Math. Proc. Camb. Phil. Soc. (1977), **82**, 131
Printed in Great Britain

Bounds for the solution of a non-linear parabolic initial-boundary value problem

By M. R. CARTER
Massey University, Palmerston North, New Zealand

(*Received* 6 *December* 1976, *Revised* 8 *February* 1977)

1. *Introduction.* A number of papers have appeared over the past decade or so which study questions of the existence and stability of positive steady-state solutions for parabolic initial-boundary value problems of the general form

$$\left.\begin{aligned} &\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i\,\partial x_j}+\sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i}+c(x,t)\,u-\frac{\partial u}{\partial t}+\lambda f(x,t,u)=0 \\ &\text{for } x=(x_1,\ldots x_n) \text{ in an } n\text{-dimensional bounded, open, connected region } V \text{ and } t>0, \\ &d_0(x,t)\,u+d_1(x,t)\frac{\partial u}{\partial \nu}=0 \\ &\text{for } x \text{ on the boundary } \partial V \text{ of } V \text{ and } t>0 \\ &u(x,0)=u_0(x) \quad \text{for } x \text{ on the closure } \bar{V}. \end{aligned}\right\} \tag{1}$$

Such problems arise in the theories of thermal explosions, of nuclear reactors and of electrical heating (see, for example, (1), (2), (8), (9), (10), (11), (14)). The method of approach has been to study directly the steady-state version of (1), usually under the assumptions that the problem is self-adjoint and that the coefficients belong to appropriate function spaces. An extensive bibliography is given in the survey paper by Amann (3). The effect of different assumptions concerning the nature of the function f has attracted particular attention (see (1), (10), (11) in particular).

However, at least in the case where the coefficients in (1) are time-independent, monotone iteration methods may be used, along the lines employed by Sattinger ((15), especially theorems 3·6, 4·1), to show that, under rather general conditions, the existence of a positive steady-state solution of (1) is equivalent to the existence and uniform boundedness over all positive time of the solution of (1) in the particular case $u_0 = 0$ (see (6) for details of the proof). In this paper we shall investigate these latter properties of the solution of (1), under various assumptions about the nature of f, by constructing upper and lower bounds for the solution. Our conclusions will be related to those obtained by studying the corresponding steady-state problem.

2. *Notation.* We shall write

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i\,\partial x_j}+\sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i}+c(x,t)\,u$$

https://doi.org/10.1017/S0305004100053767 Published online by Cambridge University Press

and $$Bu \equiv d_0(x,t)\,u + d_1(x,t)\frac{\partial u}{\partial \nu}.$$

We shall denote by D_T the set $\{(x,t): x \in V,\ 0 < t \leqslant T\}$. For the problem (1), we shall suppose that

(i) $a_{ij}, b_i (i,j = 1, \ldots, n)$ and c are real functions, continuous for $x \in \overline{V}$ and $t \geqslant 0$, and $a_{ij} = a_{ji}$ for $i, j = 1, \ldots, n$.

(ii) for each $T > 0$, the differential operator L is uniformly elliptic on

$$\{(x,t): x \in \overline{V}, 0 \leqslant t \leqslant T\}.$$

(iii) ∂V is of class $C^{2+\alpha}$ for some α with $0 < \alpha < 1$.

(iv) d_0 and d_1 are real functions, continuous and non-negative for $x \in \partial V$ and $t \geqslant 0$.

(v) ν denotes an outwardly directed, nowhere tangential unit vector field on ∂V of the form $\nu(x) = (\nu_1(x), \ldots, \nu_n(x))$, where $\nu_1, \ldots, \nu_n$ are continuous on ∂V, and $\partial u/\partial \nu$ denotes the directional derivative $\sum_{i=1}^{n} \nu_i(x)\,\partial u/\partial x_i$.

(vi) u_0 has continuous second-order partial derivatives on $\overline{V}$.

(vii) f is continuous for $x \in \overline{V}$, $t \geqslant 0$ and $u \geqslant 0$, and λ is a positive parameter.

