CONTINUANTS AND PRECONTINUANTS

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1. It is here expedient to ignore the continued fractions from which they are usually derived and to define the simple continuants determined by the parameters $a_1, a_2, a_3, \ldots, a_n, \ldots$ as given by the chain of difference equations

$$u_{n+1} = a_n u_n + u_{n-1}, \tag{1}$$

together with the initial values $u_0 = 0$, $u_1 = 1$: and these data lead successively to the functions

$$u_2 = a_1, \quad u_3 = a_1 a_2 + 1, \quad u_4 = a_1 a_2 a_3 + a_1 + a_3 \tag{2}$$

and

$$u_{n+1} = K(a_1, a_2, a_3, \ldots a_n). \tag{3}$$

If each continuant is briefly denoted by its first and last parameters, or more briefly still by their ordinal suffixes, placed in brackets, then

$$K(a_1, a_2, a_3, \ldots a_n) \equiv (a_1, a_n) \equiv (1, n).$$

The chain of equations is then given by

$$(1, n) = a_n(1, n-1) + (1, n-2), \tag{4}$$

and the series of continuants is

$$u_0 = 0, \quad u_1 = 1, \quad u_2 = (1, 1), \quad u_3 = (1, 2), \quad u_4 = (1, 3), \ \ldots, \quad u_{n+1} = (1, n). \tag{5}$$

If these same arbitrary values 0 and 1 are taken one place later in the series, so that $u_1 = 0$ and $u_2 = 1$, then from $u_2 = a_1 u_1 + u_0$ it follows that $u_0 = 1$, and the later equations give $u_3 = (2, 2)$, $u_4 = (2, 3)$, etc., so that the series of continuants becomes

$$u_0 = 1, \quad u_1 = 0, \quad u_2 = 1, \quad u_3 = (2, 2), \quad u_4 = (2, 3), \ \ldots, \quad u_{n+1} = (2, n).$$

Similarly, if $u_2 = 0$ and $u_3 = 1$, the series is

$$u_0 = -(1, 1), \quad u_1 = 1, \quad u_2 = 0, \quad u_3 = 1, \quad u_4 = (3, 3), \ \ldots, \quad u_{n+1} = (3, n),$$

and the next, for $u_3 = 0$, $u_4 = 1$, is

$$u_0 = (1, 2), \quad u_1 = -(2, 2), \quad u_2 = 1, \quad u_3 = 0, \quad u_4 = 1, \ \ldots, \quad u_{n+1} = (4, n),$$

and, generally, $u_m = 0$, $u_{m+1} = 1$ gives the series

$$u_0 = (-)^{m-1}(1, m-1), \ \ldots, \quad u_{m-2} = -(m-1, m-1), \quad u_{m-1} = 1,$$
$$u_m = 0, \quad u_{m+1} = 1, \ \ldots, \quad u_{n+1} = (m+1, n).$$

These *precontinuants*, as they may be called, which precede the zero value $u_m = 0$ in this last series of continuants, have all the same terminal parameter a_{m-1}, whereas all those which follow have the same initial parameter a_{m+1}, and the parameter a_m does not occur. The precontinuants have signs alternately

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minus and plus according as their order is odd or even. If the series were taken as running backwards the equation (4) would be used in the form

$$(1, n-2) = (-a_n)(1, n-1) + (1, n),$$

which is tantamount to making all the a's negative.

If a square table is made, showing in rows the complete set of series of continuants and precontinuants corresponding to each position of the zero value, these zero values lie on the main diagonal and the unit values lie in the diagonals next adjacent. The other entries are, by pairs, images in the main diagonal, with a reversal of sign for the continuants of odd order. In any diagonal parallel to the main diagonal the continuants are all of the same degree. The table is here shown extending as far as continuants of order 6.

	0	1	2	3	4	5	6	7
0	0	1	(11)	(12)	(13)	(14)	(15)	(16)
1	1	0	1	(22)	(23)	(24)	(25)	(26)
2	−(11)	1	0	1	(33)	(34)	(35)	(36)
3	(12)	−(22)	1	0	1	(44)	(45)	(46)
4	−(13)	(23)	−(33)	1	0	1	(55)	(56)
5	(14)	−(24)	(34)	−(44)	1	0	1	(66)
6	−(15)	(25)	−(35)	(45)	−(55)	1	0	1
7	(16)	−(26)	(36)	−(46)	(56)	−(66)	1	0

If the rows and columns are numbered 0, 1, 2, 3, 4, ..., then in the right upper region the entry in row m column n (with $m \leqslant n-2$) is $(m+1, n-1)$; and in the left lower region (with $m \geqslant n+2$) the entry is $(-)^{m-n+1}(n+1, m-1)$. Or, as an equivalent statement, the continuant (p, q) appears in row $p-1$ column $q+1$, and also in row $q+1$ column $p-1$ with prefixed sign $(-)^{q-p+1}$.

