Ergod. Th. & Dynam. Sys. (2006), **26**, 353–377
doi:10.1017/S0143385705000441
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Printed in the United Kingdom

Hyperbolic non-autonomous Julia sets

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(*Received* 12 *January* 2005 *and accepted in revised form* 15 *September* 2005)

Abstract. We consider properties of Julia sets arising from composition sequences for arbitrarily chosen polynomials with uniformly bounded degrees and coefficients. This is a slight generalization of the situation first considered by Fornaess and Sibony in Random iterations of rational functions. *Ergod. Th. & Dynam. Sys.* **11** (1991), 687–708. The classical definition of hyperbolicity has a natural extension to our setting, and we show that many of the classical results concerning hyperbolic Julia sets can indeed be generalized. We introduce two important ideas in order to prove these results—the use of Cantor diagonalization, and how the Julia set moves in the Hausdorff topology as the sequence generating it changes.

1. *Introduction*

1.1. *Basic definitions and results.* We start by considering the space $\mathbb{N}_0 \times \overline{\mathbb{C}}$, which we will regard as countably many copies of $\overline{\mathbb{C}}$ fibred over $\mathbb{N}_0$. Let $\{R_m\}_{m=1}^{\infty} = \{R_1, R_2, R_3, \dots\}$ be a sequence of rational functions where, for each m, we require the degree d_m of R_m to be at least 2. This induces a mapping $\mathbf{R}$ of $\mathbb{N}_0 \times \overline{\mathbb{C}}$ into itself via the formula $\mathbf{R}(m, z) = (m + 1, R_{m+1}(z))$. We then have a discrete dynamical system on $\mathbb{N}_0 \times \overline{\mathbb{C}}$, which can also be viewed as a non-autonomous dynamical system on $\overline{\mathbb{C}}$.

For each $m \geq 1$, let $Q_m = R_m \circ R_{m-1} \circ \cdots \circ R_2 \circ R_1$ be the composition in the natural order for iteration of the first m functions in the sequence. In addition, for each $0 \leq m \leq n$, let $Q_{m,n} = R_n \circ R_{n-1} \circ \cdots \circ R_{m+2} \circ R_{m+1}$ (so that, for $m = n$, we have $Q_{m,m} = \mathrm{Id}$, and, for $m = 0$, $Q_{0,n} = Q_n$). Let D_m and $D_{m,n}$ denote the degrees of Q_m and $Q_{m,n}$ respectively, so that $D_m = \prod_{i=1}^{m} d_i$ and $D_{m,n} = \prod_{i=m+1}^{n} d_i$.

We then define, for each $m \geq 0$, the *Fatou set in the mth fibre* $\mathbf{F}_m = (m, \mathcal{F}_m)$, where $\mathcal{F}_m$ is called the *mth iterated Fatou set* and is given by

$$\mathcal{F}_m = \{z \in \overline{\mathbb{C}} : \{Q_{m,n}\}_{n=m}^{\infty} \text{ is a normal family on some neighbourhood of } z\},$$

and where our neighbourhoods are taken from the spherical topology on $\overline{\mathbb{C}}$. Given this, we can then define the *mth iterated Julia set* $\mathcal{J}_m$ to be the complement of $\mathcal{F}_m$ in $\overline{\mathbb{C}}$ and the *Julia set in the mth fibre* as simply $\mathbf{J}_m := (m, \mathcal{J}_m)$. If $m = 0$ we will refer to $\mathcal{F}_0$ and $\mathcal{J}_0$ as simply the Fatou and Julia sets for $\{R_m\}_{m=1}^{\infty}$ and denote them by $\mathcal{F}$ and $\mathcal{J}$ respectively.

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Except for the fact that they require the degrees of the functions to be constant, these definitions are the same as those given by Fornaess and Sibony in their paper [**9**] and subsequently used by Brück and others [**1, 2, 3**]. Note that if $\{R_m\}_{m=1}^{\infty}$ is a constant sequence $\{R, R, R, \dots\}$, then these definitions coincide with the standard ones and so non-autonomous iteration is therefore a generalization of standard complex dynamics. It is also immediate that the Fatou set is open and the Julia set is closed, and one can also adapt the proof of the classical case to show that the Julia set must be non-empty. The use of the word 'iterated' in our definitions of Julia and Fatou sets is justified in view of the following analogue of complete invariance for non-autonomous iteration. The proof is a straightforward adaptation of the standard classical proof.

THEOREM 1.1. *For any $0 \leq m \leq n$, $Q_{m,n}(\mathcal{J}_m) = \mathcal{J}_n$ and $Q_{m,n}(\mathcal{F}_m) = \mathcal{F}_n$, with Fatou components of $\mathcal{F}_m$ being mapped surjectively onto those of $\mathcal{F}_n$ by $Q_{m,n}$.*

From this result, we see that the Julia and Fatou sets in the different fibres 'match up' in the sense that, if $m \leq n$, then $\mathbf{F}_n = (n, Q_{m,n}(\mathcal{F}_m))$ and $\mathbf{J}_n = (n, Q_{m,n}(\mathcal{J}_m))$. In other words, the sequence of mappings send the Julia and Fatou sets in one fibre to the corresponding sets in the next. This means that we can speak of the Julia and Fatou sets for the entire dynamical system where we consider all the fibres simultaneously. Moving along successive fibres clearly corresponds to the notion of iteration for standard dynamical systems, and with this in view, we introduce some alternative notation which is both more intuitive and more compact than the fibre notation we have above. Given a point (m, z), where z lies in the mth fibre, for $n \geq m$ we denote the corresponding image $(n, Q_{m,n}(z))$ in the nth fibre by $z_{m,n}$ (and hence when $n = m$ we have $z_{m,m} = z$). Similarly for some subset U of $\overline{\mathbb{C}}$, if we consider U as a subset of the mth fibre, i.e. the set (m, U), we denote the set $(n, Q_{m,n}(U))$ by $U_{m,n}$. If a point or a set is introduced in the mth fibre $(m, \overline{\mathbb{C}})$ as above, we say it is introduced at *stage m* or at *time m*.

It turns out that the above situation is too far from standard complex dynamics to obtain significant results. For example, one can construct examples where the Julia set has non-empty interior without being all of $\overline{\mathbb{C}}$, and also where the Julia set (which as we have seen must be non-empty) consists of a single point (e.g. [**7**]). A natural restriction on the class of functions one considers is to look at sequences $\{P_m\}_{m=1}^{\infty}$ of polynomials of the form

$$P_m(z) = a_{d_m,m} z^{d_m} + a_{d_m-1,m} z^{d_m-1} + \cdots + a_{1,m} z + a_{0,m},$$

for which there exist constants $d \geq 2$, $K \geq 1$ and $M \geq 0$ such that the degree d_m of each P_m satisfies $2 \leq d_m \leq d$, and the coefficients of each P_m are bounded according to

$$1/K \leq |a_{d_m,m}| \leq K, \quad |a_{k,m}| \leq M, \ 0 \leq k < d_m.$$

We will refer to such sequences as *bounded sequences of polynomials* or, where the context is clear, simply as *bounded sequences*.

As in the classical case, there exists $R > 0$ which depends only on d, K and M such that, for any sequence with these bounds, if $|z| > R$, then $|Q_m(z)| \to \infty$ as $m \to \infty$. We call R an *escape radius* for these bounds, and we see that, just as in the standard case, we obtain a *basin at infinity* $\mathcal{A}_\infty$ on which the iterates Q_m converge to ∞, and that, for each $m \geq 0$, we have an *iterated basin at infinity* $\mathcal{A}_{\infty,m}$. The complements of these sets

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are then called the *filled Julia set* and *iterated filled Julia sets* respectively, and are denoted by $\mathcal{K}$ and $\mathcal{K}_m$. Note that the maximum principle shows that, just as in the classical case, there can be only one component on which ∞ is a limit function and so $\mathcal{A}_\infty$ is completely invariant in the sense given in Theorem 1.1.

The main interest of this paper lies in the appropriate definition and characterization of hyperbolicity for non-autonomous iteration. We first recall the following well-known classical result.

THEOREM 1.2. *Let P be any polynomial which gives rise to a Julia set $\mathcal{J}$. Then the following are equivalent:*

(1′) *P is hyperbolic on $\mathcal{J}$;*

(2′) *the closure of the postcritical set of P is disjoint from $\mathcal{J}$; and*

(3′) *all critical points of P lie in the Fatou set and eventually get iterated to attracting basins.*

Our aim is to give the correct generalization of this result, and we start by defining hyperbolicity in the non-autonomous context.

Definition 1.1. Let $\{P_m\}_{m=1}^{\infty}$ as usual denote a bounded sequence of polynomials. We say that $\{P_m\}_{m=1}^{\infty}$ is *hyperbolic* if there exist $\lambda > 1$, $C > 0$ such that for any $m, n \geq 0$

$$|Q'_{m,m+n}(z)| > C\lambda^n$$

on the mth iterated Julia set $\mathcal{J}_m$ for $\{P_m\}_{m=1}^{\infty}$. Alternatively, we say that a Julia set $\mathcal{J}$ is *hyperbolic* if it is generated by a sequence which satisfies the above condition.

This definition is clearly an extension of the classical one. If a sequence $\{P_m\}_{m=1}^{\infty}$ is hyperbolic with constants d, K, M, C and λ as above, we say that $\{P_m\}_{m=1}^{\infty}$ is (d, K, M, C, λ)-*hyperbolic*.

For a bounded polynomial sequence, hyperbolicity is clearly equivalent to the existence of a natural number m_0 for which we have that

$$|Q'_{m,m+m_0}(z)| > 2, \quad m \geq 0,\ z \in \mathcal{J}_m.$$

That this is equivalent to hyperbolicity follows from the equicontinuity of the normal family $\{P_m\}_{m=1}^{\infty}$ on some $\overline{D(0, R)}$, where R is an escape radius for the bounds d, K and M. We will call m_0 the *doubling time* for the hyperbolic sequence $\{P_m\}_{m=1}^{\infty}$.

A nice example of a hyperbolic sequence is any sequence of quadratic polynomials of the form $P_m(z) = z^2 + c_m$ where $M = \|c_m\|_\infty < 1/4$. In this case one can check easily that if $|c_m| \leq M$, then $z^2 + c_m$ will map the open disc $D(0, (M+1)/2)$ into itself. On the other hand, one also checks easily that 2 is an escape radius for such a sequence. It then follows that all the iterated Julia sets $\mathcal{J}_m$ lie inside the annulus $A(0, (M+1)/2, 2)$ from which $|Q'_{m,n}| \geq (M+1)^{n-m}$ on $\mathcal{J}_m$, whence $\{P_m\}_{m=1}^{\infty}$ is hyperbolic. One can actually prove more concerning such sequences, namely that all the iterated Julia sets for such a sequence are quasicircles. See [**3**] for details.

Let $\{P_m\}_{m=1}^{\infty}$ be a bounded sequence and, for each $m \geq 0$, let C_m denote the set of critical points of P_{m+1}. Then C_m naturally belongs in the fibre $(m, \overline{\mathbb{C}})$, and using our earlier notation, for $n \geq m$, we denote $Q_{m,n}(C_m)$ by $C_{m,n}$. With this notation, it is easy

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to see that the following definition is the appropriate generalization of condition (2′) in Theorem 1.2.

Definition 1.2. Let $\{P_m\}_{m=1}^{\infty}$ be a bounded sequence of polynomials and let $\delta > 0$. We say that $\{P_m\}_{m=1}^{\infty}$ has *postcritical distance* $\geq \delta$ if

$$\inf_{m\geq 0,\, n\geq m} (\operatorname{dist}(C_{m,n}, \mathcal{J}_n)) \geq \delta$$

and we write $\mathrm{PD}(\{P_m\}_{m=1}^{\infty}) \geq \delta$.