3. *Comparison theorems.* We require two comparison theorems related to those proved by McNabb(13).

THEOREM 1. *Suppose that (a) the functions u_1, u_2 are defined and continuous in $\overline{D}_T$, their second-order x_i-derivatives exist and are continuous in D_T, their first-order x_i-derivatives exist in $\overline{D}_T$, and their first-order t-derivatives exist in D_T;*

(b) for all $(x,t) \in D_T$,

$$Lu_1 - \frac{\partial u_1}{\partial t} + f(x,t,u_1) > Lu_2 - \frac{\partial u_2}{\partial t} + f(x,t,u_2);$$

(c) $u_1(x,0) < u_2(x,0)$ for all $x \in \overline{V}$;

(d) for all $x \in \partial V$, $0 < t \leqslant T$, we have $Bu_1 < Bu_2$.

Then $u_1(x,t) < u_2(x,t)$ for all $(x,t) \in \overline{D}_T$.

Proof. Suppose, on the contrary, that there is a point $P \in \overline{D}_T$, where $u_1(P) \geqslant u_2(P)$. Then, by the continuity of u_1 and u_2, there is a point $(x', T') \in \overline{D}_T$ such that $u_1 \leqslant u_2$ in $\overline{D}_{T'}$, while $u_1 = u_2$ at (x', T'), $T' > 0$. So $u_1 - u_2$ has a maximum of zero in $\overline{D}_{T'}$ at the point (x', T').

If we can choose $(x', T') \in D_{T'}$, then it follows by the same method of proof as that used by McNabb(13) that at (x', T')

$$Lu_1 - \frac{\partial u_1}{\partial t} + f(x,t,u_1) \leqslant Lu_2 - \frac{\partial u_2}{\partial t} + f(x,t,u_2)$$

contradicting hypothesis (b). If, however, x' cannot be chosen away from ∂V, then we must have $u_1(x', T') = u_2(x', T')$ with $x' \in \partial V$, and $u_1 < u_2$ in $D_{T'}$. Thus $\partial/\partial \nu(u_2 - u_1) \leqslant 0$ at (x', T'), so $Bu_2 \leqslant Bu_1$ at (x', T'), contradicting hypothesis (d). This completes the proof.

https://doi.org/10.1017/S0305004100053767 Published online by Cambridge University Press

THEOREM 2. *Suppose that* (*a*) *holds as for Theorem* 1, *and also*:

(*b*) *for all* $(x,t)\in D_T$, $Lu_1-(\partial u_1/\partial t)+f(x,t,u_1)\geqslant Lu_2-(\partial u_2/\partial t)+f(x,t,u_2)$;

(*c*) $u_1(x,0)\leqslant u_2(x,0)$ *for all* $x\in\overline{V}$;

(*d*) $d_0(x,t)>0$ *for all* $x\in\partial V$, $0<t\leqslant T$;

(*e*) *for all* $x\in\partial V$, $0<t\leqslant T$, *we have* $Bu_1\leqslant Bu_2$;

(*f*) *on any finite interval* $[a,b]$, *the function* f *satisfies a uniform Lipschitz condition in* u, *i.e. there exists* $M_{[a,b]}>0$ (*depending on* $[a,b]$) *such that*

$$|f(x,t,u_1)-f(x,t,u_2)|\leqslant M_{[a,b]}|u_1-u_2|$$

for all $u_1, u_2\in[a,b]$ *and all* $(x,t)\in\overline{D}_T$.

Then $u_1(x,t)\leqslant u_2(x,t)$ *for all* $(x,t)\in\overline{D}_T$.

Proof. Let $\quad m_1=\inf_{(x,t)\in\overline{D}_T}u_2(x,t)\quad$ and $\quad m_2=\sup_{(x,t)\in\overline{D}_T}u_2(x,t)$.

Choose $\quad M'>\sup_{(x,t)\in\overline{D}_T}\{M_{[m_1,m_2+1]}+c(x,t)\}$.