A simple and permissible change in the table is to reverse the signs of all the entries in all the oddly numbered rows: for the generation of the continuant series in each row by use of the linear difference equations remains unaltered. The entry in row m column n then becomes $(-)^m(m+1, n-1)$ in the right upper region and $(-)^{n+1}(n+1, m-1)$ in the left lower; and this is a simple exchange of m and n, together with an invariable reversal of sign in addition. The entries

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which occupy image-positions in regard to the main diagonal are now equal with reversed signs. Each diagonal of units has now alternate signs for the units, and the two diagonals are images, with reversed signs, in the main diagonal. The modified diagram has thus become skew-symmetric about the main diagonal.

2. The modified diagram supplies a notation which is apt and effective in presenting the algebra of the continuant functions. Let the entry in row m column n be denoted by m and n, in that order, placed in square brackets. Then the algebra prescribed by the rules

$$[n,n]=0, \quad [n,n+1]=(-1)^n, \quad [n,m]=-[m,n]$$
$$\text{and} \quad [m,n]=(-)^m(m+1,n-1) \quad (6)$$

supplies all the continuant equivalents of the symbolic tabular entries. The last equation provides the continuant value for the entry $[m,n]$ when m is the smaller; and if n is the smaller $[m,n]=-[n,m]=(-)^{n+1}(n+1,m-1)$. The suffixes of the two terminal parameters of the continuant lie always between the arithmetical values of m and n, and differ by one unit each from m and n.

The consecutive linear operations which depend on the parameters $a_1, a_2, a_3, \ldots, a_n, \ldots$ and produce the different series of continuants are the same for each series. It follows that a linear relation (not yet determined), with functions of the a's as coefficients, connects any three entries in any one row, and that the same relation connects the corresponding entries in any other row. If therefore any three rows are taken, and any three columns, the determinant of the third order formed by the entries is zero. Suppose the rows to be l,m,n, and the columns p,q,r; then

$$\begin{vmatrix} [l,p] & [l,q] & [l,r] \\ [m,p] & [m,q] & [m,r] \\ [n,p) & [n,q] & [n,r] \end{vmatrix} = 0. \quad (7)$$

If, specially, $p=l$ and $q=m$, then since $[l,l]=0$ and $[m,m]=0$ the determinant becomes

$$\begin{vmatrix} 0 & [l,m] & [l,r] \\ [m,l] & 0 & [m,r] \\ [n,l] & [n,m] & [n,r] \end{vmatrix} = 0. \quad (8)$$

Since $[m,l]=-[l,m]$ this equation has $[l,m]$ as a factor, and since $[n,m]=-[m,n]$ the equation reduces to the form

$$[m,n][l,r]+[n,l][m,r]+[l,m][n,r]=0. \quad (9)$$

By virtue of the six conditions such as $[l,m]=-[m,l]$ this equation repeats itself identically for all permutations of l,m,n,r and is thus wholly symmetric in regard to the four ordinals, and entirely independent of any arithmetical inequalities affecting them. It is thus a completely algebraic representation of Euler's continuant identity. To make a specimen conversion, suppose l,m,n,r

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to be in ascending order of magnitude with consecutive differences not less than 2, and put $l = L-1, m = M-1, n = N+1, r = R+1$ so that $L+2 \leqslant M \leqslant N \leqslant R-2$; then

$$\left.\begin{aligned} [l,r] &= (-)^{L+1}(L,R), & [m,n] &= (-)^{M+1}(M,N), \\ [m,r] &= (-)^{M+1}(M,R), & [n,l] &= (-)^{L}(L,N), \\ [n,r] &= (-)^{N+1}(N+2,R), & [l,m] &= (-)^{L+1}(L,M-2). \end{aligned}\right\} \tag{10}$$

So $$(L,R)(M,N)-(L,N)(M,R)+(-)^{N-M}(L,M-2)(N+2,R) = 0, \tag{11}$$

which is the customary form of Euler's equation. In it each round bracket represents a simple continuant whose first and last parameters have the ordinal suffixes shown in the brackets.

The symmetrized equivalent (9) of Euler's equation here presented will be referred to as (E).