Saying that the postcritical distance is bounded below by δ is simply saying that the critical points and their orbits all stay at least δ away from the iterated Julia sets. With these definitions in hand, we can now turn to stating our generalization of the classical result. We remind the reader that, given two conditions with constants attached to them, we say that the first condition implies the second up to constants if it implies the second condition and in addition gives non-trivial bounds on the constants attached to that condition. Also, we recall that for an analytic or meromorphic function f the spherical derivative $f^{\#}$ of f is given by

$$f^{\#}(z) = \frac{f'(z)}{1 + |f(z)|^2}.$$

THEOREM 1.3. *Let $\{P_m\}_{m=1}^{\infty}$ be a bounded polynomial sequence with coefficient bound M which gives rise to a Julia set $\mathcal{J}$. Then the following are equivalent up to constants:*

(1) *$\{P_m\}_{m=1}^{\infty}$ is hyperbolic on $\mathcal{J}$;*
(2) *$\mathrm{PD}(\{P_m\}_{m=1}^{\infty}) \geq \delta$ for some $\delta > 0$; and*
(3) *there exist $K_0 > 0$ and $0 < \theta < 1$ such that, for each $m \geq 0$ and each critical point c of P_{m+1}, we have*

$$|Q^{\#}_{n,n+k}(c_{m,n})| < K_0\theta^k, \quad n \geq m,\ k \geq 0.$$

Before we turn to the proof of this result, which will occupy the rest of this paper, we first need to see that it is indeed a generalization of Theorem 1.2, namely that, in the context of a constant sequence $\{P, P, P, \dots\}$, each of the conditions above coincides with the corresponding classical one. As already observed, for the first two conditions this is immediate and it is also easy to see that (3′) implies (3). Hence it only remains to show that (3) implies (3′).

Condition (3) implies that, if we let R be an escape radius for P and look at second derivatives both inside and outside $D(0, R)$, then we can find a natural number m_0 and a (spherical) radius $r > 0$ which depend only on the coefficients and the constants K_0 and θ such that, for any critical point c of P and any $m \geq 0$, we have that $P^{\circ m_0}(D(P^{\circ m}(c), r)) \subset D(P^{\circ m_0+m}(c), r/2)$ (here we use the standard notation in complex dynamics where $P^{\circ m}$ denotes P composed with itself m times). It follows that such discs of spherical radius r must lie in the Fatou set as the iterates clearly form a normal family on them and so we see that all critical points will indeed lie in the Fatou set.

To see that all critical points get iterated to attracting basins, note first that, from above, the orbit of c can only lie in Fatou components which contain a disc of

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spherical radius r. Since the compactness of $\overline{\mathbb{C}}$ with respect to the spherical metric implies that there can only be finitely many such components, c will eventually get iterated to a periodic cycle of Fatou components and we can assume without loss of generality that c and its orbits lie in a particular Fatou component U which is invariant under P. The orbit of c must lie at least distance r away from the boundary of U, and if we let $\rho(.,.)$ denote the hyperbolic metric on U, it then follows from what we said above that P is not an isometry of the hyperbolic metric on U. Hence we can find $0 < k < 1$ such that $\rho(P^{\circ m+1}(c), P^{\circ m}(c)) < k\rho(P^{\circ m}(c), P^{\circ m-1}(c))$. From this one sees easily that any accumulation point of the orbit of c must be a fixed point for P which must clearly be attracting. Since P does not induce an isometry of the hyperbolic metric on U, there can be no other fixed points or periodic points of P in U. Therefore U is the immediate basin of an attracting fixed point and, in view of this, we have what we want.

Before we close this section, we mention two further advantages of working with bounded sequences of polynomials in addition to the existence of the escape radius. Since all the coefficients of our polynomials lie in a compact set, we can carry out Cantor diagonalization. Given $d \geq 2$, $K \geq 1$ and $M > 0$ as above, consider a sequence $\{\{P_m^n\}_{m=1}^{\infty}\}_{n=1}^{\infty}$ each of whose members is itself a sequence of polynomials whose degrees and coefficients are bounded as specified above. If we restrict ourselves to the first member of each sequence of polynomials, we see that we can find a subsequence n_k such that the coefficients of the polynomials $P_1^{n_k}$ converge to those of some limit polynomial P_1. Taking a further subsequence if needed, we can ensure that the coefficients of the second polynomials in each sequence also converge, and if we then apply the standard Cantor diagonalization argument, we obtain a sequence n_k so that, for each fixed m, the coefficients of the polynomials $P_m^{n_k}$ converge to those of some limit polynomial P_m. In this case we say that $\{P_m^n\}_{m=1}^{\infty}$ *converges pointwise* to $\{P_m\}_{m=1}^{\infty}$. This procedure will be very important to us later as it is one of the essential tools in obtaining results when working with hyperbolic behaviour in non-autonomous iteration.

Pointwise convergence is not the only type of convergence we need to consider. If we consider the space of coefficients of all sequences of polynomials as above, then this is clearly a subset of $l_\infty(\mathbb{C}^{d+1})$, the space of uniformly bounded sequences whose members are vectors in $\mathbb{C}^{d+1}$. Pointwise convergence is then precisely convergence with respect to the product topology for this space. However, it also has the natural topology of l_∞ and we need to consider convergence of sequences of polynomials with respect to this topology also. Just which topology is more useful at any given time will depend on the particular context.

The third advantage of working in this setting is that, as Fornaess and Sibony showed, one can construct the Green's function and make use of the machinery of potential theory. Although our definitions are slightly more general than theirs, the arguments used are identical, and so we merely present the statements and for details will refer the interested reader to their paper [**9**].

THEOREM 1.4. *Let* $\{P_m\}_{m=1}^{\infty}$ *be a bounded polynomial sequence. One then has the following.*

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(1) *The Green's function with pole at infinity for the basin at infinity $\mathcal{A}_\infty$ is given by*

$$G(z) = \lim_{m\to\infty} \frac{1}{D_m} \log|Q_m(z)|.$$

(2) *The Julia set $\mathcal{J}$ is perfect and regular in the sense of potential theory.*

(3) *$G(z)$ is continuous and plurisubharmonic as a function jointly of z and the sequence of coefficients of $\{P_m\}_{m=1}^\infty$ (where the sequence of coefficients lies in $l_\infty(\mathbb{C}^{d+1})$ and we give this space the topology of l_∞).*

(4) *If $G_m(z)$ denotes the Green's function with pole at infinity for $\mathcal{A}_{\infty,m}$ which is the basin at infinity for the sequence $\{P_n\}_{n=m+1}^\infty$, then for $n \geq m$,*

$$G_n(Q_{m,n}(z)) = D_{m,n} G_m(z).$$

(5) *The logarithmic capacity of the Julia set is given by the expression*

$$C = \prod_{m=1}^{\infty} |a_{d_m,m}|^{1/D_m}.$$

This is bounded below by $1/K^{1/(d-1)}$ and above by K.

(6) *For $|z|$ large,*

$$G(z) = \log|z| + c + \mathcal{O}(1/|z|)$$

where $c = \log C$, C being the capacity above, and the size of the $\mathcal{O}(1/|z|)$ term is uniformly bounded above in terms of the bounds d, K and M.

2. *Dependence on parameters*

In this section we investigate the dependence of Julia sets on the polynomial sequences which give rise to them. The two main results are that Hausdorff limits of Julia sets are completely invariant and that the dependence of the Julia set and filled Julia set have the same properties of semicontinuity as is observed in the classical case. We recall that, for two subsets A and B of $\mathbb{C}$, we define the maximum distance from A to B, $\mathrm{d}(A, B)$, by

$$\mathrm{d}(A, B) = \sup_{a\in A} \mathrm{dist}(a, B).$$

The quantity $\mathrm{d_H}(A, B) = \max\{\mathrm{d}(A, B), \mathrm{d}(B, A)\}$ is known as the Hausdorff distance between A and B. The compact subsets of $\mathbb{C}$ together with $\mathrm{d_H}(.\,,.)$ give us a complete metric space, and the Blaschke selection theorem says that any sequence of uniformly bounded compact subsets has a subsequence which converges with respect to this metric.

2.1. *Grand orbits.*

We start with the definition of a grand orbit, which we will need later.

Definition 2.1. Let $\{P_m\}_{m=1}^\infty$ be a bounded sequence of polynomials and $U \subset \overline{\mathbb{C}}$ be a set which appears at time $m \geq 0$ as (m, U) in the mth fibre. We define the *grand orbit* of U, $\mathbf{G}(U)$, by

$$\mathbf{G}(U) := \bigcup_{n\geq 0} \{(n, V) : V_{n,i} = U_{m,i} \quad \text{for some } i \geq \max(m, n)\}.$$

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For a fixed fibre $(n, \overline{\mathbb{C}})$, we define the *grand orbit in the nth fibre* of U, $\mathbf{G}_n(U) := (n, \mathcal{G}_n(U))$, where $\mathcal{G}_n(U)$ is called the *grand orbit at stage n* of U and is given by

$$\mathcal{G}_n(U) := \{V : V_{n,i} = U_{m,i} \text{ for some } i \geq \max(m, n)\}.$$

If $n = m$, we call $\mathcal{G}_m(U)$ the *immediate grand orbit* of U (at stage m).

Basically, saying two sets U and V are in the same grand orbit means that at some stage they get mapped to the same set and so the dynamical behaviour on each of them is eventually the same. In [**8**], one finds an example of a sequence that has infinitely many grand orbits of Fatou components, which is something that cannot occur in the classical case. With this definition in hand, we can now state a useful result due to Büger [**5**].

THEOREM 2.1. *Let $\{P_m\}_{m=1}^{\infty}$ be a bounded polynomial sequence with Julia set $\mathcal{J}$. Then for any point $x \in \mathcal{J}$ and any neighbourhood U of x, we have that $\mathcal{G}_0(U) \supset \mathcal{J}$. Equivalently, we can find m_0 such that $Q_m(U) \supset \mathcal{J}_m$ for all $m \geq m_0$.*

For classical rational iteration, the conclusion of this result follows more or less immediately from the density of repelling fixed points in the Julia set. Hence, we may view the above result as an analogue of the density of repelling fixed points. We note also that Fornaess and Sibony [**9**, Theorem 2.3] have also proved an analogue of the density of repelling fixed points for small non-autonomous perturbations of a hyperbolic classical Julia set. Brück [**4**] has also obtained similar conclusions for sequences of quadratic polynomials of the form $P_m(z) = z^2 + c_m$ where, as in our earlier example, the l_∞ coefficient bound on the sequence $\{c_m\}_{m=1}^{\infty}$ is less than $1/4$.

The above result can also be seen as a statement concerning self-similarity of Julia sets. Since any neighbourhood of any point in the Julia set will eventually expand under iteration to cover the whole of the iterated Julia sets, one will see the same basic features appear if one zooms in on different parts of the Julia set.

Finally, we remark that the conclusion of Büger's result does not hold in general for sequences of polynomials with unbounded coefficients. The two counterexamples at the end of Büger's paper [**5**] both exhibit this pathological behaviour where we can find a neighbourhood of some point in the Julia set whose iterates will not expand to eventually cover the whole of the iterated Julia sets.

2.2. *Hausdorff limits.* The aim of this section is to establish complete invariance of Hausdorff limits for Julia sets generated by sequences which converge to a pointwise limit. We begin with a technical lemma which uses Büger's result to show that a neighbourhood of a point in the Julia set eventually covers 'almost all' of the filled Julia set. This can therefore be viewed as a strengthening of Büger's theorem.

PROPOSITION 2.1. *Let $\{P_m\}_{m=1}^{\infty}$ denote a bounded sequence of polynomials which gives rise to a Julia set $\mathcal{J}$. Let x be a point in $\mathcal{J}$ and U be a neighbourhood of x. Then there exists a natural number N such that, for any $\varepsilon > 0$, there exists $m(\varepsilon, U)$ such that, for all $m \geq m(\varepsilon, U)$, $\mathcal{K}_m \backslash Q_m(U)$ is contained in the union of at most N discs of radius ε.*

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Proof. We start by noting that Büger's result together with compactness implies that the immediate grand orbit $\mathcal{G}_0(U)$ of U contains a δ-neighbourhood of $\mathcal{J}$ for some number $\delta > 0$.

