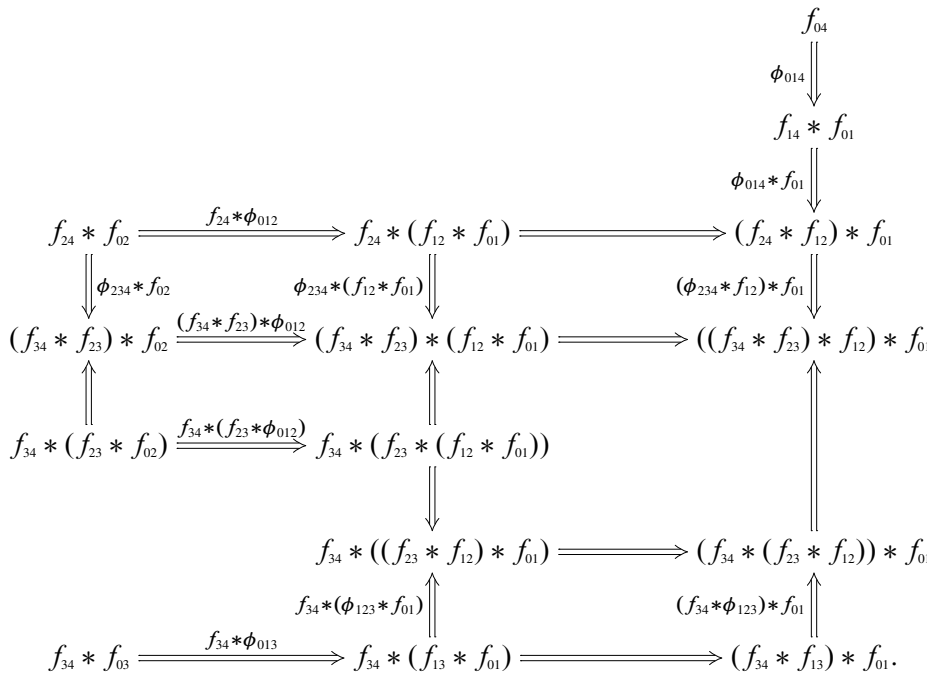
Define the *valence* of the symbol ϕ_{ijk}, $0<i<j<k<n$, to be the difference $j-i$. Observe that the flag contains all ϕ_{ijk} of valence 1. A subtetrahedron, of the n-simplex under consideration, is determined by indices $0 \le i < j < k < l \le n$. Rewrite the coherence condition for such a subtetrahedron as the equation

$$\phi_{ikl} = \left(f_{kl} * \phi_{ijk}\right)^{-1} \underline{\alpha}\left(\phi_{jkl} * f_{ij}\right)\phi_{ijl}.$$

Here $(-)^{-1}$ is the inverse 2-morphism map, and $\underline{\alpha}$ is the natural associativity isomorphism. Observe that the valence of the 2-morphisms on the right hand side is strictly less than the valence of ϕ_{ikl}. Inductively, we use this equation as a definition of ϕ_{ikl}. A calculation is needed to check that ϕ_{ikl} does not depend on a particular

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choice of j. For $i = 0$, $k = 3$, and $l = 4$, inspect the following diagram:



The unmarked arrows are natural associativity isomorphisms. Going from f_{04} to $f_{34} * f_{03}$ on the left side corresponds to defining ϕ_{034} with $j = 2$, while the composition on the right side corresponds to $j = 1$. Since the diagram commutes, the two possible definitions agree. □

Proof of Theorem 5.6: It is enough to show that for all n the space of n-simplices, $(\Delta 2\mathcal{C})_n$ is a LEC space.

By assumption **G3** it follows that $2\mathcal{C}_2$ is LEC. Observe that both $2\mathcal{C}_0$ and $2\mathcal{C}_1$ are retracts of $2\mathcal{C}_2$, where the inclusions are provided by identity maps and the retractions by source maps. Hence, $2\mathcal{C}_0$ and $2\mathcal{C}_1$ are also LEC spaces.

This provides the start of an induction on the simplicial degree n. Assume that $(\Delta 2\mathcal{C})_{n-1}$ is LEC. We can construct $(\Delta 2\mathcal{C})_n$ from the $(n-1)$-simplices by iterated pullback, i.e. we have a tower

$$(\Delta 2\mathcal{C})_n = P_n \twoheadrightarrow \cdots \twoheadrightarrow P_1 \twoheadrightarrow P_0 = (\Delta 2\mathcal{C})_{n-1}.$$

The steps of the tower correspond to the entries in the last column of the flag describing n-simplices. P_1 is the pullback of

$$P_0 \xrightarrow{x_{n-1}} 2\mathcal{C}_0 \xleftarrow{s_1} 2\mathcal{C}_1.$$

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Here we adjoin the 1-morphism f_{n-1n} along x_{n-1}. The map s_1 is a Hurewicz fibration by assumption **G1**. Hence P_1 is LEC by Theorem 5.3. The space P_k, $k > 1$, is the pullback of

$$P_{k-1} \xrightarrow{f_{n-k+1,n} * f_{n-k,n-1+1}} 2\mathcal{C}_1 \xleftarrow{t_2} 2\mathcal{C}_2.$$

Here we adjoin the 2-morphisms $\phi_{n-k,n-k+1,n}$ along $f_{n-k+1,n} * f_{n-k,n-1+1}$. By assumption **G2**, t_2 is a Hurewicz fibration. So P_k is LEC by Heath's theorem. □

Let $2\mathcal{C}$ be a strict discrete 2-category. For each pair of objects x, y in $2\mathcal{C}_0$ we have a morphism category $2\mathcal{C}(x,y)$. Applying the nerve to each of these morphism categories, we get a simplicial category $N_\bullet 2\mathcal{C}$. Apply the nerve construction again to produce a bisimplicial set $N_\bullet N_\bullet 2\mathcal{C}$. Bullejos and Cegarra [9] prove that the geometric nerve $\Delta 2\mathcal{C}$ is naturally weakly equivalent to the double nerve $N_\bullet N_\bullet 2\mathcal{C}$.

Inspired by this, we now want to study the geometric nerve of a strict topological 2-category $2\mathcal{C}$ via the nerves $N_\bullet 2\mathcal{C}(x,y)$ of its morphism categories. To do this we introduce $\mathrm{Ar}^{\|}(2\mathcal{C})$, the subspace of $2\mathcal{C}_1 \times 2\mathcal{C}_1$ consisting of a pair (f,g) of parallel 1-morphisms, i.e. the source of f is also the source of g and the target of f is also the target of g. Observe that we have a pullback diagram

$$\begin{array}{ccc} \mathrm{Ar}^{\|}(2\mathcal{C}) & \xrightarrow{f} & 2\mathcal{C}_1 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle (s_1,t_1)} \\ 2\mathcal{C}_1 & \xrightarrow{(s_1,t_1)} & 2\mathcal{C}_0 \times 2\mathcal{C}_0. \end{array}$$

Next we strengthen the assumptions **G1** and **G2** as follows:

G1' The source-target map of 1-morphisms, $2\mathcal{C}_1 \xrightarrow{(s_1,t_1)} 2\mathcal{C}_0 \times 2\mathcal{C}_0$, is a Hurewicz fibration.

G2' The source-target map of 2-morphisms, $2\mathcal{C}_2 \xrightarrow{(s_2,t_2)} \mathrm{Ar}^{\|}(2\mathcal{C})$, is a Hurewicz fibration.

Together with **G3** and **G4**, these assumptions simplify comparison of two geometric nerves of strict 2-categories:

Lemma 5.8 *Let $2\mathcal{C}$ and $2\mathcal{D}$ be strict topological 2-categories that satisfy the assumptions **G1'**, **G2'**, **G3** and **G4**. Furthermore, let $F\colon 2\mathcal{C} \to 2\mathcal{D}$ be a continuous strict 2-functor. If the map on objects $F_0\colon 2\mathcal{C}_0 \to 2\mathcal{D}_0$ is a weak equivalence, and for each pair of objects $x, y \in 2\mathcal{C}_0$ the induced map between the nerves of morphism categories $N_\bullet 2\mathcal{C}(x,y) \to N_\bullet 2\mathcal{D}(F(x),F(y))$ is a weak equivalence, then F induces a weak equivalence of the geometric nerves, $\Delta 2\mathcal{C} \xrightarrow{\simeq} \Delta 2\mathcal{D}$.*

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Proof: We will prove the lemma via a series of claims:

First claim. There is a natural weak equivalence $|\Delta 2\mathcal{C}| \simeq |\mathrm{Sing}_\bullet N_\bullet N_\bullet 2\mathcal{C}|$.

Since $2\mathcal{C}$ is strict, we may form the double nerve $N_\bullet N_\bullet 2\mathcal{C}$, as well as the geometric nerve $\Delta 2\mathcal{C}$. Because $\Delta 2\mathcal{C}$ is a good simplicial space, by Theorem 5.6, we have a weak equivalence $|\mathrm{Sing}_\bullet \Delta 2\mathcal{C}| \simeq |\Delta 2\mathcal{C}|$. Notice that the functor $\mathrm{Sing}_\bullet$ commutes with nerve constructions. Fix a simplicial degree k for the $\mathrm{Sing}_\bullet$-direction. Observe that $\mathrm{Sing}_k 2\mathcal{C}$ is a strict discrete 2-category. By Bullejos and Cegarra's theorem we get a natural weak equivalence

$$|\Delta \mathrm{Sing}_k 2\mathcal{C}| \simeq |N_\bullet N_\bullet \mathrm{Sing}_k 2\mathcal{C}|.$$

This proves the claim since a map between simplicial spaces, which in each simplicial degree is a weak equivalence, is itself a weak equivalence.

Second claim. The simplicial space $N_\bullet 2\mathcal{C}$ is good, and, moreover, for each pair of objects x, y in $2\mathcal{C}$ the simplicial space $N_\bullet 2\mathcal{C}(x,y)$ is good.

The simplicial space $N_\bullet 2\mathcal{C}$ is the nerve construction applied to the vertical composition of 2-morphisms. Thus $N_k 2\mathcal{C}$ is topologized as a subspace of the k-fold product $2\mathcal{C}_2 \times \cdots \times 2\mathcal{C}_2$. Taking the source and target object, we get a map $N_\bullet 2\mathcal{C} \to 2\mathcal{C}_0 \times 2\mathcal{C}_0$, with the fiber over (x,y) being $N_\bullet 2\mathcal{C}(x,y)$.

By Corollary 5.4 it is sufficient to check that $2\mathcal{C}_2$ is LEC and the source map $s_2 \colon 2\mathcal{C}_2 \to 2\mathcal{C}_1$ is a Hurewicz fibration. The former is assumption **G3**, while the latter holds because of **G1'** and **G2'**.

Since the source-target map $2\mathcal{C}_1 \to 2\mathcal{C}_0 \times 2\mathcal{C}_0$ is a Hurewicz fibration, by **G1'**, it follows by Heath's theorem that the source map $2\mathcal{C}_2(x,y) \to 2\mathcal{C}_1(x,y)$ is a Hurewicz fibration between LEC spaces for every pair of objects x, y in $2\mathcal{C}_0$. Thus $N_\bullet 2\mathcal{C}(x,y)$ also is good.

Third claim. The map $|\mathrm{Sing}_\bullet N_\bullet 2\mathcal{C}| \to |\mathrm{Sing}_\bullet (2\mathcal{C}_0 \times 2\mathcal{C}_0)|$ is a Serre fibration with fiber $|\mathrm{Sing}_\bullet N_\bullet 2\mathcal{C}(x,y)|$ over the path component corresponding to (x,y) in $2\mathcal{C}_0 \times 2\mathcal{C}_0$.

Given that the map is a Serre fibration, the statement about the fiber is obvious. Since the geometric realization of a Kan fibration is a Serre fibration, we can use Corollary 4.6. Hence, we need to show that the map $p_0 \colon N_0 2\mathcal{C} \to 2\mathcal{C}_0 \times 2\mathcal{C}_0$ is a Serre fibration, and that $N_\bullet 2\mathcal{C}$ satisfies the topological Kan condition. Now observe that p_0 is the source-target map $2\mathcal{C}_1 \xrightarrow{(s_1,t_1)} 2\mathcal{C}_0 \times 2\mathcal{C}_0$, whence a Serre fibration by **G1'**.

The maps $c_k^1 \colon N_1 2\mathcal{C} \to \Lambda_k^1(N_\bullet 2\mathcal{C})$ are, for $k = 0$ and $k = 1$, the target and source maps $t_2, s_2 \colon 2\mathcal{C}_2 \to 2\mathcal{C}_1$. These maps are Serre fibrations by **G1'** and **G2'**, and they are surjective by the vertical identity map $2\mathcal{C}_1 \to 2\mathcal{C}_2$.

The maps $c_k^2 \colon N_2 2\mathcal{C} \to \Lambda_k^2(N_\bullet 2\mathcal{C})$ are homeomorphisms by **G4**, while the maps c_k^n, for $n \geq 3$, are homeomorphisms because the nerve is 2-coskeletal.

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Fourth claim. For each k the map $|\mathrm{Sing}_\bullet N_k N_\bullet 2\mathcal{C}| \to |\mathrm{Sing}_\bullet 2\mathcal{C}_0^{\times(k+1)}|$ is a Serre fibration with fiber $|\mathrm{Sing}_\bullet N_\bullet 2\mathcal{C}(x_0,x_1)| \times \cdots \times |\mathrm{Sing}_\bullet N_\bullet 2\mathcal{C}(x_{k-1},x_k)|$ over the path component corresponding to $(x_0,x_1,\ldots,x_k)$ in $2\mathcal{C}_0^{\times(k+1)}$.

Under consideration here is a category which has $2\mathcal{C}_0$ as its space of objects and where the collection of 1-morphisms forms a simplicial space $N_\bullet 2\mathcal{C}$. We denote this category by its morphisms, namely $N_\bullet 2\mathcal{C}$. Observe that the horizontal composition in $2\mathcal{C}$ induces the composition in $N_\bullet 2\mathcal{C}$.

Since both $\mathrm{Sing}_\bullet$ and $|-|$ commute with finite limits, the following diagram is a pullback

$$\begin{array}{ccc} |\mathrm{Sing}_\bullet N_{k+1} N_\bullet 2\mathcal{C}| & \longrightarrow & |\mathrm{Sing}_\bullet N_k N_\bullet 2\mathcal{C}| \times |\mathrm{Sing}_\bullet N_\bullet 2\mathcal{C}| \\ \downarrow & & \downarrow \\ |\mathrm{Sing}_\bullet 2\mathcal{C}_0^{\times(k+2)}| & \longrightarrow & |\mathrm{Sing}_\bullet 2\mathcal{C}_0^{\times(k+1)}| \times |\mathrm{Sing}_\bullet (2\mathcal{C}_0 \times 2\mathcal{C}_0)|. \end{array}$$

The claim now follows by induction on k.

Final claim. The map $|\mathrm{Sing}_\bullet N_\bullet N_\bullet 2\mathcal{C}| \to |\mathrm{Sing}_\bullet N_\bullet N_\bullet 2\mathcal{D}|$ is a weak equivalence.

Observe that the previous claims apply to $2\mathcal{D}$ as well. Fix a simplicial degree k for the nerve corresponding to horizontal composition. Let $x_0,x_1,\ldots,x_k$ be any choice of objects in $2\mathcal{C}$, and set $y_i = F(x_i) \in 2\mathcal{D}_0$. We get a map of fiber sequences

$$\begin{array}{ccc} |\mathrm{Sing}_\bullet N_\bullet (2\mathcal{C}(x_0,x_1) \times \cdots \times 2\mathcal{C}(x_{k-1},x_k))| & \longrightarrow & |\mathrm{Sing}_\bullet N_\bullet (2\mathcal{D}(y_0,y_1) \times \cdots \times 2\mathcal{D}(y_{k-1},y_k))| \\ \downarrow & & \downarrow \\ |\mathrm{Sing}_\bullet N_k N_\bullet 2\mathcal{C}| & \longrightarrow & |\mathrm{Sing}_\bullet N_k N_\bullet 2\mathcal{D}| \\ \downarrow & & \downarrow \\ 2\mathcal{C}_0^{\times(k+1)} & \longrightarrow & 2\mathcal{D}_0^{\times(k+1)}. \end{array}$$

By the assumptions of the lemma and the second claim, the top and bottom maps are weak equivalences, and it follows that the middle map is a weak equivalence. Since this holds for all k, the final claim holds.

Together, the first and the final claims prove the lemma. □

We will now give criteria on $2\mathcal{C}$ to ensure that $\Delta 2\mathcal{C}$ satisfies the topological Kan condition. Recall from [11] that a 2-groupoid is a 2-category whose 2-morphisms are isomorphisms, and whose 1-morphisms $x_0 \xrightarrow{f} x_1$ induce equivalences of categories, internally in $2\mathcal{C}$:

$$\begin{aligned} -*f &: 2\mathcal{C}(x_1,y) \to 2\mathcal{C}(x_0,y), \\ f*- &: 2\mathcal{C}(y,x_0) \to 2\mathcal{C}(y,x_1). \end{aligned}$$

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We need to make the equivalence of categories continuous. We first remark that an equivalence of categories is the same as a pair of adjoint equivalences, see [21, Chapter IV].

Our approach is to define a space AdEq($2\mathcal{C}$) whose points are tuples (f,g,η,ϵ) consisting of 1-morphisms $f\colon x_0 \to x_1$ and $g\colon x_1 \to x_0$ together with 2-morphisms $\eta\colon id_{x_0} \Rightarrow g * f$ and $\epsilon\colon f * g \Rightarrow id_{x_1}$ such that η and ϵ are isomorphisms which satisfy

$$(\epsilon * f)(f * \eta) = id_f \quad \text{and} \quad (g * \epsilon)(\eta * g) = id_g.$$

We give AdEq($2\mathcal{C}$) the subspace topology with respect to the inclusion into $2\mathcal{C}_1 \times 2\mathcal{C}_1 \times 2\mathcal{C}_2 \times 2\mathcal{C}_2$. Thus the map $\pi\colon \mathrm{AdEq}(2\mathcal{C}) \to 2\mathcal{C}_1$ sending (f,g,η,ϵ) to f is continuous.

We arrive at the following reasonable set of fibrancy conditions:

SF1 $2\mathcal{C}$ is a topological 2-groupoid.

SF2 The source and target maps for the 1-morphisms are Serre fibrations.

SF3 The maps from $2\mathcal{C}_2$ to $2\mathcal{C}_1$ given by the source and by the target of 2-morphisms are both Serre fibrations.

SF4 The assignment sending a 2-morphism to its vertical inverse is continuous.

SF5 The map $\pi\colon \mathrm{AdEq}(2\mathcal{C}) \to 2\mathcal{C}_1$ is a Serre fibration.

Lemma 5.9 *Assume that $2\mathcal{C}$ is a topological 2-category satisfying **SF1**, **SF4**, and **SF5**. Let $\psi\colon D^n \to 2\mathcal{C}_2$ and $f,h_s,h_t\colon D^n \to 2\mathcal{C}_1$ be maps such that the target of f is the source of both h_s and h_t, and $\psi\colon h_s * f \Rightarrow h_t * f$. Then there exists a unique map $\phi\colon D^n \to 2\mathcal{C}_2$ such that*

$$\phi * f = \psi$$

for all points in D^n. Similarly, if $\psi\colon D^n \to 2\mathcal{C}_2$ and $f',h'_s,h'_t\colon D^n \to 2\mathcal{C}_1$ are such that the source of f is equal to the target of both h'_s and h'_t, and $\psi\colon f'*h'_s \Rightarrow f'*h'_t$, then there is a unique $\phi'\colon D^n \to 2\mathcal{C}_2$ such that $f' * \phi' = \psi$.

Proof: We prove the first part, the last statement is dual. We view $\phi * f = \psi$ as an equation parameterized over D^n, where f and ψ are known, and ϕ is the unknown to be solved for.