3. The second-order determinants formed by any two rows and two columns of the skew-symmetric table have reducible values. If the rows are m and n and the columns p and q, the determinant is

$$\begin{vmatrix} [m,p] & [m,q] \\ [n,p] & [n,q] \end{vmatrix} = [m,p][n,q]+[p,n][m,q] = [m,n][p,q], \tag{12}$$

by reason of (E). Every second-order determinant is thus equal to the product of two continuants, one given by the ordinals of the rows and the other by the ordinals of the columns. If, specially, the determinant is compact (formed by four neighbouring elements), so that $n = m+1$ and $q = p+1$, then $[m,m+1] = (-1)^m$ and $[p,p+1] = (-1)^p$; hence

$$\begin{vmatrix} [m,p] & [m,p+1] \\ [m+1,p] & [m+1,p+1] \end{vmatrix} = (-1)^{m+p}, \tag{13}$$

so that any compact second-order minor has as its value either plus or minus unity.

If a fourth-order determinant is taken from the modified table with rows l,m,n,r and columns also l,m,n,r, it is skew-symmetric and of even order, and therefore a perfect square. It is identically

$$\{[m,n][l,r]+[n,l][m,r]+[l,m][n,r]\}^2 = 0, \tag{14}$$

the square of the equation (E) itself.

The equation (E) if written in the form

$$[m,n][r,l]+[n,l][r,m]+[l,m][r,n] = 0 \tag{15}$$

shows the entries $[r,l]$, $[r,m]$, $[r,n]$ in row r and columns l,m,n to be linearly related, with multipliers $[m,n]$, $[n,l]$, $[l,m]$ as coefficients. This makes explicit

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the linear relation invoked in § 2. For any other row s the multipliers remain the same, for the same columns l, m, n, by virtue of

$$[m,n][s,l]+[n,l][s,m]+[l,m][s,n]=0, \tag{16}$$

which is also (E). The coefficients, therefore, of the linear relation between any three continuants of a series depend only on the columns selected and are independent of the rows.

If (E) is written

$$[m,n][l,r]+[n,l][m,r]+[l,m][n,r]=0, \tag{17}$$

then the entries $[l,r]$, $[m,r]$, $[n,r]$ in column r for rows l, m, n are linearly related, with the multipliers $[m,n]$, $[n,l]$, $[l,m]$, and this holds for all values of r. Hence the three entire continuant series in any three rows l, m, n are linearly related, term by term, with the multipliers $[m,n]$, $[n,l]$, $[l,m]$.

4. The equation (E) reduces to special forms when two or more of the four parameters have consecutive values.

If two of the four ordinals l, m, n, r differ by unity, say $r=n+1$, then as $[n,n+1]=(-1)^n$ the formula becomes

$$[l,n][m,n+1]-[m,n][l,n+1]=(-)^n[l,m]. \tag{18}$$

Only this single type of equation thus arises; but the l, m, n may be permuted, and three different continuant formulae are derived, namely,

$$(l,n)=(l,m)(m+1,n)+(l,m-1)(m+2,n), \tag{19}$$

$$(l,n)(m,n-1)-(l,n-1)(m,n)=(-)^{n-m}(l,m-2), \tag{20}$$

$$(l,n)(l+1,m)-(l,m)(l+1,n)=(-)^{m-l}(m+2,n), \tag{21}$$

all subject to $$l+2\leqslant m\leqslant n-2.$$

Equation (19) has four of the six terminal parameters as consecutives, and is notable as having no critical sign. The reason for this is that the tabular determinant which gives rise to it is

$$\begin{vmatrix} (l,m-1) & (l,m) & (l,n) \\ 0 & 1 & (m+1,n) \\ 1 & 0 & (m+2,n) \end{vmatrix}=0, \tag{22}$$

as taken from rows $l-1, m, m+1$ and columns $m, m+1, n+1$ of the table, and that only the left lower unit comes from the left lower side of the main diagonal of the table.

The equation

$$(m,m)(l,n)=(l,m)(m,n)-(l,m-2)(m+2,n), \tag{23}$$

with $l+2\leqslant m\leqslant n-2$ has the same absence of a power of -1 and for a similar

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reason. It has five terminal parameters of which the three intermediates are alternates. It is given by the determinant

$$\begin{vmatrix} (l,m-2) & (l,m) & (l,n) \\ 0 & (m,m) & (m,n) \\ -(m,m) & 0 & (m+2,n) \end{vmatrix} = 0. \tag{24}$$

Its derivation from (E) is made by taking a difference of two units for two of the four ordinals.

From either (19) or (23) is derivable the equation

$$(l,n) = (m,m)(l,m-1)(m+1,n) + (l,m-1)(m+2,n) + (l,m-2)(m+1,n), \tag{25}$$

which isolates the particular parameter $a_m = (m,m)$ occurring linearly in (l,n); showing the cofactor of the parameter a_m to be $(l,m-1)(m+1,n)$ and giving the aggregate of the terms independent of a_m in the symmetric form

$$(l,m-1)(m+2,n) + (l,m-2)(m+1,n).$$

If equations (19), (20), (21) are derived directly from the original form of the Euler equation (11), there arises the need to suppose that the symbol $(n+1,n)$ stands for $+1$, and this is in conflict with the identity $(n+1,n) = (n,n+1)$ given by the simple exchange of a_n and a_{n+1}. The escape from this confusion is one of the results of the introduction and use of the precontinuants.