Let V be a Fatou component on which there is a non-constant normal limit function. We claim that, for such a component, we can find N_0 similar to N in the statement above but with regard to covering V rather than $\mathcal{K}$, i.e. we can find N_0 such that for any $\varepsilon > 0$ there exists $m_0(\varepsilon, U)$ such that, for all $m \geq m_0(\varepsilon, U)$, $V_m \setminus Q_m(U)$, is contained in the union of at most N_0 discs of radius ε.

We first consider the case where there are only finitely many critical points occurring on the orbit of V. Here there are infinitely many times m where there is a critical point of P_{m+1} which is not in V_m, the iterate of V at time m. This means that, at such a time m, there is another Fatou component W_m which gets mapped to V_{m+1} by P_{m+1}, and if we consider the inverse image of this picture at time 0, we have a component W_0 which is in the grand orbit $\mathcal{G}_0(V)$. These components are distinct from V, and components arising in this way at different iterative times m will also be distinct from one another. Hence $\mathcal{G}_0(V)$ will consist of infinitely many Fatou components and, since only finitely many of them can contain a ball of spherical radius $\delta/2$, all but finitely many of their closures will lie inside $\mathcal{G}_0(U)$. However, it is quite easy to see that if W is a component of $\mathcal{G}_0(V)$ for which $\overline{W} \subset \mathcal{G}_0(U)$, then we can find m_0 for which $W_m \subset \mathcal{G}_m(U)$ for all $m \geq m_0$, and the desired conclusion for V will be true in this case.

Hence we can assume that infinitely many critical points will occur on the orbit of V. In this case, we note that if $y \in V$ is such that $\mathcal{G}_0(y) \cap \mathcal{G}_0(U) = \emptyset$, then $\mathcal{G}_0(y) \cap V$ is a finite set. Otherwise, if we let ψ denote the non-constant limit function on V, then $\psi(z) = \psi(y)$ on a set of points in V which is distance at least δ away from ∂V. However, since ψ is non-constant, this is impossible by the identity principle. Note that the same argument implies that there can only be one such grand orbit, since all but finitely many of the critical points on the orbit of V must in fact lie on the orbit of y. This means that, for any other point $z \in V$ with $z \notin \mathcal{G}_0(y)$, $\mathcal{G}_0(z) \cap V$ must be infinite and therefore meet $\mathcal{G}_0(U)$ by the argument above.

Let $V(y)$ denote the finite set $\mathcal{G}_0(y) \cap V$. Since $V(y)$ is at least δ away from $\mathcal{J}$, by equicontinuity there exists η such that for each point z in this set

$$\operatorname{diam} Q_m(D(z, \eta)) < \varepsilon \quad \text{for all } m \geq 1.$$

It follows from what we said above concerning points in $V \setminus \mathcal{G}_0(y)$ that, if $w \in V \setminus \bigcup_{z \in V(y)} D(z, \eta)$, then since $\mathcal{G}_0(w)$ meets $\mathcal{G}_0(U)$, we can find a small neighbourhood W of w and some time m_0 such that $W_m \subset U_m$ for all $m \geq m_0$. Using this, by compactness, we can then find $m_0(\eta)$ such that for all $m \geq m_0(\eta)$

$$V_m \setminus \bigcup_{z \in V(y)} D(Q_m(z), \varepsilon) \subset V_m \setminus \bigcup_{z \in V(y)} Q_m(D(z, \eta)) \subset Q_m(U),$$

whence the claim concerning covering V follows in this case.

Since $\mathcal{F} \setminus \mathcal{G}_0(U)$ is contained in the union of only finitely many Fatou components, we may assume from now on that all limit functions on Fatou components we are concerned with are constant. But in this case, $\mathcal{K} \setminus \mathcal{G}_0(U)$ is contained in only finitely

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many Fatou components and, since the limit functions on these components are constant, under sufficiently many iterations $\mathcal{K}\backslash\mathcal{G}_0(U)$ will eventually consist of a bounded number of pieces (one for each such component) each having diameter less than ε. The result then follows. □

We will use this result to prove that Julia sets depend lower semicontinuously on the sequences which give rise to them. We start by proving a result on the complete invariance of Hausdorff limits.

THEOREM 2.2. *Let $\{\{P_m^n\}_{m=1}^{\infty}\}_{n=1}^{\infty}$ be a sequence of bounded sequences of polynomials all of which have the same bound d on the degrees and the same coefficient bounds $K \geq 1$, $M > 0$, and which converges pointwise to some limit sequence $\{P_m\}_{m=1}^{\infty}$. Fix $m_0 \geq 0$ and let $\{P_m^n\}_{m=m_0+1}^{\infty}$ denote the sequence obtained from $\{P_m^n\}_{m=1}^{\infty}$ by removing the first m_0 members. Let $\mathcal{J}^n$ and $\mathcal{J}_{m_0}^n$ be the Julia sets for $\{P_m^n\}_{m=1}^{\infty}$ and $\{P_m^n\}_{m=m_0+1}^{\infty}$ respectively, so that by complete invariance we have*

$$Q_{m_0}^n(\mathcal{J}^n) = \mathcal{J}_{m_0}^n \quad \text{and} \quad (Q_{m_0}^n)^{-1}(\mathcal{J}_{m_0}^n) = \mathcal{J}^n.$$

Then the sets $\mathcal{J}^n$ converge in the Hausdorff topology if and only if the sets $\mathcal{J}_{m_0}^n$ do, and if we let $\mathcal{H}$ and $\mathcal{H}_{m_0}$ denote the corresponding Hausdorff limits, then

$$Q_{m_0}(\mathcal{H}) = \mathcal{H}_{m_0} \quad \text{and} \quad Q_{m_0}^{-1}(\mathcal{H}_{m_0}) = \mathcal{H}.$$

Proof. To start, we note first that the general case will follow immediately if we prove the case $m_0 = 1$. To prove the first half of each of the two statements above which concerns forward invariance, suppose that $\mathcal{J}^n \to \mathcal{H}$ and let $\varepsilon > 0$. Let R be an escape radius for the bounds d, K and M. Then we can choose n_0 large enough so that $|P_1^{n_1}(z) - P_1^{n_2}(z)| < \varepsilon/2$ on $\overline{D}(R, 0)$ for any $n_1, n_2 \geq n_0$. Also, by equicontinuity we can find $\delta > 0$ such that, whenever z and w are points such that $z, w \in \overline{D}(R, 0)$, $|z - w| < \delta$, we have

$$|P_1^n(z) - P_1^n(w)| < \varepsilon/2, \quad \text{for every } n \geq 1.$$

Now let z^{n_1} and z^{n_2} be points in $\mathcal{J}^{n_1}$ and $\mathcal{J}^{n_2}$ respectively. Then

$$\begin{aligned} |P_1^{n_1}(z^{n_1}) - P_1^{n_2}(z^{n_2})| &\leq |P_1^{n_1}(z^{n_1}) - P_1^{n_1}(z^{n_2})| + |P_1^{n_1}(z^{n_2}) - P_1^{n_2}(z^{n_2})| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

provided $|z^{n_1} - z^{n_2}| < \delta$ and n_1 and n_2 are chosen large enough so that $n_1, n_2 \geq n_0$. This shows that the sets $P_1^n(\mathcal{J}^n) = \mathcal{J}_1^n$ converge in the Hausdorff topology. For the forward invariance of $\mathcal{H}$, a similar argument easily shows that $P_1(\mathcal{H}) \subset \mathcal{H}_1$.

For the second half of the theorem concerning backwards invariance, suppose now that $\mathcal{J}_1^n \to \mathcal{H}_1$ and again let $\varepsilon > 0$ and R be an escape radius for d, K and M. By the compactness of the disc $\overline{D}(0, R)$ and the identity principle, we can find $\eta > 0$ such that, for any $z \in \overline{D}(R, 0)$, we can find ε_z' with $0 < \varepsilon_z' < \varepsilon$ and such that $|P_1(z) - P_1(w)| > \eta$ whenever $|z - w| = \varepsilon_z'$.

Let z^1 be a point in $\mathcal{J}^{n_1}$ and let z_1^1 be the image of z^1 under $P_1^{n_1}$. By the Hausdorff convergence of the sets $\mathcal{J}_1^n$, by taking n_1 and n_2 sufficiently large, we can find a point z_1^2 (where the 2 is an index and not a square) in $\mathcal{J}_1^{n_2}$ so that $|z_1^1 - z_1^2| < \eta/4$.

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Since the polynomials P_1^n converge uniformly on compact sets to P_1, it then follows that we can choose n_0 such that, if $n_1, n_2 \geq n_0$, then $|P_1^{n_1}(z) - P_1^{n_2}(z)| < \eta/4$ on $\overline{D}(0, R)$ which contains every $\mathcal{J}^n$. It then follows for n_1, n_2 sufficiently large, that for $z \in \overline{D}(0, R)$,

$$|(P_1^{n_1}(z) - z_1^1) - (P_1^{n_2}(z) - z_1^2)| < \eta/2,$$

whence by Rouché's theorem we can find a point z^2 in $D(z^1, \varepsilon'_{z^1}) \subset D(z^1, \varepsilon)$ such that $P_1^{n_2}(z^2) = z_1^2$. By complete invariance (Theorem 1.1), z^2 lies in $\mathcal{J}^{n_2}$ and, since ε was arbitrarily chosen, the Hausdorff convergence of the sequence $\mathcal{J}^n$ follows.

A similar argument shows that if we let $\mathcal{H}$ denote the limit of the sequence $\mathcal{J}^n$ and we take a point z such that $P_1(z) \in \mathcal{H}_1$, then we must have that $z \in \mathcal{H}$ and from this and what we deduced earlier we have that

$$P_1(\mathcal{H}) \subset \mathcal{H}_1 = P_1(P_1^{-1}(\mathcal{H}_1)) \subset P_1(\mathcal{H})$$

and

$$\mathcal{H} \subset P_1^{-1}(P_1(\mathcal{H})) \subset P_1^{-1}(\mathcal{H}_1) \subset \mathcal{H}.$$

Hence $P_1(\mathcal{H}) = \mathcal{H}_1$ and $P_1^{-1}(\mathcal{H}_1) = \mathcal{H}$ and the proof of Theorem 2.2 is complete. □

2.3. *Semicontinuity.* Suppose we are in the classical situation where we iterate with only one polynomial P. The following result is well known.

THEOREM 2.3. *The filled Julia set $\mathcal{K}$ depends upper semicontinuously on the coefficients of the polynomial P and the Julia set $\mathcal{J}$ depends lower semicontinuously on the coefficients of P.*

Both statements may be generalized to the context of non-autonomous sequences with bounded coefficients. However, the classical proof of the lower semicontinuous dependence of $\mathcal{J}$ relies on the density of repelling periodic orbits in $\mathcal{J}$ which we do not have here, and we therefore need a new proof for this half of the result.

THEOREM 2.4. *Let $\{\{P_m^n\}_{m=1}^\infty\}_{n=1}^\infty$ be a sequence of bounded sequences of polynomials all of which have the same bound d on the degree and the same bounds K, M on the coefficients. For each $n \geq 1$, let $\mathcal{J}^n$ and $\mathcal{K}^n$ be the Julia and filled Julia sets respectively for $\{P_m^n\}_{m=1}^\infty$. Suppose that the sequences $\{P_m^n\}_{m=1}^\infty$ tend pointwise to some limit sequence $\{P_m\}_{m=1}^\infty$ with Julia set $\mathcal{J}$ and filled Julia set $\mathcal{K}$. Then*

$$\mathrm{d}(\mathcal{K}^n, \mathcal{K}) \to 0 \quad \textit{and} \quad \mathrm{d}(\mathcal{J}, \mathcal{J}^n) \to 0 \quad \textit{as } n \to \infty.$$

Hence, if $\mathcal{H}$ denotes a Hausdorff limit of some subsequence of the sets $\mathcal{J}^n$, we have

$$\mathcal{J} \subset \mathcal{H} \subset \mathcal{K}.$$

Proof. Let R be an escape radius for the bounds d, K, M above. Let z be a point in $\mathcal{A}_\infty$, the basin at infinity for the limit sequence, and pick a natural number m_0 and a positive constant $\delta > 0$ such that

$$|Q_m(w)| > 2R, \quad m \geq m_0, \ |w - z| < \delta.$$

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By pointwise convergence, we can then find n_0 such that

$$|Q_m^n(w)| > R, \quad n \geq n_0,\ m \geq m_0,\ |w - z| < \delta.$$

It then follows by an easy compactness argument that $\mathrm{d}(\mathcal{K}^n, \mathcal{K}) \to 0$, from which it follows immediately that $\mathcal{H} \subset \mathcal{K}$.