By the assumptions **SF1** and **SF5** there are maps $g\colon D^n \to 2\mathcal{C}_1$ and $\eta,\epsilon\colon D^n \to 2\mathcal{C}_2$ such that (f,g,η,ϵ) is an adjoint equivalence for all points in D^n. The following commutative diagram shows that ϕ can be expressed uniquely in terms of $\psi * g =$

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$(\phi * f) * g$, the adjoint equivalence, and other known quantities:

$$\begin{array}{ccccccc}
(h_s * f) * g & \xleftarrow{\underline{\alpha}} & h_s * (f * g) & \xrightarrow{h_s * \epsilon} & h_s * id & \xrightarrow{\underline{\rho}} & h_s \\
\Big\Downarrow{\scriptstyle \psi * g} & & \Big\Downarrow{\scriptstyle \phi * (f * g)} & & \Big\Downarrow{\scriptstyle \phi * id} & & \Big\Downarrow{\scriptstyle \phi} \\
(h_t * f) * g & \xleftarrow{\underline{\alpha}} & h_t * (f * g) & \xrightarrow{h_t * \epsilon} & h_t * id & \xrightarrow{\underline{\rho}} & h_t .
\end{array}$$

Here $\underline{\alpha}$ and $\underline{\rho}$ denote the natural associativity isomorphism and natural right unit isomorphism of $2\mathcal{C}$ respectively. Also, when writing out the expression for ϕ, we observe that assumption **SF4** is used to invert some 2-morphisms. This completes the proof. □

We introduced our fibrancy conditions to prove the following theorem:

Theorem 5.10 *If **SF1**, **SF2**, **SF3**, **SF4**, and **SF5** holds for a topological* 2*-category* $2\mathcal{C}$*, then the geometric nerve* $\Delta 2\mathcal{C}$ *satisfies the topological Kan condition.*

Proof: We have to show that all fill-horn-maps

$$c_k \colon (\Delta 2\mathcal{C})_n \to \Lambda^k_n(\Delta 2\mathcal{C})$$

are surjective Serre fibrations. By [11, Theorem 8.6] all maps c_k are surjective. It remains to show that they are Serre fibrations. Since $\Delta 2\mathcal{C}$ is 3-coskeletal, the c_k's are homeomorphisms in dimensions $n > 3$. For $n = 1$ the fill-horn-maps c_0 and c_1 are the source and target maps $2\mathcal{C}_1 \to 2\mathcal{C}_0$, which are Serre fibrations by condition **SF2**.

For $n = 3$ there are essentially two different cases to consider. For $k = 1$ and $k = 2$ the fill-horn-map c_k asks us to solve the coherence equation

$$(\phi_{123} * f_{01})\phi_{013} = \underline{\alpha}(f_{23} * \phi_{012})\phi_{023}$$

with respect to ϕ_{023} and ϕ_{013} given all other 1- and 2-morphisms. This can be done continuously using assumption **SF4** alone. Hence c_1 and c_2 are homeomorphisms. Next consider the case $k = 0$. In order to show that c_0 is a Serre fibration we view the coherence condition above as an equation, parameterized over some disk $D^m \times I$, and with ϕ_{123} as the unknown. We want to extend a solution $\phi_{123} \colon D^m \times \{0\} \to 2\mathcal{C}_2$ to all parameters $D^m \times I$. Using the continuous inverse we reduce to a parameterized equation of the form $\phi * f = \psi$. This has a unique solution by Lemma 5.9. The case $k = 3$ is similar.

The rest of the proof concerns the case $n = 2$, i.e. we are trying to find lifts for

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all diagrams of the form

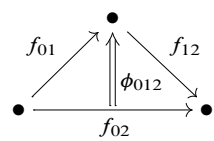
$$\begin{array}{ccc} D^m \times \{0\} & \longrightarrow & \Delta(2\mathcal{C})_2 \\ \downarrow & \nearrow & \downarrow{\scriptstyle c_k} \\ D^m \times I & \longrightarrow & \Lambda^k_2(\Delta 2\mathcal{C}). \end{array}$$

In the case $k = 1$ observe that the following diagram is a pullback:

$$\begin{array}{ccc} \Delta(2\mathcal{C})_2 & \xrightarrow{\phi_{012}} & 2\mathcal{C}_2 \\ \downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle \text{target}} \\ \Lambda^1_2(\Delta 2\mathcal{C}) & \xrightarrow{f_{12} * f_{01}} & 2\mathcal{C}_1. \end{array}$$

Thus c_1 is a Serre fibration by assumption **SF3**.

Next we consider the case $k = 0$. The map $D^m \times I \to \Lambda^0_2(\Delta 2\mathcal{C})$ corresponds to families of 1-morphisms f_{01} and f_{02}, parameterized over $D^m \times I$, such that the source of f_{01} equals the source of f_{02}. The map $D^m \to \Delta 2\mathcal{C}_2$ corresponds to two additional maps $f_{12}\colon D^m \to 2\mathcal{C}_1$ and $\phi_{012}\colon D^m \to 2\mathcal{C}_2$ such that the following diagram commutes for all parameters in D^m:



We can choose $g\colon D^m \times I \to 2\mathcal{C}_1$ and $\eta, \epsilon\colon D^m \times I \to 2\mathcal{C}_2$ such that $(f_{01}, g, \eta, \epsilon)$ is an adjoint equivalence for all parameters. By assumption **SF3** we may solve the lifting problem

$$\begin{array}{ccc} D^m \times \{0\} & \xrightarrow{(f_{12} * \epsilon)(\phi_{012} * g)} & 2\mathcal{C}_2 \\ \downarrow & \overset{\psi}{\nearrow} & \downarrow{\scriptstyle \text{source}} \\ D^m \times I & \xrightarrow{f_{02} * g} & 2\mathcal{C}_1. \end{array}$$

Furthermore, by Lemma 5.9 and assumption **SF4** the equation

$$(f_{12} * \epsilon)(\phi_{012} * g) = \psi$$

has a unique solution for ϕ_{012} for all parameters $D^m \times I$, and by uniqueness this solution extends the given ϕ_{012} defined for parameters $D^m \times \{0\}$.

The case $k = 2$ is dual to the case $k = 0$. This finishes the proof. □

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6. Concordance theory

Let us fix a simplicial space $Z_\bullet$. In this section we will define the notion of a $Z_\bullet$-bundle over a topological space X, and study its homotopy properties up to concordance. In the case $Z_\bullet = \Delta 2\mathcal{C}$ this recovers the notion of principal $2\mathcal{C}$-bundles.

Definition 6.1 A *$Z_\bullet$-bundle* $\mathcal{E}$ over X consists of an ordered open cover $\mathcal{U}$ together with a simplicial map $\phi\colon U_\bullet \to Z_\bullet$, where $U_\bullet$ is the ordered Čech complex of $\mathcal{U}$.

We have to be precise about our conventions and definitions regarding covers. For every cover $\mathcal{U}$ we will assume that each $U_\alpha \in \mathcal{U}$ is non-empty. We say that $\mathcal{U}'$ is a *refinement* of $\mathcal{U}$ if there exists a *carrier function* $c\colon \mathcal{I}' \to \mathcal{I}$ between the respective indexing sets such that for each $\alpha \in \mathcal{I}'$ the set U'_α is contained in $U_{c(\alpha)}$. Whenever the indexing sets $\mathcal{I}$ and $\mathcal{I}'$ are partially ordered, we will demand that any carrier function is order-preserving. Observe that to any refinement together with a choice of carrier function there is a canonical way of associating a simplicial map $U'_\bullet \to U_\bullet$ between the corresponding (ordered) Čech complexes. Specializing the notion of a refinement with carrier function, we say that a cover $\mathcal{U}'$ is a *shrinking* of $\mathcal{U}$, and we write $\mathcal{U}' \subseteq \mathcal{U}$, if the indexing set $\mathcal{I}'$ is a subset of $\mathcal{I}$ and for each $\alpha \in \mathcal{I}'$ we have $U'_\alpha \subseteq U_\alpha$. A cover $\mathcal{U}$ is *good* if each finite intersection $U_{\alpha_0 \cdots \alpha_k} = \bigcap_{i=0}^k U_{\alpha_i}$ is either empty or contractible. Given a map $f\colon X \to Y$, we may pull back a $Z_\bullet$-bundle $\mathcal{E}$ over Y to X as follows: let $\mathcal{U}'$ be the ordered open cover of X consisting of the non-empty open sets having the form $f^{-1}(U_\alpha)$ where $U_\alpha \in \mathcal{U}$. There is a natural map of ordered Čech complexes $U'_\bullet \to U_\bullet$. Define $f^*\mathcal{E}$ to be $\mathcal{U}'$ together with the composition $U'_\bullet \to U_\bullet \xrightarrow{\phi} Z_\bullet$. The pullback to a subspace $A \subseteq X$ is called the *restriction* and will be denoted $\mathcal{E}|_A$.

Definition 6.2 We say that $\mathcal{E}_0$ and $\mathcal{E}_1$ over X are *concordant* if there exists a $Z_\bullet$-bundle $\mathcal{E}$ over $X \times I$ such that the restrictions to $X \times \{0\}$ and $X \times \{1\}$ give $\mathcal{E}_0$ and $\mathcal{E}_1$ respectively. Let $\mathrm{Con}(X, Z_\bullet)$ be the set of all concordance classes of $Z_\bullet$-bundles over X. The class in $\mathrm{Con}(X, Z_\bullet)$ corresponding to $\mathcal{E}$ will be written as $[\mathcal{E}]$.

Observe that concordance is an equivalence relation.

Before proceeding with building the concordance theory of $Z_\bullet$-bundles, we shall present an interesting example of $Z_\bullet$-bundles where $Z_\bullet$ is not the geometric nerve of a 2-category:

Example 6.3 Wirth and Stasheff, [37], describe locally homotopy trivial fibrations with fiber F. It seems plausible that we can define a simplicial space with k-simplices, Z_k, the space of all homotopy coherent functors from $[k]$ to the topological monoid of homotopy equivalences of F, $H(F)$. In that case a $Z_\bullet$-

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bundle should be the same as a homotopy transition cocycle, see [37, Definition 2.5]. Moreover, if $Z_\bullet$ is good, then our Theorem 7.12 shows that these fibrations are classified by $|Z_\bullet|$. Furthermore, $|Z_\bullet|$ should be weakly equivalent to $BH(F)$ for reasonable F.

In the theory that follows we will also consider classes of $Z_\bullet$-bundles over X fixed on some subspace A. We say that two $Z_\bullet$-bundles $\mathcal{E}_0$ and $\mathcal{E}_1$ over X are *equal on* A if the restriction $\mathcal{E}_0|_A$ is equal to the restriction $\mathcal{E}_1|_A$. We will use the notation $X \times I/A$ for the quotient space of $X \times I$ where we identify (x,t) and (x,t') for all $x \in A$ and any t, t'. Let i_0 and i_1 denote the inclusions of X into $X \times I/A$ given by sending $x \in X$ to $(x,0)$ and $(x,1)$ respectively. If $\mathcal{E}_0$ and $\mathcal{E}_1$ over X are equal on A, then they are *concordant relative to* A if there exists a $Z_\bullet$-bundle $\mathcal{E}'$ over $X \times I/A$ such that $i_0^*\mathcal{E}' = \mathcal{E}_0$ and $i_1^*\mathcal{E}' = \mathcal{E}_1$. Fixing a $Z_\bullet$-bundle $\mathcal{E}_A$ over A, we denote by

$$\mathrm{Con}_{Z_\bullet}(X, A; \mathcal{E}_A)$$

the set of all $\mathcal{E}$ over X whose restriction to A equals $\mathcal{E}_A$ modulo the equivalence relation of concordance relative to A.

Taking a more categorical approach to concordance, we now consider *presheaves on spaces*, i.e. functors $\mathcal{F}\colon \mathcal{T}op^{\mathrm{op}} \to \mathcal{S}et$. Sending a space X to the set of $Z_\bullet$-bundles over X defines our first example of a presheaf on spaces, and we let $\mathcal{F}_{Z_\bullet}$ denote this presheaf. More generally, we will consider presheaves, $\mathcal{F}\colon \mathcal{T}^{\mathrm{op}} \to \mathcal{S}et$, defined on categories $\mathcal{T}$ satisfying the following conditions:

i) There is a faithful functor $V\colon \mathcal{T} \to \mathcal{T}op$ such that

ii) The functor V creates pushouts in $\mathcal{T}$, i.e. whenever given a span $B \leftarrow A \to X$ in $\mathcal{T}$, the pushout Y' of

$$V(B) \leftarrow V(A) \to V(X)$$

in $\mathcal{T}op$ has a unique lifting to a pushout Y of the original span in $\mathcal{T}$.

iii) Moreover, if $A \hookrightarrow X$ in $\mathcal{T}$ maps to an inclusion $V(A) \hookrightarrow V(X)$ in $\mathcal{T}op$, then the diagram

$$\begin{array}{ccc} V(A) & \hookrightarrow & V(X) \\ \downarrow & & \downarrow{\scriptstyle i_0} \\ V(X) & \xrightarrow{i_1} & V(X) \times I/V(A) \end{array}$$

lifts to $\mathcal{T}$, i.e. the space $X \times I/A$ represents a well-defined object in $\mathcal{T}$ and the inclusions $i_0, i_1\colon X \to X \times I/A$ are also well-defined in $\mathcal{T}$.

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Since V is a faithful functor, we will usually omit it from our notation. Thus we think of $\mathcal{T}$ as a subcategory of $\mathcal{T}op$. Moreover, if $i\colon A \to X$ in $\mathcal{T}$ is mapped by V to an inclusion in $\mathcal{T}op$ we will abuse notation and identify A with the corresponding subspace $i(A) \subseteq X$. Consequently, the map i itself is often omitted from the notation.

For later use, here are three examples of categories $\mathcal{T}$ satisfying conditions i)–iii):

Example 6.4

i) Fix a space A. Let $\mathcal{T}$ be the subcategory of $\mathcal{T}op$ consisting of spaces X containing A and maps $X \to X'$ restricting to the identity on A. We call this $\mathcal{T}$ the category of *spaces containing* A.

ii) Fix a space A. Let $\mathcal{T}$ be the category having as objects spaces X containing A together with the structure of a relative CW-complex on (X, A). The morphisms of $\mathcal{T}$ are defined to be the inclusion of subcomplexes, i.e. maps $X \hookrightarrow X'$ where the image of X is a union of A and some of the cells of (X', A). We call this $\mathcal{T}$ the category of *CW-complexes relative to* A.

iii) Let $\mathcal{T}$ be the full subcategory of *compact spaces*. Condition ii) holds since any quotient of a compact space is compact.

If $\mathcal{F}$ is a presheaf on $\mathcal{T}$, we define relative concordance as follows:

Definition 6.5 Let $i\colon A \to X$ in $\mathcal{T}$ be the lift of an inclusion $A \hookrightarrow X$ in $\mathcal{T}op$. Fix some $s_A \in \mathcal{F}(A)$. On the set of all s in $\mathcal{F}(X)$ that restrict to s_A on A we introduce the equivalence relation that $s_0 \sim s_1$ if there exists an element $s' \in \mathcal{F}(X \times I/A)$ with $i_0^* s' = s_0$ and $i_1^* s' = s_1$. Define $\mathcal{F}[X, A; s_A]$ to be the quotient set of this equivalence relation. We call $[s] \in \mathcal{F}[X, A; s_A]$ the *concordance class* of s.

Actually, the notation $\mathcal{F}[X, A; s_A]$ is not entirely precise since the choice of lift $i\colon A \to X$ in $\mathcal{T}$ has been left out.

Suppose that $\mathcal{F}$ is a presheaf on spaces and $\mathcal{T}$ is a category satisfying the conditions above. With the same data as in the definition above, we observe that condition iii) ensures that concordance of $\mathcal{F}|_{\mathcal{T}}$ equals concordance on $\mathcal{F}$, and we may write

$$\mathcal{F}[X, A; s_A] = \mathcal{F}|_{\mathcal{T}}[X, A; s_A].$$

Definition 6.6 We say that a presheaf $\mathcal{F}$ on $\mathcal{T}$ is *excisive* if it satisfies the following

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condition. Consider a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{j} & Y \end{array}$$

in $\mathcal{T}$ where i is a closed cofibration on $\mathcal{T}op$, and elements $s_B \in \mathcal{F}(B)$ and $s_X \in \mathcal{F}(X)$ that restrict to the same element $s_A = f^*s_B = i^*s_X$ in $\mathcal{F}(A)$, then we require that there exists a unique element s_Y of $\mathcal{F}(Y)$ such that $g^*s_Y = s_X$ and $j^*s_Y = s_B$. We say that $\mathcal{F}$ is *strongly excisive* if such an element s_Y exists uniquely whenever i is an inclusion. We call $\mathcal{F}$ *weakly excisive* if s_Y exists uniquely whenever (X, A) is a relative CW-complex.

Observe that condition ii) on $\mathcal{T}$ ensures that the restriction of an excisive presheaf on spaces to a presheaf on $\mathcal{T}$ also is excisive.

A map between presheaves on $\mathcal{T}$ is a natural transformation $\nu\colon \mathcal{F} \to \mathcal{F}'$. Clearly ν induces maps of relative concordance classes $\nu_{[X,A;s_A]}\colon \mathcal{F}[X,A;s_A] \to \mathcal{F}'[X,A;\nu(s_A)]$. We will also use ν_* as notation for the induced map instead of the cumbersome $\nu_{[X,A;s_A]}$. A useful insight in the theory of concordance is that surjectivity often implies injectivity, compare [23, Proposition 2.18]. We prove a similar result:

Proposition 6.7 *Let $\mathcal{F}$ and $\mathcal{F}'$ be excisive presheaves on a category $\mathcal{T}$ satisfying conditions i)–iii). Consider a map of presheaves $\nu\colon \mathcal{F} \to \mathcal{F}'$. Assume that for all closed cofibrations $C \hookrightarrow X$ with choice of lift to $\mathcal{T}$ and s_C in $\mathcal{F}(C)$ the induced maps*

$$\nu_{[X,C;s_C]}\colon \mathcal{F}[X,C;s_C] \to \mathcal{F}'[X,C;\nu(s_C)].$$

are surjective. Then all these $\nu_{[X,C;s_C]}$ are bijections.

If $\mathcal{F}$ and $\mathcal{F}'$ are strongly excisive and $\nu_{[X,C;s_C]}$ are surjective for all inclusions $C \hookrightarrow X$, then all $\nu_{[X,C;s_C]}$ are bijections.

Proof: Let s_0 and s_1 be elements of $\mathcal{F}(X)$ whose restrictions to C both equal s_C. Since $\mathcal{F}$ is excisive there exists an element $s_{0,1}$ in $\mathcal{F}(X \amalg_C X)$ that restricts to s_0 and s_1 on the two copies of X in $X \amalg_C X$. If $\nu(s_0)$ and $\nu(s_1)$ are concordant relative to C there exists an element $s' \in \mathcal{F}'(X \times I/C)$ with $i_0^*s' = \nu(s_0)$ and $i_1^*s' = \nu(s_1)$. Observe that the uniqueness part of excision for $\mathcal{F}'$ implies that $s'|_{X \amalg_C X} = \nu(s_{0,1})$. Since the map

$$\mathcal{F}[X \times I/C, X \amalg_C X; s_{0,1}] \xrightarrow{\nu_*} \mathcal{F}'[X \times I/C, X \amalg_C X; \nu(s_{0,1})]$$

is surjective, we may lift s' to an element $\bar{s}$ in $\mathcal{F}(X \times I/C)$. This element $\bar{s}$ is a concordance between s_0 and s_1 relative to C.