If l,m,n,r have three consecutive values with unit differences, the equation (E) reduces to the form

$$[m-1,n] - [m+1,n] = (-)^m [m-1,m+1][m,n], \tag{26}$$

and this, in terms of the continuants, leads to the two equations

$$(m,n) = (n,n)(m,n-1) + (m,n-2), \tag{27}$$

$$(m,n) = (m,m)(m+1,n) + (m+2,n), \tag{28}$$

with $n-m \geqslant 2$.

In (27) the initial parameter is the same in the consecutive continuants, and the equation is identical with the difference equation (4) defining the continuants. In (28) the final parameter is the same in the three consecutive continuants. If the series of parameters is reversed in its sequence this equation is consistent with the fact (or proves) that any continuant is unaltered by reversal of the sequence of parameters.

If two pairs of the parameters l,m,n,r have each a unit difference the equation (E) takes the form

$$[m,n][m+1,n+1] - [m+1,n][m,n+1] = (-1)^{m+n}, \tag{29}$$

which is identical with (13). The continuant equivalent is

$$(m,n)(m+1,n-1) - (m+1,n)(m,n-1) = (-1)^{n-m+1} \tag{30}$$

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with $n-m \geqslant 2$. It is the equation familiar as relating the numerators and denominators of two consecutive convergents

$$(m+1, n-1)/(m, n-1) \quad \text{and} \quad (m+1, n)/(m, n)$$

of the continued fraction $\dfrac{1}{a_m+}\ \dfrac{1}{a_{m+1}+}\ \ldots\ \dfrac{1}{a_n+}\ \ldots$.

5. The (E) equation connects any four numbers. If any five are taken, say p, q, r, s, t, then an (E) equation connects any four, and fresh relations may be obtained from these five equations. As an instance, the (E) equations which omit in turn r and s may be written*

[ ]
$$qs\,.\,pt + sp\,.\,qt = -pq\,.\,st,$$
$$qr\,.\,pt + rp\,.\,qt = -pq\,.\,rt.$$

These may be squared and the product term $pt\,.\,qt$ may be eliminated, to produce an equation linear in the squares pt^2, qt^2, rt^2, st^2. The (E) equation which omits t simplifies the eliminant equation and makes it divisible by pq, and the final form is

[ ] $$qr\,.\,rs\,.\,sq\,.\,pt^2 - pr\,.\,rs\,.\,sp\,.\,qt^2 + pq\,.\,qs\,.\,sp\,.\,rt^2 - pq\,.\,qr\,.\,rp\,.\,st^2 = 0. \qquad (31)$$

The signs are self-verifying on taking any two of p, q, r, s to be equal. An equivalent, more obviously symmetrical in p, q, r, s, is

[ ] $$\Sigma pt^2/pq\,.\,pr\,.\,ps = 0. \qquad (32)$$

Any four continuants of a series thus have their squares linearly related; the coefficients being continuant products depending only on the intervals between the terminal parameters p, q, r, s and independent of t.

The equation is more summarily obtainable by writing the three (E) equations which omit in turn p, q, r in the form

[ ]
$$qr\,.\,st + sq\,.\,rt + rs\,.\,qt = 0,$$
$$ps\,.\,rt + rp\,.\,st + sr\,.\,pt = 0,$$
$$sp\,.\,qt + qs\,.\,pt + pq\,.\,st = 0,$$

and taking the zero determinant (of which each row has a zero sum) presented by the equations, namely,

[ ] $$\begin{vmatrix} qr\,.\,st & sq\,.\,rt & rs\,.\,qt \\ ps\,.\,rt & rp\,.\,st & sr\,.\,pt \\ sp\,.\,qt & qs\,.\,pt & pq\,.\,st \end{vmatrix} = 0. \qquad (33)$$

Of the six product terms in its expansion two only have st missing, and these two cancel: the remaining four terms divide by st and give the above equation (31).

* Here and later where no ambiguity would arise the tabular continuant symbol $[p, q]$ will be written in the attenuated form pq. The empty square brackets are reminders of the omission.

