$HS < E_7(5)$

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Introduction

In this paper we describe our discovery that the sporadic simple groups HS and M_{22} are contained in the simple Chevalley group $E_7(5)$.

The work of [**9**] produces a short list of the possibilities for a sporadic simple subgroup of an exceptional group of Lie type. Apart from possible embeddings of M_{22} and HS in groups of type E_7 in characteristic 5, all of the embeddings of [**9**] are already known to occur. Thus our paper completes the classification of sporadic simple subgroups of exceptional groups of Lie type.

We give two proofs of the embedding $HS < E_7(5)$. The first of these proofs is entirely computer free, while the second proof makes some use of machine calculations. As a step in our hand proof of $HS < E_7(5)$ we establish the embedding $M_{22} < E_7(5)$: of course, since M_{22} is a subgroup of HS, this result also follows as a consequence of our computer proof of $HS < E_7(5)$.

We were led to conjecture the inclusion $HS < E_7(5)$ for the following reasons (similar arguments are presented in [**9**]). The double cover $2.HS$ has a faithful 56-dimensional character, whose values are compatible with the character values of groups of type E_7 acting on their natural 56-dimensional module. Now HS and $2.HS$ contain a subgroup $5^2{:}20$, an elementary abelian group of order 25 extended by a cyclic group of order 20 acting faithfully on the 5^2. Since 20 is not the order of an element in the Weyl group $W(E_7) = 2 \times S_6(2)$, it can be shown that $5^2{:}20$ does not embed in groups of type $E_7(K)$, where K is a field of characteristic prime to 5. Thus HS embeds in $E_7(q)$ only if $5 \mid q$. On the other hand, all local subgroups of $2.HS$ embed in $2.E_7(5)$, where our conjecture $HS < E_7(5)$.

Throughout, G denotes the double cover $2.E_7(5)$, $\bar{G}$ denotes the simple group $E_7(5)$, and V denotes the natural 56-dimensional module for G over GF(5). Most of our notation follows that of the atlas [**4**]. The two sections of our paper are completely independent.

1. *A computer-free construction of the Higman–Sims group as a subgroup of* $E_7(5)$

In this section we will prove the following theorem.

THEOREM 1.1. *$E_7(5)$ contains subgroups isomorphic to M_{22} and the Higman–Sims group.*

We start with some of the properties of M_{22} we will need in the proof of the theorem.

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LEMMA 1.2. *Let $M = M_{22}$ act faithfully on $\Omega = \{1, 2, \ldots, 22\}$, let $\omega \in \Omega$ and let D be the stabilizer of a hexad $\mathscr{H}$ in Ω.*

(a) *M_ω acts transitively on $\Omega\backslash\{\omega\}$ and $M_\omega \cong L_3(4)$.*

(b) *$D \sim 2^4.\mathrm{Alt}(6)$, D acts transitively on $\Omega\backslash\mathscr{H}$ and any subgroup of shape $2^4\,\mathrm{Alt}(6)$ in M is conjugate to D.*

(c) *Let $E \leqslant M_\omega$ with $E \sim 2^4\,\mathrm{Alt}(5)$. Then E has orbits of lengths 1, 5 and 16 or of lengths 1, 1 and 20 on Ω. In the first case E stabilizes a hexad, and in the second case $N_M(E) \sim 2^4\,\mathrm{Sym}(5)$.*

(d) *Let $A \leqslant B \leqslant M$ with $A \cong \mathrm{Alt}(6)$ and $B \cong \mathrm{Alt}(7)$. Then A is not contained in a conjugate of D.*

(e) *Let $A \leqslant D$ such that $A \cong \mathrm{Alt}(5)$ and such that $O_2(D)$ is a natural $\mathrm{SL}_2(4)$-module for A. Then A has orbits of lengths 1 and 5 on $\mathscr{H}$ and orbits of lengths 1 and 15 or of lengths 6 and 10 on $\Omega\backslash\mathscr{H}$.*

(f) *Let $A \leqslant D$ such that $A \cong \mathrm{Alt}(5)$ and such that $O_2(D)$ is a natural $\Omega_4^-(2)$-module for A. Then A acts transitively on $\mathscr{H}$ and has orbits of lengths 1, 5 and 10 on $\Omega\backslash\mathscr{H}$.*

(g) *If $A \leqslant D$ with $A \cong \mathrm{Alt}(6)$, then A has orbits of lengths 1 and 15 or of lengths 6 and 10 on $\Omega\backslash\mathscr{H}$.*

(h) *If $A \leqslant M_\omega$ with $A \cong \mathrm{Alt}(6)$, then A has orbits of lengths 1, 6 and 15 on Ω.*

(i) *M has no subgroup of index 56.*

Proof. The maximal subgroups of M_{22} and their orbits on Ω are listed in [**3**, Table 10.3, p. 285]. We use this table without further reference. In particular, (a) and (b) hold. From the definition of a Steiner system, the set $\Omega\backslash\{\omega\}$ together with the set of hexads containing ω form a projective plane of order 4. In particular both the two-point stabilizer and the stabilizer of an incident point hexad pair have shape $2^4\mathrm{SL}_2(4) = 2^4\,\mathrm{Alt}(5)$ where the 2^4 is a natural $\mathrm{SL}_2(4)$-module. Let E^* be the normalizer of a pair of points. Then $E^* \cong 2^4\,\mathrm{Sym}(5)$ and (c) holds.

To prove (h), let f be an element of order 5 in A. Then f has exactly one fixed point η on $\Omega\backslash\{\omega\}$. Clearly A does not fix η and $|\eta^A|$ divides $|A|$ and is congruent to 1 modulo 5. Thus $|\eta^A| = 6$. As f acts fixed-point freely on the remaining 15 points and A has no orbit of length 5, (h) holds.

(g): By (h) we may assume that A has no fixed points on Ω. Then by the same argument as in the proof of (h), A has orbits of length 6 and 10 on $\Omega\backslash\mathscr{H}$.

For (e) and (f) we note that if A fixes a point η outside $\mathscr{H}$, then the action of A on $\Omega\backslash(\mathscr{H} \cup \{\omega\})$ is isomorphic to the action of A on $O_2(D)^\#$. Since in case (f) there exists a unique class of subgroups Alt(5) in $O_2(D)\,A$, (f) holds. In case (e), we need to rule out the possibility that the orbits of A on $\Omega\backslash\mathscr{H}$ have lengths 5, 5 and 6. In this case, the elements of order 3 in A would have 4 fixed points on $\Omega\backslash\mathscr{H}$, but only one fixed point on $O_2(D)$, a contradiction.

(d): Note that B is unique up to conjugation in $\mathrm{Aut}(M_{22})$ an has an orbit Ξ of length 7. Hence A has orbits of length 1 and 6 on Ξ and so is transitive on $\Omega\backslash\Xi$ by (h). As no two hexads can intersect in a set of size 5, the orbit of length 6 is not a hexad. Thus (d) holds.

Since M has no maximal subgroup of index dividing 56, (i) holds. □

For the convenience of the reader we recall the definition of the Higman–Sims group as found [**6**]. Let $(S, \mathscr{B})$ be the Steiner system of type $(3, 6, 22)$. Let $\mathscr{G}$ be the (undirected) graph with vertex set $\{*\} \cup S \cup \mathscr{B}$, where $*$ is a new symbol. In $\mathscr{G}$

(a) the vertex $*$ is joined to each point in S;

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(b) each point $\alpha \in S$ is joined to the 21 hexads containing α;
(c) two hexads are joined if and only if they are disjoint.

Then by [**6**], $\mathrm{Aut}(\mathscr{G})$ is transitive on $\mathscr{G}$ and $\mathrm{Aut}(\mathscr{G})$ has a simple subgroup of index 2 and order 44 352 000, now called the Higman–Sims group HS. We refer to $\mathscr{G}$ as the Higman–Sims graph.

The next two lemmas characterize M_{22} and HS in terms of certain subgroups.