Now let z be any point in $\mathcal{J}$ and let U be a neighbourhood of z. By Proposition 2.1, there exists N such that, given $\varepsilon > 0$, we can find m_0 such that, $K_{m_0} \setminus Q_{m_0}(U)$ is contained in at most N discs of radius ε. Since capacity is subadditive, by making ε small enough, this collection of balls can have arbitrarily small capacity. If we then remember that in view of Theorem 1.4 all Julia sets concerned have logarithmic capacity bounded below by $1/K^{1/(d-1)}$, we see that $Q_{m_0}(U)$ must contain points of $\mathcal{J}_{m_0}^n$ for n sufficiently large. It then follows by complete invariance (Theorem 1.1) that U contains points of $\mathcal{J}^n$ for n sufficiently large. It then follows by compactness of $\mathcal{J}$ that $\mathrm{d}(\mathcal{J}, \mathcal{J}^n) \to 0$ and from this we also see that $\mathcal{J} \subset \mathcal{H}$, whence the proof is complete. □

The fact that, for the filled Julia sets $\mathcal{K}^n$ of each sequence $\{P_m^n\}_{m=1}^\infty$, $\mathrm{d}(\mathcal{K}^n, \mathcal{K}) \to 0$ as $n \to \infty$ and that hence $\mathcal{H} \subset \mathcal{K}$ has also been noted by Brück [**4**] for polynomial sequences $\{P_m\}_{m=1}^\infty$ of the form $P_m(z) = z^2 + c_m$. He then observed that if the corresponding filled Julia set $\mathcal{K}$ has empty interior which forces $\mathcal{K} = \mathcal{J}$, then the Julia sets $\mathcal{J}^n$ for each of the sequences $\{P_m^n\}_{m=1}^\infty$ will converge in the Hausdorff topology to $\mathcal{J}$, which is immediate from the result above.

3. *Proof of the main result*

We now turn to proving Theorem 1.3. The proof of the classical result is short and fairly easy. However, it relies heavily on the classification of periodic Fatou components and also on the fact that, for iteration with a constant sequence using a single polynomial, the iterated Julia sets at all stages are the same. Clearly neither of these things are available to us and we will need a new proof.

The strategy will be as follows. We will first establish the equivalence of (1) and (2) in Theorem 1.3. We do this by first showing that (1) $\Rightarrow$ (2), which is fairly easy. Next we establish the converse, which will take a good deal more work and is where much of the interest of this result lies. We then proceed to show that (1) $\Rightarrow$ (3) and then finally (3) $\Rightarrow$ (2). We remark at this point that the equivalence of (1) and (2) has also been established for the case of polynomial skew products on $\mathbb{C}^2$ by Jonsson [**10**, Theorem 4.8], which is a special case of what we consider. The two main tools we will need to prove Theorem 1.3 are Cantor diagonalization and the stability of the Julia set with respect to the Hausdorff topology as the sequence which gives rise to it is varied. Deriving the necessary results concerning the latter will be carried out in this section of the paper and, once we have accomplished this, we can turn to finishing the proof of Theorem 1.3 in §4. We now begin by showing (1) $\Rightarrow$ (2).

THEOREM 3.1. *Let $\{P_m\}_{m=1}^\infty$ be a hyperbolic bounded polynomial sequence with bounds (d, K, M, C, λ) as in the definitions. Then* $\mathrm{PD}(\{P_m\}_{m=1}^\infty) \geq \delta$ *for some $\delta > 0$ which depends only on these bounds.*

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Proof. Suppose $\{P_m\}_{m=1}^{\infty}$ is (d, K, M, C, λ)-hyperbolic and let R be an escape radius for the bounds d, K and M. By hyperbolicity we can find $\alpha > 0$ such that $|P'_{m+1}| > \alpha$ on the iterated Julia sets $\mathcal{J}_m$ for all $m \geq 0$. On the other hand, $\{P_m\}_{m=1}^{\infty}$ is clearly a normal family on any subset of $\overline{\mathbb{C}}$ and in particular on $D(2R, 0)$ which compactly contains all of the sets $\mathcal{J}_m$. It is then easy to see that there exists $\delta > 0$ such that, for all $m \geq 0$,

$$\operatorname{dist}(C_m, \mathcal{J}_m) > \delta,$$

where C_m denotes the set of critical points of P_{m+1} as before.

Having proved that the critical points stay away from the iterated Julia sets, it remains to show that their orbits also do. Let U be a Fatou component for $\{P_m\}_{m=1}^{\infty}$. Since $\{P_m\}_{m=1}^{\infty}$ is hyperbolic, we can find $m_0 \geq 1$ such that

$$|Q'_{m+m_0}| > 3 \text{ on } \mathcal{J}_m \text{ for every } m \geq 0.$$

We start by observing that, by the normality of the family $\{Q_{m,m+m_0}\}_{m=0}^{\infty}$, we can make δ smaller if needed so that for any $m \geq 0$, if $x \in U_m$ with $\delta/2 \leq \mathrm{d}(x, \partial U_m) \leq \delta$, then $\mathrm{d}(Q_{m,m+m_0}(x), \partial U_{m+m_0}) \geq \delta$. To see this, note that by the normality the image of $D(x, \delta)$ under $Q_{m,m+m_0}$ will be approximately a disc, and the desired estimate follows on considering the appropriate inverse branch of this mapping. Hence if we take a simple smooth closed curve γ in U_m such that

$$\frac{\delta}{2} \leq \mathrm{d}(\gamma, \partial U_m) < \delta,$$

then the image of γ under $Q_{m,m+m_0}$ is a closed curve in U_{m+m_0} for which

$$\mathrm{d}(Q_{m,m+m_0}(\gamma), \partial U_{m+m_0}) > \delta.$$

The argument principle tells us that the image under $Q_{m,m+m_0}$ of the interior component of $\overline{\mathbb{C}}\backslash\gamma$ avoids the unbounded component of $\overline{\mathbb{C}}\backslash Q_{m,m+m_0}(\gamma)$. It then follows from what we proved earlier that the orbits of the critical points will indeed avoid the iterated Julia sets and hence it follows from the normality of $\{P_m\}_{m=1}^{\infty}$ that $\mathrm{PD}(\{P_m\}_{m=1}^{\infty}) \geq \delta$ as required.

For the last part of the result concerning the dependence of δ on d, K, M, C and λ, let $\mathcal{P}$ be the family of polynomials which have the bounds d, K and M as in the definition of bounded sequences, and let R be an escape radius for all sequences with these bounds. Then $\mathcal{P}$ is normal on $D(0, 2R)$ and, since all estimates involving the number δ appearing in the argument above depended only on the equicontinuity of $\mathcal{P}$ and the number m_0, which we specified to be the number of iterations required for the absolute value of the derivative of the iterates to exceed 3 on the Julia set, the proof is now complete. □

We continue our collection of facts by proving the following result which, though we will not be needing it later, is a good illustration of how hyperbolic behaviour in non-autonomous iteration resembles its classical counterpart.

THEOREM 3.2. *Let $\{\mathbf{c}_m\}_{m=1}^{\infty}$ be a sequence in $l_\infty(\mathbb{C}^{d+1})$ which gives rise to a (unique) bounded sequence $\{P_m\}_{m=1}^{\infty}$ whose coefficients are given by this sequence and which is hyperbolic on the corresponding Julia set $\mathcal{J}$. Then there is a neighbourhood of $\{\mathbf{c}_m\}_{m=1}^{\infty}$ in $l_\infty(\mathbb{C}^{d+1})$ in which every sequence gives rise to a hyperbolic Julia set and in which the Julia set moves continuously with respect to the Hausdorff topology.*

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Proof. Let $\{\mathbf{c}_m\}_{m=1}^{\infty}$ be as above, let $\{P_m\}_{m=1}^{\infty}$ be the corresponding sequence of polynomials with this sequence of coefficients, and for $m \geq 0$ let $\mathcal{J}_m$ denote the mth iterated Julia set for $\{P_m\}_{m=1}^{\infty}$. Let m_0 be such that $|Q'_{m,m+m_0}(z)| > 4$ for every $m \geq 0$ and for every point $z \in \mathcal{J}_m$. For the sake of simplicity of notation, we assume $m_0 = 1$. The proof for $m_0 > 1$ is essentially the same. Let $\varepsilon > 0$ be arbitrary. We will have proved both of the claims in the statement of the theorem if we can find $\delta > 0$ so that all sequences in a δ-neighbourhood of $\{\mathbf{c}_m\}_{m=1}^{\infty}$ give rise to Julia sets which are hyperbolic and less than ε away from $\mathcal{J}$ in terms of Hausdorff distance.

Since the sequence $\{P_m\}_{m=1}^{\infty}$ is a normal family, it follows that we may make ε smaller if we need so that $|P'_m(z)| > 3$ on an ε-neighbourhood of every iterated Julia set $\mathcal{J}_m$. From this it follows that we can find $\delta > 0$ such that, if we change the coefficients of our sequence by at most δ and denote the new sequence by $\{\tilde{P}_m\}_{m=1}^{\infty}$, then $|\tilde{P}'_m(z)| > 2$ on an ε-neighbourhood of every iterated Julia set $\mathcal{J}_m$ (where $\mathcal{J}_m$ is still the Julia set for the old sequence $\{P_m\}_{m=1}^{\infty}$).

Now let δ_0 be the postcritical distance for $\{P_m\}_{m=1}^{\infty}$ whose existence is naturally guaranteed in view of Theorem 3.1. For $m \geq 0$ and a point z within distance $\varepsilon/2$ of $\mathcal{J}_m$, let $C(z, \varepsilon/4)$ denote the circle with centre z and radius $\varepsilon/4$. By making ε smaller if necessary, we may ensure that $\varepsilon < \delta_0/4$ (we will make use of this later for estimates using the standard distortion theorems for univalent mappings). It follows again by the normality of $\{P_m\}_{m=1}^{\infty}$ that, if we take ε sufficiently small, the image of any point on this circle is at least $\varepsilon/2$ away from $P_{m+1}(z)$. If we now apply Rouché's theorem, we see that (independently of m) we can make δ smaller if needed so that, if we change the coefficients of P_{m+1} by at most δ, then again denoting the new polynomial by $\tilde{P}_{m+1}$, we have on choosing an appropriate branch of $\tilde{P}_{m+1}^{-1}$ that

$$|\tilde{P}_{m+1}^{-1}(P_{m+1}(z)) - z| < \varepsilon/4.$$

We now argue by induction on the number of members of our sequence which we have changed in this way. We claim that after changing the first n members of $\{P_m\}_{m=1}^{\infty}$, all the resulting iterated Julia sets which we call $\tilde{\mathcal{J}}_m^n$ are less than ε away from $\mathcal{J}_m$. Clearly, in view of what we just proved above, the claim is true for $n = 1$.

Now suppose we have changed the first $n-1$ members of our sequence in this way and denote the iterated Julia sets obtained by modifying these first $n-1$ polynomials by $\tilde{\mathcal{J}}_m^{n-1}$. Our induction assumption is that the Hausdorff distance between $\tilde{\mathcal{J}}_m^{n-1}$ and $\mathcal{J}_m$ is less than $(8/9)\sum_{i=1}^{n-m-1} 2^{-i}\varepsilon$ for every $0 \leq m \leq n-2$.