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If $C \hookrightarrow X$ is a closed cofibration, then $X \amalg_C X \hookrightarrow X \times I/C$ is also a closed cofibration. □

Let us now verify that the set of $Z_\bullet$-bundles satisfies strong excision:

Proposition 6.8 *Let Y be the pushout of $f\colon A \to B$ along an inclusion $A \hookrightarrow X$. Given $Z_\bullet$-bundles $\mathcal{E}$ over X and $\mathcal{E}_B$ over B with $f^*\mathcal{E}_B = \mathcal{E}|_A$, there exists a unique $Z_\bullet$-bundle $\mathcal{E}'$ over Y such that $\mathcal{E}'|_B = \mathcal{E}_B$ and $\mathcal{E}'$ pulls back to $\mathcal{E}$ over X.*

Proof: Without loss of generality we may assume that A is a subspace of X. We explicitly construct $\mathcal{E}'$. Let $\mathcal{I}$ and $\mathcal{I}_B$ denote the indexing sets of the ordered open covers $\mathcal{U}$ and $\mathcal{U}_B$ associated to $\mathcal{E}$ and $\mathcal{E}_B$. Define $\mathcal{I}'$ as the union $\mathcal{I} \cup \mathcal{I}_B$, and extend the notation of $U_\alpha \in \mathcal{U}$ and $U_\alpha^B \in \mathcal{U}_B$ so that $U_\alpha = \emptyset$ for $\alpha \notin \mathcal{I}$ and $U_\alpha^B = \emptyset$ for $\alpha \notin \mathcal{I}_B$. The equation $f^{-1}(U_\alpha^B) = U_\alpha \cap A$ is satisfied for all $\alpha \in \mathcal{I}'$. Hence, we may define an ordered open cover $\mathcal{U}' = \{U'_\alpha\}_{\alpha \in \mathcal{I}'}$ of Y by declaring U'_α to be the image of $U_\alpha \amalg U_\alpha^B$ under the quotient map $X \amalg B \to Y$. We see that the corresponding ordered Čech complex $U'_\bullet$ is the pushout of $U_\bullet^B \leftarrow U_\bullet \cap A \hookrightarrow U_\bullet$, and we define the simplicial map $\phi^Y\colon U_\bullet^Y \to Z_\bullet$ as the uniquely defined extension of the simplicial maps corresponding to $\mathcal{E}_B$ and $\mathcal{E}$. □

Corollary 6.9 *Let Y be the pushout of $f\colon A \to B$ along an inclusion $A \hookrightarrow X$. Fix a $Z_\bullet$-bundle $\mathcal{E}_B$ over B, and let $\mathcal{E}_A$ be $f^*\mathcal{E}_B$. Pullback of $Z_\bullet$-bundles induces a bijection*

$$\mathrm{Con}_{Z_\bullet}(Y, B; \mathcal{E}_B) \xrightarrow{\cong} \mathrm{Con}_{Z_\bullet}(X, A; \mathcal{E}_A).$$

Proof: Keep A fixed and let $\mathcal{T}$ be the category of spaces containing A. Let $\mathcal{F}(X)$ and $\mathcal{F}'(X)$ be the sets of $Z_\bullet$-bundles over $Y = X \amalg_f B$ and X respectively. Observe that $\mathcal{F}$ and $\mathcal{F}'$ are strongly excisive presheaves on $\mathcal{T}$. Pullback of $Z_\bullet$-bundles induces a map $\nu\colon \mathcal{F} \to \mathcal{F}'$ of presheaves. For any inclusion $C \hookrightarrow X$ in $\mathcal{T}$ and any $Z_\bullet$-bundle $\mathcal{E}_{C \amalg_f B}$ over $C \amalg_f B$ extending $\mathcal{E}_B$ it follows from Proposition 6.8 that the induced map $\nu_*\colon \mathcal{F}[X, C; \mathcal{E}_{C \amalg_f B}] \to \mathcal{F}'[X, C; \nu(\mathcal{E}_{C \amalg_f B})]$ is surjective. By Proposition 6.7 such ν_* are bijections. Taking $C = A$ we get the bijection $\mathrm{Con}_{Z_\bullet}(Y, B; \mathcal{E}_B) \xrightarrow{\cong} \mathrm{Con}_{Z_\bullet}(X, A; \mathcal{E}_A)$. □

Lemma 6.10 *Let $A \hookrightarrow X$ be an inclusion, and fix a $Z_\bullet$-bundle $\mathcal{E}_A$ over A. Then the inclusion $i_1\colon X \hookrightarrow X \times I/A$ induces a bijection*

$$i_1^*\colon \mathrm{Con}_{Z_\bullet}(X \times I/A, A; \mathcal{E}_A) \xrightarrow{\cong} \mathrm{Con}_{Z_\bullet}(X, A; \mathcal{E}_A).$$

Proof: Keep A fixed and let $\mathcal{T}$ be the category of spaces containing A. Two strongly excisive presheaves on $\mathcal{T}$ are defined by letting $\mathcal{F}(X)$ and $\mathcal{F}'(X)$ be the sets of $Z_\bullet$-bundles over $X \times I/A$ and X respectively. Pullback by the inclusion $i_1\colon X \hookrightarrow X \times I/A$ induces a map of presheaves $i_1^*\colon \mathcal{F} \to \mathcal{F}'$ inducing

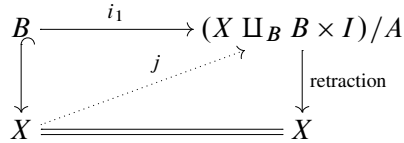
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$\mathrm{Con}_{Z_\bullet}(X\times I/A, A;\mathcal{E}_A)\to \mathrm{Con}_{Z_\bullet}(X,A;\mathcal{E}_A)$. By Proposition 6.7 all i_1^* are bijections if i_1^* induces a surjection $\mathcal{F}[X,C;\mathcal{E}_{C\amalg_f B}]\to \mathcal{F}'[X,C;\nu(\mathcal{E}_{C\amalg_f B})]$ for all inclusions $C\hookrightarrow X$ in $\mathcal{T}$ and $Z_\bullet$-bundles $\mathcal{E}_{C\amalg_f B}$ over $C\amalg_f B$. The projection map $\pi\colon X\times I/A\to X$ induces on concordance classes a map π^* such that $i_1^*\pi^*$ is the identity. Consequently i_1^* is surjective. Taking $C=A$ completes the proof. □

Corollary 6.11 *$\mathrm{Con}_{Z_\bullet}(X,A;\mathcal{E}_A)$ is a homotopy functor of X relative to A.*

Lemma 6.12 *Let $A\hookrightarrow B\hookrightarrow X$ be closed cofibrations. Given $Z_\bullet$-bundles $\mathcal{E}$ over X and $\mathcal{E}_B$ over B that are equal over A, and such that the restriction $\mathcal{E}|_B$ is concordant to $\mathcal{E}_B$ relative to A, then there exists a $Z_\bullet$-bundle $\mathcal{E}'$ over X concordant to $\mathcal{E}$ relative to A such that the restriction $\mathcal{E}'|_B$ equals $\mathcal{E}_B$.*

Proof: Represent the concordance between $\mathcal{E}|_B$ and $\mathcal{E}_B$ by a $Z_\bullet$-bundle $\mathcal{E}''$ over $B\times I/A$. Gluing $\mathcal{E}$ and $\mathcal{E}''$ we get a $Z_\bullet$-bundle $\bar{\mathcal{E}}$ over $(X\amalg_B B\times I)/A$. By the homotopy extension lifting property, there exists a map $j\colon X\to (X\amalg_B B\times I)/A$ such that in the following diagram



the upper triangle commutes and the lower diagram commutes up to homotopy relative to B. Let $\mathcal{E}'=j^*\bar{\mathcal{E}}$. Clearly $\mathcal{E}'_B=\mathcal{E}_B$. Moreover, j is homotopic to the inclusion $i_X\colon X\hookrightarrow (X\amalg_B B\times I)/A$ relative to A. Hence,

$$\left[\mathcal{E}'\right]=\left[j^*\bar{\mathcal{E}}\right]=\left[i_X^*\bar{\mathcal{E}}\right]=[\mathcal{E}]$$

in $\mathrm{Con}_{Z_\bullet}(X,A;\mathcal{E}|_A)$. □

Corollary 6.13 *The functor $\mathrm{Con}_{Z_\bullet}$ is half exact in the following sense: let $A\hookrightarrow B\hookrightarrow X$ be closed cofibrations, and fix some $Z_\bullet$-bundle $\mathcal{E}_B$ over B. Let $\mathcal{E}_A=\mathcal{E}_B|_A$. In the sequence*

$$\mathrm{Con}_{Z_\bullet}(X,B;\mathcal{E}_B)\to \mathrm{Con}_{Z_\bullet}(X,A;\mathcal{E}_A)\xrightarrow{\text{restriction}} \mathrm{Con}_{Z_\bullet}(B,A;\mathcal{E}_A)$$

the image of the first map is precisely the classes in $\mathrm{Con}_{Z_\bullet}(X,A;\mathcal{E}_A)$ that restrict to $[\mathcal{E}_B]$ in $\mathrm{Con}_{Z_\bullet}(B,A;\mathcal{E}_A)$.

Proposition 6.14 *For any relative CW-pair (X,A) and $Z_\bullet$-bundle $\mathcal{E}_A$ over A, the map*

$$\mathrm{Con}_{Z_\bullet}(X,A;\mathcal{E}_A)\xrightarrow{\cong}\lim_k \mathrm{Con}_{Z_\bullet}(X^k,A;\mathcal{E}_A)$$

is a bijection. Here X^k is the k-skeleton of X.

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Proof: Keep A fixed and let $\mathcal{T}$ be the category of CW-complexes relative to A. There is a map $\nu\colon \mathcal{F} \to \mathcal{F}'$ of excisive presheaves on $\mathcal{T}$ inducing $\mathrm{Con}_{Z_\bullet}(X,A;\mathcal{E}_A) \to \lim_k \mathrm{Con}_{Z_\bullet}(X^k,A;\mathcal{E}_A)$. Here $\mathcal{F}$ is the presheaf of $Z_\bullet$-bundles restricted to $\mathcal{T}$, while $\mathcal{F}'(X)$ is the set of sequences $(\mathcal{E}_0,\mathcal{E}_1,\ldots)$ where $\mathcal{E}_k$ is a $Z_\bullet$-bundle over the k-skeleton X^k such that the restriction of $\mathcal{E}_k$ to X^{k-1} is concordant to $\mathcal{E}_{k-1}$ relative to A. The map ν is given by sending $\mathcal{E} \in \mathcal{F}(X)$ to $(\mathcal{E}|_{X^0},\mathcal{E}|_{X^1},\ldots)$ in $\mathcal{F}'(X)$. By Proposition 6.7, showing that ν induces surjections $\nu_*\colon \mathcal{F}[X,C;\mathcal{E}_C] \to \mathcal{F}'[X,C;\nu(\mathcal{E}_C)]$ for all inclusions $C \hookrightarrow X$ in $\mathcal{T}$ and all $Z_\bullet$-bundles $\mathcal{E}_C$ over C, will imply that all ν_* are bijections.

Fix $C \hookrightarrow X$ and $\mathcal{E}_C$, and take a sequence $(\mathcal{E}_0,\mathcal{E}_1,\ldots)$ in $\mathcal{F}'(X)$ that restricts to $\nu(\mathcal{E}_C)$ in $\mathcal{F}'(C)$. By definition of $\mathcal{T}$, see Example 6.4, $C \hookrightarrow X$ is the inclusion of a subcomplex. Hence $C^k \hookrightarrow X^{k-1} \cup C^k \hookrightarrow X^k$ are closed cofibrations. By induction we construct a sequence $\mathcal{E}'_k$ over X^k such that the restriction of $\mathcal{E}'_k$ to X^{k-1} is equal to $\mathcal{E}'_{k-1}$, and $\mathcal{E}'_k$ is concordant to $\mathcal{E}_k$ relative to C^k. We start by taking $\mathcal{E}'_0 = \mathcal{E}_0$. The inductive step uses Lemma 6.12 with $\mathcal{E}_C|_{C^k}$ over C^k, the union of $\mathcal{E}'_{k-1}$ and $\mathcal{E}_C|_{C^k}$ over $X^{k-1} \cup C^k$ and $\mathcal{E}_k$ over X^k. This constructs an honest $Z_\bullet$-bundle $\mathrm{colim}_k \mathcal{E}'_k$ over X mapping to the concordance class represented by the original sequence $(\mathcal{E}_0,\mathcal{E}_1,\ldots)$. Thus all ν_* are surjective. □

So far we have studied how $\mathrm{Con}_{Z_\bullet}(X,A;\mathcal{E}_A)$ depends on the space X. The next step is to understand what happens when $\mathcal{E}_A$ varies. To see this we now define a category as follows:

Definition 6.15 Define $\mathscr{C}_{Z_\bullet}(A)$ to be the category with objects all $Z_\bullet$-bundles over A, and with morphisms from $\mathcal{E}_A$ to $\mathcal{E}'_A$ being the set $\mathrm{Con}_{Z_\bullet}(A \times I, A \amalg A; \mathcal{E}_A \amalg \mathcal{E}'_A)$. The source and target of a morphism represented by $\mathcal{E}$ over $A \times I$ are given as the restrictions $i_0^*\mathcal{E}$ and $i_1^*\mathcal{E}$ respectively. Composition is defined by gluing.

Associativity of gluing $Z_\bullet$-bundles over $A \times I$ holds up to concordance, so $\mathscr{C}_{Z_\bullet}(A)$ is a well defined category.

Lemma 6.16 *The category $\mathscr{C}_{Z_\bullet}(A)$ is a groupoid.*

Proof: We will show that a morphism represented by an arbitrary concordance $\mathcal{E}$ has an inverse. Define $\mathcal{E}^{-1}$ by pulling back $\mathcal{E}$ over the flip $A \times I \to A \times I$ sending (a,t) to $(a,1-t)$. The composition of $\mathcal{E}$ with $\mathcal{E}^{-1}$ equals the pullback of $\mathcal{E}$ over the fold $A \times I \to A \times I$ sending (a,t) to $(a,|2t-1|)$. A contraction θ from the fold to the map $(a,t) \mapsto (a,1)$ is given by $\theta(a,s,t) = (a,(1-s)+s|2t-1|)$. The pullback $\theta^*\mathcal{E}$ is a concordance relative to $A \amalg A$ from the identity on $i_1^*\mathcal{E}$ to the composition of $\mathcal{E}$ and $\mathcal{E}^{-1}$. □

Proposition 6.17 *For closed cofibrations $A \hookrightarrow X$ the expression* $\mathrm{Con}_{Z_\bullet}(X,A;-)$ *defines a functor from $\mathscr{C}_{Z_\bullet}(A)$ to the category of sets.*

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Proof: By the homotopy extension lifting property there exists a map $j\colon X \to X \amalg_{A\times\{0\}} A\times I$ such that $j(a) = (a,1)$ for every $a \in A \subseteq X$ and the composition

$$X \xrightarrow{j} X \amalg_{A\times\{0\}} A\times I \to X$$

is homotopic to the identity on X relative to A.

Given a $Z_\bullet$-bundle $\mathcal{E}$ representing a morphism from $\mathcal{E}_A$ to $\mathcal{E}'_A$ in $\mathscr{C}_{Z_\bullet}(A)$ and a $Z_\bullet$-bundle $\mathcal{E}'$ representing a class of $\mathrm{Con}_{Z_\bullet}(X,A;\mathcal{E}_A)$ we can glue them together and produce a $Z_\bullet$-bundle $\bar{\mathcal{E}}$ over $X \amalg_{A\times\{0\}} A\times I$. The functor is defined by declaring $[j^*\bar{\mathcal{E}}]$ in $\mathrm{Con}_{Z_\bullet}(X,A;\mathcal{E}'_A)$ to represent the image of $[\mathcal{E}']$ under the map induced by the morphism $[\mathcal{E}]$. □

Let $[\mathcal{E}]$ be a morphism from $\mathcal{E}_A$ to $\mathcal{E}'_A$ in $\mathscr{C}_{Z_\bullet}(A)$. It follows from Lemma 6.16 that $\mathcal{E}$ induces a bijection

$$[\mathcal{E}]_*\colon \mathrm{Con}_{Z_\bullet}(X,A;\mathcal{E}_A) \to \mathrm{Con}_{Z_\bullet}(X,A;\mathcal{E}'_A).$$

An open cover $\mathcal{U}$ is *finite and totally ordered* if its indexing set $\mathcal{I}$ is finite and totally ordered. We call a $Z_\bullet$-bundle *finite* if its associated open cover is finite and totally ordered. The finite $Z_\bullet$-bundles form a strongly excisive presheaf $\mathcal{F}^f_{Z_\bullet}$ on spaces. Consequently there is a notion of *finite concordance*, and we denote by

$$\mathrm{Con}^f_{Z_\bullet}(X) \quad \text{and} \quad \mathrm{Con}^f_{Z_\bullet}(X,A;\mathcal{E}_A)$$

the classes of finite $Z_\bullet$-bundles over X modulo finite concordance and finite $Z_\bullet$-bundles over X equal to $\mathcal{E}_A$ over a subset A modulo finite concordance relative to A respectively. Over compact spaces there is no difference between Con and Con^f:

Lemma 6.18 *Let A be a compact subset of a compact space X. Then the natural map $\mathrm{Con}^f_{Z_\bullet}(X,A;\mathcal{E}_A) \to \mathrm{Con}_{Z_\bullet}(X,A;\mathcal{E}_A)$ is a bijection for all finite $Z_\bullet$-bundles $\mathcal{E}_A$ over A.*

Proof: Let $\mathcal{T}$ be the full subcategory of compact space. Inclusion of finite $Z_\bullet$-bundles into all $Z_\bullet$-bundles gives a map of strongly excisive presheaves $\nu\colon \mathcal{F}^f_{Z_\bullet} \to \mathcal{F}_{Z_\bullet}$ on $\mathcal{T}$. By Proposition 6.7 it is sufficient to show that the map $\mathrm{Con}^f_{Z_\bullet}(X,A;\mathcal{E}_A) \to \mathrm{Con}_{Z_\bullet}(X,A;\mathcal{E}_A)$ is surjective for all inclusions $A \hookrightarrow X$ in $\mathcal{T}$ and all finite $\mathcal{E}_A$ over A.

Let $\mathcal{E}$ be a $Z_\bullet$-bundle over X equal to $\mathcal{E}_A$ on A. By compactness of X, the ordered open cover $\mathcal{U}$ associated to $\mathcal{E}$ has a finite subcover $\mathcal{U}^f$. Without loss of generality we may assume that the restriction of $\mathcal{U}^f$ to A equals the cover associated to $\mathcal{E}_A$. Choose a total ordering on $\mathcal{U}^f$ that extends the partial ordering. Let $\mathcal{E}^f$ be the restriction of $\mathcal{E}$ to the smaller covering $\mathcal{U}^f$. Observe that the finite $Z_\bullet$-bundle $\mathcal{E}^f$ is concordant to $\mathcal{E}$ relative to A. □

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Proposition 6.19 *Let $q\colon Z_\bullet \to W_\bullet$ be a map of simplicial spaces. If*

$$\mathrm{Con}^f_{Z_\bullet}(D^n, S^{n-1}; \mathcal{E}_{S^{n-1}}) \to \mathrm{Con}^f_{W_\bullet}(D^n, S^{n-1}; q_*\mathcal{E}_{S^{n-1}})$$

is surjective for all $n \geq 0$ and finite $Z_\bullet$-bundles $\mathcal{E}_{S^{n-1}}$ over S^{n-1}, then

$$\mathrm{Con}_{Z_\bullet}(X, A; \mathcal{E}_A) \to \mathrm{Con}_{W_\bullet}(X, A; q_*\mathcal{E}_A)$$

is a bijection for all pairs (X, A) of the homotopy type of a CW-pair and all $Z_\bullet$-bundles $\mathcal{E}_A$ over A.

Proof: By homotopy invariance, Corollary 6.11, it is sufficient to consider relative CW-complexes (X, A). Fix A and let $\mathcal{T}$ be the category of CW-complexes relative to A. The simplicial map q gives a map $q_*\colon \mathcal{F}_{Z_\bullet} \to \mathcal{F}_{W_\bullet}$ of strongly excisive presheaves on $\mathcal{T}$ inducing $\mathrm{Con}_{Z_\bullet}(X, A; \mathcal{E}_A) \to \mathrm{Con}_{W_\bullet}(X, A; q_*\mathcal{E}_A)$. By Proposition 6.7 this map of concordance classes is a bijection if $\mathcal{F}_{Z_\bullet}[X, C; \mathcal{E}_C] \to \mathcal{F}_{W_\bullet}[X, C; q_*\mathcal{E}_C]$ is surjective for all subcomplex inclusions $C \subseteq X$ and $Z_\bullet$-bundles $\mathcal{E}_C$ over C.

By Proposition 6.14 we can represent an element $\mathcal{E}'$ of $\mathrm{Con}_{W_\bullet}(X, C; q_*\mathcal{E}_C)$ as a sequence $(\mathcal{E}'_0, \mathcal{E}'_1, \ldots)$ where $\mathcal{E}'_k$ is a $W_\bullet$-bundle over X^k, such that $\mathcal{E}'_k|_{X^{k-1}}$ is concordant to $\mathcal{E}'_{k-1}$ relative to C^{k-1} and $\mathcal{E}'_k|_{C^k} = (q_*\mathcal{E}_C)|_{C^k}$. By induction on k we will construct a sequence $(\mathcal{E}_0, \mathcal{E}_1, \ldots)$ where $\mathcal{E}_k$ is a $Z_\bullet$-bundle over X^k, such that $\mathcal{E}_k|_{X^{k-1}} = \mathcal{E}_{k-1}$, $\mathcal{E}_k|_{C^k} = \mathcal{E}_C|_{C^k}$, and $q_*\mathcal{E}_k$ is concordant to $\mathcal{E}'_k$ relative to C^k. This new sequence will represent an element $\mathcal{E}$ of $\mathrm{Con}_{Z_\bullet}(X, C; \mathcal{E}_C)$ mapping to the class $[\mathcal{E}']$ under q_*.

We start the induction at the (-1)-skeleton, i.e. $X^{-1} = C^{-1} = A$, $\mathcal{E}_{-1} = \mathcal{E}_A$, and $\mathcal{E}'_{-1} = q_*\mathcal{E}_A$. By the induction hypothesis there is a concordance between $\mathcal{E}'_k|_{X^{k-1} \cup C^k}$ and $q_*(\mathcal{E}_{k-1} \cup \mathcal{E}_C|_{C^k})$ relative to C^k. By Lemma 6.12 we can find a $W_\bullet$-bundle $\tilde{\mathcal{E}}'_k$ over X^k concordant to $\mathcal{E}'_k$ relative to C^k such that $\tilde{\mathcal{E}}'_k|_{X^{k-1} \cup C^k} = q_*(\mathcal{E}_{k-1} \cup \mathcal{E}_C|_{C^k})$. We claim that

$$q_*\colon \mathrm{Con}_{Z_\bullet}(X^k, X^{k-1} \cup C^k; \mathcal{E}_{k-1} \cup \mathcal{E}_C|_{C^k}) \to \mathrm{Con}_{W_\bullet}(X^k, X^{k-1} \cup C^k; q_*(\mathcal{E}_{k-1} \cup \mathcal{E}_C|_{C^k}))$$

is surjective. If the claim holds, then choose $\mathcal{E}_k$ to be any $Z_\bullet$-bundle over X^k equal to $\mathcal{E}_{k-1} \cup \mathcal{E}_C|_{C^k}$ on $X^{k-1} \cup C^k$ such that $q_*[\mathcal{E}_k] = [\tilde{\mathcal{E}}'_k]$. This completes the inductive step.