LEMMA 1.3. *Let M be a group and L, M_1 and M_2 be subgroups of M such that $L \cong L_3(4)$, $M_1 \sim 2^4\,\mathrm{Sym}(5)$, $M_2 \sim 2^4\,\mathrm{Alt}(6)$, $L \cap M_1 \cong L \cap M_2 \sim 2^4\,\mathrm{Alt}(5)$, $M_1 \cap M_2 \sim 2^4\,\mathrm{Sym}(4)$, $L \cap M_1 \cap M_2 \sim 2^4\,\mathrm{Alt}(4)$ and $M = \langle L, M_1, M_2\rangle$. Then $M \cong M_{22}$.*

Proof. Put $\Gamma = M/L$ and $\alpha = L \in \Gamma$. Let $t \in M_1 \cap M_2 \backslash L$. Since $M_1 \cap L$ is normal in M_1, we have $M_1 \cap L \leqslant L^t$. As $M_1 \cap M_2$ does not normalize $L \cap M_2$, the element t does not normalize L and so $L \cap L^t = L \cap M_1$. Put $\Gamma_0 = \{\alpha\} \cup \alpha^{tL}$. Then $|\Gamma_0| = 1 + 21 = 22$.

Now $|\alpha^{M_2}| = |M_2/M_2 \cap L| = 6$ and $\alpha^{M_2} = \{\alpha\} \cup \alpha^{t(M_2 \cap L)} \subset \Gamma_0$. Further, as $M_1 = \langle t\rangle (M_1 \cap L)$, we have

$$\alpha^{M_2 M_1} = \alpha^{M_2 \langle t\rangle (M_1 \cap L)} = \alpha^{M_2 (M_1 \cap L)} = \alpha \cup \alpha^{t(M_2 \cap L)(M_1 \cap L)}.$$

Since $M_1 \cap L$ acts transitively on $\alpha^{tL} \backslash \{\alpha\}$ we get $\alpha^{M_2 M_1} = \Gamma_0$. Hence M_1 and L normalize Γ_0. Note that $M_2 = \langle M_2 \cap L, M_2 \cap M_1\rangle$ and so $M = \langle M_1, M_2, L\rangle = \langle M_1, L\rangle$. Thus M normalizes Γ_0, also $\Gamma = \Gamma_0$ and $|M/L| = 22$. Put $B = \alpha^{M_2}$ and $\mathscr{B} = \{B^m \mid m \in M\}$. We claim that $(\Gamma, \mathscr{B})$ is a Steiner system of type $(3, 6, 22)$. Since L is doubly transitive on $\Gamma \backslash \{\alpha\}$, M is triply transitive on Γ. Hence each set of three elements in Γ lies in e elements of $\mathscr{B}$ where e is a positive integer independent of the set of three. Counting tuples (H, a, b, c) such that $H \in \mathscr{B}$ and a, b, c are pairwise different elements of H we get

$$|\mathscr{B}| \cdot 6 \cdot 5 \cdot 4 = 22 \cdot 21 \cdot 20 \cdot e.$$

As $|\mathscr{B}| = |M/M_2| = 22 \cdot |L|/|M_2| = 77$ we get $e = 1$ and the claim is established.

Since $M \leqslant \mathrm{Aut}(\Gamma, \mathscr{B}) \cong \mathrm{Aut}(M_{22})$ and $|M| = 22 \cdot |L| = |M_{22}|$, we deduce that $M \cong M_{22}$. □

LEMMA 1.4. *Let H be a group, M and D be subgroups of H, and L be a subgroup of M. Suppose that each of the following holds:*

(i) *$M \cong M_{22}$, $L \cong L_3(4)$ and $D \sim 2^4\,\mathrm{Sym}(6)$;*

(ii) *there exists $t \in N_H(L)$ with $t^2 \in L$ such that $D \cap M \sim 2^4\,\mathrm{Sym}(5)$, and $D \cap M^t \sim 2^4\,\mathrm{Alt}(6)$;*

(iii) *$H = \langle M, D, t\rangle$.*

Then H is isomorphic to HS, $C_2 \times HS$ or $\mathrm{Aut}(HS)$.

Proof. Let Γ be the graph whose vertices are the right cosets of M in H, and whose edges are the sets $\{Mh, Mth\}$ for $h \in H$. Put $\alpha = M$ and $\beta = Mt$. Then $\beta = \alpha^t$ and $\beta^t = \alpha^{t^2} = \alpha$.

Note that $\langle H_\alpha, H_{\{\alpha,\beta\}}\rangle = \langle M, L\langle t\rangle\rangle$ and $D = \langle D \cap M, D \cap M^t\rangle$, so $\langle H_\alpha, H_{\{\alpha,\beta\}}\rangle = H$ and Γ is connected.

Let $\triangle^i(\alpha)$ be the set of vertices at distance exactly i from α and put $\triangle(\alpha) = \triangle^1(\alpha)$. Then $|\triangle(\alpha)| = 22$, $L = H_{\alpha\beta}$ and L acts transitively on $\triangle(\beta) \backslash \{\alpha\}$. Let $r \in D \cap M \backslash L$ and put $\gamma = \alpha^{r^t} = \alpha^{trt}$. Then $\{\beta, \gamma\} = \{\alpha, \beta\}^{rt}$ and $\{\beta, \gamma\}$ is an edge. Since $r \in D$ and $D \cap M^t$

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is normal in D, r normalizes $D \cap M^t$. Moreover, $t^2 \in L \leqslant M$ and so r^t normalizes $D^t \cap M$. Thus $D^t \cap M \leqslant H_\alpha \cap H_\alpha^{r^t} = H_{\alpha\gamma}$. In particular γ is not adjacent to α. Also if $\gamma = \alpha$ then $\alpha^{tr} = \alpha^t$ and so $r \in M \cap M^t = L$, a contradiction. Thus $\gamma \in \triangle^2(\alpha)$.

Suppose that $H_{\alpha\gamma} \neq D^t \cap M$. As $D^t \cap M$ is maximal in M we get $H_{\alpha\gamma} = H_\alpha = H_\gamma$, and $\triangle(\alpha) \cap \triangle(\gamma) = \beta^{H\alpha} = \triangle(\alpha) = \triangle(\gamma)$, and since Γ is connected, $\Gamma = \{\alpha, \gamma\} \cup \triangle(\alpha)$. Thus L fixes exactly the vertices α, β and γ. Hence $\gamma = \gamma^t$ and $t \in H_\gamma = H_\alpha$, a contradiction. We have proved the following statement.

(1) H acts transitively on geodesics of length 2, $H_{\alpha\gamma} \sim 2^4 \mathrm{Alt}(6)$, $|\triangle^2(\alpha)| = 77$, $|\triangle(\alpha) \cap \triangle(\gamma)| = 6$ and $H_{\alpha\beta\gamma} \sim 2^4 \mathrm{Alt}(5)$.

By Lemma 1.2(c), $H_{\alpha\beta\gamma}$ acts transitively on $\triangle(\gamma) \backslash \triangle(\alpha)$. Let $\delta \in \triangle(\gamma) \backslash \triangle(\alpha)$. Then $H_{\alpha\gamma\delta} \cong \mathrm{Alt}(6)$ and $H_{\alpha\beta\gamma\delta} \cong \mathrm{Alt}(5)$. Suppose that δ is at distance 2 from α. Then $\triangle^3(\alpha) = \varnothing$; thus $\Gamma = \{\alpha\} \cup \triangle(a) \cup \triangle^2(\alpha)$ and $|\Gamma| = 100$. It is now easy to see that Γ is isomorphic to the Higman–Sims graph (see for example [**17**] for a formal proof). As $|H| = 100 \cdot |M_{22}| = |HS|$, $H \cong HS$.

Hence we may assume from now on that δ is not at distance 2 from α. Statement 2 follows.

(2) $\delta \in \triangle^3(\alpha)$ and H acts transitively on geodesics of length 3.

By Lemma 1.2(h), $H_{\alpha\gamma\delta}$ has orbits of lengths 1, 6 and 15 on $\triangle(\delta)$. Further by statement (2), $H_{\alpha\delta}$ acts transitively on $\triangle^2(\alpha) \cap \triangle(\delta)$. Since $\triangle(\beta) \cap \triangle(\delta) \subset \triangle^2(\alpha) \cap \triangle(\delta)$, we get $|\triangle^2(\alpha) \cap \triangle(\delta)|$ is 7, 16 or 22.