Suppose now we change the coefficients of P_n by at most δ. The standard distortion theorems for univalent functions (e.g. [**6**, Theorem 1.6]) applied to the polynomials P_m, $1 \leq m \leq n$, and the fact that $\varepsilon < \delta_0/4$, show that, if $z \in \mathcal{J}_{m+1}$, any component of the inverse image of $D(z, \varepsilon)$ under P_m will have diameter at most $(4/9)\varepsilon$ since $|P'_m| > 4$ on $\mathcal{J}_{m-1}$. If we combine this with the estimate using Rouché's theorem we obtained earlier, this means that components of the inverse images of the same disc under $\tilde{P}_m$ will have diameter at most $(4/9 + 1/4)\varepsilon < \varepsilon$. Since $|\tilde{P}'_m(z)| > 2$ on an ε-neighbourhood of every iterated Julia set $\mathcal{J}_m$, it then follows easily that the inverse branches of $\tilde{Q}_{m,n}$, $0 \leq m \leq n$, will all be univalent on an ε-neighbourhood of $\mathcal{J}_n^n = \mathcal{J}_n$. It then follows from what we proved earlier together with the induction hypothesis and again the standard distortion

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theorems that, for any $m \leq n$, the preimage of $\mathcal{J}_n^n$ under $\tilde{Q}_{m,n}$ (i.e. the set $\tilde{\mathcal{J}}_m^n$) is at most $(16/9)(\varepsilon/4)2^{-n+m+1} = (8/9)\varepsilon 2^{-n+m}$ away from $\tilde{\mathcal{J}}_m^{n-1}$ in terms of Hausdorff distance. From this we see that the distance between $\mathcal{J}_m$ and $\tilde{\mathcal{J}}_m^n$ is at most

$$\frac{8}{9}\varepsilon \sum_{i=1}^{n-m} 2^{-i} < \varepsilon,$$

and in view of this, our induction is complete.

Now let n tend to infinity and let $\tilde{\mathcal{J}}_m$ denote the iterated Julia sets for the new sequence $\{\tilde{P}_m\}_{m=1}^{\infty}$. If we let $\tilde{\mathcal{H}}$ denote a Hausdorff limit of the sets $\tilde{\mathcal{J}}^n$, then $\tilde{\mathcal{H}}$ is forward and backward invariant under $\{\tilde{P}_m\}_{m=1}^{\infty}$ in the sense given in the statement of Theorem 2.2, and it follows by our induction that, for any $m \geq 0$, we have

$$\mathrm{d}(\tilde{\mathcal{H}}_m, \mathcal{J}_m) < \frac{8}{9}\varepsilon < \varepsilon.$$

From this it follows that $|\tilde{Q}'_{m,m+i}| > 2^i$ on $\tilde{\mathcal{H}}_m$, from which we can deduce that $\tilde{\mathcal{H}} \subset \tilde{\mathcal{J}}$. But by Theorem 2.4, $\tilde{\mathcal{J}} \subset \tilde{\mathcal{H}}$ and the result follows. □

The above result tells us that hyperbolicity is an open condition which is robust under perturbation in the sense both that perturbations of a hyperbolic sequence will remain hyperbolic and that the Julia set for the perturbed sequence will be near that of the original sequence. This is, of course, completely analogous to the behaviour of classical hyperbolic polynomials. Another result which echoes the classical behaviour is the small but important lemma below. This result can be viewed as a weak version of (2) ⇒ (1) in Theorem 1.3 and in fact use will be made of it for the purpose of proving this implication.

LEMMA 3.1. *Let* $\{P_m\}_{m=1}^{\infty}$ *be a bounded sequence with* $\mathrm{PD}(\{P_m\}_{m=1}^{\infty}) \geq \delta$ *and let* $B > 0$. *Then there exists a natural number* m_0 *such that, for any* $m \geq m_0$,

$$|Q'_m(z)| \geq B, \quad z \in \mathcal{J}.$$

Proof. Applying Büger's result and compactness shows that we can choose m_0 such that, for any point z of $\mathcal{J}$ and any $m \geq m_0$, $Q_m(D(z, \delta/(8B)))$ will contain the iterated Julia set $\mathcal{J}_m$. Let $m \geq m_0$. Since all critical points and their iterates stay a distance of at least δ away from the iterated Julia sets, any branch of the inverse $Q_m^{\circ -1}$ will be univalent on $D(Q_m(z), \delta/2)$. Let U denote the component of the inverse image of $D(Q_m(z), \delta/2)$ containing z. Since $Q_m(D(z, \delta/(8B)))$ is obviously connected, it follows that, provided $\delta \leq \operatorname{diam} \mathcal{J}_m$ (which is bounded below in view of the capacity estimates of Theorem 1.2), we will have points on $C(Q_m(z), \delta/2)$ which lie inside $Q_m(D(z, \delta/(8B)))$. From this it immediately follows that $\operatorname{dist}(z, \partial U) \leq \delta/(8B)$. The Koebe one-quarter theorem then implies that the derivative $Q'_m(z)$ must satisfy $Q'_m(z) \geq B$ and the result follows. □

It is important to note that this result is *a priori* a good deal weaker than saying that (2) ⇒ (1) in Theorem 1.3. Since we have no bound on the number m_0, although we know that the derivatives of the iterates on the iterated Julia sets become large, we have no control on the rate at which this happens. What is necessary, therefore, is to obtain bounds on m_0

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which depend only on the bounds d, K and M for the sequence of polynomials under consideration, and this therefore becomes our aim for the immediate future. Our first step in this direction continues our collection of facts which show how hyperbolic behaviour for non-autonomous iteration resembles the classical version.

LEMMA 3.2. *Let* $\{P_m\}_{m=1}^{\infty}$ *be a sequence with* $\mathrm{PD}(\{P_m\}_{m=1}^{\infty}) \geq \delta$ *for some* $\delta > 0$. *Then infinitely many critical points must lie on the orbit of any Fatou component.*

Proof. Let U be a Fatou component which has only finitely many critical points on its orbit. Since $\infty \in \mathcal{A}_{\infty,m}$ for every m, then U must be a bounded Fatou component. By Lemma 3.1 and the minimum principle, the derivatives of the iterates Q_m will get arbitrarily large in absolute value at every point of U with at most finitely many exceptions. This clearly contradicts the fact that U is a bounded Fatou component, in view of which we are done. □

This motivates the following definition. We say that a Fatou component U is *mature* if U contains the orbit of a critical point. One of the important properties of such a component is that the iterated components U_m must contain a disc of radius δ for every $m \geq 0$. We can therefore restate the result we just proved by saying that, for a bounded sequence $\{P_m\}_{m=1}^{\infty}$ with $\mathrm{PD}(\{P_m\}_{m=1}^{\infty}) \geq \delta$ for some $\delta > 0$, every Fatou component will eventually become mature under iteration.

In the case of hyperbolic sequences, essentially the same argument as given above allows us to say a little more.

LEMMA 3.3. *Let* $\{P_m\}_{m=1}^{\infty}$ *be a* (d, K, M, C, λ)*-hyperbolic bounded sequence. Then there exists a natural number* $D = D(d, K, M, C, \lambda)$ *such that, for any Fatou component* U *and any* $m \geq 0$, *at least one of the iterated Fatou components* $Q_i(U)$, $m \leq i < m+D$, *contains a critical point of the corresponding polynomial* P_{i+1}.

Finally, we have the following corollary.

COROLLARY 3.1. *Let* $\{P_m\}_{m=1}^{\infty}$ *be a bounded sequence of polynomials with the usual bounds* d, K *and* M, *and suppose* $\mathrm{PD}(\{P_m\}_{m=1}^{\infty}) \geq \delta$ *for some* $\delta > 0$. *Then the number of grand orbits of Fatou components for* $\{P_m\}_{m=1}^{\infty}$ *is finite and bounded. The upper bound is* $\mathcal{O}(1/\delta^2)$ *and depends only on* δ, d, K *and* M.

Proof. Let R be an escape radius for d, K and M. By Lemma 3.2, for each Fatou component U, we can find m_0 such that $Q_m(U)$ contains a disc of radius δ for each $m \geq m_0$. As we saw earlier, all the iterated Julia sets and hence iterated filled Julia sets have diameter less than or equal to $2R$ and so can contain only finitely many such discs. The result then follows. □

Corollary 3.1 is no longer true when we do not assume anything about the sequence other than that the coefficients be bounded. In fact, it is possible to construct a sequence whose Fatou set has infinitely many grand orbits of Fatou components. The proof relies heavily on properties of parabolic fixed points and is given in [**8**]. For now, let us state the following result, which again illustrates how bounded sequences with positive postcritical distance give rise to behaviour similar to that of hyperbolic polynomials in classical complex dynamics.

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THEOREM 3.3. *Let* $\{P_m\}_{m=1}^{\infty}$ *be a bounded sequence of polynomials for which* $\mathrm{PD}(\{P_m\}_{m=1}^{\infty}) \geq \delta$ *for some constant* $\delta > 0$. *Then all limit functions on any Fatou component are constant.*

Before beginning the proof, which is rather long and involved, we make some remarks. Like Lemma 3.1, this result can in fact be viewed as a weak version of (2) $\Rightarrow$ (1) in Theorem 1.3 and is in fact the key result needed to establish this implication. The proof borrows from ideas in the argument given in Steinmetz's book [**11**] for the proof of the classification of periodic Fatou components for standard rational iteration. The main idea is that for a (bounded) Fatou component U on which some subsequence of iterates Q_{m_k} converges to a non-constant limit function, the differences between the iterates for k large should be small. Put another way, if $k \leq l$ are both large, then Q_{m_k,m_l} should be close to the identity on the corresponding iterated Fatou component U_{m_k}. On the other hand Lemma 3.1 tells us that the derivatives of the iterates get large on the iterated Julia sets and in particular on the boundary of U_{m_k}. It is then plausible to guess that these two behaviours are incompatible and the technical details of the proof are then concerned with making this precise and forcing a contradiction out of this situation.

Proof of Theorem 3.3. Suppose the result is false. Then there exists a Fatou component U on which there is a non-constant limit function f. Let $\{m_k\}_{k=1}^{\infty}$ be a subsequence so that Q_{m_k} converges uniformly to f on compact subsets of U. Since the only limit function on $\mathcal{A}_{\infty}$ is ∞, U must be a bounded Fatou component. In addition, in view of Lemma 3.2, we may assume without loss of generality that U contains a critical point c of P_1. The sets $\overline{U}_{m_k}$ are compact and uniformly bounded, and so by taking a further subsequence if necessary we can assume that the sets $\overline{U}_{m_k}$ and ∂U_{m_k} converge in the Hausdorff topology in view of the Blaschke selection theorem. Let us denote their Hausdorff limits by $\mathcal{H}$ and $\mathcal{L}$ respectively. By the open mapping theorem, if we let $V = f(U)$, then V will be a non-empty connected open set. If we now let z be any point of V and let $\varepsilon > 0$ so that $D(z, \varepsilon) \subset V$, it follows easily from Rouché's theorem and the argument principle that we can find k_0 large enough so that, for all $w \in D(z, \varepsilon/2)$ and all $k \geq k_0$, $w \in U_{m_k}$. Using this it is then easy enough to deduce that V must lie in a component W of $\mathrm{int}(\mathcal{H})\backslash\mathcal{L}$. Note that since $\mathrm{PD}(\{P_n\}_{n=1}^{\infty}) \geq \delta$, V and hence W will contain a disc of radius δ about $f(c)$.