To prove the claim observe that X^k can be constructed from $X^{k-1} \cup C^k$ by attaching k-cells. By Corollary 6.9 and the fact that Con turns a disjoint union of spaces into a product, it is sufficient to prove that

$$\mathrm{Con}_{Z_\bullet}(D^k, S^{k-1}; \mathcal{E}_{S^{k-1}}) \to \mathrm{Con}_{W_\bullet}(D^k, S^{k-1}; q_*\mathcal{E}_{S^{k-1}})$$

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is surjective for all k and $Z_\bullet$-bundles $\mathcal{E}_{S^{k-1}}$ over S^{k-1}. By Lemma 6.18 $\mathcal{E}_{S^{k-1}}$ is concordant to a finite $Z_\bullet$-bundle. By Proposition 6.17 nothing is lost by assuming that $\mathcal{E}_{S^{k-1}}$ actually is finite. Applying Lemma 6.18 we see that the claim holds if

$$\mathrm{Con}^f_{Z_\bullet}(D^k, S^{k-1}; \mathcal{E}_{S^{k-1}}) \to \mathrm{Con}^f_{W_\bullet}(D^k, S^{k-1}; q_* \mathcal{E}_{S^{k-1}})$$

is surjective, which is precisely the hypothesis of the proposition we are proving. □

We now simplify by noticing that any ordered Čech complex $U_\bullet$ has *free degeneracies*, see [10, Definition A.4]. This means that $U_\bullet$ is the image of a presimplicial space $N_\bullet$ under the left adjoint to the forgetful functor from simplicial spaces to presimplicial spaces. It is easy to see what $N_\bullet$ must be; we define the k-simplices by

$$N_k = \coprod_{\alpha_0 < \cdots < \alpha_k} U_{\alpha_0 \cdots \alpha_k},$$

and we call $N_\bullet$ the *non-degenerate part* of $U_\bullet$. Define $L^k N_\bullet$ to be subspace of N_k consisting of points in the image of some face map d_i, i.e. $L^k N_\bullet = \bigcup_i d_i(N_{k+1})$

Because of the proposition above we are led to study simplicial maps whose domains are ordered Čech complexes of finite and totally ordered covers $\mathcal{U}$. Let us make the necessary preparations for the construction of maps $N_\bullet \to Z_\bullet$ by downward induction on simplicial degree:

Lemma 6.20 *Let $N_\bullet$ be the non-degenerate part of the ordered Čech complex corresponding to a finite and totally ordered open covering. The following properties are satisfied:*

i) If k is greater than the number of elements in the cover, then $N_k = \emptyset$.

ii) If $d_i(x) = d_j(y) \in N_k$ for some $i < j$, then there exists some $z \in N_{k+2}$ such that $y = d_i(z)$ and $x = d_{j+1}(z)$.

iii) The obvious map

$$\coprod_{0 \le i \le k+1} N_{k+1} \xrightarrow{\coprod d_i} L^k N_\bullet$$

is a quotient map.

Proof: Property i) is obvious. Property ii) is a general fact of the non-degenerate part of an ordered Čech complex.

Addressing the last property, we consider a subset V of N_k such that all $d_i^{-1}(V)$ are open in N_{k+1}. We have to show that $V \cap L^k N_\bullet$ is open. We can assume that V is a subset of some $U_{\alpha_0 \cdots \alpha_k}$. Notice that all $V \cap U_\beta$ are open for $\beta \neq \alpha_i$. The result follows since $V \cap L^k N_\bullet$ equals the finite union $\bigcup_{\beta \neq \alpha_i} (V \cap U_\beta)$. □

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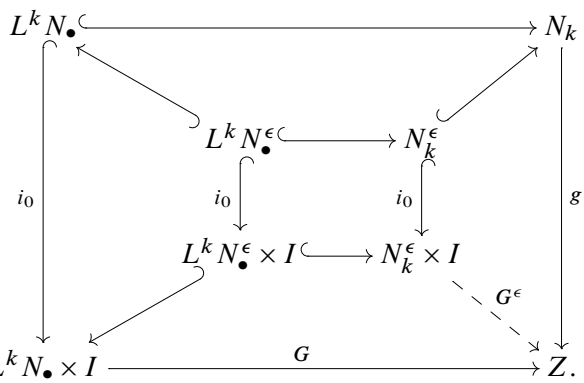
Remark 6.21 In the Reedy formalism, see [17, Chapter 15], one considers the category of diagrams, $\mathbf{X}$, indexed by a Reedy category $\mathcal{C}$. Each object in $\mathcal{C}$ has a degree, and the usual way of constructing a map $f\colon \mathbf{X} \to \mathbf{Y}$ between diagrams is by increasing induction on the degree. Associated to a diagram $\mathbf{X}$ and an index $\alpha \in \mathcal{C}$ there are the so called latching and matching objects, $L_\alpha \mathbf{X} \to \mathbf{X}_\alpha \to M_\alpha \mathbf{X}$. If f is already defined in degrees below α, this fixes maps of latching and matching objects, and an extension of f to $\mathbf{X}_\alpha$ must respect this.

In our case the presimplicial indexing category clearly is a Reedy category. However, the union $L^k N_\bullet \subseteq N_k$ is neither a latching nor a matching object. This is not so strange; we are preparing for downward induction on degree, and $L^k N_\bullet$ plays the role analogous of a latching object. The analogy for the matching object is the terminal space.

The following lemma plays the role of "a homotopy extension property" for the inclusion $L^k N_\bullet \hookrightarrow N_k$. In a vague sense this is dual to Segal's definition of a good simplicial space, [29].

Lemma 6.22 *Let $N_\bullet$ be the non-degenerate part of the ordered Čech complex corresponding to a finite and totally ordered open covering $\mathcal{U}$ of a normal space X and let Z be a topological space. Given $g\colon N_k \to Z$ and $G\colon L^k N_\bullet \times I \to Z$ with $g(x) = G(x,0)$ for all x in $L^k N_\bullet$, there exists a shrinking $\mathcal{U}^\epsilon$ of $\mathcal{U}$, inducing an inclusion $N_\bullet^\epsilon \hookrightarrow N_\bullet$, and a homotopy $G^\epsilon\colon N_k^\epsilon \times I \to Z$ such that $g(x) = G^\epsilon(x,0)$ for all $x \in N_k^\epsilon$ and $G(x,t) = G^\epsilon(x,t)$ for all $x \in L^k N_\bullet^\epsilon$ and $t \in I$. Moreover, if $t \mapsto G(x,t)$ is constant for some $x \in N_k^\epsilon \cap L^k N_\bullet$, then we may choose G^ϵ such that also $t \mapsto G^\epsilon(x,t)$ is constant.*

The conditions imposed on the maps are encoded by commutativity of the following diagram:



Proof: Let K be the number of elements in the cover. Choose a partition of unity $\{\psi_\alpha\}$ subordinate to $\mathcal{U}$. For $\epsilon > 0$ define $\mathcal{U}^\epsilon$ be the family of open sets $\{\psi_\alpha^{-1}(\epsilon,1]\}$.

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If ϵ is small, $\epsilon < \frac{1}{K}$, then $\mathcal{U}^\epsilon$ is also a cover. Let $\psi\colon N_k \to I$ be the map given on each intersection $U_{\alpha_0\cdots\alpha_k}$ as the sum $\sum_{\beta\neq\alpha_i}\psi_\beta$. Notice that the support of ψ is contained in $L^k N_\bullet$, and that $\psi^{-1}(\epsilon,1]$ equals $L^k N^\epsilon_\bullet$.

Define a homotopy $\hat{G}^\epsilon\colon N_k \times I \to Z$ by the formula:

$$\hat{G}^\epsilon(x,t) = \begin{cases} g(x) & \text{if } x \notin L^k N_\bullet, \\ G(x,t) & \text{if } x \in L^k N_\bullet \text{ and } \psi(x) \geq \epsilon, \text{ and} \\ G(x, \frac{\psi(x)}{\epsilon}t) & \text{if } x \in L^k N_\bullet \text{ and } \psi(x) \leq \epsilon. \end{cases}$$

Let G^ϵ be the restriction of $\hat{G}^\epsilon$ to $N^\epsilon_k \times I$. □

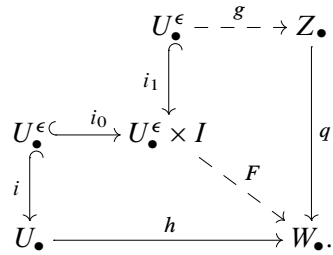
By shrinking the cover we also get a "lifting property" up to homotopy:

Proposition 6.23 *Let $h\colon U_\bullet \to W_\bullet$ and $q\colon Z_\bullet \to W_\bullet$ be maps of simplicial spaces. Suppose that each $q_k\colon Z_k \to W_k$ is the inclusion of a strong deformation retract, and that $U_\bullet$ is the Čech complex of a finite and totally ordered open cover $\mathcal{U}$ of a normal space X. There exists a shrinking $\mathcal{U}^\epsilon$ of $\mathcal{U}$, inducing an inclusion $i\colon U^\epsilon_\bullet \hookrightarrow U_\bullet$, together with simplicial maps $g\colon U^\epsilon_\bullet \to Z_\bullet$ and $F\colon U^\epsilon_\bullet \times I \to W_\bullet$ such that*

$$F_k(x,0) = h_k(i_k(x)) \quad \textit{and} \quad F_k(x,1) = q_k(g_k(x))$$

for all k and $x \in U^\epsilon_k$. Moreover, if $h_k(x)$ is contained in the image of q_k, then we may choose F such that $t \mapsto F_k(x,t)$ is constant.

The conditions imposed on the maps are encoded by commutativity of the following diagram:



Proof: It is enough to consider the non-degenerate part $N_\bullet$ of $U_\bullet$. We construct g and F by downward induction on n. Assume that there exists shrinking $\mathcal{U}^\epsilon$ of $\mathcal{U}$, inducing $i\colon N^\epsilon_\bullet \to N_\bullet$ on the non-degenerate part of the ordered Čech complexes, and a presimplicial map

$$F^n\colon N^\epsilon_\bullet \times I \to W_\bullet$$

such that $F^n_k(x,0) = h_k(i_k(x))$ and $F^n_k(x,1) \in q_k(Z_k)$ for all $k > n$ and $x \in N^\epsilon_k$. To complete the inductive step and define F^{n-1}, it suffices to construct a shrinking $\mathcal{U}^{\epsilon'}$ of $\mathcal{U}^\epsilon$ and a presimplicial map $G\colon N^{\epsilon'}_\bullet \times I \to W_\bullet$ with the following properties:

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$G_k(x,0) = F_k^n(x,1)$ for all k and $x \in N_k^{\epsilon'}$,

$G_k(x,t) = F_k^n(x,1)$ if $k > n$, $x \in N_k^{\epsilon'}$, and $t \in I$, and

$G_n(x,1) \in q_n(Z_n) \subset W_n$ for all $x \in N_k^{\epsilon'}$.

Then we obtain F^{n-1} by composing the homotopy F^n with the homotopy G.

We construct G by downward induction on k. For $k > n$, the conditions already define G_k. We start the induction at $k = n$ and take $\mathcal{U}^{\epsilon'} = \mathcal{U}^{\epsilon}$. Let $H\colon W_n \times I \to W_n$ be the deformation retraction into $q_n(Z_n)$, and define G_n by the formula $G_n(x,t) = H(F_n^n(x,1),t)$. Observe that the image of G_{n+1} is contained in $q_{n+1}(Z_{n+1})$, whence the definition of G_n ensures commutativity of the diagram

$$\begin{array}{ccccc} N_{n+1}^{\epsilon'} \times I & & \xrightarrow{\quad d_i \quad} & & N_n^{\epsilon'} \times I \\ \downarrow{\scriptstyle G_{n+1}} & & & & \downarrow{\scriptstyle G_n} \\ q_{n+1}(Z_{n+1}) & \xrightarrow{\subseteq} & W_{n+1} & \xrightarrow{d_i} & W_n. \end{array}$$

Thus G satisfies the presimplicial relations in simplicial degrees $k \geq n$. It remains to define G_k for $k < n$.

For each step in the inductive construction of G we will shrink the cover $\mathcal{U}^{\epsilon'}$ to a smaller cover $\mathcal{U}^{\epsilon''}$. So we implicitly restrict the domains of the G_k's already constructed. Assuming that $G_{k+1}\colon N_{k+1}^{\epsilon'} \times I \to W_{k+1}$ already has been defined we proceed to define G_k.

Because of Lemma 6.20.ii), there is a well defined set map $\tilde{G}_k\colon L^k N_\bullet^{\epsilon'} \times I \to W_k$ such that the following diagram commutes

$$\begin{array}{ccc} N_{k+1}^{\epsilon'} \times I & \xrightarrow{d_i} & L^k N_\bullet^{\epsilon'} \times I \\ \downarrow{\scriptstyle G_{k+1}} & & \downarrow{\scriptstyle \tilde{G}_k} \\ W_{k+1} & \xrightarrow{d_i} & W_k \end{array}$$

for all d_i. According to Lemma 6.20.iii), $\tilde{G}_k$ is continuous. By Lemma 6.22, we can find a shrinking $\mathcal{U}^{\epsilon''}$ of the cover $\mathcal{U}^{\epsilon'}$, inducing an inclusion $N_\bullet^{\epsilon''} \subseteq N_\bullet^{\epsilon'}$, together with a map $G_k\colon N_k^{\epsilon''} \times I \to W_k$ such that $G_k(x,0) = F_k^n(x,1)$ for $x \in N_k^{\epsilon''}$, and $G_k(x,t) = \tilde{G}_k(x,t)$ for $x \in L^k N_\bullet^{\epsilon''}$ and $t \in I$.

This completes the inductive step, and the proof of everything except the last part. However, tracing the proof it is easily seen that all homotopies fix points $x \in N^\epsilon$ having $h_k(x) \in q_k(Z_k)$. □

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Corollary 6.24 *Let X be a space and A a subspace such that (X,A) has the homotopy type of a CW pair. Let $q\colon Z_\bullet \hookrightarrow W_\bullet$ be a simplicial map which in each simplicial degree is the inclusion of a strong deformation retract. Let $\mathcal{E}_A$ be any $Z_\bullet$-bundle over A. Then*

$$\mathrm{Con}_{Z_\bullet}(X,A;\mathcal{E}_A) \to \mathrm{Con}_{W_\bullet}(X,A;q_*\mathcal{E}_A)$$

is a bijection.

Proof: According to Proposition 6.19 it is sufficient to show that the induced map

$$q_*\colon \mathrm{Con}^f_{Z_\bullet}(D^n,S^{n-1};\mathcal{E}_{S^{n-1}}) \to \mathrm{Con}^f_{W_\bullet}(D^n,S^{n-1};q_*\mathcal{E}_{S^{n-1}})$$

is surjective when fixing an arbitrary integer n and a finite $Z_\bullet$-bundle $\mathcal{E}_{S^{n-1}}$ over the boundary sphere. Let $\mathcal{E}'$ be a finite $W_\bullet$-bundle over D^n whose restriction to S^{n-1} equals $q_*\mathcal{E}_{S^{n-1}}$. Let $\mathcal{U}$ denote the corresponding finite and totally ordered open cover, and let $h\colon U_\bullet \to W_\bullet$ be the simplicial map associated to $\mathcal{E}'$. Applying Proposition 6.23 we get a shrinking $\mathcal{U}^\epsilon$ of $\mathcal{U}$, an induced inclusion of ordered Čech complexes $U^\epsilon_\bullet \hookrightarrow U_\bullet$, a simplicial map $g\colon U^\epsilon_\bullet \to Z_\bullet$, and a homotopy $F\colon U^\epsilon_\bullet \times I \to W_\bullet$ between the restriction of h to $U^\epsilon_\bullet$ and qg.

The shrinking $\mathcal{U}^\epsilon$ of $\mathcal{U}$ gives us a family of inclusions $\{U^\epsilon_\alpha \hookrightarrow U_\alpha\}$. For each α define U''_α to be the open subset $U^\epsilon_\alpha \times I \cup U_\alpha \times \left[0,\frac{1}{2}\right)$ of $D^n \times I$. Together all the U''_α constitute a cover $\mathcal{U}''$ of $D^n \times I$. Let $\phi''\colon U''_\bullet \to W_\bullet$ be the simplicial map given in simplicial degree k as

$$\phi''(x,t) = \begin{cases} h(x) & \text{for } x \in U_k \text{ and } 0 \le t < \frac{1}{2}, \text{ and} \\ F(x,2t-1) & \text{for } x \in U^\epsilon_k \text{ and } \frac{1}{2} \le t \le 1. \end{cases}$$

This simplicial map explicitly defines a $W_\bullet$-bundle $\mathcal{E}''$ over $D^n \times I$. The restriction of $\mathcal{E}''$ to $D^n \times \{0\}$ lifts to a $Z_\bullet$-bundle $\mathcal{E}$ represented by the simplicial map $g\colon U^\epsilon_\bullet \to Z_\bullet$. Moreover, the restriction of $\mathcal{E}''$ to $S^{n-1} \times I$ lifts to a concordance between the restriction $\mathcal{E}|_{S^{n-1}}$ and our fixed $Z_\bullet$-bundle $\mathcal{E}_{S^{n-1}}$. Denote this concordance by $\mathcal{E}_{S^{n-1}\times I}$.

Now consider the diagram:

$$\begin{array}{ccc} \mathrm{Con}^f_{Z_\bullet}(D^n,S^{n-1};\mathcal{E}|_{S^{n-1}}) & \longrightarrow & \mathrm{Con}^f_{W_\bullet}(D^n,S^{n-1};q_*\mathcal{E}|_{S^{n-1}}) \\ \downarrow{\scriptstyle [\mathcal{E}_{S^{n-1}\times I}]_*} & & \downarrow{\scriptstyle [q_*\mathcal{E}_{S^{n-1}\times I}]_*} \\ \mathrm{Con}^f_{Z_\bullet}(D^n,S^{n-1};\mathcal{E}_{S^{n-1}}) & \xrightarrow{q_*} & \mathrm{Con}^f_{W_\bullet}(D^n,S^{n-1};q_*\mathcal{E}_{S^{n-1}}). \end{array}$$

The vertical maps, see Proposition 6.17, correspond to gluing the concordances $\mathcal{E}_{S^{n-1}\times I}$ and $q_*\mathcal{E}_{S^{n-1}\times I}$ along the boundary sphere. They are bijections by

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Lemma 6.16. We started by picking a $W_\bullet$-bundle $\mathcal{E}'$ representing an element from the lower right corner. The $Z_\bullet$-bundle $\mathcal{E}$ represents an element in $\mathrm{Con}^f_{Z_\bullet}(D^n, S^{n-1}; \mathcal{E}|_{S^{n-1}})$, and $\mathcal{E}''$ produces a concordance between the image of $q_*[\mathcal{E}_{S^{n-1}\times I}]_*\mathcal{E}$ and $\mathcal{E}'$. Hence q_* is surjective. □

We say that two simplicial maps $p,q\colon Z_\bullet \to W_\bullet$ are *topologically homotopic* if there exists a simplicial map $F\colon Z_\bullet \times I \to W_\bullet$ with $F_k(x,0) = p_k(x)$ and $F_k(x,1) = q_k(x)$ for each k and all $x \in Z_k$.

Proposition 6.25 *Let $A \hookrightarrow X$ be a closed cofibration, and suppose that $p,q\colon Z_\bullet \to W_\bullet$ are topologically homotopic. Given any $Z_\bullet$-bundle $\mathcal{E}_A$ over A there exists a $W_\bullet$-bundle $\mathcal{E}'_{A\times I}$ over $A \times I$, restricting to $p_*\mathcal{E}_A$ and $q_*\mathcal{E}_A$ on $A \times \{0\}$ and $A \times \{1\}$ respectively, making the following triangle commute*

$$\begin{array}{ccc} & \mathrm{Con}_{Z_\bullet}(X,A;\mathcal{E}_A) & \\ {\scriptstyle p_*}\swarrow & & \searrow{\scriptstyle q_*} \\ \mathrm{Con}_{W_\bullet}(X,A;p_*\mathcal{E}_A) & \xrightarrow{[\mathcal{E}'_{A\times I}]_*} & \mathrm{Con}_{W_\bullet}(X,A;q_*\mathcal{E}_A). \end{array}$$

Proof: Let $F\colon Z_\bullet \times I \to W_\bullet$ denote a homotopy between p and q.

Given an ordered open cover $\mathcal{U}$ of X we define a cover $\mathcal{U}'$ of $X \times I$ consisting of the open sets $U'_\alpha = U_\alpha \times I$ where $U_\alpha \in \mathcal{U}$. The corresponding ordered Čech complexes satisfy $U'_\bullet = U_\bullet \times I$.