Suppose that $|\triangle^2(\alpha) \cap \triangle(\delta)| = 7$. Then $|H_{\alpha\delta}| = 7 \cdot |H_{\alpha\gamma\delta}|$ and $H_{\alpha\delta} \cong \mathrm{Alt}(7)$. Now $H_{\alpha\gamma} \sim 2^4.\mathrm{Alt}(6)$ and $H_{\alpha\gamma} \cap H_{\alpha\delta} = H_{\alpha\gamma\delta} \cong \mathrm{Alt}(6)$. This contradicts Lemma 1.2(d).

Suppose that $|\triangle^2(\alpha) \cap \triangle(\delta)| = 22$. Then $|H_{\alpha\delta}| = 22 \cdot |H_{\alpha\gamma\delta}| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$ and $|H_\alpha / H_{\alpha\delta}| = 56$, a contradiction to Lemma 1.2(i).

Thus $|\triangle^2(\alpha) \cap \triangle(\delta)| = 16$ and $|H_{\alpha\delta}| = 16 \cdot |\mathrm{Alt}(6)|$. Since $H_{\alpha\delta}$ acts non-trivially on the six points in $\triangle(\delta) \backslash \triangle^2(\alpha)$, we conclude that $H_{\alpha\delta}$ has a factor group Alt(6) or Sym(6). This implies that $H_{\alpha\delta} \sim 2^4 \mathrm{Alt}(6)$ and $\triangle(\delta) \backslash \triangle^2(\alpha)$ is a hexad in the H_δ-invariant Steiner system on $\triangle(\delta)$. Since $H_{\alpha\gamma\delta}$ has orbits of lengths 1 and 15 on $\triangle^2(\alpha) \cap \triangle(\delta)$, the group $H_{\alpha\beta\gamma\delta}$ ($\cong \mathrm{Alt}(5)$) fixes a unique point in $\triangle^2(\alpha) \cap \triangle(\delta)$, namely γ. Note that $H_{\alpha\beta\delta} \cong H_{\alpha\gamma\delta} \cong \mathrm{Alt}(6)$ and that $H_{\alpha\beta\delta}$ does not fix γ. Therefore $H_{\alpha\beta\delta}$ fixes no point in $\triangle^2(\alpha) \cap \triangle(\delta)$ and by Lemma 1.2(g), $H_{\alpha\beta\delta}$ has orbits of lengths 6 and 10 on $\triangle^2(\alpha) \cap \triangle(\delta)$. Now parts (e) and (f) of Lemma 1.2 imply that $H_{\alpha\beta\gamma\delta}$ has orbits of lengths 1, 5 and 10 on $\triangle^2(\alpha) \cap \triangle(\delta)$ and acts transitively on $\triangle(\delta) \backslash \triangle^2(a)$. Let $\varepsilon \in \triangle(\delta) \backslash \triangle^2(\alpha)$. Then $H_{\alpha\delta\varepsilon} \sim 2^4 \mathrm{Alt}(5)$. Since $H_{\alpha\delta\varepsilon}$ lies in a unique subgroup $2^4.\mathrm{Alt}(6)$ of H_α and since the stabilizer in H_α of points at distance 3 from α are $2^4 \mathrm{Alt}(6)$, we get that δ is the unique point in $\triangle^3(\alpha)$ fixed by $H_{\alpha\delta\varepsilon}$. In particular $\varepsilon \notin \triangle^3(\alpha)$. We have proved statement 3.

(3) $H_{\alpha\delta} \sim 2^4 \mathrm{Alt}(6)$, $|\triangle^3(\alpha)| = 77$, $\varepsilon \in \triangle^4(\alpha)$, $H_{\alpha\delta\varepsilon} \sim 2^4 \mathrm{Alt}(5)$, $H_{\alpha\beta\gamma\delta\varepsilon} \cong D_{10}$ and H acts transitively on geodesics of length 4.

By statement 1 and Lemma 1.2(b), a subgroup $2^4 \mathrm{Alt}(5)$ of $H_{\beta\gamma}$ which has orbits of lengths 1, 1 and 20 on $\triangle(\beta)$ has orbits of lengths 1, 5 and 16 on $\triangle(\gamma)$. Note that $\triangle(\delta) \backslash \triangle^2(\alpha)$ is an orbit of length 16 for $H_{\alpha\delta\varepsilon}$ on $\triangle(\delta)$, and so $H_{\alpha\delta\varepsilon}$ has orbits of lengths 1, 1 and 20 on $\triangle(\varepsilon)$. Since $H_{\alpha\varepsilon}$ acts transitively on $\triangle^3(\alpha) \cap \triangle(\varepsilon)$ and since $\triangle(\gamma) \cap \triangle(\varepsilon)$

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$\subset \triangle^3(\alpha) \cap \triangle(\varepsilon)$, we get that $|\triangle^3(\alpha) \cap \triangle(\varepsilon)|$ is 21 or 22. Suppose that $|\triangle^3(\alpha) \cap \triangle(\varepsilon)| = 22$, then $|H_{\alpha\varepsilon}| = 22 \cdot |H_{\alpha\delta\varepsilon}| = 22 \cdot |2^4 \mathrm{Alt}(5)|$ and thus $|H_\alpha / H_{\alpha\varepsilon}| = 21$, a contradiction. Thus $|\triangle^3(\alpha) \cap \triangle(\varepsilon)| = 21$ and $\triangle(\varepsilon) \setminus \triangle^3(\alpha) = \{\eta\}$ for some η. Thus $H_{\alpha\varepsilon} = H_{\alpha\varepsilon\eta} \cong L_3(4)$. Thus $|\triangle^4(\alpha)| = 22$ and ε is the unique point in $\triangle^4(\alpha)$ fixed by $H_{\alpha\varepsilon\eta}$ and so $\eta \notin \triangle^4(\alpha)$. Hence we have statement 4.

(4) $H_{\alpha\varepsilon} \cong L_3(4)$, $|\triangle^4(\alpha)| = 22$, $\eta \in \triangle^5(\alpha)$, $H_{\alpha\varepsilon\eta} = H_{\alpha\varepsilon}$ and H is transitive on geodesics of length 5.

Since $H_{\alpha\eta}$ is transitive on $\triangle^4(\alpha) \cap \triangle(\eta)$ and $\triangle(\delta) \cap \triangle(\eta) \subset \triangle^4(\alpha) \cap \triangle(\eta)$ we conclude that $\triangle^4(\alpha) \cap \triangle(\eta) = \triangle(\eta)$ and $H_{\alpha\eta} = H_\alpha \cong M_{22}$. Thus we have statement 5.

(5) $H_{\alpha\eta} = H_\alpha$, $\triangle^5(\alpha) = \{\eta\}$, $\Gamma = \sum_{0 \leqslant i \leqslant 5} \triangle^i(\alpha)$ and $|\Gamma| = 200$.