Let K be a compact subset of W. In view of the fact that W does not intersect with $\mathcal{L}$, we can find k_0 so that $Q_{m_k}(U) = U_{m_k}$ contains a relatively compact open neighbourhood of K for any $k \geq k_0$. This follows from a similar argument using Rouché's theorem as above and then applying the compactness of K to the resulting cover by the balls $D(z, \varepsilon/2)$ obtained as above. If we now fix $k_1 \geq k_0$, then the fact that Q_{m_k} converges uniformly to f on compact subsets of U guarantees that the sequence $Q_{m_{k_1},m_k}$ will converge on compact subsets of the set $U_{m_{k_1}}$, which contains our neighbourhood of K, to some non-constant function φ_{k_1}. The sequence $\{\varphi_{k_1}\}_{k_1 \geq k_0}$ is itself clearly a normal family as all the functions have the same range as f and are therefore uniformly bounded. By passing to another subsequence if necessary, we can therefore find a limit function φ so that $\varphi_{k_1} \to \varphi$ uniformly on compact subsets of this neighbourhood of K (K itself being one of these compact subsets). However, since K was arbitrarily chosen, φ can be defined on all of W and $\varphi_{k_1} \to \varphi$ uniformly on compact subsets of W.

https://doi.org/10.1017/S0143385705000441 Published online by Cambridge University Press

We claim that φ is the identity function on W. To do this we first show that $\varphi^{\circ 2} = \varphi$ on W. Let $\varepsilon > 0$, fix a compact subset K_1 of W, and let k_1 be such that all functions Q_{m_k,m_l} and φ_k are defined on some relatively compact open neighbourhood W_1 of K_1 for $k \geq k_1$.

Now consider the set $K_1 \cup f(c)$. The fact that $\mathrm{PD}(\{P_m\}_{m=1}^{\infty}) \geq \delta$ implies that the iterate c_{m_k} of the critical point c which converges to $f(c)$ is always at least distance δ away from ∂U_{m_k}. Let $z \in K_1$ and join z to $f(c)$ by a hyperbolic geodesic γ in W. It follows that for k_1 large enough, and for $k \geq k_1$, γ is bounded away from ∂U_{m_k}. Since all bounded Fatou components must be simply connected in view of the maximum principle, the standard estimates for the hyperbolic metric for simply connected domains (e.g. [**6**, Theorem 4.3]) imply that, for $k \geq k_1$, the hyperbolic distance in U_{m_k} between these points is uniformly bounded. An easy compactness argument then shows that we may in fact take k_1 large enough so that the whole set $K_1 \cup f(c)$ and hence the set $K_1 \cup c_{m_k}$ has uniformly bounded diameter in the hyperbolic metric in U_{m_k} for $k \geq k_1$.

Now pick $z \in K_1$, let $k_1 \leq k \leq l$, and join z to c_{m_k} by a hyperbolic geodesic γ in U_{m_k}. From the Schwarz lemma for the hyperbolic metric combined with the fact that $\mathrm{PD}(\{P_m\}_{m=1}^{\infty}) \geq \delta$ again implies that c_{m_k} is always at least distance δ away from ∂U_{m_k}, it follows that we can find $\alpha > 0$ independent of z, k and l for which U_{m_l} contains an α-neighbourhood of the curve $Q_{m_k,m_l}(\gamma)$ which connects $Q_{m_k,m_l}(z)$ to $Q_{m_l}(c)$.

We claim that if we take k_1 large enough, then an open (connected) $(\alpha/4)$-neighbourhood of $Q_{m_k,m_l}(\gamma)$ will lie in W. Pick k_1 large enough so that for all $k \geq k_1$, $Q_{m_k}(c) \in W$ and the sets U_{m_k} and ∂U_{m_k} are within distance $\alpha/4$ of their respective Hausdorff limits $\mathcal{H}$ and $\mathcal{L}$. Let w^1 be a point within distance $\leq \alpha/4$ of $Q_{m_k,m_l}(\gamma)$ so that $w^1 \in U_{m_l}$ (the reader should note that here the 'l' is an index rather than a power). We claim that if $l' > l$, then $w^1 \in \overline{U}_{m_{l'}}$. Otherwise, if we can find $l' > l$ such that $w^1 \notin \overline{U}_{m_{l'}}$, then (by moving w^1 closer to $Q_{m_k,m_l}(\gamma)$ and then if necessary along $Q_{m_k,m_l}(\gamma)$ towards $Q_{m_l}(c) \in U_{m_{l'}}$) we may assume that $w^1 \in \partial U_{m_{l'}}$. By the Hausdorff convergence of the sets ∂U_{m_k}, we can find another point w^2 in ∂U_{m_l}, with $|w^1 - w^2| \leq \alpha/4 + \alpha/4 = \alpha/2$. However, since w^1 is within distance $\alpha/4$ of $Q_{m_k,m_l}(\gamma)$, we would then have a point w^3 on $Q_{m_k,m_l}(\gamma)$ with $|w^3 - w^2| \leq (3/4)\alpha < \alpha$, which clearly contradicts what we have above. This shows that our $(\alpha/4)$-neighbourhood of $Q_{m_k,m_l}(\gamma)$ lies within the interior of $\mathcal{H}$, and that it also misses $\mathcal{L}$. Since $Q_{m_l}(c) \in W$ by our choice of k_1, it follows that this (connected) neighbourhood must be contained in W.

Now let K_2 be the set $\{z \in W : \mathrm{d}(z, \partial W) \geq \alpha/4\}$ and $\mathcal{F}_1$ denote the family

$$\{Q_{m_k,m_l}\}_{l \geq k \geq k_1} \cup \{\varphi_k\}_{k \geq k_1} \cup \{\varphi\}.$$

It follows from our claim above that K_2 is a compact subset of W for which $g(K_1) \subset K_2$ for every $g \in \mathcal{F}_1$. Now choose k_2 large enough so that all the functions

$$\{Q_{m_k,m_l}\}_{l \geq k \geq k_2} \cup \{\varphi_k\}_{k \geq k_2} \cup \{\varphi\}$$

are defined on some relatively compact open neighbourhood W_2 of K_2 and let us denote this new family by $\mathcal{F}_2$. By making k_2 large enough, we can guarantee that

$$|\varphi(z) - \varphi_k(z)| < \varepsilon, \quad z \in K_2,\ k \geq k_2.$$

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Also, $\mathcal{F}_2$ is clearly normal on W_2 and so, by equicontinuity on K_2, we can find a constant η so that, if z and w are points in K_2 with $|z-w| < \eta$, then $|g(z)-g(w)| < \varepsilon$ for any $g \in \mathcal{F}_2$. Now make k_1 larger if needed so that

$$|\varphi_k(z) - \varphi(z)| < \min(\varepsilon, \eta), \quad z \in K_1,\ k \geq k_1,$$

and make k_2 large enough so that

$$|Q_{m_{k_1},m_k}(z) - \varphi_{k_1}(z)| < \min(\varepsilon, \eta), \quad z \in K_1,\ k \geq k_2.$$

Note that we will need to have $k_2 \geq k_1$ and so we may need to make k_2 larger at this stage to preserve this. However, the convergence and equicontinuity estimates above will not be affected and in particular we can still use the same value of η. Lastly, we pick $k_3 > k_2$ so that

$$|Q_{m_{k_2},m_k}(z) - \varphi_{k_2}(z)| < \varepsilon, \quad z \in K_2,\ k \geq k_3.$$

Then on K_1, we have that

$$\begin{aligned}
|\varphi^{\circ 2} - \varphi| &\leq |\varphi^{\circ 2} - \varphi_{k_2} \circ \varphi_{k_1}| + |\varphi_{k_2} \circ \varphi_{k_1} - \varphi_{k_1}| + |\varphi_{k_1} - \varphi| \\
&\leq |\varphi^{\circ 2} - \varphi_{k_2} \circ \varphi| + |\varphi_{k_2} \circ \varphi - \varphi_{k_2} \circ \varphi_{k_1}| \\
&\quad + |\varphi_{k_2} \circ \varphi_{k_1} - \varphi_{k_2} \circ Q_{m_{k_1},m_{k_2}}| \\
&\quad + |\varphi_{k_2} \circ Q_{m_{k_1},m_{k_2}} - Q_{m_{k_2},m_{k_3}} \circ Q_{m_{k_1},m_{k_2}}| \\
&\quad + |Q_{m_{k_1},m_{k_3}} - \varphi_{k_1}| + |\varphi_{k_1} - \varphi| \\
&< 6\varepsilon.
\end{aligned}$$

Since ε and K_1 were arbitrarily chosen, we see that $\varphi^{\circ 2} = \varphi$ on W as required.

If φ is non-constant, then by the open mapping theorem, $\varphi(V)$ is a non-empty open set. Because $\varphi^{\circ 2} = \varphi$ on V, $\varphi = \mathrm{Id}$ on $\varphi(V)$. We now show that φ is indeed non-constant and that $\varphi(V) \cap V \neq \emptyset$. If this is the case, then by the identity principle $\varphi = \mathrm{Id}$ on V and hence on W, and our claim will be proved.

To see this let z be any point in U. Choose $\varepsilon > 0$ so that the open disc $D(z, 2\varepsilon) \subset U$ (we choose a disc of radius 2ε so that we can ensure that the convergence of the iterates Q_{m_k} is uniform on the smaller disc $D(z, \varepsilon)$). By the open mapping theorem, there exists $\eta > 0$ with $D(f(z), \eta) \subset f(D(z, \varepsilon)) \subset V$. Now let X^1 be the component of $f^{\circ -1}(D(f(z), \eta))$ which contains z. The function f must be proper on X^1 as this set is contained in $D(z, \varepsilon)$ and f can have only finite degree on this disc as it is compactly contained in U. It then follows that, by making η slightly smaller if needed to avoid critical values of f, we may assume that the boundary of X^1, which we will denote by Γ^1, is a simple closed curve.

Now apply uniform convergence and Rouché's theorem (applying Rouché's theorem was why we needed to construct a domain whose boundary is a simple closed curve) to see that for k_1 large enough and $k \geq k_1$, $D(f(z), \eta/2) \subset Q_{m_k}(X^1) \subset Q_{m_k}(D(z, \varepsilon))$. On the other hand, we see from our earlier reasoning that on $D(f(z), (3/4)\eta)$ (which we make use of as it is a relatively compact open neighbourhood of $\overline{D}(f(z), \eta/2)$), for fixed k sufficiently large Q_{m_k,m_l} converges uniformly to φ_k as l tends to infinity, while $\varphi_k \to \varphi$ as k tends to infinity.

https://doi.org/10.1017/S0143385705000441 Published online by Cambridge University Press

Let X^2 be the component of $f^{\circ -1}(D(f(z), \eta/2))$ which also contains z. As before we may assume that the boundary of X^2 is again a simple closed curve which we call Γ^2. Recall the family

$$\mathcal{F}_1 = \{Q_{m_k,m_l}\}_{l \geq k \geq k_1} \cup \{\varphi_k\}_{k \geq k_1} \cup \{\varphi\},$$

which for k_1 sufficiently large will be defined and normal on $D(f(z), (3/4)\eta)$. By equicontinuity, we can find $\mu > 0$ such that, if $g \in \mathcal{F}_1$ and a and b are points in $D(f(z), (3/4)\eta)$ with $|a - b| < \mu$, then $|g(a) - g(b)| < \eta/8$. Now pick k_1 large enough so that $|Q_{m_k} - f| < \min\{\eta/8, \mu\}$ on $D(z, \varepsilon)$ and hence on the closure of X^2 for all $k \geq k_1$. Hence for $k_1 \leq k < l$ and $w \in Q_{m_k}(\Gamma^2)$, $|Q_{m_k,m_l}(w) - f(z)| > (3/8)\eta$ provided k_1 is sufficiently large. If we then combine this with the equicontinuity above, we see that, if w is within less than μ of $Q_{m_k}(\Gamma^2)$, then $|Q_{m_k,m_l}(w) - f(z)| > \eta/4$ for $k_1 \leq k < l$. It is important to note here that $C(f(z), \eta/2)$, the boundary of $D(f(z), \eta/2)$, is homotopic to $Q_{m_k}(\Gamma^2)$ in this μ-neighbourhood of $Q_{m_k}(\Gamma^2)$, and that from above $|Q_{m_k,m_l}(w) - f(z)| > \eta/4$ holds on this circle.