Let $\mathcal{E}$ be a $Z_\bullet$-bundle over X equal to $\mathcal{E}_A$ on A, and consider the associated simplicial map $\phi\colon U_\bullet \to Z_\bullet$. Subordinate to the cover $\mathcal{U}'$ we define a $W_\bullet$-bundle $\mathcal{E}'$ over $X \times I$ having as its simplicial map the composition

$$U'_\bullet = U_\bullet \times I \xrightarrow{\phi\times id} Z_\bullet \times I \xrightarrow{F} W_\bullet.$$

The restriction of $\mathcal{E}'$ to $A \times I$ defines the concordance $\mathcal{E}'_{A\times I}$ between $p_*\mathcal{E}_A$ and $q_*\mathcal{E}_A$. Moreover, $\mathcal{E}'$ yields a concordance between $[\mathcal{E}'_{A\times I}]_* p_*\mathcal{E}$ and $q_*\mathcal{E}$. □

Theorem 6.26 *Let $A \hookrightarrow X$ be a closed cofibration such that (X,A) has the homotopy type of a CW pair. Let $f\colon Z_\bullet \to W_\bullet$ be a simplicial map such that each f_k is a homotopy equivalence. Let $\mathcal{E}_A$ be any $Z_\bullet$-bundle over A. Then f induces a bijection*

$$f_*\colon \mathrm{Con}_{Z_\bullet}(X,A;\mathcal{E}_A) \to \mathrm{Con}_{W_\bullet}(X,A;f_*\mathcal{E}_A).$$

Proof: Let $M_\bullet$ be the degreewise mapping cylinder of f. There are inclusions of degreewise strong deformation retracts $i\colon Z_\bullet \hookrightarrow M_\bullet$ and $j\colon W_\bullet \hookrightarrow M_\bullet$. Moreover, the map jf is topologically homotopic to i. According to Proposition 6.25 there exists a concordance $\mathcal{E}'_{A\times I}$ of $M_\bullet$-bundles between $i_*\mathcal{E}_A$ and $j_*f_*\mathcal{E}_A$ making the

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following diagram commute

$$\begin{array}{ccc} \mathrm{Con}_{Z_\bullet}(X,A;\mathcal{E}_A) & \xrightarrow{f_*} & \mathrm{Con}_{W_\bullet}(X,A;f_*\mathcal{E}_A) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle j_*} \\ \mathrm{Con}_{M_\bullet}(X,A;i_*\mathcal{E}_A) & \xrightarrow{[\mathcal{E}'_{A\times I}]_*} & \mathrm{Con}_{M_\bullet}(X,A;j_*f_*\mathcal{E}_A). \end{array}$$

The vertical maps are bijections by Corollary 6.24. The bottom map is a bijection by Lemma 6.16. □

In our next line of argument we will show that filling a horn in $Z_\bullet$ does not affect the concordance classes of $Z_\bullet$-bundles.

A shrinking $\mathcal{U}'$ of $\mathcal{U}$ induces an inclusion of ordered Čech complexes $i : U'_\bullet \hookrightarrow U_\bullet$. Given a simplicial map $U_\bullet \xrightarrow{\phi} Z_\bullet$, we define the *cover-restriction* of ϕ to $\mathcal{U}'$ to be the composition $U'_\bullet \xrightarrow{i} U_\bullet \xrightarrow{\phi} Z_\bullet$. Similarly, if $\mathcal{E}$ is the $Z_\bullet$-bundle associated to ϕ, then we denote the $Z_\bullet$-bundle associated to ϕi by $i^*\mathcal{E}$, and call it a *cover-restriction* of $\mathcal{E}$.

Proposition 6.27 *Let $h\colon \Lambda^{n,k}_\bullet \to Z_\bullet$ be a k-horn of dimension n in a simplicial space $Z_\bullet$, and define $q\colon Z_\bullet \hookrightarrow W_\bullet$ by filling this horn, i.e. $W_\bullet$ is the pushout of $\Delta^n_\bullet \hookleftarrow \Lambda^{n,k}_\bullet \xrightarrow{h} Z_\bullet$. Let $\phi\colon U_\bullet \to W_\bullet$ be a simplicial map from the ordered Čech complex associated to a finite and totally ordered open cover $\mathcal{U}$ of a normal space X. Then there exists a zigzag chain of finite and totally ordered open covers*

$$\mathcal{U} = \mathcal{U}_0 \subseteq \mathcal{U}'_1 \supseteq \mathcal{U}_1 \subseteq \mathcal{U}'_2 \supseteq \ldots \subseteq \mathcal{U}'_{s-1} \supseteq \mathcal{U}_{s-1} \subseteq \mathcal{U}'_s \supseteq \mathcal{U}_s$$

together with simplicial maps ϕ_i and ϕ'_i from the various ordered Čech complexes into $W_\bullet$ such that each ϕ_i is equal to the cover-restriction of both ϕ'_{i-1} and ϕ'_i, and the last simplicial map ϕ_s lifts through $q\colon Z_\bullet \to W_\bullet$. Moreover, if A is a subset of X such that ϕ restricted to $\mathcal{U} \cap A$ already lifts to $Z_\bullet$, then we may assume that the restriction to A of all ϕ'_i and ϕ_i also lifts to $Z_\bullet$.

Roughly, the proof is by downward induction on the number of points in the image of the non-degenerate part $N_\bullet$ mapping into $W_\bullet$ which obstructs a lifting to $Z_\bullet$. In the simplest possible setup $Z = \Delta^0_\bullet$, $W = \Delta^1_\bullet$, X is a one point space with a cover $\mathcal{U}$ consisting of one subset U_0, and $\phi\colon U_\bullet \to W_\bullet$ has image in $W_\bullet \smallsetminus Z_\bullet$. In this case the proof constructs a new cover $\mathcal{U}'$ with two subsets U_0 and U_β together with a map $\phi'\colon U'_\bullet \to W_\bullet$ extending ϕ and such that ϕ' is an isomorphism. The next step of the proof is to shrink $\mathcal{U}'$ to a smaller cover $\mathcal{U}^\epsilon$ by removing U_0 from the cover. Thus the restriction $\phi^\epsilon = \phi'|_{U^\epsilon_\bullet}$ has image in $Z_\bullet$.

At the center of this long and complicated proof there is an unpolished gem; the emptying of the horn by modifications of the cover.

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Proof: It is enough to consider presimplicial maps from the non-degenerate part of the various ordered Čech complexes.

Our focus will be to move $\phi\colon N_\bullet \to W_\bullet$ away from the kth face of the filled horn. Recall that the r-simplices of $\Delta^n_\bullet$ correspond to order-preserving maps $\theta\colon [r] \to [n]$. Denote by w_θ the r-simplex in W_r corresponding to θ under the simplicial map $\Delta^n_\bullet \to W_\bullet$. Let Θ be the set of all θ that factor as a surjective map $[r] \twoheadrightarrow [n-1]$ followed by the coface map $\delta_k\colon [n-1] \to [n]$ that omits k. Here we allow r to vary. Choose a total ordering on Θ such that $\theta_1 < \theta_2$ whenever the domain of θ_1 contains fewer elements than the domain of θ_2. Each $\theta \in \Theta$ corresponds to a degenerate simplex above the kth face of $\Delta^n_\bullet$. Furthermore, for all $\theta \in \Theta$ the simplex w_θ is an isolated point of W_r.

The cover $\mathcal{U}$ has a finite and totally ordered indexing set $\mathcal{I}$. An order-preserving injection $\rho\colon [r] \hookrightarrow \mathcal{I}$ determines a subset $U_\rho = U_{\rho(0)} \cap \cdots \cap U_{\rho(r)}$. Moreover, U_ρ is a summand of N_r. If the presimplicial map $\phi\colon N_\bullet \to W_\bullet$ does not factor through $q\colon Z_\bullet \hookrightarrow W_\bullet$, then some degenerate simplex above the kth face of $\Delta^n_\bullet$ lies in the image of ϕ. From now on reserve θ as a symbol for the maximal element of Θ such that w_θ is contained in $\phi_r(N_r)$ for some r.

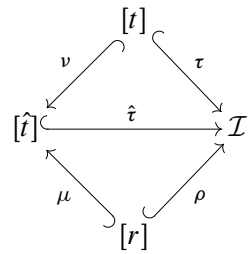
Define the *θ-valence* of a presimplicial map $\phi\colon N_\bullet \to W_\bullet$ to be the number of order-preserving injections $\rho\colon [r] \hookrightarrow \mathcal{I}$ such that $\phi_r(U_\rho)$ contains w_θ. We will soon construct a finite and totally ordered cover $\mathcal{U}'$ containing $\mathcal{U}$ as a subcover together with an extension $\phi'\colon N'_\bullet \to W_\bullet$ of ϕ such that there exists a shrinking $\mathcal{U}^\epsilon$ of $\mathcal{U}'$ where the cover-restriction $\phi^\epsilon\colon N^\epsilon_\bullet \to W_\bullet$ of ϕ' has smaller θ-valence than ϕ. Repeating this construction we will eventually reach a presimplicial map of θ-valence zero, i.e. the point w_θ is no longer in its image. By downward induction on $\theta \in \Theta$ the forthcoming construction will establish the lemma.

For such a maximal θ choose any ρ with $w_\theta \in \phi_r(U_\rho)$. Let j be the largest integer such that $\theta(j-1) < k$. Define the indexing set $\mathcal{I}'$ by inserting into $\mathcal{I}$ a new index β as the successor of $\rho(j-1)$, i.e. we have $\mathcal{I}' = \mathcal{I} \cup \{\beta\}$. Since w_θ is an isolated point of W_r the subset of U_ρ mapping to w_θ is both closed and open, and we define $U_\beta = U_\rho \cap \phi^{-1}(w_\theta)$. Now view U_β as a subspace of X, and consider the restriction of the presimplicial map ϕ to U_β. We claim that there is a factorization of $\phi|_{U_\beta}$ as a presimplicial map $N_\bullet \cap U_\beta \to \Delta^n_\bullet$ followed by the simplicial map $\Delta^n_\bullet \to W_\bullet$.

Let us verify this claim. Given an order-preserving injection $\tau\colon [t] \hookrightarrow \mathcal{I}$ let $\hat{t}$ be the smallest integer such that there exists order-preserving injections $\hat{\tau}$, μ, and ν

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making the diagram



commute. Observe that $\hat{\tau}$, μ, and ν are unique. Informally, $\hat{\tau}$ is the union of τ and ρ.

By the constructions above ϕ_r maps any x in $U_\rho \cap U_\beta$ to w_θ in W_r. Suppose that $U_{\hat{\tau}}$ meets U_β and take some x in $U_{\hat{\tau}} \cap U_\beta$. Observe that $\phi_{\hat{t}}(x) \in W_{\hat{t}}$ is sent to w_θ by the face operator μ^* corresponding to $\mu\colon [r] \to [\hat{t}]$. Inspecting the definition of $W_\bullet$ we see that all simplices mapping to w_θ by some face operator lie in the image of $\Delta^n_\bullet \to W_\bullet$. Hence $\phi_{\hat{t}}(x) = w_{\hat{\theta}}$ for some $\hat{\theta}\colon [\hat{t}] \to [n]$. The equation $\theta = \hat{\theta}\mu$ is satisfied, and for any $i \in [\hat{t}]$ outside the image of μ we must have $\hat{\theta}(i) = k$. Otherwise we get a contradiction, i.e. some $\theta_2 > \theta$ in Θ would be in the image of ϕ. Thus $\hat{\theta}$ is uniquely determined. Lastly we see that ϕ_t maps any $x \in U_\tau \cap U_\beta$ to $w_{\hat{\theta}\nu}$ in W_t. This completes the verification of the claim.

Now define the cover $\mathcal{U}'$ as the union $\mathcal{U} \cup \{U_\beta\}$. Any extension $\phi'\colon N'_\bullet \to W_\bullet$ is completely determined by the restrictions of ϕ'_{t+1} to $U_{\tau'}$ for the order-preserving injections $\tau'\colon [t+1] \hookrightarrow \mathcal{I}'$ containing β in their image. For any such τ' define τ and δ_l by the pullback diagram

$$\begin{array}{ccc} [t] & \xrightarrow{\delta_l} & [t+1] \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau'} \\ \mathcal{I} & \hookrightarrow & \mathcal{I}'. \end{array}$$

Applying the notation introduced above, we define $\theta'\colon [t+1] \to [n]$ by $\theta'(l) = k$ and $\theta'\delta_l = \hat{\theta}\nu$. For x in $U_{\tau'} = U_\tau \cap U_\beta$ define ϕ'_{t+1} by the formula

$$\phi'_{t+1}(x) = w_{\theta'}.$$

Since $d_l\phi'_{t+1}(x) = w_{\theta'\delta_l} = w_{\hat{\theta}\nu} = \phi_t(x)$ it follows that ϕ' satisfies the presimplicial identities. Furthermore, ϕ and ϕ' have the same θ-valence since w_θ is not contained in any $\phi'_{t+1}(U_{\tau'})$. What remains to be proved is that we can find a shrinking $\mathcal{U}^\epsilon$ of $\mathcal{U}'$ such that the cover-restriction ϕ^ϵ of ϕ' has θ-valence smaller than the θ-valence of ϕ and ϕ'.

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Since $\mathcal{U}$ is a finite open cover of a normal space, we can choose a partition of unity $\{\psi_\alpha\}$ subordinate to the cover $\mathcal{U}$. Let C be the closure of U_β in X. Define B to be the set of all points $x \in X \smallsetminus U_\beta$ with $\psi_{\rho(i)}(x) = \frac{1}{r+1}$ for $i = 0,\ldots,r$. Because $U_\rho \smallsetminus U_\beta$ is an open neighborhood of B in X, we see that B and C are disjoint closed subsets in X. By the Tietze extension theorem there exists a continuous function $\epsilon\colon X \to \left[0, \frac{1}{r+1}\right]$ such that $\epsilon(x) = 0$ for all $x \in C$ and $\epsilon(x) = \frac{1}{r+1}$ for all x in B. Informally, we think about the points x with all $\psi_{\rho(i)}(x) = \frac{1}{r+1}$ as the bad points. The role of ϵ is to separate the bad points inside U_β from those outside.

We will now define a shrinking $\mathcal{U}^\epsilon$ of $\mathcal{U}'$ by shrinking the $U_{\rho(i)}$'s while leaving the other open sets unchanged. Define $U^\epsilon_{\rho(i)}$ to be the set of all points $x \in U_{\rho(i)}$ satisfying the inequality

$$r \cdot \psi_{\rho(i)}(x) + \epsilon(x) > \psi_{\rho(0)}(x) + \cdots + \psi_{\rho(i-1)}(x) + \psi_{\rho(i+1)}(x) + \cdots + \psi_{\rho(r)}(x).$$

Explicitly, the cover $\mathcal{U}^\epsilon$ is given as the union $\{U^\epsilon_{\rho(i)}\}_{i=0}^r \cup \{U_\alpha\}_{\alpha \neq \rho(i),\beta} \cup \{U_\beta\}$. Let us verify that $\mathcal{U}^\epsilon$ really is a cover of X. Take an arbitrary point x in X.

i) If $\sum_{i=0}^r \psi_{\rho(i)}(x) < 1$, then $x \in U_\alpha$ for some $\alpha \in \mathcal{I}$ outside the image of ρ.

ii) If $\psi_{\rho(j)}(x) > \frac{1}{r+1}$ for some j, then

$$(r+1) \cdot \psi_{\rho(j)}(x) > 1 \geq \sum_{i=0}^{r} \psi_{\rho(i)}(x).$$

Consequently, x lies in $U^\epsilon_{\rho(j)}$.

iii) If $x \in B$, then $\psi_{\rho(i)}(x) = \frac{1}{r+1}$ for $i = 0,\ldots,r$ and $\epsilon(x) = \frac{1}{r+1}$. Hence,

$$r \cdot \psi_{\rho(0)}(x) + \epsilon(x) = 1 > \frac{r}{r+1} = \sum_{i=1}^{r} \psi_{\rho(i)}(x).$$

So, x lies in $U^\epsilon_{\rho(0)}$.

iv) Otherwise, all $\psi_{\rho(i)}(x) = \frac{1}{r+1}$ and $x \notin B$, whence $x \in U_\beta$.

This shows that $\mathcal{U}^\epsilon$ indeed is a cover of X.

Define $\phi^\epsilon\colon N^\epsilon_\bullet \to W_\bullet$ to be the cover-restriction of ϕ' over $\mathcal{U}^\epsilon \subseteq \mathcal{U}'$. We claim that the θ-valence of ϕ^ϵ is less than the θ-valence of ϕ'. More precisely, w_θ is contained in $\phi'(U'_\rho)$, but not in $\phi^\epsilon(U^\epsilon_\rho)$. To see this, assume for a contradiction that $\phi^\epsilon(x) = w_\theta$ for some $x \in U^\epsilon_\rho$. By definition of U_β, we see that such x must

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be contained in U_β, and consequently $\epsilon(x) = 0$. Moreover x satisfies each of the inequalities defining $U^\epsilon_{\rho(i)}$ for $i = 0,\ldots,r$. Adding all these inequalities we get

$$\sum_{j=0}^{r}(r\cdot\psi_{\rho(j)}(x)) + r\cdot\epsilon(x) > \sum_{j=0}^{r}\sum_{i\neq j}\psi_{\rho(i)}(x),$$

which contradicts $\epsilon(x) = 0$.

Addressing the last statement of the lemma, we consider a subset $A \subseteq X$ such that the restriction of ϕ to $N_\bullet \cap A$ already lifts through $q\colon Z_\bullet \hookrightarrow W_\bullet$. Let θ, ρ, and U_β be defined as above. Notice that $A \cap U_\beta$ must be empty, otherwise w_θ would be in the image of $\phi|_A$. Consequently, the restriction of $\mathcal{U}' = \mathcal{U} \cup \{U_\beta\}$ to A equals $\mathcal{U} \cap A$, and similarly $\phi|_A = \phi'|_A$. It follows that $\phi'|_A$ lifts through $Z_\bullet$. Since $\phi^\epsilon|_A$ is a cover-restriction of $\phi'|_A$, the restriction of ϕ^ϵ to A also lifts through $Z_\bullet$, and we are done. □

Corollary 6.28 *Let X be a space and A a subspace such that (X,A) has the homotopy type of a CW pair. Let $h\colon \Lambda^{n,k}_\bullet \to Z_\bullet$ be a horn in a simplicial space $Z_\bullet$, and define $q\colon Z_\bullet \hookrightarrow W_\bullet$ by filling this horn, i.e. let $W_\bullet$ be the pushout of $\Delta^n_\bullet \hookleftarrow \Lambda^{n,k}_\bullet \xrightarrow{h} Z_\bullet$. Let $\mathcal{E}_A$ be any $Z_\bullet$-bundle over A. Then*

$$\mathrm{Con}_{Z_\bullet}(X,A;\mathcal{E}_A) \to \mathrm{Con}_{W_\bullet}(X,A;q_*\mathcal{E}_A)$$

is a bijection.

Proof: By Proposition 6.19 it is enough to show that q induces surjections

$$q_*\colon \mathrm{Con}^f_{Z_\bullet}(D^m,S^{m-1};\mathcal{E}_{S^{m-1}}) \to \mathrm{Con}^f_{W_\bullet}(D^m,S^{m-1};q_*\mathcal{E}_{S^{m-1}})$$

for arbitrary integers m and finite $Z_\bullet$-bundles $\mathcal{E}_{S^{m-1}}$ over the boundary sphere. Let $\mathcal{E}_0$ be a finite $W_\bullet$-bundle over D^m whose restriction to S^{m-1} equals $q_*\mathcal{E}_{S^{m-1}}$. Let $\mathcal{U}_0$ denote the corresponding finite and totally ordered open cover, and let $\phi_0\colon U^0_\bullet \to W_\bullet$ be the simplicial map associated to $\mathcal{E}_0$. By Proposition 6.27 there exists a zigzag chain of finite and totally ordered open covers

$$\mathcal{U}_0 \subseteq \mathcal{U}'_1 \supseteq \mathcal{U}_1 \subseteq \mathcal{U}'_2 \supseteq \ldots \subseteq \mathcal{U}'_{s-1} \supseteq \mathcal{U}_{s-1} \subseteq \mathcal{U}'_s \supseteq \mathcal{U}_s$$

together with simplicial maps ϕ_i and ϕ'_i from the various ordered Čech complexes into $W_\bullet$ such that each ϕ_i is equal to the cover-restriction of both ϕ'_{i-1} and ϕ'_i, and the last simplicial map ϕ_s lifts through $q\colon Z_\bullet \to W_\bullet$. Moreover, the restriction to A of each ϕ'_i and ϕ_i lifts to $Z_\bullet$.