Let ϕ be the map that sends a vertex μ in Γ to the unique point at distance 5 from μ. Then ϕ is obviously a bijection and $(\mu^\phi)^g = (\mu^g)^\phi$ for all μ in Γ and g in $\mathrm{Aut}(\Gamma)$. In particular $\{\alpha, \beta, \alpha^\phi, \beta^\phi\}$ is the set of fixed points of $H_{\alpha\beta}$ on Γ, so β^ϕ is adjacent to α^ϕ and therefore ϕ is a graph automorphism of Γ. Put $\Gamma_0 = \{\{\mu, \mu^\phi\} \mid \mu \in \Gamma\}$ and let $\{\mu, \mu^\phi\}$ be adjacent to $\{\lambda, \lambda^\phi\}$ if μ is adjacent to λ or λ^ϕ. Then $|\Gamma_0| = 100$, M has orbits of lengths 1, 22 and 77 on Γ_0 and H acts transitively on Γ_0. Therefore as above Γ_0 is the Higman–Sims graph and $\mathrm{Aut}(\Gamma_0) \cong \mathrm{Aut}(HS)$. Let N be the kernel of the action of $\mathrm{Aut}(\Gamma)$ on Γ_0. We claim that $N = \{1, \phi\}$. Indeed, let $n \in N$. Then $\alpha^n \in \{\alpha, \alpha^\phi\}$ and replacing n by $n\phi$ if necessary we may assume that $\alpha^n = \alpha$. Since β^ϕ is not adjacent to α, we have $\beta^n = \beta$, so n fixes the neighbors of all its fixed points and since Γ is connected, we conclude that $n = 1$. Thus $N = \{1, \phi\}$ and $\mathrm{Aut}(\Gamma) \cong C_2 \times \mathrm{Aut}(HS)$. Further, $|H| = 200 \cdot |M| = 2 \cdot |HS|$ and so $H \cong \mathrm{Aut}(HS)$ or $C_2 \times HS$ and the lemma is proved. □

LEMMA 1.5. (a) *G has a subgroup T of order 3 with $N_G(T) \sim (\mathrm{SU}_6(5) \circ \mathrm{SU}_3(5)).\mathrm{Sym}(3)$.*

(b) *Put $U = N_G(T)$, $U_1 = [V, T]$ and $U_2 = C_V(T)$. Then $V = U_1 \oplus U_2$, moreover, U_1 and U_2 are irreducible as U modules, $U_1 \cong W_6 \otimes_{\mathrm{GF}(25)} W_3$ and $\mathrm{GF}(25) \otimes_{\mathrm{GF}(5)} U_2 \cong \bigwedge^3 W_6$, where W_i is the natural i-dimensional $\mathrm{GF}(25)$-module for $\mathrm{SU}_i(5)$, for $i = 3, 6$.*

(c) *$N_G(U_1) = U$.*

Proof. It is clear from the extended Dynkin diagram of type E_7 that $2.E_7(K)$ has a subgroup $H \cong \mathrm{SL}_3(K) \circ \mathrm{SL}_6(K)$. Moreover, the central involution of the Weyl group induces a graph automorphism on both of the factors and so an application of Lang's theorem yields a subgroup $(\mathrm{SU}_6(5) \circ \mathrm{SU}_3(5)).\mathrm{Sym}(3)$ in $E_7(5)$ (see [**5**, entry 29, p. 34]). Using the embedding of $K^{1+56}2.E_7(K)$ in $E_8(K)$, the Steinberg relations and weight theory it is easy to check that the 56-dimensional $KE_7(K)$-module is as an H-module the direct sum of $X_3 \otimes X_6$, $X_3^* \otimes X_6^*$ and $\bigwedge^3 X_6$ where X_i is a natural module for $\mathrm{SL}_i(K)$, $i = 3, 6$. As H is a maximal connected closed subgroup of $2.E_7(K)$ and is of index 2 in its normalizer, all the statements of the lemma are now readily verified. □

LEMMA 1.6. *There exists an involution $\bar{f}$ in $\bar{U} \setminus \bar{U}'$ such that $C_{\bar{U}}(\bar{f}) \sim L_4(5).2^2 \times \mathrm{Sym}(5) \times \langle \bar{f} \rangle$. Moreover, for any such $\bar{f}$,*

$$C_{\bar{G}}(\bar{f}) \sim (C_2 \times L_8(5)).2$$

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and $C_{\bar{U}}(\bar{f})$ acts transitively on the elementary abelian subgroups of order 16 *contained in the normal subgroup* $L_4(5)$ *in* $C_{\bar{U}}(f)$.

Proof. It is easy to verify that $\mathrm{Aut}(U_6(5))$ has three classes of involutions that do not induce a diagonal automorphism on $U_6(5)$. The derived groups of the respective centralizers in $U_6(5)$ are $P\mathrm{Sp}_6(5)$, $D_3(5)$ and ${}^2D_3(5)$, and the first two of these three classes of involutions lie in the same coset of $\mathrm{Inn}(U_6(5))$.

Since $\Omega_6^-(5)$ ($\sim 2.{}^2D_3(5)$) acts absolutely irreducibly on the exterior cube of its natural module, we conclude that the centralizer of $\Omega_6^-(5)$ acts as scalars on the 20-dimensional module. Hence, any involution of $\bar{U}\backslash\bar{U}'$ that centralizes a ${}^2D_3(5)$ would also centralize the $U_6(5)$, a contradiction, as it induces a graph automorphism thereon. It follows that $\bar{U}\backslash\bar{U}'$ contains no involution centralizing a ${}^2D_3(5)$ in $U_6(5)$ and so contains an involution centralizing a $D_3(5)$ in $U_6(5)$.

Let $\bar{f}$ be any such involution. Note that the normalizer of $D_3(5)$ in $U_6(5)$ is $\mathrm{PSO}_6^-(5)$ extended by an involution which multiplies the quadratic form associated to $\mathrm{PSO}_6^-(5)$ by a fourth root of unity. So $C_{U_6(5)}(\bar{f}) \sim L_4(5).2^2$ and the two classes of elementary abelian subgroups of order 16 in $C_{U_6(5)}(\bar{f})'$ are fused in $C_{U_6(5)}(\bar{f})$. Moreover, $\mathrm{Aut}(U_3(5))$ has exactly one class of involutions outside $\mathrm{Inn}(U_3(5))$ and the corresponding centralizers are $\mathrm{Sym}(5)$. Therefore $C_{\bar{U}'}(\bar{f}) \sim L_4(5).2^2 \times \mathrm{Sym}(5)$.

It remains to determine $C_{\bar{G}}(\bar{f})$. Note that $\bar{G}$ has three classes of involutions whose centralizers have shapes $2.(L_2(5) \times D_6(5)).2$, $(C_2 \times E_6(5)).2$ and $(C_2 \times L_8(5)).2$ [**7**]. Under the actions of the derived groups of these centralizers V decomposes into direct sums of irreducible modules of dimensions 24 and 32; 1, 1, 27, and 27; and 28 and 28, respectively. On the other hand, from the action on the 20- and 36-spaces, we know that V is the direct sum of irreducible modules of dimensions 18, 18, 10 and 10 for $C_U(f)'$. It follows that $C_{\bar{G}}(\bar{f}) \sim (C_2 \times L_8(5)).2$, and the lemma is proved. □

From now on let T, U be as in Lemma 1.5 and let f be a preimage in G of the involution $\bar{f}$ of Lemma 1.6. Note that f^2 is the central involution of G. Put $N = C_U(f)$, $F = \langle f \rangle$ and $R = N_G(F)$. Then $\bar{R} = C_{\bar{G}}(\bar{f})$ and so by Lemma 1.6, $R \sim C_4 \circ 2 \cdot L_8(5).2$. Let $\hat{R}'$ be a group with $\hat{R}' \cong \mathrm{SL}_8(5)$ so that $\hat{R}'$ has R' as a quotient group. For X in R', let $\hat{X}$ be the inverse image of X in $\hat{R}'$. As $Z(\bar{N}') = 1$, it is easy to see that the natural 8-dimensional module for $\hat{R}'$ is, as a module for $\hat{N}'$, the tensor product of a 4-dimensional module for $\mathrm{SL}_4(5)$ and a 2-dimensional module for $\mathrm{SL}_2(5)$. Let $H = N_{\mathrm{GL}_8(5)}(\hat{N}') \cong \mathrm{GL}_4(5) \otimes \mathrm{GL}_2(5)$. If $a \in \mathrm{GL}_4(5)$, and $b \in \mathrm{GL}_2(5)$, then $\det(a \otimes b) = \det(a)^2 \cdot \det(b)^4$ is a square. Therefore all elements of H have determinant ± 1. So $H\mathrm{SL}_8(5) \neq \mathrm{GL}_8(5)$. Now, the Frattini argument implies that $\mathrm{SL}_8(5)$ does not act transitively on the orbit of $\hat{N}'$ under $\mathrm{GL}_8(5)$. Hence, $\bar{R}'$ has two classes of subgroups isomorphic to $L_4(5) \times L_2(5)$. Since $R = NR'$, R fixes these classes. Moreover, as the element inverting f can be chosen to invert a Cartan subgroup of R', R does not induce an outer diagonal automorphism on R'.