Clearly the winding number of $C(f(z), \eta/2)$ about $f(z)$ is 1. Since Q_{m_k} converges uniformly to f on $D(z, \varepsilon)$, it follows using homotopy and Cauchy's theorem that for k and l large enough, with $k \leq l$, the winding number of $Q_{m_l}(\Gamma^2)$ about $f(z)$ is 1. Hence the integral of $Q'_{m_k,m_l}/(Q_{m_k,m_l} - f(z))$ along $Q_{m_k}(\Gamma^2)$ is also 1. From above, $Q_{m_k}(\Gamma^2)$ and $C(f(z), \eta/2)$ are homotopic in an open set on which $|Q_{m_k,m_l} - f(z)| > \eta/4$, and if we make use of this homotopy to apply Cauchy's theorem and then apply the argument principle, we then see that $Q_{m_k,m_l}(D(f(z), \eta/2))$ will contain the point $f(z)$. In addition, again from above, the minimum value of $|Q_{m_k,m_l}(w) - f(z)|$ on the boundary $C(f(z), \eta/2)$ will be uniformly bounded below by $\eta/4$ (independently of k and l). It then follows from Hurwitz's theorem on letting l go to infinity that $\varphi_k(D(f(z), \eta/2))$ will also contain $f(z)$ and again the minimum value of $|\varphi_k(w) - f(z)|$ on the boundary of this disc will also be uniformly bounded below by $\eta/4$. Now let k tend to infinity and conclude that φ is non-constant and that $f(z)$ is contained in $\varphi(D(f(z), \eta/2))$. Hence $f(z) \in V \cap \varphi(V)$ and our original claim that φ is the identity function on V and hence on W is now established.

It is a consequence of the fact that W must contain a disc of radius δ together with the Hausdorff convergence of the sets ∂U_{m_k} that we can find a point x in W and k_1 such that $(3/8)\delta < \mathrm{d}(x, \partial U_{m_k}) < (5/8)\delta$ for all $k \geq k_1$. (The reader should note that here x is *not* being iterated under our sequence!) Also, by making k_1 larger if needed, we can ensure by the claim we just proved that for $k \geq k_1$ we have $|\varphi_k(x) - x| < \delta/8$. Next choose k_2 such that, for any $k \geq k_2$, we have $|Q_{m_{k_1},m_k}(x) - \varphi_{k_1}(x)| < \delta/8$. Hence

$$\frac{\delta}{8} < \mathrm{d}(Q_{m_{k_1},m_k}(x), \partial U_{m_k}) < \frac{7}{8}\delta, \quad k \geq k_2.$$

Now Lemma 3.1 implies that $|Q'_{m_{k_1},m_k}| \to \infty$ on $\partial U_{n_{k_1}}$. Since we have that $\mathrm{d}(Q_{m_{k_1},m_k}(x), \partial U_{m_k}) < (7/8)\delta$ for $k \geq k_2$, the fact that inverse images are univalent on a disc of radius δ about any point in any of the iterated Julia sets combined with the standard distortion estimates for univalent functions allows us to conclude that, by taking k_2 large enough, we can guarantee that $\mathrm{d}(x, \partial U_{m_{k_1}}) < \delta/8$. This contradiction then finishes the proof of Theorem 3.3. □

https://doi.org/10.1017/S0143385705000441 Published online by Cambridge University Press

Before proving that (2) $\Rightarrow$ (1) in Theorem 1.3, we prove a stability result concerning sequences with a uniform lower bound on the postcritical distance.

PROPOSITION 3.1. *Let $\{\{P_m^n\}_{m=1}^\infty\}_{n=1}^\infty$ be a sequence each of whose members is a bounded sequence of polynomials. Suppose there exist constants d, K, M, $\delta > 0$ such that for each n, $\{P_m^n\}_{m=1}^\infty$ has bounds d, K and M as in the definition, and postcritical distance bounded below by δ. Also, suppose that, as n tends to infinity, $\{P_m^n\}_{m=1}^\infty$ tends pointwise to a limit sequence $\{P_m\}_{m=1}^\infty$. Then* $\mathrm{PD}(\{P_m\}_{m=1}^\infty) \geq \delta$.

Proof. For a fixed stage m, Hurwitz's theorem implies that the critical points of P_{m+1}^n converge to those of P_{m+1}, and if we combine this with the pointwise convergence of $\{\{P_m^n\}_{m=1}^\infty\}_{n=1}^\infty$ to $\{P_m\}_{m=1}^\infty$, we see that the critical values of Q_m^n converge to those of Q_m. Now pick a subsequence n_k such that the sets $\mathcal{J}^{n_k}$ converge to some limit $\mathcal{H}$ in the Hausdorff topology. Then by Theorem 2.2 on the complete invariance of Hausdorff limits, $\mathcal{J}_m^{n_k}$ converges to $\mathcal{H}_m = Q_m(\mathcal{H})$. Since at each stage m the set of critical values of $Q_{m-1}^{n_k}$ and critical points of $P_m^{n_k}$ is at least δ away from $\mathcal{J}_m^{n_k}$ in terms of Hausdorff distance, it follows that the set of critical values of Q_m and critical points of P_{m+1} is at least δ away from $\mathcal{H}_m$. But $\mathcal{J}_m$ is contained in $\mathcal{H}_m$ by Theorem 2.4, and the result is proved. □

The following result concerning the stability in the Hausdorff topology of Julia sets for sequences with positive postcritical distance is of central importance in our exposition and will be made use of extensively in what follows.

THEOREM 3.4. *Let $\{\{P_m^n\}_{m=1}^\infty\}_{n=1}^\infty$ be a sequence of bounded polynomial sequences each of which has bounds d, K and M as in the definition, and suppose in addition that there exists $\delta > 0$ such that each member of this sequence has postcritical distance bounded below by δ. For each n, let $\mathcal{J}^n$ be the Julia set for $\{P_m^n\}_{m=1}^\infty$. Suppose also that $\{P_m^n\}_{m=1}^\infty$ tends pointwise to a limit sequence $\{P_m\}_{m=1}^\infty$ as n tends to infinity, and let $\mathcal{J}$ denote the Julia set for $\{P_m\}_{m=1}^\infty$. Then $\mathcal{J}^n \to \mathcal{J}$ in the Hausdorff topology.*

Proof. Let R be an escape radius for the bounds d, K and M. We start by noting that, since all the Julia sets $\mathcal{J}^n$ lie in $\overline{D}(0, R)$ and are therefore uniformly bounded, using the Blaschke selection theorem we can take a subsequence n_k so that the Julia sets $\mathcal{J}^{n_k}$ will converge in the Hausdorff topology to some limit $\mathcal{H}$. We will therefore be done if we can show that $\mathcal{H} = \mathcal{J}$. This we will prove by contradiction and so let us assume that it is false. From Theorem 2.4, $\mathcal{H} \subset \mathcal{K}$ where $\mathcal{K}$ denotes the filled Julia set for the limit sequence $\{P_m\}_{m=1}^\infty$, and so there must be a bounded Fatou component U for the limit sequence such that $\mathcal{H} \cap U \neq \emptyset$. By Proposition 3.1, $\mathrm{PD}(\{P_m\}_{m=1}^\infty) \geq \delta$, and applying Lemma 3.2, we see that a critical point c will eventually occur at some stage on the orbit of U under the polynomials Q_m. It then follows from our result on the complete invariance of Hausdorff limits (Theorem 2.2) that we may without loss of generality assume that c occurs on the orbit of U at stage 0 so that c is a critical point of P_1.

Now let $y \in U \cap \mathcal{H}$. Then we can find Fatou components U^{n_k}, critical points c^{n_k} of $P_1^{n_k}$ with $c^{n_k} \in U^{n_k}$, and points $y^{n_k} \in \mathcal{J}^{n_k}$ such that $y^{n_k} \to y$ and $c^{n_k} \to c$ as $k \to \infty$. We then have that, since $\mathrm{PD}(\{P_m^{n_k}\}_{m=1}^\infty) \geq \delta$ for all k,

$$|Q_m^{n_k}(y^{n_k}) - Q_m^{n_k}(c^{n_k})| \geq \delta, \quad m \geq 0.$$

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On the other hand, by Theorem 3.3, it follows that we can find m_0 such that for all $m \geq m_0$,

$$|Q_m(y) - Q_m(c)| < \delta/5.$$

By equicontinuity of the family $\mathcal{Q} = \{Q_{m_0}^{n_k}\}_{k=1}^{\infty}$ on $\overline{D}(0, R)$, we can find $\eta > 0$ such that $x, y \in \overline{D}(0, R)$ and $|x - y| < \eta$ implies that $|f(x) - f(y)| < \delta/5$ for any member f of $\mathcal{Q}$. Also, by uniform convergence of these polynomials on $\overline{D}(0, R)$ we can find k_0 such that

$$|Q_{m_0}^{n_k}(z) - Q_{m_0}(z)| < \delta/5, \quad z \in \overline{D}(0, R),\ k \geq k_0,$$

and by the fact that $c^{n_k} \to c$, we can also choose k_0 such that

$$|c^{n_k} - c| < \eta, \quad k \geq k_0.$$

Lastly, by making k_0 larger if necessary, the convergence of the sequence $\{y^{n_k}\}_{k=1}^{\infty}$ guarantees that

$$|y^{n_k} - y| < \eta, \quad k \geq k_0.$$

With this in hand, we see that

$$\begin{aligned} |Q_{m_0}^{n_k}(y^{n_k}) - Q_{m_0}^{n_k}(c^{n_k})| &\leq |Q_{m_0}^{n_k}(y^{n_k}) - Q_{m_0}^{n_k}(y)| + |Q_{m_0}^{n_k}(y) - Q_{m_0}(y)| \\ &\quad + |Q_{m_0}(y) - Q_{m_0}(c)| + |Q_{m_0}(c) - Q_{m_0}^{n_k}(c)| \\ &\quad + |Q_{m_0}^{n_k}(c) - Q_{m_0}^{n_k}(c^{n_k})| \\ &< \delta/5 + \delta/5 + \delta/5 + \delta/5 + \delta/5 = \delta, \end{aligned}$$

which is clearly a contradiction, and so $\mathcal{H} = \mathcal{J}$ as required. □

A remark worth making is that the fact that there is a uniform lower bound on the postcritical distance for all the sequences in the statement of the above result is essential. Without this we cannot hope to say anything, even in the classical case. One sees this in light of the discontinuous dependence on parameters of classical Julia sets as shown by Lavaurs, Douady, Hubbard, Shishikura and others where one sees 'parabolic implosion' in the neighbourhood of the cusp located at $1/4$ in the Mandelbrot set for quadratic polynomials of the form $z^2 + c$. We are now finally in a position to show that (2) $\Rightarrow$ (1) in Theorem 1.3.

THEOREM 3.5. *Let d be an integer with $d \geq 2$, let $K > 1$, and let $M, \delta > 0$. Then there exists a natural number m_0, depending only on d, K, M and δ, with the following property. Given any bounded sequence $\{P_m\}_{m=1}^{\infty}$ of polynomials with coefficient bounds d, K, M and for which $\mathrm{PD}(\{P_m\}_{m=1}^{\infty}) \geq \delta$, then if we denote the Julia set for such a sequence by $\mathcal{J}$, it follows that for all $z \in \mathcal{J}$ there exists $m_z \leq m_0$ such that*

$$|Q'_{m_z}(z)| > 2.$$

Hence any such sequence $\{P_m\}_{m=1}^{\infty}$ is hyperbolic.