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If other values u and v replace t, (32) gives

$$[\ ] \qquad \Sigma pu^2./pq.pr.ps = 0 \quad \text{and} \quad \Sigma pv^2./pq.pr.ps = 0; \qquad (34)$$

and pt, pu, pv are linearly related by $uv.pt + vt.pu + tu.pv = 0$. Hence

$$uv^2.pt^2 = vt^2.pu^2 + ut^2.pv^2 - 2ut.vt.pu.pv,$$

and similarly for q, r, s in place of p: hence by substitution of pt^2, qt^2, rt^2, st^2 in (32) it follows from (34) that

$$[\ ] \qquad \Sigma pu.pv./pq.pr.ps = 0, \qquad (35)$$

summed for p, q, r, s with u and v stationary. This equation is more general than (32): it gives a linear relation connecting the four products of corresponding continuants of any two series generated by the same system of parameters, the coefficients being products of continuants which depend only on the terminals p, q, r, s and are independent of the terminals u and v.

It may be noticed that, by analogy with the form of the equations (32) and (35), the equation (E) itself may be written

$$[\ ] \qquad pt./pq.pr + qt./qr.qp + rt./rp.rq = 0. \qquad (36)$$

From this last form of (E) the equation (35) may be obtained alternatively and more directly. The (E) equations in p, q, r, u and p, q, s, u may be written

$$[\ ] \qquad pu./pq.pr + qu./qr.qp + ru./rp.rq = 0,$$

$$pu./pq.ps + qu./qs.qp + su./sp.sq = 0.$$

If these are multiplied respectively by rv and vs and added the terms in pu and qu simplify by use respectively of the (E) equations for p, r, s, v and q, r, s, v and the whole, divided by rs, gives equation (35).

This last process serves to produce a similar formula of the next higher order. If in (35) s is replaced by t, then

$$[\ ] \qquad \Sigma pu.pv./pq.pr.pt = 0.$$

If this equation is multiplied by wt and (35) by sw, the coefficients in the sum are reducible by quotation of (E) for $pstw, qstw, rstw$; and if the whole is then divided by st the resulting equation is

$$[\ ] \qquad \Sigma pu.pv.pw./pq.pr.ps.pt = 0 \qquad (37)$$

summed for p, q, r, s, t with u, v, w stationary.

If in (37) two of the three parameters u, v, w are made identical there come formulae of the type

$$[\ ] \qquad \Sigma pu^2.pv./pq.pr.ps.pt = 0, \qquad (38)$$

summed for p, q, r, s, t with u and v stationary.

And if all three parameters become identical then

$$[\ ] \qquad \Sigma pu^3./pq.pr.ps.pt = 0. \qquad (39)$$

It may be remarked that if any two of p, q, r, s, t are made identical (say p and q,

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after multiplication by pq) the equation becomes a mere identity. And if one of p, q, r, s, t is made equal to any one of u, v, w in (37), the equation reverts to the next lower order.

The formulae (37), (38), (39) here found may obviously be extended, by the same method, to any higher order. The symbols in the denominators may be of any number, and the number of fractions is that same number: the number of factors in each denominator is one less, and the number of factors (whether different or equal) in the numerators is less by two.

6. An extension of a different kind may be made of the (E) equation and its derivatives. The occurrence of the value zero as a term in each series of continuants, insisted upon for purposes of the table, may be abandoned, while preserving the same difference equations. If the *subcontinuants* (say) thus produced are $v_0, v_1, v_2, v_3, \ldots, v_n, \ldots$, they obey the equations $v_{n+1} = a_n v_n + v_{n-1}$ but involve two arbitrary constants, which might, specially, be the values of any two of the v's themselves. The symbolic notation $[m, n]$ for the tabular entries may be preserved by dropping the row symbol m and keeping only the column symbol n, so that $[n] = v_n$ with a_{n-1} as the parameter of highest order in the subcontinuant v_n.

The subcontinuant series may be taken as a linear function of any two of the series of continuants, say those in rows l and m; so that, in columns p, q, r,

$$\left.\begin{aligned} [p] &= \lambda[l,p] + \mu[m,p], \\ [q] &= \lambda[l,q] + \mu[m,q], \\ [r] &= \lambda[l,r] + \mu[m,r], \end{aligned}\right\} \tag{40}$$

giving

$$[q,r][p] + [r,p][q] + [p,q][r] = 0 \tag{41}$$

in place of equation (E): and, in place of (36),

$$[p]/[p,q][p,r] + [q]/[q,r][q,p] + [r]/[r,p][r,q] = 0. \tag{42}$$

If any two of the subcontinuants $[p], [q], [r]$ have arbitrarily assigned values, then (41) or (42) gives the value of the third (any other of the series), the coefficients being supplied by the continuants corresponding to the intervening sets of parameters. The denominators in (42) are still continuants, but the numerators are subcontinuants of the same rank (in terms of the parameter of highest order) as the continuants they replace.