The character tables of $L_3(4)$ and its covers (see [**8**, p. 56]) show that there is a group of shape $4.L_3(4)$ which has a faithful irreducible character χ of degree 8. Hence R contains a subgroup L of shape $2.L_3(4)$. We plan to extend L to a subgroup M_{22} and then to a subgroup HS of G. Let S be a Sylow 2-subgroup of L, let $B = N_L(S)$ and let A_1 and A_2 be the two elementary abelian groups of order 2^5 in S. Note that $\hat{L}$ is perfect and that $\hat{A}_i$ is the central product of a cyclic group of order 4 with an extra-special group of order 2^5. Moreover, a natural 8-dimensional module for $\hat{R}'$ is

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as an $\hat{A}_i$-module the direct sum of two isomorphic irreducible modules of dimension 4. Thus $C_{\hat{R}'}(\hat{A}_i) \cong \mathrm{GL}_2(5)$. Note that A_1 is conjugate under $\mathrm{GL}_8(5)$ to a subgroup of the normal $L_4(5)$ in N. Thus replacing L by a conjugate under $\mathrm{GL}_8(5)$ we may assume that $\bar{A}_1$ is contained in the normal $L_4(5)$ in $\bar{N}$. It follows that $N_{\bar{R}'}(\bar{A}_1) \leqslant N$ and has shape $2^4\,\mathrm{Sym}(6) \times \mathrm{Sym}(5)$. Let $\bar{B}_2$ be the projection of A_2 onto the $L_4(5)$ in N. Put $L_i = N_L(A_i)$. Our goal is to show that both L_i can be extended to a subgroup $2^5\,\mathrm{Sym}(6)$ in U. Inside $2^5\,\mathrm{Sym}(6)$ we will choose appropriate subgroups $2^5\,\mathrm{Sym}(5)$ and $2^5\,\mathrm{Alt}(6)$ which will allow us to apply Lemma 1.3 and Lemma 1.4 to find M_{22} and HS in $\bar{G}$.

LEMMA 1.7. *In R', the subgroup A_2 is conjugate to B_2 but not to A_1.*

Proof. Put $C_i = O^2(C_{\hat{R}'}(\hat{A}_i))$. Then $C_i \cong \mathrm{SL}_2(5)$. Suppose that $[S, C_1] = 1$. Then $C_1 \leqslant C_{\hat{R}'}(\hat{A}_2)$ and $C_1 = C_2$. Since $L = \langle L_1, L_2 \rangle$, the group L normalizes C_1, a contradiction. Hence $[S, C_1] \neq 1$ and similarly $[S, C_2] \neq 1$. As $B/A_i \cong \mathrm{Alt}(4)$, we have $B = O^2(B)\,A_i$. Since $\mathrm{Out}(C_i)$ is a 2-group and A_i centralizes C_i we conclude that S induces inner automorphisms on C_i. Hence S normalizes a unique subgroup Q_8 in C_i. Denote this Q_8 by Q_i and note that S induces inner automorphisms on Q_1. In particular $[\hat{A}_2, Q_1] \leqslant Z(Q_1) \cdot Z(\hat{R}')$. Thus Q_1 induces inner automorphisms on $\hat{A}_2$. Put $R_i = Q_1\,\hat{A}_i$ (where we really mean Q_1 and not Q_i). Then R_1 and R_2 are both the central product of an extra-special group of order 2^7 with a cyclic group of order 4. Put $X = C_{R_2}(\hat{A}_2)$. It follows that $R_2 = X\hat{A}_2$, that S normalizes X and that X contains a Q_8 invariant under S. Thus $Q_2 \leqslant X$, $R_2 = Q_2\,\hat{A}_2 = Q_2\,B_2$ and $\hat{B}_2 = C_{R_2}(Q_1)$.

Note that $N_{\hat{R}'}(R_2)/R_2 \cong \mathrm{Sp}_6(2)$. Also $\bar{A}_2$ and $\bar{B}_2$ are both non-degenerate 4-spaces in the 6-dimensional symplectic space $R_2/Z(R_2)$ (where the symplectic form is given by the commutator map). Hence by Witt's theorem, A_2 and B_2 are conjugate under $N_{\hat{R}'}(R_2)$, proving the first statement of the lemma.

Since $C_{\hat{R}'}(\hat{A}_i)$ acts transitively on the Q_8 in $C_{\hat{R}'}(\hat{A}_i)$ and since $N_{\hat{R}'}(A_1) \cap N_{\hat{R}'}(Q_1)$ induces the full automorphism group of Q_1 on Q_1, we conclude that A_1 and B_2 are conjugate in R' if and only if they are conjugate under $Y = C_{\hat{R}'}(Q_1)$. Note that Y is isomorphic to the subgroup of index 2 in $\mathrm{GL}_4(5)$. Since B normalizes Y and $[A_1, B] = A_1$, A_1 and hence also B_2 are contained in Y'. Put $S_0 = A_1\,B_2$. Since A_1 and B_2 normalize each other we conclude that S_0 is a Sylow 2-subgroup of $Y/Z(Y)$ and that A_1 and B_2 are the two maximal elementary abelian subgroups of S_0. It follows that A_1 and B_2 are not conjugate in Y'. Moreover, $\mathrm{Out}(Y') \cong D_8$, $\mathrm{Out}(Y')$ acts on $\{A_1^{Y'}, B_2^{Y'}\}$ and $\mathrm{Out}(Y')'$ fixes $A_1^{Y'}$ and $B_2^{Y'}$. As the group of outer automorphisms of Y' induced by Y is $\mathrm{Out}(Y')'$, A_1 and B_2 are not conjugate in Y. Hence A_1 and B_2 are not conjugate in R' and lemma is established. □

LEMMA 1.8. *$N_{\bar{R}'}(\bar{A}_1)$ acts transitively on $L_3(4)$ in $\bar{R}'$ containing $\bar{A}_1$ and on subgroups $2^4\,\mathrm{Alt}(5)$ in $N_{\bar{R}'}(\bar{A}_1)$ which can be extended to an $L_3(4)$ in $\bar{R}'$.*

Proof. Recall that $N_{\bar{R}'}(\bar{A}_1) \sim 2^4\,\mathrm{Sym}(6) \times \mathrm{Sym}(5)$ and, as $[S, C_i] \neq 1$ in the notation of Lemma 1.7, any $2^4\,\mathrm{Alt}(5)$ in $N_{\bar{R}'}(\bar{A}_1)$, which can be extended to an $L_3(4)$ in $\bar{R}'$, projects non-trivially on the Sym(5). Hence $N_{\bar{R}'}(\bar{A}_1)$ acts transitively on such subgroups. Note that $\bar{L}_1$ is such a subgroup and that

$$L_1^* \overset{\text{def}}{=} N_{\bar{R}'}(\bar{L}_1) \sim 2^4\,\mathrm{Sym}(5).$$

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Put $\bar{W} = N_{\bar{R}'}(\bar{L}_1) \cap N_{\bar{R}'}(\bar{A}_2)$. As $\bar{A}_1$ and $\bar{A}_2$ are the only maximal elementary abelian subgroups of $\bar{S} = \bar{A}_1\bar{A}_2$, we have $N_{L_1^*}(A_2) = N_{L_1^*}(S)$. Since S/A_1 is the Sylow 2-subgroup of L_1/A_1, we have $\bar{W} \sim 2^4\,\mathrm{Sym}(4)$. We claim that W is contained in a $2^4\,\mathrm{Alt}(6) \times \mathrm{Sym}(5)$ subgroup of $N_{\bar{R}'}(\bar{A}_1)$. For this let i be an involution in $\bar{W}\backslash\bar{L}_1$. Then it is enough to show that $[\hat{A}_2, i]$ is not elementary abelian. Let C_2 be the projection of A_2 onto the normal Sym(5) in $\bar{N}$. Then $[\hat{B}_2, i]$ is elementary abelian, while $[\hat{C}_2, i]$ is not, so also $[\hat{A}_2, i]$ is not elementary abelian, and the claim is proved.