We will now build a concordance $\mathcal{E}_{D^m\times I}$ between $W_\bullet$-bundles such that $i_0^*\mathcal{E}_{D^m\times I} = \mathcal{E}_0$ while the restrictions of $\mathcal{E}_{D^m\times I}$ to $D^m \times \{1\}$ and $S^{m-1}\times I$ lifts

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to $Z_\bullet$-bundles. Let $\mathcal{I}'_i$ be the indexing set of $\mathcal{U}'_i$. The elements of $\mathcal{U}_i$ and $\mathcal{U}'_i$ will be denoted by $U_{i,\alpha}$ and $U'_{i,\alpha}$ respectively, and we extend the notation by letting $U_{i,\alpha} = \emptyset$ and $U'_{i,\alpha} = \emptyset$ whenever the index α lies outside the respective indexing sets. Now define a cover $\mathcal{U}$ of $X \times I$ indexed by the union $\mathcal{I} = \bigcup_{i=1}^{s} \mathcal{I}'_i$ by defining the open set U_α to be the union

$$\left(\bigcup_{i=0}^{s} U_{i,\alpha} \times \left\{\frac{i}{s}\right\}\right) \cup \left(\bigcup_{i=1}^{s} U'_{i,\alpha} \times \left(\frac{i-1}{s}, \frac{i}{s}\right)\right).$$

We define a simplicial map ϕ from the ordered Čech complex of $\mathcal{U}$ into $W_\bullet$ by defining each restriction $\phi|_{X\times\{\frac{i}{s}\}}$ to be equal to ϕ_i whereas each restriction $\phi|_{X\times\{t\}}$ is defined equal to ϕ'_i for all t between $\frac{i-1}{s}$ and $\frac{i}{s}$. Let $\mathcal{E}_{D^m\times I}$ be the $W_\bullet$-bundle corresponding to $\phi\colon U_\bullet \to W_\bullet$.

Recall that $D^m \times I/S^{m-1}$ denotes the quotient space of $D^m \times I$ where we have identified $(\mathbf{x},t_0)$ with $(\mathbf{x},t_1)$ for $\mathbf{x} \in S^{m-1}$ and any two $t_0,t_1 \in I$. Let $H\colon D^m \times I/S^{m-1} \to D^m \times I$ be a map with $H(\mathbf{x},1) \in S^{m-1} \times I \cup D^m \times \{1\}$ and $H(\mathbf{x},0) = (\mathbf{x},0)$ for all $\mathbf{x} \in D^m$. Then $H^*\mathcal{E}_{D^m\times I}$ gives a concordance relative to $q_*\mathcal{E}_{S^{m-1}}$ between the $W_\bullet$-bundle $\mathcal{E}_0$ and another $W_\bullet$-bundle $\mathcal{E}'$ such that $\mathcal{E}' = q_*\mathcal{E}$ for some $Z_\bullet$-bundle $\mathcal{E}$ with $\mathcal{E}|_{S^{m-1}} = \mathcal{E}_{S^{m-1}}$. Thus we have shown that the map $\mathrm{Con}^f_{Z_\bullet}(D^m,S^{m-1};\mathcal{E}_{S^{m-1}}) \to \mathrm{Con}^f_{W_\bullet}(D^m,S^{m-1};q_*\mathcal{E}_{S^{m-1}})$ is surjective. □

We will now show how Quillen's small object argument, see for example [12, Proposition 7.17], can be use to replace an arbitrary simplicial space $Z_\bullet$ with a simplicial space $\tilde{Z}_\bullet$ satisfying the topological Kan condition, Definition 4.1. First observe that $\tilde{Z}_\bullet$ satisfies the topological Kan condition if and only if all horn filling problems of the forms

$$\begin{array}{ccc} \Lambda^{n,k}_\bullet & \longrightarrow & \tilde{Z}_\bullet \\ \downarrow & \nearrow & \\ \Delta^n_\bullet & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} D^l \times I \times \Lambda^{n,k}_\bullet \cup D^l \times \{0\} \times \Delta^n_\bullet & \longrightarrow & \tilde{Z}_\bullet \\ \downarrow & \nearrow & \\ D^l \times I \times \Delta^n_\bullet & & \end{array}$$

can be solved. Therefore we define J to be the union of the sets of simplicial maps

$$\{\Lambda^{n,k}_\bullet \hookrightarrow \Delta^n_\bullet\} \cup \{D^l \times I \times \Lambda^{n,k}_\bullet \cup D^l \times \{0\} \times \Delta^n_\bullet \hookrightarrow D^l \times I \times \Delta^n_\bullet\}$$

where the integers k, l and n vary freely.

A *relative J-cell complex* is a map $q\colon Z_\bullet \to W_\bullet$ that can be constructed as a transfinite composition of pushouts of elements of J, see [17, Definition 10.5.8]. A *presentation* of a relative J-cell complex $q\colon Z_\bullet \to W_\bullet$ is a particular choice of how to construct $W_\bullet$ from $Z_\bullet$, see [17, Section 10.6]. Moreover, a *subcomplex*

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of a presented relative J-cell complex $q\colon Z_\bullet \to W_\bullet$ is a presented relative J-cell complex $q'\colon Z_\bullet \to W'_\bullet$ together with a map $W'_\bullet \to W_\bullet$ sending each J-cell of $W'_\bullet$ to a J-cell of $W_\bullet$. A relative J-cell complex is *finite* if it can be constructed by attaching only finitely many J-cells, i.e. if the transfinite composition is actually finite.

Lemma 6.29 *Let $C_\bullet$ be a presimplicial space such that each C_k is compact and there exists an integer K such that $C_k = \emptyset$ for all $k > K$. Let $q\colon Z_\bullet \to W_\bullet$ be a relative J-cell complex. Then for any presimplicial map $f\colon C_\bullet \to W_\bullet$ there is a finite subcomplex $q'\colon Z_\bullet \to W'_\bullet$ such that f factors as a presimplicial map $f'\colon C_\bullet \to W'_\bullet$ followed by the subcomplex inclusion $W'_\bullet \to W_\bullet$.*

Proof: Fix a presentation of $q\colon Z_\bullet \to W_\bullet$, and observe that each $q_k\colon Z_k \to W_k$ is a relative cell complex of topological spaces. By [17, Proposition 10.8.7] it follows that $f_k\colon C_k \to W_k$ intersects the interiors of only finitely many cells. Using that C_k is non-empty in only finitely many simplicial degrees, we see that the image of f intersects the interiors of only finitely many J-cells of the given presentation of $Z_\bullet \to W_\bullet$.

To finish the proof it is enough to show that each J-cell of $W_\bullet$ is contained in a finite subcomplex. The argument uses a transfinite induction over the given presentation of the relative J-cell complex. Observe that the non-degenerate part of the domain of a J-cell satisfies the same assumptions as the presimplicial space $C_\bullet$. It follows from the first part of this proof that any attaching map of a J-cell intersects the interiors of only finitely many J-cells. By the induction hypothesis it follows that the attaching map of a J-cell has image contained in a finite subcomplex, and this completes the proof. □

Proposition 6.30 *Let X be a space and A a subspace such that (X, A) has the homotopy type of a CW pair. Let $q\colon Z_\bullet \to W_\bullet$ be a relative J-cell complex. Then for any $Z_\bullet$-bundle $\mathcal{E}_A$ over A the induced map*

$$\mathrm{Con}_{Z_\bullet}(X, Z; \mathcal{E}_A) \to \mathrm{Con}_{W_\bullet}(X, A; q_*\mathcal{E}_A)$$

is a bijection.

Proof: Using Proposition 6.19 we reduce the problem to that of showing that q induces surjections

$$q_*\colon \mathrm{Con}^f_{Z_\bullet}(D^m, S^{m-1}; \mathcal{E}_{S^{m-1}}) \to \mathrm{Con}^f_{W_\bullet}(D^m, S^{m-1}; q_*\mathcal{E}_{S^{m-1}})$$

for any m and finite $Z_\bullet$-bundle $\mathcal{E}_{S^{m-1}}$ over the boundary sphere. Let $\mathcal{E}_0$ be a $W_\bullet$-bundle that equals $q_*\mathcal{E}_{S^{m-1}}$ over S^{m-1}, and let $\mathcal{U}$ be the finite and totally ordered cover associated to $\mathcal{E}_0$. Choose a partition of unity $\{\psi_\alpha\}$ subordinate to $\mathcal{U}$. Denote by $\phi\colon U_\bullet \to W_\bullet$ the simplicial map associated to $\mathcal{E}$.

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Since $\mathcal{U}$ is finite there is an $\epsilon > 0$ such that $\mathcal{U}^\epsilon$ defined as $\{\phi_\alpha^{-1}(\epsilon,1]\}$ is a cover of D^m. Between $\mathcal{U}$ and its shrinking $\mathcal{U}^\epsilon$ there is a cover $\{\phi_\alpha^{-1}[\epsilon,1]\}$ consisting of compact subsets. Associated to these three covers we have the non-degenerate part of the ordered Čech complexes and natural inclusions

$$N_\bullet^\epsilon \to C_\bullet \to N_\bullet.$$

We observe that $C_\bullet$, associated to the compact cover, satisfies the condition of Lemma 6.29. Consequently, the restriction of ϕ to $C_\bullet$ factors through some finite subcomplex $q'\colon Z_\bullet \to W'_\bullet$ of the relative J-cell complex $q\colon Z_\bullet \to W_\bullet$. Hence, the restriction of ϕ to $U_\bullet^\epsilon$ factors as a map $U_\bullet^\epsilon \to W'_\bullet$ followed by the subcomplex inclusion $W'_\bullet \to W_\bullet$. Using the technique from the proof of Corollary 6.28 we can now construct a finite $W'_\bullet$-bundle $\mathcal{E}'$ with $\mathcal{E}'|_{S^{m-1}} = q'_*\mathcal{E}_{S^{m-1}}$ such that the associated $W_\bullet$-bundle is concordant to $\mathcal{E}_0$ relative to S^{m-1}. Thus $[\mathcal{E}_0]$ is in the image of

$$\mathrm{Con}^f_{W'_\bullet}(D^m,S^{m-1};q'_*\mathcal{E}_{S^{m-1}}) \to \mathrm{Con}^f_{W_\bullet}(D^m,S^{m-1};q_*\mathcal{E}_{S^{m-1}}).$$

Since $q'\colon Z_\bullet \to W'_\bullet$ is a finite relative J-cell complex there is a finite chain of simplicial spaces

$$Z_\bullet = W_\bullet^0 \to W_\bullet^1 \to W_\bullet^2 \to \cdots \to W_\bullet^{s-1} \to W_\bullet^s = W'_\bullet$$

such that each $W_\bullet^i \to W_\bullet^{i+1}$ is the pushout of some element from J. Let q^i denote the composition $Z_\bullet \to W_\bullet^i$. We claim that for each i the induced map of concordance classes

$$\mathrm{Con}^f_{W_\bullet^i}(D^m,S^{m-1};q^i_*\mathcal{E}_{S^{m-1}}) \xrightarrow{\cong} \mathrm{Con}^f_{W_\bullet^{i+1}}(D^m,S^{m-1};q^{i+1}_*\mathcal{E}_{S^{m-1}})$$

is a bijection. There are two cases to verify. First, if $W_\bullet^{i+1}$ is the pushout of $D^l \times I \times \Delta_\bullet^n \hookleftarrow D^l \times I \times \Lambda_\bullet^{n,k} \cup D^l \times \{0\} \times \Delta_\bullet^n \to W_\bullet^i$, then $W_\bullet^i \to W_\bullet^{i+1}$ is in each simplicial degree a homotopy equivalence, whence the map above is a bijection by Theorem 6.26. Secondly, if W^{i+1} is the pushout of $\Delta_\bullet^n \hookleftarrow \Lambda_\bullet^{n,k} \to W_\bullet^i$, the map above is a bijection by Corollary 6.28.

Composing these bijections, we get a bijection

$$\mathrm{Con}^f_{Z_\bullet}(D^m,S^{m-1};\mathcal{E}_{S^{m-1}}) \xrightarrow{\cong} \mathrm{Con}^f_{W'_\bullet}(D^m,S^{m-1};q'_*\mathcal{E}_{S^{m-1}}).$$

Thus there is a $Z_\bullet$-bundle $\mathcal{E}$ whose restriction to S^{m-1} equals $\mathcal{E}_{S^{m-1}}$ and such that $[q_*\mathcal{E}] = [\mathcal{E}_0]$. This finish the proof. □

Lemma 6.31 *If $q\colon Z_\bullet \to W_\bullet$ is a relative J-cell complex, then its geometric realization $|Z_\bullet| \to |W_\bullet|$ is a weak equivalence.*

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Proof: Pushouts commute with geometric realization, and the geometric realization of any element of J is both a relative CW complex and a weak equivalence. The conclusion follows. □

By Quillen's small object argument we now get:

Theorem 6.32 *Let $Z_\bullet$ be a simplicial space, let X be a space, and let A be a subspace such that (X, A) has the homotopy type of a CW pair. Then there exists a simplicial space $\tilde{Z}_\bullet$ together with a simplicial map $q\colon Z_\bullet \to \tilde{Z}_\bullet$ such that*

i) $\tilde{Z}_\bullet$ satisfies the topological Kan condition,

ii) for any $Z_\bullet$-bundle $\mathcal{E}_A$ over A the induced map

$$\mathrm{Con}_{Z_\bullet}(X, A; \mathcal{E}_A) \xrightarrow{q_*} \mathrm{Con}_{\tilde{Z}_\bullet}(X, A; q_*\mathcal{E}_A)$$

is a bijection, and

iii) the geometric realization $|q|\colon |Z_\bullet| \to |\tilde{Z}_\bullet|$ is a weak equivalence.

Moreover, if $Z_\bullet$ is a good simplicial space, then $\tilde{Z}$ can also be taken to be a good simplicial space.

Proof: The set J of simplicial maps permits the small object argument. By the small object argument, see for example [17, Proposition 10.5.16], it is possible to construct a simplicial map $q\colon Z_\bullet \to \tilde{Z}_\bullet$ such that q is a relative J-cell complex and $\tilde{Z}_\bullet \to *$ has the right lifting property with respect to any element of J. The last statement immediately implies that $\tilde{Z}_\bullet$ satisfies the topological Kan condition. By Lemma 6.31 the geometric realization $|q|$ is a weak equivalence. Property iii) is implied by Proposition 6.30. □

Remark 6.33 It is natural to ask for a model structure on simplicial spaces such that the weak equivalences are the maps $Z_\bullet \to W_\bullet$ that induce bijections $q\colon \mathrm{Con}_{Z_\bullet}(D^m, S^{m-1}; \mathcal{E}_{S^{m-1}}) \to \mathrm{Con}_{W_\bullet}(D^m, S^{m-1}; q_*\mathcal{E}_{S^{m-1}})$ for all m and $\mathcal{E}_{S^{m-1}}$, and fibrant objects are the simplicial spaces satisfying the topological Kan condition. Such a model structure would not be identical to the realization model structure of [27]. We would like more fibrant objects, compare [27, Lemma 8.10]. However, answering this question is beyond the scope of the present paper.

7. Proof of Theorem 1.1

We want to relate concordance classes of $Z_\bullet$-bundles to homotopy classes of maps into the geometric realization $|Z_\bullet|$. Intuitively, a $Z_\bullet$-bundle over X is a simplicial map $\phi\colon U_\bullet \to Z_\bullet$, and realizing gives a map $|U_\bullet| \to |Z_\bullet|$. As explained in [10]

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there is a natural weak equivalence $X \xleftarrow{\simeq} |U_\bullet|$ for any ordered open cover $\mathcal{U}$. Whenever X has the homotopy type of a CW-complex each $Z_\bullet$-bundle yields a well-defined homotopy class of maps from X into $|Z_\bullet|$. Actually, we will not use Dugger and Isaksen's weak equivalence, but rather we compare homotopy classes and concordance classes by more direct means.

Given a topological space Y there are several ways to functorially associate a simplicial space. We have the *constant simplicial space* $C_\bullet(Y)$ defined in each simplicial degree as $C_k(Y) = Y$ and with all face and degeneracy maps equal to the identity map on Y. On the other hand we have the *continuous singular simplicial space* $S_\bullet(Y)$ whose space of k-simplices is the mapping space $\mathcal{T}op(\Delta^k, Y)$. Observe that $C_\bullet(-)$ is the left adjoint to the space of 0-simplices functor $Z_\bullet \mapsto Z_0$ and that $S_\bullet(-)$ is the right adjoint to geometric realization of simplicial spaces. Since we have a homeomorphism $Y \xrightarrow{\cong} S_0(Y)$ there is a natural map $C_\bullet(Y) \to S_\bullet(Y)$. Inspecting the definitions, we see that this natural map assigns to a point y in $C_k(Y) = Y$ the constant map sending all of Δ^k to $y \in Y$. We may also consider the discrete topology on the set of maps $\Delta^k \to Y$. This defines the *singular simplicial set* $\mathrm{Sing}_\bullet Y$. There is a natural simplicial map $\kappa\colon \mathrm{Sing}_\bullet Y \to S_\bullet(Y)$ given as the identity on the underlying sets.

Lemma 7.1 *The simplicial spaces $C_\bullet(Y)$, $S_\bullet(Y)$, and $\mathrm{Sing}_\bullet Y$ are good, the geometric realization $|C_\bullet(Y)|$ is naturally homeomorphic to Y, and the natural maps induces weak equivalences*

$$Y \cong |C_\bullet(Y)| \xrightarrow{\simeq} |S_\bullet(Y)| \xleftarrow{\simeq} |\mathrm{Sing}_\bullet Y|.$$

Proof: Clearly $C_\bullet(Y)$ is a good simplicial space, i.e. all degeneracies are closed cofibrations, and we see that the geometric realization of $C_\bullet(Y)$ is naturally homeomorphic to Y. Furthermore, it is not difficult to show explicitly that the map $\mathcal{T}op(\Delta^{k-1}, Y) \to \mathcal{T}op(\Delta^k, Y)$, induced by $\sigma_i\colon \Delta^k \to \Delta^{k-1}$, is the inclusion of a deformation retract. Consequently, $S_\bullet(Y)$ is also good. The singular simplicial set is in each simplicial degree discrete, hence $\mathrm{Sing}_\bullet Y$ is good.

Let $q\colon Z_\bullet \to W_\bullet$ be a simplicial map between good simplicial spaces. It is a classical result that the geometric realization $|q|$ is a weak equivalence if each $q_k\colon Z_k \to W_k$ is a weak equivalence, see for example [22, Section 2.2]. This applies to the simplicial map $C_\bullet(Y) \to S_\bullet(Y)$, i.e. we have a weak equivalence

$$Y \cong |C_\bullet(Y)| \xrightarrow{\simeq} |S_\bullet(Y)|.$$

It is well-known that the counit of the adjunction between $|-|$ and $\mathrm{Sing}_\bullet$ is a natural weak equivalence $|\mathrm{Sing}_\bullet Y| \xrightarrow{\simeq} Y$. Now observe that the counit of the

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adjunction between geometric realization and the functor $S_\bullet(-)$ is a retraction to the weak equivalence $|C_\bullet(Y)| \xrightarrow{\simeq} |S_\bullet(Y)|$. The result now follows by commutativity of the diagram

$$\begin{array}{ccc} |S_\bullet(Y)| & \xleftarrow{\kappa} & |\mathrm{Sing}_\bullet Y| \\ & {\scriptstyle\simeq}\searrow \quad \swarrow{\scriptstyle\simeq} & \\ & Y. & \end{array}$$

□

Let A be a subspace of X, and fix a continuous map $g\colon A \to Y$. Denote by $[X,A;Y,g]$ the set of maps $f\colon X \to Y$ whose restriction to A equals the fixed map f modulo homotopy relative to A, i.e. two such maps f_0 and f_1 represent the same class if there is a homotopy between through maps equal to g on A.

Let $\phi\colon U_\bullet \to C_\bullet(Y)$ be the simplicial map of a $C_\bullet(Y)$-bundle $\mathcal{E}$ over X subordinate to some ordered open cover $\mathcal{U}$. At the level of 0-simplices, ϕ maps each open set $U_\alpha \in \mathcal{U}$ into Y. Moreover, these maps agree on each intersection $U_\alpha \cap U_\beta$. Hence ϕ comes from a continuous map $X \to Y$, and we call this map the *underlying map* of the $C_\bullet(Y)$-bundle. We now have the following important, but easy observation:

Proposition 7.2 *Let X and Y be topological spaces, and let A be a subspace of X. Let $g\colon A \to Y$ be the underlying map of a $C_\bullet(Y)$-bundle $\mathcal{E}_A$ over A. Then the underlying map of $C_\bullet(Y)$-bundles defines a natural bijection*

$$\mathrm{Con}_{C_\bullet(Y)}(X,A;\mathcal{E}_A) \xrightarrow{\cong} [X,A;Y,g].$$

Proof: We compare the notion of a map $X \to Y$ to the notion of a $C_\bullet(Y)$-bundle over X. The only additional piece of data contained in the latter definition is the choice of an ordered open cover $\mathcal{U}$ over X. Clearly any two ways of making this choice is equivalent up to concordance. The result follows. □

Let us compare $C_\bullet(Y)$-bundles and $S_\bullet(Y)$-bundles.

Proposition 7.3 *Let $A \hookrightarrow X$ be a closed cofibration such that (X,A) has the homotopy type of a CW pair. Let Y be a topological space, and let $\mathcal{E}_A$ be any $C_\bullet(Y)$-bundle over A. Then the natural map $i : C_\bullet(Y) \to S_\bullet(Y)$ induces a bijection*

$$\mathrm{Con}_{C_\bullet(Y)}(X,A;\mathcal{E}_A) \xrightarrow{\cong} \mathrm{Con}_{S_\bullet(Y)}(X,A;i_*\mathcal{E}_A).$$

Proof: This follows immediately from Theorem 6.26 since the simplicial map i in each simplicial degree k is a homotopy equivalence $Y = C_k(Y) \to S_k(Y) = \mathcal{T}op(\Delta^k, Y)$. □

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Let us also compare $S_\bullet(-)$-bundles for weakly equivalent spaces Y and Y'.