By reason of (32) and (35) equation (32) becomes

$$\Sigma[p]^2/[p,q][p,r][p,s] = 0, \tag{43}$$

and similarly (39) becomes

$$\Sigma[p]^3/[p,q][p,r][p,s][p,t] = 0. \tag{44}$$

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If different arbitrary constants λ', μ' give a subcontinuant series v', with $[p]' = \lambda'[l,p] + \mu'[m,p]$, then (35) becomes

$$\Sigma [p][p]'/[pq][pr][ps] = 0, \tag{45}$$

as relating the four products $[p][p]'$, $[q][q]'$, $[r][r]'$, $[s][s]'$ of corresponding terms in two subcontinuant series: and similarly (37) becomes

$$\Sigma [p][p]'[p]''/[p,q][p,r][p,s][p,t] = 0 \tag{46}$$

for products of corresponding terms in three subcontinuant series. This last equation, extended to any higher order, comprehends in itself the symmetrized equivalent of Euler's equation and all the derivatives above indicated.

But it may be finally remarked that the numerator product of subcontinuant factors in (46), and the like equations of higher order, is itself a special case of any homogeneous polynomial function (of the same proper order) of corresponding continuants of any two or more series: though the factors of such a polynomial would not usually be rational nor calculable. The denominator factors remain strictly and simply the continuants determined by the intervals alone.

7. By way of illustration the general formulae may be applied to the special case in which the parameters are all equal in value and unlimited in number. If the common value is a then $u_{n+1} = au_n + u_{n-1}$ for all values of n, and the continuant values are

$$u_0 = 0, \quad u_1 = 1, \quad u_2 = a, \quad u_3 = a^2 + 1, \quad u_4 = a^3 + 2a, \quad \text{etc.},$$

and the precontinuants are given by

$$u_{-n} = (-)^{n+1} u_n. \tag{47}$$

The polynomial in a giving u_{n+1} is readily seen to be

$$u_{n+1} = \Sigma_{n-s} C_s a^{n-2s} \tag{48}$$

by virtue of the combinatorial identity

$${}_{m+1}C_s = {}_mC_s + {}_mC_{s-1}. \tag{49}$$

Since all the parameters are identical it follows that for any two terminals the value of any continuant depends only on its range. Hence

$$(m,n) = (1, n-m+1) = u_{n-m+2}, \quad \text{where } m \leqslant n, \tag{50}$$

and

$$[m,n] = (-)^{n+1} u_{m-n} = (-)^m u_{n-m}. \tag{51}$$

The equation (E) therefore takes the cyclically symmetric form

$$(-)^n u_{m-n} u_l + (-)^l u_{n-l} u_m + (-)^m u_{l-m} u_n = 0, \tag{52}$$

and this remains true when the second factors are replaced by subcontinuants v_l, v_m, v_n taken from any series in which $v_{n+1} = av_n + v_{n-1}$, so that

$$(-)^n u_{m-n} v_l + (-)^l u_{n-l} v_m + (-)^m u_{l-m} v_n = 0. \tag{53}$$

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Of these last two equations (53) and (52) a few noteworthy special cases, written in pairs, may be recorded.

$$\left.\begin{aligned}
&(\mathrm{i}a)\quad u_{n-1}v_{n+1}-u_nv_n=(-)^nv_1,\\
&(\mathrm{ii}a)\quad v_{2n}=u_{n+1}v_n+u_nv_{n-1},\\
&(\mathrm{iii}a)\quad u_2v_{2n}=u_{n+1}v_{n+1}-u_{n-1}v_{n-1},\\
&(\mathrm{iv}a)\quad v_{2n+1}=u_{n+1}v_{n+1}+u_nv_n,\\
&(\mathrm{v}a)\quad v_{m+n+1}=u_{m+1}v_{n+1}+u_mv_n,\\
&(\mathrm{vi}a)\quad u_2v_{m+n}=u_{m+1}v_{n+1}-u_{m-1}v_{n-1},\\
&(\mathrm{vii}a)\quad v_{m+2n}=(u_{n+1}+u_{n-1})v_{m+n}+(-)^{n+1}v_m.
\end{aligned}\right\}\qquad(54a)$$

$$\left.\begin{aligned}
&(\mathrm{i}b)\quad u_{n-1}u_{n+1}-u_n^2=(-1)^n,\\
&(\mathrm{ii}b)\quad u_{2n}=u_n(u_{n+1}+u_{n-1}),\\
&(\mathrm{iii}b)\quad u_2u_{2n}=u_{n+1}^2-u_{n-1}^2,\\
&(\mathrm{iv}b)\quad u_{2n+1}=u_{n+1}^2+u_n^2,\\
&(\mathrm{v}b)\quad u_{m+n+1}=u_{m+1}u_{n+1}+u_mu_n,\\
&(\mathrm{vi}b)\quad u_2u_{m+n}=u_{m+1}u_{n+1}-u_{m-1}u_{n-1},\\
&(\mathrm{vii}b)\quad u_{m+2n}=(u_{n+1}+u_{n-1})u_{m+n}+(-)^{n+1}u_m.
\end{aligned}\right\}\qquad(54b)$$