It is now easy to see that $\bar{W} \cap \bar{L}_1$ is contained in exactly two subgroups $2^4\,\mathrm{Alt}(5)$ of $N_{\bar{R}'}(\bar{A}_2)$ and these two subgroups are interchanged by W. Since any $L_3(4)$ in $\bar{R}'$ containing $\bar{L}_1$ is generated by $\bar{L}_1$ and a $2^4\,\mathrm{Alt}(5)$ in $N_{\bar{R}'}(\bar{A}_2)$ containing $\bar{W} \cap \bar{L}_1$ the lemma is proved. □

LEMMA 1.9. *Put $L_0 = N_R(L)$. Then $|L_0/LF| = 2$ and A_1 and A_2 are conjugate in L_0.*

Proof. Since $\mathrm{Out}(R'/Z(R')) \cong D_8$, since R' has four orbits on $A_1^{\mathrm{Aut}(R')}$ and since $\mathrm{PGL}_8(5)$ acts transitively on those four orbits, the normalizer of any such orbit in $\mathrm{Out}(R')$ is a group of order 2 that is not contained in the center of $\mathrm{Out}(R')$. It follows that R fixes two of these orbits and interchanges the remaining two. Moreover, by Lemma 1.8, $L_8(5)$ has four orbits on $(A_1, L)^{\mathrm{Aut}(R')}$ and, by Lemma 1.7, each orbit of subgroups $L_3(4)$ in $\bar{R}'$ leads to two orbits of R' on $(A_1, L)^{\mathrm{Aut}(R')}$. It follows that $\bar{R}'$ has exactly two classes of subgroups $L_3(4)$. Suppose that R interchanges those two classes. Then R could not normalize any of the orbits of R' on $A_1^{\mathrm{Aut}(R')}$, a contradiction. Hence $|L_0/LF| = 2$. Suppose that A_1 and A_2 are not conjugate in L_0. Then they are also not conjugate in R. By Lemma 1.7, we conclude that A_1 and B_2 are not conjugate in R, and this contradicts Lemma 1.6. Hence A_1 and A_2 must be conjugate in L_0. □

LEMMA 1.10. *Put $X = N_G(A_1)$. Then $X \leqslant U$ and $X \sim 6.(2^4.3^4.\mathrm{Sym}(6) \times U_3(5)).\mathrm{Sym}(3)$.*

Proof. Let $X_0 = N_U(A_1)$. By the action of A_1 on the natural 6-dimensional GF(25)-module W for $\mathrm{SU}_6(5)$ we see that, modulo the normal $\mathrm{SU}_3(5)$ in U, X_0 is a full monomial subgroup of $U_6(5).\mathrm{Sym}(3)$. It follows that $X_0 \sim 6.(2^4.3^4.\mathrm{Sym}(6) \times U_3(5)).\mathrm{Sym}(3)$. Moreover, X_0 has two orbits Σ_1 and Σ_2 on the hyperplanes in A_1 which do not contain $Z(G)$. Choose notation so that $|\Sigma_1| = 6$ and $|\Sigma_2| = 10$. Recall that $V \cong U_1 \oplus U_2$ where U_1 and U_2 are defined and described in Lemma 1.5.

Let H_1, H_2, H_3 be three different elements of Σ_1. Then it is easy to check that $H_1 \cap H_2 \not\leqslant H_3$. As $W = \bigoplus_{H \in \Sigma_1}(C_W(H))$, this implies that H_1 acts fixed-point freely on U_2. Since U_1 is a direct sum of copies of W as an A_1-module, we get $\bigoplus_{H \in \Sigma_1} C_V(H) = U_1$.

Also $C_V(H)$ is either 6-dimensional, if $H \in \Sigma_1$, or $20/10 = 2$-dimensional, if $H \in \Sigma_2$. Thus $N_G(A_1)$ normalizes Σ_1 and so also U_1. Since U is maximal in G, we conclude that $X \leqslant U$. Hence $X = X_0$. □

LEMMA 1.11. *There exist $t \in N_R(L)$ and $D \leqslant U$ such that $t^2 \in L$, $A_1^t = A_2$, $L_1 \leqslant D$, $D \sim 2^5\,\mathrm{Sym}(6)$, t normalizes $N_D(B)$ and t does not normalize $N_{D'}(B)$.*

Proof. Let $Y = X/A_1$. Then $Y \sim 3.(3^4\mathrm{Sym}(6) \times U_3(5))\mathrm{Sym}(3)$. Let K be the image of L_1 in Y. Then $K \cong \mathrm{Alt}(5)$ and we are looking for subgroups Sym(6) of Y containing

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K. Let Y_1 be the normal subgroup $3.3^4\mathrm{Sym}(6)$ of Y, let Y_2 be the normal subgroup $\mathrm{SU}_3(5)$ of Y and let K_i be the projection of K onto Y_i. Then Y acts transitively on subgroups $3\cdot\mathrm{Alt}(6)$ (the triple cover of Alt(6)) and transitively on subgroups Alt(5) in Y_2. Moreover, the normalizer of an Alt(5) in $U_3(5).\mathrm{Sym}(3)$ is a $C_2\times\mathrm{Sym}(5)$ and the normalizer of an Alt(6) is Aut(Alt(6)). It follows that K_2 can be embedded into exactly two subgroups $3\cdot\mathrm{Alt}(6)$ of Y_2 and f interchanges these two $3\cdot\mathrm{Alt}(6)$. Furthermore, Y has one orbit on subgroups $3\cdot\mathrm{Alt}(6)$ in Y_1 and one orbit on subgroups Alt(5) in Y_1 for which $\bar{A}_1$ is a natural $\mathrm{SL}_2(4)$-module. (Note here that $3\cdot\mathrm{Alt}(6)$ exists in Y_1, since $3^4\,\mathrm{Alt}(6)$ has nine classes of Alt(6) while $3^{1+4}\,\mathrm{Alt}(6)$ has only three classes.) Now $Y/Y_2\sim 3^{4+1}(C_2\times\mathrm{Sym}(6))$. The normalizer of the image of K in Y/Y_2 is a $C_2\times\mathrm{Sym}(5)$ and the normalizer of the image of a $3\cdot\mathrm{Alt}(6)$ of Y_1 in Y/Y_2 is a Sym(6). It follows that K_1 can be embedded into exactly two subgroups $3\cdot\mathrm{Alt}(6)$ of Y_1 and f interchanges these two $3\cdot\mathrm{Alt}(6)$. Let D_i be any of the two subgroups $3\cdot\mathrm{Alt}(6)$ in Y_i with $K_i\leqslant D_i$. Then there exists precisely one subgroup A in D_1D_2 such that $K\leqslant A$, $A=A'$ and $A/Z(A)\cong\mathrm{Alt}(6)$. Similarly there exists precisely one subgroup $\hat{A}$ in $D_1^fD_2$ such that $K\leqslant\hat{A}$, $\hat{A}=\hat{A}'$ and $\hat{A}/Z(\hat{A})\cong\mathrm{Alt}(6)$. We claim that exactly one of A and $\hat{A}$ is an Alt(6), while the other is the triple cover. Indeed, let x_i, y_i be elements of order 3 in D_i such that $\langle x_1x_2, y_1y_2\rangle Z(Y)$ is a Sylow 3-subgroup of $AZ(Y)$. Then $\langle {x_1}^f{x_2}^f, y_1y_2\rangle Z(Y)$ is a Sylow 3-subgroup of $\hat{A}Z(Y)$. Note that

$$[x_1x_2, y_1y_2]=[x_1,y_1][x_2,y_2],\quad [{x_1}^f{x_2}^f, y_1y_2]=[x_1,y_1]^f[x_2,y_2]. \tag{1}$$

Since $[x_1,y_1]$ and $[x_2,y_2]$ are both contained in $Z(Y)$ and unequal to 1 and since f inverts $Z(Y)$, we see that exactly one of the two expressions in (1) is equal to 1. This proves our claim.