Corollary 7.4 *Let $A \hookrightarrow X$ be a closed cofibration such that A has the homotopy type of a CW complex and (X, A) has the homotopy type of a CW pair. Let $f : Y \to Y'$ be a weak equivalence, and let $\mathcal{E}_A$ be any $S_\bullet(Y)$-bundle over A. Then f induces a bijection*

$$\mathrm{Con}_{S_\bullet(Y)}(X, A; \mathcal{E}_A) \xrightarrow{\cong} \mathrm{Con}_{S_\bullet(Y')}(X, A; f_*\mathcal{E}_A).$$

Proof: By Proposition 7.3 there is a $C_\bullet(Y)$-bundle $\mathcal{E}'_A$ over A and a concordance $\mathcal{E}_{A\times I}$ between the $S_\bullet(Y)$-bundles $i_*\mathcal{E}'_A$ and $\mathcal{E}_A$. By Lemma 6.16 and Proposition 6.17 gluing of $\mathcal{E}_{A\times I}$ gives bijections. Hence, we have a diagram

$$\begin{array}{ccccc}
\mathrm{Con}_{C_\bullet(Y)}(X, A; \mathcal{E}'_A) & \xrightarrow{②} & \mathrm{Con}_{S_\bullet(Y)}(X, A; i_*\mathcal{E}'_A) & \xrightarrow{④} & \mathrm{Con}_{S_\bullet(Y)}(X, A; \mathcal{E}_A) \\
\downarrow{\scriptstyle ①} & & \downarrow & & \downarrow{\scriptstyle ⑥} \\
\mathrm{Con}_{C_\bullet(Y')}(X, A; f_*\mathcal{E}'_A) & \xrightarrow{③} & \mathrm{Con}_{S_\bullet(Y')}(X, A; f_*i_*\mathcal{E}'_A) & \xrightarrow{⑤} & \mathrm{Con}_{S_\bullet(Y')}(X, A; f_*\mathcal{E}_A)
\end{array}$$

where ④ and ⑤ are bijections. Furthermore, ① is a bijection by Proposition 7.2, and ② and ③ are bijections by Proposition 7.3. It follows that ⑥ is a bijection. □

Let $Z_\bullet$ be a simplicial space. Since the diagram $C_\bullet(-) \to S_\bullet(-) \leftarrow \mathrm{Sing}_\bullet(-)$ is natural, we can insert $Z_\bullet$ and produce a diagram of bisimplicial spaces

$$C_\bullet(Z_\bullet) \to S_\bullet(Z_\bullet) \leftarrow \mathrm{Sing}_\bullet(Z_\bullet).$$

Applying the diagonal we recover from $C_\bullet(Z_\bullet)$ the simplicial space $Z_\bullet$. Define the functors T and D by sending $Z_\bullet$ to $\mathrm{diag}S_\bullet(Z_\bullet)$ and $\mathrm{diag}\mathrm{Sing}_\bullet(Z_\bullet)$ respectively. The underlying sets of $D(Z_\bullet)$ and $T(Z_\bullet)$ are identical. In both cases a k-simplex is represented by a continuous map $\Delta^k \to Z_k$. The difference is the topology we place on this set. In the case of $D(Z_\bullet)$ we take the discrete topology. In the case of $T(Z_\bullet)$ we take the compact-open topology on each of the mapping spaces. Let $\kappa : D(Z_\bullet) \to T(Z_\bullet)$ denote the natural map which is the identity on the underlying sets.

Lemma 7.5 *Assume that $Z_\bullet$ is a good simplicial space. The geometric realization of $Z_\bullet \to T(Z_\bullet) \leftarrow D(Z_\bullet)$ gives a diagram of natural weak equivalences*

$$|Z_\bullet| \xrightarrow{\simeq} |T(Z_\bullet)| \xleftarrow{\simeq} |D(Z_\bullet)|.$$

Proof: Suppose that $W_{\bullet,\bullet}$ is a bisimplicial space. We can take the geometric realization in two steps; first by defining a simplicial space $[k] \mapsto |W_{k,\bullet}|$ and then realizing again to get $\big|[k] \mapsto |W_{k,\bullet}|\big|$, or we can take the geometric realization of

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the diagonal, $|\mathrm{diag}\, W_{\bullet,\bullet}|$. It is a classical result that these two construction produce naturally homeomorphic spaces.

By Lemma 7.1 we get weak equivalences

$$Z_k \xrightarrow{\simeq} |S_\bullet(Z_k)| \xleftarrow{\simeq} |\mathrm{Sing}_\bullet Z_k| \qquad \text{for each } k.$$

The simplicial space $Z_\bullet$ is good by assumption, and it is not hard to see that $[k] \mapsto |\mathrm{Sing}_\bullet Z_k|$ is good. Observing that each degeneracy $s_i\colon Z_{k-1} \to Z_k$ is a closed cofibration, we use [33, Lemma 4] to conclude that the induced map $\mathcal{T}op(\Delta^l, Z_{k-1}) \to \mathcal{T}op(\Delta^l, Z_k)$ is also a closed cofibration. It follows that $[k] \mapsto |S_\bullet(Z_k)|$ is a good simplicial space.

The statement of the lemma follows since $|Z_\bullet| \to |T(Z_\bullet)| \leftarrow |D(Z_\bullet)|$ is homeomorphic to the geometric realization of simplicial maps which are weak equivalences in each simplicial degree. □

Given a simplicial space $Z_\bullet$, let $\eta\colon Z_\bullet \to S_\bullet(|Z_\bullet|)$ denote the unit of the adjunction between the geometric realization and the continuous singular simplicial space. We use η to relate $Z_\bullet$-bundles to homotopy classes of maps into $|Z_\bullet|$. To be precise, if $\mathcal{E}$ is a $Z_\bullet$-bundle over a CW-complex X represented by a simplicial map $U_\bullet \xrightarrow{\phi} Z_\bullet$, then composing with η yields $U_\bullet \to S_\bullet(|Z_\bullet|)$, and this simplicial map corresponds to a well-defined homotopy class of maps $X \to |Z_\bullet|$ by Proposition 7.2 and Proposition 7.3. Our aim is to prove that η, under reasonable assumptions, induces a bijection between concordance classes. To handle a technical point in this proof we need:

Theorem 7.6 (Whitney) *Every open subset of S^{n-1} is a CW-complex.*

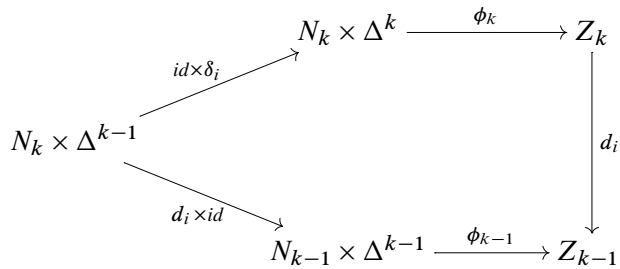
Proof: Every proper subset U of S^{n-1} embeds as an open subset of $\mathbb{R}^{n-1}$. Consider meshes $\mathscr{M}_s$ in $\mathbb{R}^{n-1}$ of cubes with side length 2^{-s}, such that $\mathscr{M}_{s+1}$ is a subdivision of $\mathscr{M}_s$. Following [36, Section 8], we can write U as a union $\bigcup_s K_s$ such that K_s consists of cubes in $\mathscr{M}_s$, the interiors of the K_s's are mutually disjoint, and K_s meets only K_{s-1} and K_{s+1}. This is a CW-structure on U. □

Lemma 7.7 *Assume that $Z_\bullet$ satisfies the topological Kan condition and $\mathcal{U}$ is a good ordered open cover of S^{n-1}. For every simplicial map $\phi\colon U_\bullet \to T(Z_\bullet)$ there exists a simplicial map $\Phi\colon U_\bullet \times I \to T(Z_\bullet)$ such that $\phi = \Phi(-,1)$ and such that $\Phi(-,0)$ factors as $U_\bullet \to D(Z_\bullet) \xrightarrow{\kappa} T(Z_\bullet)$.*

Proof: Let $N_\bullet$ be the non-degenerate part of the ordered Čech complex $U_\bullet$ associated to the good ordered open cover $\mathcal{U}$. It is enough to define Φ as a presimplicial map $N_\bullet \times I \to T(Z_\bullet)$. Inspecting the definition of $T(Z_\bullet)$, we see that maps $N_\bullet \to T(Z_\bullet)$ can be described by continuous maps $\phi_k\colon N_k \times \Delta^k \to Z_k$ such

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that the following *presimplicial coherence diagram* commutes for all face maps d_i:



Observe that a map $N_\bullet \to T(Z_\bullet)$ factors through $\kappa\colon D(Z_\bullet) \to T(Z_\bullet)$ if and only if for each k and $\mathbf{t} \in \Delta^k$ the function sending $x \in N_k$ to $\phi_k(x,\mathbf{t})$ is a locally constant on N_k.

By definition the space of non-degenerate k-simplices is $N_k = \coprod_{\alpha_0 < \cdots < \alpha_k} U_{\alpha_0 \cdots \alpha_k}$. Since $\mathcal{U}$ is a good cover, each connected component of N_k is contractible. Hence, there exist local contractions $H^k\colon N_k \times I \to N_k$, i.e. homotopies such that for $x \in N_k$ we have $H^k(x,1) = x$, and $H^k(x,0)$ locally constant.

Let $\phi_k\colon N_k \times \Delta^k \to Z_k$ be the collection of maps that represents the given simplicial map $\phi\colon U_\bullet \to T(Z_\bullet)$. By induction on k we will construct maps $\Phi_k\colon N_k \times I \times \Delta^k \to Z_k$ such that

i) $\Phi_k(x,1,\mathbf{t}) = \phi_k(x,\mathbf{t})$ for all $x \in N_k$ and $\mathbf{t} \in \Delta^k$,

ii) for $\mathbf{t} \in \Delta^k$ fixed, the expression $\Phi_k(x,s,\mathbf{t})$ is locally constant as a function $N_k \times \left[0, \frac{1}{k+2}\right] \to Z_k$, and

iii) all presimplicial coherence diagrams for Φ commute.

We begin by defining Φ_0. For $x \in N_0$, $s \in I$ and $1 \in \Delta^0$ we define

$$\Phi_0(x,s,1) = \begin{cases} \phi_0(H^0(x,2s-1),1) & \text{if } s \geq \frac{1}{2} \\ \phi_0(H^0(x,0),1) & \text{if } s \leq \frac{1}{2}. \end{cases}$$

Next, we construct Φ_k from Φ_i, $i < k$. This is done in four steps:

First step. In this step we will extend $\phi_k\colon N_k \times \{1\} \times \Delta^k \to Z_k$ to a map

$$\Phi_k\colon \left(N_k \times \left[\frac{1}{k+1}, 1\right] \times \partial\Delta^k\right) \cup \left(N_k \times \{1\} \times \Delta^k\right) \to Z_k.$$

We will do this by induction over the dimension of proper faces σ of Δ^k. Given a face σ, let I be the subset of $[k]$ such that $n \in I$ if and only if σ is contained in the

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n'th $(k-1)$-face of Δ^k. We can, by the induction hypothesis, construct a diagram

$$\begin{array}{ccc} N_k \times \left(\left[\frac{1}{k+1},1\right] \times \partial\sigma \cup \{1\} \times \sigma\right) & \longrightarrow & Z_k \\ \downarrow & & \downarrow c_k^I \\ N_k \times \left[\frac{1}{k+1},1\right] \times \sigma & \longrightarrow & C\Lambda_k^I(Z_\bullet), \end{array}$$

where the lower map comes from the Φ_i, $i < k$. The vertical map on the left side is a cellular inclusion between CW-complexes, and a homotopy equivalence. The topological Kan condition says that c_k^I is a Serre fibration. Hence, a lift exists, and we use this to define Φ_k on $N_k \times \left[\frac{1}{k+1},1\right] \times \sigma$.

Second step. Let

$$r\colon \left[\frac{1}{k+1},1\right] \times \Delta^k \to \left(\left[\frac{1}{k+1},1\right] \times \partial\Delta^k\right) \cup \left(\{1\} \times \Delta^k\right)$$

be a retraction. For $x \in N_k$, $\frac{1}{k+1} \le s \le 1$, $\mathbf{t} \in \Delta^k$, we define

$$\Phi_k(x,s,\mathbf{t}) = \Phi_k(r(x,s,\mathbf{t})),$$

where the Φ_k on the right side is defined by the previous step.

Third step. Recall that H^k denoted a local contraction of N_k. Choose a function $f\colon \left[\frac{1}{k+2},\frac{1}{k+1}\right] \to I$ such that $f(\frac{1}{k+2}) = 0$ and $f(\frac{1}{k+1}) = 1$. For $x \in N_k$, $\frac{1}{k+2} \le s \le 1$, $\mathbf{t} \in \Delta^k$, we now define

$$\Phi_k(x,s,\mathbf{t}) = \begin{cases} \Phi_k(x,s,\mathbf{t}) & \text{if } \frac{1}{k+1} \le s, \\ \Phi_k(H^k(x,f(s)),\frac{1}{k+1},\mathbf{t}) & \text{if } \frac{1}{k+2} \le s \le \frac{1}{k+1}. \end{cases}$$

Again, the Φ_k on the right comes from the previous step. At this stage, we should verify that the presimplicial coherence diagrams commute for $s \in \left[\frac{1}{k+2},\frac{1}{k+1}\right]$, i.e. we need to check the equality

$$d_i\,\Phi_k(x,s,\delta_i\mathbf{t}) = \Phi_{k-1}(d_i x,s,\mathbf{t})$$

for all face maps d_i, $x \in N_k$, s as above and $t \in \Delta^{k-1}$. This is an easy exercise using the fact that the equality holds for $s = \frac{1}{k+1}$, and that when fixing $\mathbf{t}$ the function $\Phi_{k-1}(y,s,\mathbf{t})$ is locally constant on $N_{k-1} \times \left[0,\frac{1}{k+1}\right]$.

Final step. The steps above define Φ_k on $N_k \times \left[\frac{1}{k+2},1\right] \times \Delta^k$. Now extend the domain to all of $N_k \times I \times \Delta^k$ by

$$\Phi_k(x,s,\mathbf{t}) = \Phi_k(x,\frac{1}{k+2},\mathbf{t}), \quad \text{for } s \le \frac{1}{k+2}.$$

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This completes the construction of all Φ_ks.

Since $\Phi_k(x,0,\mathbf{t})$ is locally constant for fixed $\mathbf{t} \in \Delta^k$, it follows that $\Phi(-,0)\colon N_\bullet \to T(Z_\bullet)$ factors through $D(Z_\bullet)$. This completes the proof of the lemma. □

Lemma 7.8 *Let $Z_\bullet$ be a good simplicial space satisfying the topological Kan condition. Let $\mathcal{U}$ be a good ordered open cover of D^n such that the restriction to S^{n-1} also is good. Let $U_\bullet$ and $U_\bullet^\partial$ denote the ordered Čech complexes associated to $\mathcal{U}$ and its restriction to S^{n-1} respectively. Suppose that the pair $(|U_\bullet|,|U_\bullet^\partial|)$ is a relative CW-complex. Let ϕ^∂ be a simplicial map $U_\bullet^\partial \to D(Z_\bullet)$, and let $\tilde{\phi}$ be a continuous map $|U_\bullet| \to |T(Z_\bullet)|$ such that the diagram*

$$\begin{array}{ccc} U_\bullet^\partial & \xrightarrow{\phi^\partial} & D(Z_\bullet) \\ \downarrow & & \downarrow{\scriptstyle \eta\kappa} \\ U_\bullet & \xrightarrow{\phi} & S_\bullet(|T(Z_\bullet)|) \end{array}$$

commutes. Here ϕ is the adjoint of $\tilde{\phi}$. Then there exists a simplicial map $\phi'\colon U_\bullet \to D(Z_\bullet)$ extending ϕ^∂ such that the composition $|U_\bullet| \xrightarrow{|\phi'|} |D(Z_\bullet)| \xrightarrow{|\kappa|} |T(Z_\bullet)|$ is homotopic to $\tilde{\phi}$ relative to $|U_\bullet^\partial|$.

Proof: Since $|\kappa|\colon |D(Z_\bullet)| \to |T(Z_\bullet)|$ is a weak equivalence it follows by classical techniques, see [30, Theorem 7.6.22], that there exists a map $\tilde{\phi_1}\colon |U_\bullet| \to |D(Z_\bullet)|$ such that the composition

$$|U_\bullet| \xrightarrow{\tilde{\phi_1}} |D(Z_\bullet)| \xrightarrow{|\kappa|} |T(Z_\bullet)|$$

is homotopic to $\tilde{\phi}$ relative to $|U_\bullet^\partial|$.

A key idea in what follows is to replace the Čech complexes $U_\bullet$ and $U_\bullet^\partial$ by their simplicial sets of path components $V_\bullet$ and $V_\bullet^\partial$. To be precise, we define the k-simplices of $V_\bullet$ to be the set of path components of U_k, i.e. we have $V_k = \pi_0 U_k$. Because $\mathcal{U}$ is good the natural quotient map $q\colon U_\bullet \to V_\bullet$ is a weak equivalence in each simplicial degree. Consequently, we get a weak equivalence $|U_\bullet| \xrightarrow{\simeq} |V_\bullet|$. Similarly for $V_\bullet^\partial$.

Inside the mapping cylinder $|U_\bullet| \times I \cup_{|q|} |V_\bullet|$ we find $|U_\bullet| \times \{0\} \cup_0 |U_\bullet^\partial| \times I \cup_{|q^\partial|} |V_\bullet^\partial|$ lying as a deformation retract. The restriction of $\tilde{\phi_1}$ to $|U_\bullet^\partial|$ factors through $|V_\bullet^\partial|$ since it is the realization of the simplicial map $\phi^\partial\colon U_\bullet^\partial \to D(Z_\bullet)$ and $D(Z_\bullet)$ is a simplicial set. It follows that $\tilde{\phi_1}$ extends to a map

$$|U_\bullet| \times \{0\} \cup_0 |U_\bullet^\partial| \times I \cup_{|q^\partial|} |V_\bullet^\partial| \to |D(Z_\bullet)|$$

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being constant in the I-coordinate. Using the deformation retraction, we extend to a map from the mapping cylinder of q,

$$|U_\bullet| \times I \cup_{|q|} |V_\bullet| \to |D(Z_\bullet)|.$$

Consequently we get a continuous map $\tilde{\phi}_2 \colon |V_\bullet| \to |D(Z_\bullet)|$ whose restriction to $|V_\bullet^\partial|$ agrees with the realization of ϕ^∂ and such that there is a homotopy relative to $|U_\bullet^\partial|$ between $\tilde{\phi}_1$ and the composition

$$|U_\bullet| \xrightarrow{|q|} |V_\bullet| \xrightarrow{\tilde{\phi}_2} |D(Z_\bullet)|.$$

The adjoint of $\tilde{\phi}_2$ fits into the following solid commutative diagram of simplicial sets,

$$\begin{array}{ccc} V_\bullet^\partial & \longrightarrow & D(Z_\bullet) \\ \downarrow & \overset{\phi_2}{\nearrow} & \downarrow \eta \\ V_\bullet & \longrightarrow & \mathrm{Sing}_\bullet |D(Z_\bullet)|. \end{array}$$

Since $Z_\bullet$ satisfies the topological Kan condition, we know that $D(Z_\bullet)$ is a Kan complex by Theorem 4.4. It follows that η is a weak equivalence between fibrant and cofibrant objects. It is well-known, see for example [18, Lemma 4.6], that under these circumstances the dotted map exists such that the upper triangle commutes, while the lower triangle commutes up to homotopy relative to $V_\bullet^\partial$. This implies that there is a simplicial map $\phi_3 \colon V_\bullet \to D(Z_\bullet)$ whose restriction to $V_\bullet^\partial$ is ϕ^∂ and such that there is a homotopy relative to $|U_\bullet^\partial|$ between $\tilde{\phi}_2$. This concludes the proof. □

Lemma 7.9 *Let $\mathcal{U}'$ be any ordered open cover of D^n. Then there exists a refinement $\mathcal{U}$ of $\mathcal{U}'$ together with a carrier function, such that $\mathcal{U}$ and its restriction to S^{n-1} both are good covers and the pair $(|U_\bullet|, |U_\bullet^\partial|)$ is a relative CW-complex. Here $U_\bullet$ and $U_\bullet^\partial$ are the ordered Čech complexes associated to $\mathcal{U}$ and the restriction $\mathcal{U}^\partial$ of $\mathcal{U}$ to S^{n-1}.*

Proof: To prove the lemma we introduce *convex fragments of spherical shells*. We call a subset V of S^{n-1} *convex* if

i) V contains no pair of antipodal points, and

ii) whenever $\mathbf{x}$ and $\mathbf{y}$ lie in V then the shortest segment of the great arc between $\mathbf{x}$ and $\mathbf{y}$ is completely contained in V.