The equations (54a) show their precise derivation from (53) by the suffixes of the v's: but in the second set (54b) the derivation is obscured. Equation (vii$a$) comes initially in the form

$$u_nv_{m+2n}-u_{2n}v_{m+n}+(-)^nu_nv_m=0,\qquad(55)$$

which is peculiar in being divisible by u_n by reason of (ii$b$).

As a special case of (32), for four consecutive squares,

$$u_n^2=u_3(u_{n+1}^2+u_{n+2}^2)+u_{n+3}^2=0,\qquad(56)$$

where $u_3=a^2+1$.

Similarly from (35)

$$v_mw_n-u_3(v_{m+1}w_{n+1}+v_{m+2}w_{n+2})+v_{m+3}w_{n+3}=0,\qquad(57)$$

where the v's and w's are taken from any two subcontinuant series with the same difference equation as the u's.

Or again, for four products consecutively alternate,

$$v_{m-3}w_{n-3}-(a^2+1)(a^2+3)(v_{m-1}w_{n-1}-v_{m+1}w_{n+1})-v_{m+3}w_{n+3}=0.\qquad(58)$$

8. If in the equations of § 7 for the case of equal parameters the common value a is put equal to unity, the continuants then have as their values the series of "golden" (Fibonacci) numbers. The early values of u_n are

n ...	0	1	2	3	4	5	6	7	8	9	10	11	12
u_n ...	0	1	1	2	3	5	8	13	21	34	55	89	144

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with the difference equation $u_{n+1} = u_n + u_{n-1}$: and u_{n+1} enumerates the number of terms in a simple continuant of order n. The formulae (ib)–(vii$b$) all hold good for this subcase, with the substitution merely of the value $u_2 = 1$. The last equation (vii$b$, 54b) allows golden numbers of high order to be obtained with little labour by making n steps at a time; e.g. $n = 11$, $u_{n+1} + u_{n-1} = 199\ (= 200-1)$, and $u_{m+22} = 199u_{m+11} + u_m$. Putting $m = -2$, say, $u_{-2} = -1$ and $u_9 = 34$, and the formula gives in turn $u_{20} = 6765$ and $u_{31} = 1346269$. (This last is the second case, next beyond $u_{19} = 4181 = 37 \times 113$, for which n is prime and u_n is composite: for $u_{31} = 557 \times 2417$.)

In the peculiar and trivial case $a = 0$ the series of continuants becomes $0, 1, 0, 1, 0, \ldots$, a perpetual alternation of 0 and 1, so that $u_{n+1} = \frac{1}{2}\{1 + (-1)^n\}$. For the tabular symbol $[m, n]$

$$[m, n] = (-)^m (m+1, n-1) = \tfrac{1}{2}(-)^m \{1 - (-1)^{m+n}\} = \tfrac{1}{2}\{(-1)^m - (-1)^n\}.$$

So if $(-1)^l = \lambda$, etc., the equation (E) becomes the algebraic identity

$$(\mu - \nu)(\lambda - \rho) + (\nu - \lambda)(\mu - \rho) + (\lambda - \mu)(\nu - \rho) = 0.$$

9. In the foregoing work the continuants are all *simple* continuants: but continuants of the general type, given by a chain of difference equations of the form $u_{n+1} = a_n u_n + b_n u_{n-1}$, may readily be treated in the same way. Taking initial values $u_0 = 0$, $u_1 = 1$, the series continues with

$$u_2 = a_1 = (1, 1),\ u_3 = a_1 a_2 + b_2 = (1, 2),\ u_4 = a_1 a_2 a_3 + a_3 b_2 + a_1 b_3,\ \ldots,\ u_{n+1} = (1, n) \tag{59}$$

giving the entries in the top row of the table. The table is continued and completed by keeping zero entries in the main diagonal and units in the places next to the right. In the right upper region the entry in row m column n (with $m \leqslant n-2$) is $(m+1, n-1)$; in the diagonal next to the left of the main diagonal the entry in row m is $1/b_m$; and in the left lower region (with $m \geqslant n+2$) the precontinuant entry is $(-)^{m-n+1}(n+1, m-1)/b_{n+1} \ldots b_m$; or, as an equivalent statement, the continuant (p, q) appears in row $p-1$ column $q+1$, and also in row $q+1$ column $p-1$ multiplied by $(-)^{q-p+1}/b_p \ldots b_{q+1}$ as a precontinuant.