Choose notation so that $A\cong\mathrm{Alt}(6)$. From what we have proved so far it follows that A and A^f are the only two Alt(6) subgroups in Y which contain K. It is easy to see that $N_Y(K)/Z(Y)\cong C_2\times\mathrm{Sym}(5)$. Furthermore, there exists a subgroup of index 2 in $N_Y(K)$ which normalizes A. Since f does not normalize A, this subgroup is, modulo the center of Y, isomorphic to Sym(5). Put $E=A(N_Y(K)\cap N_Y(A))$ and let $\hat{D}$ be the inverse image of E in X. Then $E/Z(Y)\cong\mathrm{Sym}(6)$ and $\hat{D}/T\cong 2^5\,\mathrm{Sym}(6)$. (Recall that T is the cyclic group of order 3 with $U=N_G(T)$.)

By Lemma 1.9, there exists $t\in N_R(L)$ with $A_1^t=A_2$. Choose t so that t normalizes B. Then $t^2\in BF$ and since t inverts F, we have $t^2\in B$. In particular $t^2\in L$. Note that A_1 and A_2 are the only elementary abelian groups of order 2^5 in B. Put $J=N_G(B)\cap N_G(A_1)$. As $t\in N_G(B)\backslash J$, we conclude that J is of index 2 in $N_G(B)$. For $Q\subset N_G(B)$ let Q^* be the image of Q in $N_G(B)/B$. Since $J\leqslant X$, it is easily verified that J^* is an elementary abelian group of order 9 extended by an elementary abelian group of order 8. Pick a in $N_{\hat{D}}(B)\backslash BT$ and b in $N_{\hat{D}}(B)$ with $[a,B]\leqslant A_1$ and such that a^* and b^* are involutions. Since $U_3(5)$ contains no Sym(6), b induces an outer automorphism on Y_2. The same is true for f and so b and f both invert $O_3(J^*)$. Note that a centralizes T and inverts $O_3(J^*)/T$. As A_1 is in the $\mathrm{SU}_6(5)$ subgroup of U but A_2 is not, the element t is not in U. Since $T\leqslant O_3(J)$, we conclude that $O_3(J)^*=T^*T^{*t}=O_3(J^*)$. It follows that $\langle a^*,t^*\rangle$ acts as a D_8 on $O_3(J^*)$. Since $\langle a^*,t^*\rangle$ is a dihedral group, we conclude that $\langle a^*,t^*\rangle\cong D_8$. Let $x^*=[a^*,t^*]$.

We claim that $x^*T^*=b^*T^*$. Indeed, x^* inverts $O_3(J^*)$ and $|C_{N_{G'(B)^*}}(O_3(J)^*)/O_3(J^*)|=2$. Hence x^* lies in the same coset of $O_3(J^*)$ as f^* or as b^*. Since $N_{\hat{R}'}(A_1)\sim 2^4.\mathrm{Sym}(6)\times\mathrm{Sym}(5)$, $N_{R'}(B)^*\cong C_2\times C_2$. Furthermore, f and t are in $N_R(B)$ and so $N_R(B)^*$ is a Sylow 2-subgroup of $N_G(B)^*$. Since $f\notin R'$ we conclude that

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$f^* \notin N_G(B)^{*\prime} O_3(J^*) = \langle x^* \rangle O_3(J^*)$. Hence $x^* O_3(J^*) = b^* O_3(J^*)$. Moreover, a^* centralizes b^* and x^* and hence $b^* T^* = C_{x^* O_3(J^*)}(a^*) = x^* T^*$.

In particular, we can choose b so that $b^* = x^* = [a^*, t^*]$. Put $D = \hat{D}'\langle b \rangle$. Then $\bar{D} \sim 2^4 \mathrm{Sym}(6)$. In addition, $N_D(B) = B\langle a, b \rangle$ and so t normalizes $N_D(B)$. On the other hand, $N_{D'}(B) = B\langle a \rangle$ and so $N_{D'}(B)^t = B\langle ab \rangle \neq N_{D'}(B)$. □

Proof of Theorem 1.1. We are now able to construct HS and M_{22} in $\bar{G}$. For this choose L, t and D as in Lemma 1.11 and put $M_1 = N_D(L_1)$. By Lemma 1.10, $N_{D'^t}(B)$ is contained in D, but not in D'. Since $N_{D'^t}(B)/O_2(D) \cong \mathrm{Sym}(4)$, we conclude that $N_{D'^t}(B)$ is contained in M_1. Note that $L_1^t = L_2$ and so $L_2 \leqslant D'^t$. Put $M_2 = D'^t$. Then $\bar{M}_1 \sim 2^4 \mathrm{Sym}(5)$ and $\bar{M}_2 \sim 2^4 \mathrm{Alt}(6)$. Moreover,

$$M_1 \cap L = L_1, \quad M_2 \cap L = L_2, \quad M_1 \cap M_2 = N_{D'^t}(B).$$

Put $M = \langle L, M_1, M_2 \rangle$. Then by Lemma 1.3, $\bar{M} \cong M_{22}$. Now $D \cap M = M_1$, $D^t \cap M = M_2$ and $D \cap D^t = N_D(B)$. Let $H = \langle D, L, t \rangle$. Then by Lemma 1.4, $\bar{H} \cong HS$, $C_2 \times HS$ or $\mathrm{Aut}(HS)$. Thus in any case $\bar{H}' \cong HS$ and Theorem 1.1 is proved. The dedicated reader might check that actually $\bar{H}$ itself is already the Higman–Sims group. □

2. *A computational proof that HS is a subgroup of* $E_7(5)$

In this section we give a computer-dependent proof that $HS < E_7(5)$. Our strategy is to use a machine calculation to prove that HS acts (absolutely) irreducibly on a 133-dimensional, 5-modular, Lie algebra with a non-zero Killing form. We then apply the extensive theory of modular Lie algebras to deduce that the Lie algebra is simple, that it is a classical modular Lie algebra of type E_7 over GF(5), and hence $HS < E_7(5)$.

COMPUTATION 2.1. *We construct an explicit* 133*-dimensional* (*absolutely*) *irreducible matrix representation of HS over* GF(5).

Method. Although it seems natural to construct the 133-dimensional representation as a constituent of a tensor product of smaller representations of HS, all useful tensor products are too large for our implementation of MeatAxe [**10**]. For example, the 133-dimensional, 5-modular representations of HS arise as constituents in the tensor product of a 21-dimensional representation with a 55-dimensional representation and also in the symmetric cube of the 21-dimensional representation; however these representations have degrees 1155 and 1771. In order to avoid such large computations, we shall locate a 133-dimensional representation as a constituent of the symmetric square of a 28-dimensional representation of the double cover of HS; this latter representation can be found in a previously known representation of the Harada–Norton group. A 28-dimensional representation of $2.HS$ is also constructed in [**16**].

We start with the 133-dimensional matrix representation of HN over GF(5) that is constructed in [**13**]. We locate matrices, x_1 and y_1, that represent HN elements of classes $40A$ and $12C$. (Since these classes have small centralizers, we can simply use a random search to locate such elements.) Let $x = x_1^{20}$. In the group HN, every dihedral subgroup generated by the $2A$-element x and a conjugate of the $2B$-element y_1^6 has a central involution. By collecting such involutions together with the matrix

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x_1, it is an easy matter to generate enough matrices to represent the centralizer of x in HN. (We just keep adding involutions until we have a group that contains an element of order 11.)

The group $C_{HN}(x)$ has structure $2.HS.2$. We use MeatAxe [**10**] to decompose the restriction of our 133-dimensional HN-module into its irreducible constituents of degrees 1, 21, 28, 28 and 55, under the group $2.HS = (C_{HN}(x))'$. Another application of MeatAxe to the symmetric square of either of the irreducible 28-dimensional $2.HS$-modules yields an (absolutely) irreducible matrix representation of HS of degree 133. □

Let E denote the 133-dimensional HS-module afforded by the matrix representation of Computation 2.1. In our calculations with E we shall use a fixed basis $e_1, e_2, \ldots, e_{133}$ of E; moreover for technical reasons, we use a basis on which a particular subgroup

$$P \sim 2^4.2^3 < 2^4.(2 \times \mathrm{Sym}(4)) < 2^4.\mathrm{Sym}(6) < HS$$

acts monomially. There are two conjugacy classes of subgroups in $2 \times S_4$ with structure 2^3; thus there are two possibilities for a subgroup of HS with the indicated structure. We choose the possibility for P that fixes a vector in our 133-dimensional representation of HS; the corresponding subgroup of HS contains 39 elements of HS-class $2A$, eight elements of class $2B$, two elements of class $4A$, 54 elements of class $4B$, and 24 elements of class $4C$.