Proof. We will proceed by contradiction. So suppose the result is false and we can find a particular $d \geq 2$, $K \geq 1$, $M > 0$ and $\delta > 0$ such that, for each n, we can find a sequence $\{P_m^n\}_{m=1}^{\infty}$ whose coefficients are bounded in absolute value by M and with $\mathrm{PD}(\{P_m^n\}_{m=1}^{\infty}) \geq \delta$ for which there is a point z^n in the corresponding Julia set $\mathcal{J}^n$ such that

$$|(Q_m^n)'(z^n)| \leq 2, \quad m \leq n.$$

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Since the sequences $\{P_m^n\}_{m=1}^\infty$ are uniformly bounded, we can apply the standard diagonalization argument as described in §1 to obtain a diagonal subsequence n_k and a limit sequence $\{P_m\}_{m=1}^\infty$ to which the sequences $\{P_m^{n_k}\}_{m=1}^\infty$ converge pointwise in the usual sense where, for each fixed m, the coefficients of $P_m^{n_k}$ converge to those of P_m. By the previous result, the Julia sets $\mathcal{J}^{n_k}$ will converge in the Hausdorff topology and the Hausdorff limit must be $\mathcal{J}$, the Julia set for $\{P_m\}_{m=1}^\infty$.

By Proposition 3.1, $\mathrm{PD}(\{P_m\}_{m=1}^\infty) \geq \delta$ also, and so by Lemma 3.1 we can find m_0 so that

$$|Q_m'(z)| > 3, \quad z \in \mathcal{J},\ m \geq m_0.$$

However, since $\mathcal{J}^{n_k} \to \mathcal{J}$, any limit of the sequence $\{z^{n_k}\}_{k=1}^\infty$ must be a point in $\mathcal{J}$. This clearly contradicts what we have just said above and with this the existence of the number m_0 in the statement of the theorem is established.

To see that this implies hyperbolicity, note first that by normality we have an upper bound B depending only on d, K and M such that $|P_m'| \leq B$ on $\mathcal{J}_m$. Combining this with the fact that the number m_0 above depends only on d, K, M and δ, we also obtain a lower bound C depending only on these constants such that $|P_m'| \geq C$ on $\mathcal{J}_m$. Hyperbolicity then follows immediately from this and the existence of m_0. □

So we have shown that any bounded sequence with postcritical distance bounded below by some positive quantity δ is hyperbolic. The above proof also shows that the number of iterations necessary for the absolute value of the derivative to exceed a fixed value on the Julia set will depend only on d, K, M and δ. If we combine what we have proved above with Theorem 3.1, we have the following important stability result.

COROLLARY 3.2. *Let $\{\{P_m^n\}_{m=1}^\infty\}_{n=1}^\infty$ be a sequence of bounded polynomial sequences each of which is hyperbolic with associated constants d, K, C, M and λ. Suppose also that $\{P_m^n\}_{m=1}^\infty$ tends pointwise to $\{P_m\}_{m=1}^\infty$. Then $\{P_m\}_{m=1}^\infty$ is also hyperbolic with the same constants, and the Julia sets $\mathcal{J}^n$ for each $\{P_m^n\}_{m=1}^\infty$ converge in the Hausdorff topology to the Julia set $\mathcal{J}$ for $\{P_m\}_{m=1}^\infty$.*

4. *Dynamics on the Fatou set*

In this section we prove that (1) ⇒ (3) and (3) ⇒ (2) in Theorem 1.3, thereby completing the proof of this result. We start with the following theorem which shows how the behaviour of the iterates on Fatou components for a hyperbolic non-autonomous sequence is similar to (but not exactly the same as) the classical case where all Fatou components are eventually iterated to attracting basins. For a set S, let $\mathrm{diam}^\#(S)$ denote the diameter of S with respect to the spherical metric $|dz|/(1+|z|^2)$ on $\overline{\mathbb{C}}$. Also, for a given Fatou component U and a set $K \subset U$, let $\delta(K)$ denote $\mathrm{dist}^\#(K, \partial U)$, the (spherical) distance from K to the boundary of U.

THEOREM 4.1. *If $\{P_m\}_{m=1}^\infty$ is a hyperbolic bounded sequence with associated constants d, K, M, C and λ, there exists $0 < \theta_0 < 1$ such that, for any Fatou component U, any $\delta > 0$, $m \geq 0$, and any continuum $K \subset U$, where $\delta(K) \geq \delta$, we have that*

$$\mathrm{diam}^\#(Q_m(K)) \leq C_0\, \theta_0^m\, \mathrm{diam}^\#(K),$$

where θ_0 depends on d, K, M, C, λ and C_0 depends on δ and on d, K, M, C, λ.

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Proof. The hyperbolicity of $\{P_m\}_{m=1}^{\infty}$ implies that $\{P_m\}_{m=1}^{\infty}$ has some positive postcritical distance $\delta_0 > 0$ by Theorem 3.1. Hence using hyperbolicity and the distortion theorems for univalent mappings (e.g. [**6**, Theorem 1.6]), we see that we can find m_0 depending only on δ and d, K, M, C, λ such that, for any point z which is at least δ away from $\mathcal{J}$, $Q_m(z)$ will be at least $\delta_0/2$ away from the iterated Julia set $\mathcal{J}_m$ for $m \geq m_0$. Hence, we will have proved our result if we can show that, given constants d, K, M, C, λ as in the definition of hyperbolicity and $\delta > 0$, there exists $m_1 = m_1(d, K, M, C, \lambda, \delta)$ such that, for any sequence $\{P_m\}_{m=1}^{\infty}$ which is hyperbolic with these constants, any Fatou component U for this sequence, and any continuum K lying in U with $\delta(K) \geq \delta$,

$$\operatorname{diam}^{\#}(Q_m(K)) \leq \frac{\operatorname{diam}^{\#}(K)}{2}, \quad \text{for some } m \leq m_1.$$

As usual, we will argue by contradiction. So suppose the result is false. This means that we can find particular values of (d, K, M, C, λ) and $\delta > 0$ such that, for all n, we can find sequences $\{P_m^n\}_{m=1}^{\infty}$ which are (d, K, M, C, λ)-hyperbolic for which there are Fatou components U^n and continua $K^n \subset U^n$ with $\delta(K^n) \geq \delta$ but for which

$$\operatorname{diam}^{\#}(Q_m^n(K^n)) > \frac{\operatorname{diam}^{\#}(K^n)}{2}, \quad m \leq n.$$

Note that we must have that $\operatorname{diam}^{\#}(K^n) > 0$ for each n or else the result is trivially true. We then carry out the usual diagonalization argument and find a subsequence n_k such that the sequences $\{P_m^{n_k}\}_{m=1}^{\infty}$ converge pointwise to a limit sequence $\{P_m\}_{m=1}^{\infty}$ and hence the corresponding Julia sets $\mathcal{J}^{n_k}$ will converge to $\mathcal{J}$, the Julia set for $\{P_m\}_{m=1}^{\infty}$ in view of Corollary 3.2. Again, we can take a further subsequence if necessary to ensure that the uniformly bounded compact sets $\overline{U}^{n_k}$ and the continua K^{n_k} converge in the Hausdorff topology (with respect to the spherical metric). Let K denote the Hausdorff limit of the sets K^{n_k} which will itself clearly also be a continuum. Then the fact that $\delta(K^{n_k}) \geq \delta$ for all k guarantees that the Hausdorff limit of the sets $\overline{U}^{n_k}$ will have non-empty interior, and since K is connected we can therefore find a Fatou component U for the limit sequence $\{P_m\}_{m=1}^{\infty}$ such that the continuum K is contained in U. We then have the following two cases.

Case 1: $\operatorname{diam}^{\#}(K) > 0$. Using Corollary 3.2, $\{P_m\}_{m=1}^{\infty}$ is (d, K, M, C, λ)-hyperbolic and by Theorems 3.1 and 3.3, we can find m_2 such that

$$\operatorname{diam}^{\#}(Q_{m_2}(K)) < \frac{\operatorname{diam}^{\#}(K)}{3}.$$

But a standard convergence argument (similar to that used in Theorem 3.4, for example) shows that this is impossible in view of our assumption above concerning the sets K^{n_k}.

Case 2: $\operatorname{diam}^{\#}(K) = 0$. Here $K = \{x\}$ for some $x \in U$ with $\delta(x) \geq \delta$. The disc $D(x, \delta/2)$ avoids the sets $\mathcal{J}^{n_k}$ for $k > k_0$, provided k_0 is sufficiently large, and since all Julia sets involved have logarithmic capacity bounded below by Theorem 1.2, we can argue by postcomposing with a family of Möbius transformations which is equicontinuous and whose inverses are also equicontinuous, and then applying Montel's theorem that the family of functions $\{Q_m^{n_k} : m \geq 1, k \geq k_0\} \cup \{Q_m\}_{m=1}^{\infty}$ is normal on $D(x, \delta/2)$. This implies that the absolute values of the spherical derivatives of functions in this

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family will be uniformly bounded on compact subsets of $D(x, \delta/2)$, which gives us another normal family of (real-valued) functions. If we now apply this to the fact that $\operatorname{diam}^{\#}(K^{n_k}) > 0$ for all k, while $\operatorname{diam}^{\#}(K^{n_k}) \to 0$ as $k \to \infty$, we deduce that we can find k_1 sufficiently large for which we have that

$$|Q_m^{n_k\#}(x)| \geq \tfrac{1}{3}, \quad k \geq k_1,\ m \leq n_k.$$

However, in view of the fact that $\{P_m\}_{m=1}^{\infty}$ is hyperbolic, by Theorems 3.1 and 3.3, we can find m_3 such that

$$|Q_{m_3}^{\#}(x)| < \tfrac{1}{4}.$$

This clearly leads to a contradiction, in view of which Theorem 4.1 is now proved. □

One important remark worth making is that the fact that we restricted ourselves to connected sets K in the hypothesis of the theorem was essential. We needed this in Case 1 of the proof to guarantee that the Hausdorff limit of the sets K^{n_k} was connected. Otherwise, different components of K could lie in different Fatou components of $\{P_m\}_{m=1}^{\infty}$, which would prevent us from being able to obtain a contradiction. Whether or not one can in fact dispense with the assumption of connectedness turns out to be equivalent to whether or not the component one considers is a John domain. See [**7**] for details together with examples of hyperbolic sequences which have Fatou components which are not John domains.

We now continue with the proof of Theorem 1.3 and turn to establishing that (1) $\Rightarrow$ (3). So suppose $\{P_m\}_{m=1}^{\infty}$ is a hyperbolic bounded sequence with associated constants d, K, M, C, λ. Let c be a critical point which in view of Lemma 3.3 we may as well assume occurs at stage 0 (i.e. it is a critical point of P_1). By Theorem 4.1 and the fact that by Theorem 3.1 $\operatorname{PD}(\{P_m\}_{m=1}^{\infty}) \geq \delta$ for some $\delta > 0$, we can find m_0 depending on d, K, M, C, λ such that for any $m \geq 0$ and any $0 < r < \delta/2$

$$\operatorname{diam}^{\#}(Q_{m,m+m_0}(D(c, r))) < \frac{r}{3}.$$

Now the family $\{Q_{m,m+m_0}\}_{m=0}^{\infty}$ is clearly normal on $\overline{\mathbb{C}}$ which implies that, provided r was chosen small enough (the choice depending on d, K, M, C, λ), we have that for every $m \geq 0$

$$|Q_{m,m+m_0}^{\#}(c)| < 1/2.$$

The implication (1) $\Rightarrow$ (3) now follows easily from this fact and the fact that $|P_m^{\#}|$ is uniformly bounded above on $\overline{\mathbb{C}}$ in terms of the bounds d, K and M (the bound in particular not depending on m).

To prove the converse, suppose now that $\{P_m\}_{m=1}^{\infty}$ satisfies condition (3). It is fairly easy using normality to see that we can find $r > 0$ and m_1 depending only on θ and the bounds d, K and M such that, for all $m \geq 0$ and all critical points c of P_{m+1}, we have that

$$\operatorname{diam}^{\#}(Q_{n,n+m_1}(D(c_{m,n}, r))) < r/2, \quad n \geq m.$$

Such a disc must lie in the Fatou set and so

$$\operatorname{dist}(c_{m,n}, \mathcal{J}_n) \geq r.$$

Thus $\operatorname{PD}(\{P_m\}_{m=1}^{\infty}) \geq r$, whence (2) follows, and the proof of Theorem 1.3 is now finally complete.

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