If the rows of the table are multiplied by factors $1, -b_1, b_1 b_2, -b_1 b_2 b_3$, etc., so that row m is multiplied by $(-)^m b_1 b_2 \ldots b_m$, then the table becomes skew-symmetric and the entry $[m, n]$ in row m column n is given by

$$[m, n] = (-)^m b_1 b_2 \ldots b_m (m+1, n-1), \tag{60}$$

with $\quad [n, n] = 0, \quad [n, n+1] = (-)^n b_1 b_2 \ldots b_n \quad$ and $\quad [n, m] = -[m, n]. \tag{61}$

All the results derivable from the skew-symmetric table are then available for the general continuants. In particular and fundamentally the (E) equation

$$[m, n][l, r] + [n, l][m, r] + [l, m][n, r] = 0$$

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is the symmetrized equivalent of the generalized Euler's equation. The latter, in its asymmetric and usual form

$$(M,N)(L,R)-(L,N)(M,R)+(-)^{N-M}(L,M-2)(N+2,R)\,b_M \dots b_{N+1}=0, \tag{62}$$

is derivable from the symmetric form by taking

$$l=L-1, \quad m=M-1, \quad n=N+1, \quad r=R+1$$

and observing the restrictions $L+2\leqslant M\leqslant N\leqslant R-2$.

As a special case of the general continuants the a's may be all equal and the b's all equal, and each set unlimited in number, so that $u_{n+1}=au_n+bu_{n-1}$ for all values of n. Then if $u_0=0$ and $u_1=1$ there follow

$$u_2=a, \quad u_3=a^2+b, \quad u_4=a^3+2ab, \quad u_5=a^4+3a^2b+b^2, \tag{63}$$

with $$u_{n+1}=\Sigma_{n-s}C_s a^{n-2s}b^s \quad \text{and} \quad u_{-n}=(-)^{n+1}u_n/b^n, \tag{64}$$

and $$[m,n]=(-)^m b^m(m+1,n-1)=(-)^{n+1}b^n u_{m-n}, \tag{65}$$

and hence for the symmetric form (E) of the Euler equation

$$(-b)^n u_l u_{m-n}+(-b)^l u_m u_{n-l}+(-b)^m u_n u_{l-m}=0. \tag{66}$$

As for (52) of § 7 for the simple continuants, this identity gives rise to a profusion of special cases, comparable with (54b), by assignments of particular values to l, m, n.

NOTE

In biliteral algebras for pair-symbols (pq) equations such as

$$(qp)=-(pq), \tag{i}$$

$$(qr)+(rp)+(pq)=0, \tag{ii}$$

$$(qr)(ps)+(rp)(qs)+(pq)(rs)=0, \tag{iii}$$

$$(qr)(rp)(pq)+(rq)(qs)(sr)+(pr)(rs)(sp)+(qp)(ps)(sq)=0, \tag{iv}$$

or its equivalent, if (i) holds,

$$\Sigma\, 1/(pq)(pr)(ps)=0$$

variously occur. In common algebra $(pq)=p-q$ makes (i), (ii), (iii) and (iv) all true. In trigonometric algebra $(pq)=\sin(p-q)$ makes (i), (iii) and (iv) true and (ii) is not true. For the modified continuants $[pq]$ of this paper (i) and (iii) are true and (ii) and (iv) are not true. A referee remarks that a parallel case occurs in the equations of symbolic invariant theory (e.g. Grace and Young, *Algebra of invariants*, pp. 19ff. and p. 322), where $(pq)=p_1q_2-p_2q_1$.

Addendum. In checking (iv) above as untrue for continuants, by use of some Fibonacci numbers, it appeared insistently that only an alteration in signs was needed to make the equation come true: and the same slight modification served

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for other cases of equal parameters of § 7 with a other than unity. Investigation then showed as true a singular identity of the form $\Sigma 1/u_{p-q}u_{p-r}u_{p-s} = 0$. This corresponds to a continuant equation $\Sigma(-)^p/[pq][pr][ps]$ which is *not* true of the continuants in general, but has the same denominators as identity (35). In this last, however, taken as quoted for the special case of equal parameters, the non-circulating parameters in the numerator may be supposed at the plus and minus infinities: the series u_n at its two extremities becomes two geometric progressions with negative reciprocals (roots of $x^2 - ax - 1 = 0$) as the common ratios: and hence the numerator products are proportional to $(-1)^p$ and the like. The singular result thus falls into place. Its existence and source were overlooked in § 7 and $\Sigma 1/u_{p-q}u_{p-r}u_{p-s} = 0$ is noted here as a supplement.

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