We let $e_1^*, e_2^*, \ldots, e_{133}^*$ denote the dual basis of the dual module E^* (this module is isomorphic to E, but for computational purposes it is convenient to distinguish E and E^*).

COMPUTATION 2.2. *We compute an HS-invariant product*, $\circ : \Lambda^2 E \longrightarrow E$.

Method. We need to find a list of components $a_{i,j,k}$ of an invariant rank 3 tensor such that the map $e_i \wedge e_j \mapsto \sum_k a_{i,j,k} e_k$ extends to an HS-invariant multilinear map. By duality, such a map is HS-invariant if and only if the dual map $\circ^* : E^* \longrightarrow \Lambda^2 E^*$, for which $e_k^* \mapsto \sum_{i,j} a_{i,j,k} e_i^* \wedge e_j^*$ is HS-invariant. Thus, we begin by computing the essentially unique possibility for $\circ^*$.

The module $\Lambda^2 E^*$ has degree 8778. It is too large to decompose directly with our implementation of MeatAxe. In order to locate all copies of E^* in $\Lambda^2 E^*$ we use the condensation techniques described in [**12**]. Let π denote the idempotent $(\sum_{p \in P} p)/|P|$ of the group algebra $\mathrm{GF}(5)\,HS$. The condensation programs of [**12**] compute matrix representations of the Hecke algebra $\pi \mathrm{GF}(5)\,HS\pi$ on the condensed modules $E^*\pi$ and $\Lambda^2 E^*\pi$ which we find have degrees 1 and 50. (It is in the computation of these matrix representations of the Hecke algebra that our programs require the group P to act monomially on E and E^*.) A standard MeatAxe calculation locates the single copy of $E^*\pi$ in the $\pi \mathrm{GF}(5)\,HS\pi$-module $\Lambda^2 E^*\pi$. Thus, there is a single copy of E^* in the HS-module $\Lambda^2 E^*$.

The embedding of Hecke algebra modules that we have calculated gives the image, $\Pi^* \in \Lambda^2 E^*$, of an element π^*, in the 1-dimensional space of fixed points of P on E^*, under an HS-module homomorphism $\circ^*$. Now, if $h \in HS$, we have $(\pi^* h) \circ^* = (\pi^* \circ^*) h = \Pi^* h$. Moreover, since E^* is irreducible, this formula completely determines the effect of $\circ^*$ on a spanning set of E^*. We thus obtain the effect of $\circ^*$ on our chosen basis of E^* and the components of an HS-invariant tensor $a_{i,j,k}$. □

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For each $e \in E$ we write $\circ e$ for the matrix that represents the action of right multiplication by e on our HS-invariant algebra. After completing Computation 2.2, we ran a simple precautionary program to verify HS-invariance of our tensor. For each basis vector e_i and for each of generator h of HS, we checked that the matrices $(\circ e_i)^h$ and $\circ(e_i^h)$ agree.

COMPUTATION 2.3. *The HS-invariant product $\circ : \Lambda^2 E \longrightarrow E$ of Computation 2.2 is a Lie product on E. Moreover, the Killing form on $(E, \circ)$ is non-singular.*

Method. A straightforward computation shows that for each basis vector e_i we have

$$(\circ e_i)(\circ e_1) - (\circ e_1)(\circ e_i) - (\circ((e_i)(\circ e_1))) = 0.$$

Hence the Jacobi identity holds for any triple of basis vectors of the form e_1, e_i, e_j. Therefore right multiplication by e_1 is a derivation of $(E, \circ)$. Now, applying the action of HS, we deduce that $\circ(e_1^h)$ is a derivation of $(E, \circ)$ for any choice of $h \in HS$. Therefore, since HS is irreducible on E, we deduce that $\circ$ is a Lie product on E.

A random search quickly produces $e \in E$ with $\mathrm{Tr}((\circ e)(\circ e)) \neq 0$. It follows that the Killing form on E is not identically zero; hence, irreducibility of E as an HS-module shows that the Killing form is non-singular. □

We complete the proof that $HS < E_7(5)$ by applying standard results to show that the automorphism group of $(E, \circ)$ can only be $\mathrm{Aut}(E_7(5))$. The following lemma from [**11**] shows that $(E, \circ)$ is a simple Lie algebra.

LEMMA 2.4. *Suppose that $(X, \circ)$ is a finite-dimensional (non-associative) algebra and that $A \leqslant \mathrm{Aut}(X, \circ)$ acts irreducibly on X. Then one of the following holds:*

(a) *the algebra $(X, \circ)$ is simple;*
(b) *the A-module X is induced from a module of a proper subgroup of A;*
(c) *the product $\circ$ is identically zero.*

Proof. Let I be a minimal non-zero ideal of $(X, \circ)$, and let $\mathscr{I} = \{I^a \mid a \in A\}$. We say that a subset of $\mathscr{I}$ is independent if it consists of independent (vector) subspaces of X. Let $\mathscr{J} = \{I_1, I_2, \ldots, I_l\}$ be a maximal independent subset of $\mathscr{I}$. Let $Y = \bigoplus_{k=1}^{l} I_k$; then Y is an ideal of $(X, \circ)$.

Let I^a be any A-image of I. Maximality of $\mathscr{J}$ shows that $I^a \cap Y$ is a non-zero ideal; hence, since I^a is a minimal ideal, we have $I^a \subset Y$. Thus Y contains a non-zero A-submodule of X, and, since X is irreducible, we have $X = Y$.

We now suppose that neither (a) nor (b) holds; thus $I \neq X$ and hence $\mathrm{Stab}_A I$ is a proper subgroup of A. Moreover, since X is not induced, there is an $a \in A$ with $I^a \notin \mathscr{J}$. Then, $I^a \circ I_k \subset I^a \cap I_k = \{0\}$, for each $I_k \in \mathscr{J}$. Thus $I^a \circ X = I^a \circ Y = \sum I^a \circ I_k = \{0\}$. The A-invariance of $\circ$ now gives $I_k \circ X = \{0\}$ for each $I_k \in \mathscr{J}$, and thus $X \circ X = \sum I_k \circ X = \{0\}$. □

Let $(\bar{E}, \circ)$ be the Lie algebra obtained from $(E, \circ)$ by extending the scalars to the algebraic closure of GF(5). Since E is absolutely irreducible as an HS-module and it is not induced (since HS has no subgroup of index 133), Lemma 2.4 shows that the Lie algebra $(\bar{E}, \circ)$ is simple. Moreover, by Computation 2.3, the simple Lie algebra $(\bar{E}, \circ)$ has a non-singular Killing form.

A theorem of Block [**1**, Corollary 3.1] shows that over an algebraically closed field of characteristic $p > 3$, any perfect Lie algebra with a quotient trace form is a direct

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sum of classical simple Lie algebras of types $A_1, \ldots, E_8$. Our algebra $(\bar{E}, \circ)$ meets the conditions of Block's theorem; it is perfect, that is, equal to its derived algebra, since it is simple, and the non-singular Killing form gives it a quotient trace form. Hence $(\bar{E}, \circ)$ must be the classical simple Lie algebra of type E_7 (since no other classical simple Lie algebra has dimension 133). Therefore, the HS-invariant algebra $(E, \circ)$ is a GF(5)-form of E_7. By [**14**, Theorem IV.6.1] there is just one GF(5)-form of E_7; thus $(E, \circ)$ is the classical simple Lie algebra of type E_7 over GF(5). It now follows from [**15**] that $\mathrm{Aut}(E, \circ)$ has structure $E_7(5).2$ and we obtain $HS < E_7(5)$.

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