We call a subset U of D^n a *convex fragment of a spherical shell* if there is a convex subset V of S^{n-1} and an interval $J \subseteq (0,1]$ such that

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i) the set U does not contain the center of D^n, and

ii) a point $\mathbf{x}$ in D^n is contained in U if and only if there exists $\mathbf{y} \in V$ and $r \in J$ such that $\mathbf{x} = r\mathbf{y}$.

A subset U of D^n is a ball if there exists an interval $J \subseteq [0,1]$ such that $0 \in J$ and $\mathbf{x}$ lies in U if and only if $\|x\| \in J$. Observe that the intersection between a convex fragment of a spherical shell and either a ball or another convex fragment of a spherical shell is again a convex fragment of a spherical shell.

Given the ordered open cover $\mathcal{U}'$ we let $\mathcal{U}$ be an ordered open cover such that

i) each U_α in $\mathcal{U}$ is either an open ball of radius less than 1 or a open convex fragment of a spherical shell, and

ii) each U_α in $\mathcal{U}$ is contained in some U'_β in $\mathcal{U}'$.

The existence of such a cover $\mathcal{U}$ follows from the fact that the collection of all open balls and open convex fragments of spherical shells is a basis for the topology on D^n. Choose any carrier function $c\colon \mathcal{I} \to \mathcal{I}'$ between the corresponding indexing sets.

Any non-empty finite intersection $U_{\alpha_0 \ldots \alpha_k}$ of subsets in $\mathcal{U}$ is contractible since it is either a ball or a convex fragment of a spherical shell. Consequently $\mathcal{U}$ is a good cover. Moreover, the restriction of $\mathcal{U}$ to S^{n-1} is a cover $\mathcal{U}^\partial$ consisting of open convex subsets. Hence $\mathcal{U}^\partial$ is also good. To check that $(|U_\bullet|, |U^\partial_\bullet|)$ is a relative CW complex it is enough to see that for each finite intersection the pair $(U_{\alpha_0 \ldots \alpha_k}, U_{\alpha_0 \ldots \alpha_k} \cap S^{n-1})$ is a relative CW complex. Such a pair is either an open ball relative to the empty set or an open convex fragment of a spherical shell relative to an open convex subset of S^{n-1}. In both cases an argument using the technique of Theorem 7.6 shows that we indeed have a relative CW complex. This proves the lemma. □

Proposition 7.10 *Assume that $Z_\bullet$ is good and satisfies the topological Kan condition. Let $\mathcal{E}_{S^{n-1}}$ be any $T(Z_\bullet)$-bundle over S^{n-1}. Then the map*

$$\eta_*\colon \mathrm{Con}_{T(Z_\bullet)}(D^n, S^{n-1}; \mathcal{E}_{S^{n-1}}) \to \mathrm{Con}_{S_\bullet(|T(Z_\bullet)|)}(D^n, S^{n-1}; \eta_* \mathcal{E}_{S^{n-1}})$$

is surjective.

Proof: Let $\mathcal{E}'$ be any $S_\bullet(|T(Z_\bullet)|)$-bundle with $\mathcal{E}'|_{S^{n-1}} = \eta_* \mathcal{E}_{S^{n-1}}$, and let $\mathcal{U}'$ be the ordered open cover associated to $\mathcal{E}'$. Use Lemma 7.9 to choose a refinement $\mathcal{U}$ of $\mathcal{U}'$. The carrier function $c\colon \mathcal{I} \to \mathcal{I}'$ between the corresponding indexing sets induces simplicial maps $i\colon U_\bullet \to U'_\bullet$ and $i_\partial\colon U^\partial_\bullet \to U'^\partial_\bullet$ between the ordered Čech complexes of $\mathcal{U}$ and $\mathcal{U}'$ and their restriction to S^{n-1} respectively. Using i and i_∂

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we change the associated cover and get a new $S_\bullet(|T(Z_\bullet)|)$-bundle $i^*\mathcal{E}'$ and a new $T(Z_\bullet)$-bundle $i_\partial^*\mathcal{E}_{S^{n-1}}$.

We claim that there exists a concordance of $S_\bullet(|T(Z_\bullet)|)$-bundles between $\mathcal{E}'$ and $i^*\mathcal{E}'$ whose restriction to S^{n-1} lifts to a concordance of $T(Z_\bullet)$-bundles between $\mathcal{E}_{S^{n-1}}$ and $i_\partial^*\mathcal{E}_{S^{n-1}}$. Let $\phi'\colon U'_\bullet \to S_\bullet(|T(Z_\bullet)|)$ be the simplicial map corresponding to $\mathcal{E}'$. By definition, the composition $\phi' i$ corresponds to $i^*\mathcal{E}'$. To see the existence of these concordances it is sufficient to construct a new ordered open cover $\mathcal{U}''$ together with a simplicial map $\phi''\colon U''_\bullet \to Z_\bullet$ such that $\mathcal{U}$ and $\mathcal{U}'$ are subcovers and $\phi' i$ and ϕ' are the restrictions of ϕ''. Let the indexing set $\mathcal{I}''$ of $\mathcal{U}''$ be the disjoint union $\mathcal{I} \amalg \mathcal{I}'$ and give $\mathcal{I}''$ the smallest partial ordering such that the inclusions of $\mathcal{I}$ and $\mathcal{I}'$ are order preserving and $\alpha > \beta$ whenever $c(\alpha) = \beta$. We define $\mathcal{U}'' = \mathcal{U} \cup \mathcal{U}'$. Observe that c extends to a carrier function $c''\colon \mathcal{I}'' \to \mathcal{I}'$ inducing a map $r\colon U''_\bullet \to U'_\bullet$ of the corresponding ordered Čech complexes such that r is a retraction of $U'_\bullet \to U''_\bullet$. Define $\phi'' = \phi' r$. This proves the claim.

As a consequence of this claim it follows that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Con}_{T(Z_\bullet)}(D^n, S^{n-1}; i_\partial^*\mathcal{E}_{S^{n-1}}) & \xrightarrow{②} & \mathrm{Con}_{S_\bullet(|T(Z_\bullet)|)}(D^n, S^{n-1}; \eta_* i_\partial^*\mathcal{E}_{S^{n-1}}) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Con}_{T(Z_\bullet)}(D^n, S^{n-1}; \mathcal{E}_{S^{n-1}}) & \xrightarrow{①} & \mathrm{Con}_{S_\bullet(|T(Z_\bullet)|)}(D^n, S^{n-1}; \eta_* \mathcal{E}_{S^{n-1}}), \end{array}$$

where the vertical maps are gluing of concordances over S^{n-1}. These vertical maps are bijections by Lemma 6.16 and Proposition 6.17. Moreover, the class represented by $\mathcal{E}'$ corresponds to the class represented by $i^*\mathcal{E}'$. So it is enough to prove that $i^*\mathcal{E}'$ lies in the image of ②.

Since $\mathcal{U}^\partial$ is a good cover, Lemma 7.7 applies to the simplicial map $\phi_{S^{n-1}} i_\partial\colon U^\partial_\bullet \to T(Z_\bullet)$ associated to $i_\partial^*\mathcal{E}_{S^{n-1}}$. Thus we get a $D(Z_\bullet)$-bundle $\mathcal{E}^\partial$ over S^{n-1} together with a simplicial map $\Phi\colon U^\partial_\bullet \times I \to T(Z_\bullet)$ such that $\Phi(-,1) = \phi_{S^{n-1}} i_\partial$ and $\Phi(-,0) = \kappa\phi^\partial$, where $\phi^\partial\colon U^\partial_\bullet \to D(Z_\bullet)$ is associated to $\mathcal{E}^\partial$.

Let $\tilde{\phi}'\colon |U_\bullet| \to |T(Z_\bullet)|$ denote the adjoint of the simplicial map $\phi' i\colon U_\bullet \to S_\bullet(|T(Z_\bullet)|)$ associated to $i^*\mathcal{E}'$. We see that the geometric realization of Φ matches up with $\tilde{\phi}'$ and produces a continuous map

$$|U^\partial_\bullet| \times I \cup |U_\bullet| \times \{1\} \to |T(Z_\bullet)|.$$

Since $(|U_\bullet|, |U^\partial_\bullet|)$ is a relative CW complex it follows that the inclusion $|U^\partial_\bullet| \times I \cup |U_\bullet| \times \{1\} \hookrightarrow |U_\bullet| \times I$ is a trivial cofibration, and consequently we may extend the map above to a continuous map

$$\tilde{\Phi}\colon |U_\bullet| \times I \to |T(Z_\bullet)|.$$

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The simplicial map Φ immediately gives a concordance of $T(Z_\bullet)$-bundles between $i_\partial^* \mathcal{E}_{S^{n-1}}$ and $\kappa_* \mathcal{E}^\partial$. Furthermore, the adjoint of $\tilde{\Phi}$ gives a concordance of $S_\bullet(|T(Z_\bullet)|)$-bundles from $i^*\mathcal{E}'$ to a new $S_\bullet(|T(Z_\bullet)|)$-bundle $\mathcal{E}''$. The restriction of the latter concordance to S^{n-1} is the image of the first concordance under η_*. Moreover, $\mathcal{E}''$ has the following properties:

i) it is subordinate to the good cover $\mathcal{U}$,

ii) the restriction of $\mathcal{E}''$ to S^{n-1} corresponds to a simplicial map that factors as

$$U_\bullet^\partial \xrightarrow{\phi^\partial} D(Z) \xrightarrow{\eta\kappa} S_\bullet(|T(Z_\bullet)|).$$

Consequently, the gluing of these concordances gives a commutative diagram

$$\begin{array}{ccc} \mathrm{Con}_{T(Z_\bullet)}(D^n, S^{n-1}; \kappa_* \mathcal{E}^\partial) & \xrightarrow{③} & \mathrm{Con}_{S_\bullet(|T(Z_\bullet)|)}(D^n, S^{n-1}; \eta_*\kappa_* \mathcal{E}^\partial) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Con}_{T(Z_\bullet)}(D^n, S^{n-1}; i_\partial^* \mathcal{E}_{S^{n-1}}) & \xrightarrow{②} & \mathrm{Con}_{S_\bullet(|T(Z_\bullet)|)}(D^n, S^{n-1}; \eta_* i_\partial^* \mathcal{E}_{S^{n-1}}), \end{array}$$

where $i^*\mathcal{E}'$ in the lower left corner lifts to $\mathcal{E}''$ in the upper left corner.

Translating Lemma 7.8 into the language of $?_\bullet$-bundles we get a $D(Z_\bullet)$-bundle $\mathcal{E}$ over D^n whose restriction to S^{n-1} equals $\mathcal{E}^\partial$, together with a concordance relative to S^{n-1} between the $S_\bullet(|T(Z_\bullet)|)$-bundles $\mathcal{E}''$ and $\eta_*\kappa_*\mathcal{E}$. Thus the class of $\mathcal{E}''$ is in the image of ③. This implies that the class of $\mathcal{E}'$ is in the image of ①, and the proof is completed. □

Corollary 7.11 *Assume that $Z_\bullet$ is good and satisfies the topological Kan condition. Let $A \hookrightarrow X$ be a closed cofibration such that (X, A) has the homotopy type of a CW-pair. Let $\mathcal{E}_A$ be any $Z_\bullet$-bundle over A. Then the map*

$$\eta_* \colon \mathrm{Con}_{T(Z_\bullet)}(X, A; \mathcal{E}_A) \to \mathrm{Con}_{S_\bullet(|T(Z_\bullet)|)}(X, A; \eta_* \mathcal{E}_A)$$

is a bijection.

Proof: This follows directly from the proposition above together with Proposition 6.19. □

We have now almost proved the following theorem:

Theorem 7.12 *Assume that X has the homotopy type of a CW-complex, and that $Z_\bullet$ is good. Then geometric realization induces a bijection* $\mathrm{Con}_{Z_\bullet}(X) \xrightarrow{\cong} [X, |Z_\bullet|]$.

In some sense this theorem is related to [23, Proposition A.1.1]. Let us now explain how the our theory of concordances gives a proof:

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Proof: By Theorem 6.32 there is a good simplicial space $\tilde{Z}_\bullet$ satisfying the topological Kan condition together with a simplicial map $q\colon Z_\bullet \to \tilde{Z}_\bullet$ inducing bijections of concordance classes and homotopy classes of maps. Now consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{Con}_{Z_\bullet}(X) & \xrightarrow{①} & \mathrm{Con}_{\tilde{Z}_\bullet}(X) & \xrightarrow{②} & \mathrm{Con}_{T(\tilde{Z}_\bullet)}(X) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle ③} \\
\mathrm{Con}_{S_\bullet(|Z_\bullet|)}(X) & \longrightarrow & \mathrm{Con}_{S_\bullet(|\tilde{Z}_\bullet|)}(X) & \longrightarrow & \mathrm{Con}_{S_\bullet(|T(\tilde{Z}_\bullet)|)}(X) \\
\uparrow{\scriptstyle ④} & & \uparrow{\scriptstyle ④} & & \uparrow{\scriptstyle ④} \\
\mathrm{Con}_{C_\bullet(|Z_\bullet|)}(X) & \longrightarrow & \mathrm{Con}_{C_\bullet(|\tilde{Z}_\bullet|)}(X) & \longrightarrow & \mathrm{Con}_{C_\bullet(|T(\tilde{Z}_\bullet)|)}(X) \\
\downarrow{\scriptstyle ⑤} & & \downarrow{\scriptstyle ⑤} & & \downarrow{\scriptstyle ⑤} \\
[X, |Z_\bullet|] & \xrightarrow{⑦} & [X, |\tilde{Z}_\bullet|] & \xrightarrow{⑥} & [X, |T(\tilde{Z}_\bullet)|].
\end{array}$$

The maps ① and ⑦ are induced by q and they are bijections by Theorem 6.32. Since the natural map $\tilde{Z}_\bullet \to T(\tilde{Z}_\bullet)$ is a homotopy equivalence in each simplicial degree, it follows from Theorem 6.26 that ② is a bijection. Moreover, the map ⑥ is a bijection for the same reason, see Lemma 7.5. By Corollary 7.11 the map ③ is a bijection. All maps marked ④ are bijections by Proposition 7.3, and all maps marked ⑤ are bijections by Proposition 7.2. Consequently, we get our natural bijection by composing these maps. □

Observe that we easily could generalize the theorem to a CW-complex X relative to fixed data over some subcomplex A.

Of course we have the case $Z_\bullet = \Delta 2\mathcal{C}$ in mind, and when $2\mathcal{C}$ is a good topological 2-category the condition of Theorem 7.12 is satisfied. Thus we get a proof of the Theorem 1.1. Reasonable criteria implying that $2\mathcal{C}$ is good are provided by Theorem 5.6.

REFERENCES

1. C. Ausoni, 'On the algebraic K-theory of the complex K-theory spectrum', *Invent. Math.* **180** (2010), no.3, 611–668.

2. N. A. Baas, B. I. Dundas, B. Richter J. Rognes, 'Stable bundles over rig categories', J. Topology **4** (2011) 623–640.

3. N. A. Baas, B. I. Dundas, B. Richter J. Rognes, 'Ring completion of rig categories', preprint, 2010; arXiv:math/0706.0531v4.

https://doi.org/10.1017/is012001012jkt181 Published online by Cambridge University Press

4. N. A. Baas, B. I. Dundas J. Rognes, 'Two-vector bundles and forms of elliptic cohomology', *Topology, geometry and quantum field theory* (ed U. Tillmann), London Math. Soc. Lecture Note Ser. **308** (Cambridge Univ. Press, Cambridge, 2004), pp. 18–45.

5. J. C. Baez A. S. Crans, 'Higher-dimensional algebra. VI. Lie 2-algebras', *Theory Appl. Categ.* **12** (2004) 492–538 (electronic).

6. T. Beke, 'Higher Čech theory', *K-Theory* **32** (2004) 293–322.

7. J. Bénabou, 'Introduction to bicategories', *Reports of the Midwest Category Seminar*, Lecture Notes in Mathematics **47**, (Springer, Berlin, 1967), pp. 1–77.

8. E. H. Brown, Jr. R. H. Szczarba, 'Real and rational homotopy theory', *Handbook of algebraic topology* (ed I. M. James), (North-Holland, Amsterdam, 1995), pp. 867–915.

9. M. Bullejos A. M. Cegarra, 'On the geometry of 2-categories and their classifying spaces', *K-Theory* **29** (2003) 211–229.

10. D. Dugger D. C. Isaksen, 'Topological hypercovers and $\mathbb{A}^1$-realizations', *Math. Z.* **246** (2004) 667–689.

11. J. W. Duskin, 'Simplicial matrices and the nerves of weak n-categories. I. Nerves of bicategories', *Theory Appl. Categ.* **9** (2001/02), 198–308 (electronic), CT2000 Conference (Como).

12. W. G. Dwyer J. Spaliński, 'Homotopy theories and model categories', *Handbook of algebraic topology* (ed I. M. James), (North-Holland, Amsterdam, 1995), pp. 73–126.

13. E. Dyer S. Eilenberg, 'An adjunction theorem for locally equiconnected spaces', *Pacific J. Math.* **41** (1972) 669–685.

14. S. Galatius, 'Stable homology of automorphism groups of free groups', Ann. of Math (2) **173** (2011) 705–768.

15. S. Galatius, I. Madsen, U. Tillmann M. Weiss, 'The homotopy type of the cobordism category', *Acta Math.* **202** (2009), no.2, 195–239.

16. P. R. Heath, 'A pullback theorem for locally-equiconnected spaces', *Manuscripta Math.* **55** (1986) 233–237.

17. P. S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs **99**, (American Mathematical Society, Providence, RI, 2003).

18. J. F. Jardine, 'Simplicial approximation', *Theory Appl. Categ.* **12** (2004) 34–72 (electronic).

19. M. M. Kapranov V. A. Voevodsky, '2-categories and Zamolodchikov tetrahedra equations', *Algebraic groups and their generalizations: quantum and infinite-dimensional methods*, University Park, PA, 1991, (eds W. J. Haboush and B. J. Parshall), Proc. Sympos. Pure Math. **56**, (Amer. Math. Soc., Providence, RI, 1994), pp. 177–259.

https://doi.org/10.1017/is012001012jkt181 Published online by Cambridge University Press

20. J. Lurie, 'What is … an ∞-category?', *Notices Amer. Math. Soc.* (8) **55** (2008) 949–950.

21. S. Mac Lane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics **5**, (Springer-Verlag, New York, 1998).

22. I. Madsen, 'Algebraic K-theory and traces', *Current developments in mathematics, 1995*, Cambridge, MA, (eds R. Bott, M. Hopkins, A. Jaffe, I. Singer, D. Stroock and S.-T. Yau), (Internat. Press, Cambridge, MA, 1994), pp. 191–321.

23. I. Madsen M. Weiss, 'The stable moduli space of Riemann surfaces: Mumford's conjecture', *Ann. of Math.* (2) **165** (2007) 843–941.

24. I. Moerdijk, *Classifying spaces and classifying topoi*, Lecture Notes in Mathematics **1616**, (Springer-Verlag, Berlin, 1995).

25. M. K. Murray D. Stevenson, 'Bundle gerbes: stable isomorphism and local theory', *J. London Math. Soc.* (2) **62** (2000) 925–937.

26. D. Quillen, 'Higher algebraic K-theory. I', *Algebraic K-theory, I: Higher K-theories*, Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972, (ed H. Bass), Lecture Notes in Math. **341**, (Springer, Berlin, 1973), pp. 85–147.

27. C. Rezk, S. Schwede B. Shipley, 'Simplicial structures on model categories and functors', *Amer. J. Math.* **123** (2001) 551–575.

28. G. Segal, 'Classifying spaces and spectral sequences', *Inst. Hautes Études Sci. Publ. Math.* **34** (1968) 105–112.

29. G. Segal, 'Categories and cohomology theories', *Topology* **13** (1974) 293–312.

30. E. H. Spanier, *Algebraic topology*, (Springer-Verlag, New York, 1991); Corrected reprint of the 1966 original.

31. R. Street, 'The algebra of oriented simplexes', *J. Pure Appl. Algebra* **49** (1987) 283–335.

32. R. Street, 'Categorical structures', *Handbook of algebra* **1**, (North-Holland, Amsterdam, 1996), pp. 529–577.

33. A. Strøm, 'The homotopy category is a homotopy category', *Arch. Math.* (Basel) **23** (1972) 435–441.

34. U. Tillmann, 'On the homotopy of the stable mapping class group', *Invent. Math.* **130** (1997) 257–275.

35. M. Weiss, 'What does the classifying space of a category classify ?', *Homology, Homotopy and Applications* **7** (2005) 185–195; http://intlpress.com/HHA.

36. H. Whitney, 'Analytic extensions of differentiable functions defined in closed sets', *Trans. Amer. Math. Soc.* **36** (1934) 63–89.

37. J. Wirth J. Stasheff, 'Homotopy transition cocycles', *J. Homotopy Relat. Struct.* **1** (2006), no.1, 273–283 (electronic).

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