For all $(x,t)\in\overline{D}_T$ and all $y\in[0,1]$, define $U_y(x,t)=u_2(x,t)+ye^{M'(t-T)}$. Then for all $(x,t)\in\overline{D}_T$ and all $y\in(0,1]$, $m_1\leqslant u_2(x,t)<U_y(x,t)\leqslant m_2+1$. Thus, for all $(x,t)\in D_T$ and all $y\in(0,1]$, we have:

$$\begin{aligned}LU_y-\frac{\partial U_y}{\partial t}+f(x,t,U_y)-\left[Lu_2-\frac{\partial u_2}{\partial t}+f(x,t,u_2)\right]&\\=f(x,t,U_y)-f(x,t,u_2)+yc(x,t)\,e^{M'(t-T)}-yM'e^{M'(t-T)}&\\\leqslant M_{[m_1,m_2+1]}y\,e^{M'(t-T)}+yc(x,t)\,e^{M'(t-T)}-yM'e^{M'(t-T)}<0.&\end{aligned}$$

Thus, using (*b*), we have that for all $(x,t)\in D_T$ and all $y\in(0,1]$,

$$Lu_1-\frac{\partial u_1}{\partial t}+f(x,t,u_1)>LU_y-\frac{\partial U_y}{\partial t}+f(x,t,U_y).$$

Also, for all $x\in\partial V$, $0<t\leqslant T$ and $y>0$,

$$BU_y-Bu_2=d_0(x,t)\,(U_y-u_2)>0.$$

Thus, using (*e*) we have $Bu_1<BU_y$ for all $x\in\partial V$, $0<t\leqslant T$ and $y>0$. Further, $u_1(x,0)\leqslant u_2(x,0)<U_y(x,0)$ for all $x\in\overline{V}$, $y>0$. It follows by Theorem 1 that, for all $y\in(0,1]$, $u_1(x,t)<U_y(x,t)$ for all $(x,t)\in\overline{D}_T$. Since $U_y(x,t)\to u_2(x,t)$ as $y\to0+$ for each $(x,t)\in\overline{D}_T$, the required result follows.

Remark. (i) If we omit hypothesis (*f*) in the statement of Theorem 2, the theorem fails to hold, as the following counter-example shows:

$$\text{Take } Lu\equiv\frac{d^2u}{dx^2},\quad V=\{x\,|-1<x<1\},\quad Bu\equiv u+x\frac{du}{dx}$$

and $f(x,t,u)\equiv576\exp(u^{\frac{1}{2}})$. Define $u_1(x,t)\equiv t^2$, $u_2(x,t)\equiv t^2y(x)$ where

$$y(x)\equiv1-\tfrac{1}{5}(1-x^2)\,(5-x^2).$$

Then:

(*a*) $u_1(x,0)=0=u_2(x,0)$, so $u_1(x,0)\leqslant u_2(x,0)\quad$ for $\quad-1\leqslant x\leqslant1$.

https://doi.org/10.1017/S0305004100053767 Published online by Cambridge University Press

(b) $$\left[Lu_1-\frac{\partial u_1}{\partial t}+f(x,t,u_1)\right]-\left[Lu_2-\frac{\partial u_2}{\partial t}+f(x,t,u_2)\right]$$
$$=-t^2y''(x)+2t\{y(x)-1\}+576\{\exp(t^{\frac{1}{2}})-\exp(t^{\frac{1}{2}}\{y(x)\}^{\frac{1}{4}})\}$$
$$=A(x,t), \text{ say.}$$

We have $A(x,0)=0$ for $-1\leqslant x\leqslant 1$. Now suppose $t>0$ and $-1<x<1$. Then

$$\frac{\partial}{\partial t}A(x,t)=2\{y(x)-1\}-2ty''(x)+288t^{-\frac{1}{2}}[\exp(t^{\frac{1}{2}})-\{y(x)\}^{\frac{1}{4}}\exp(t^{\frac{1}{2}}\{y(x)\}^{\frac{1}{4}}y)]. \quad (2)$$

For $-1<x<1$, we have $0\leqslant y(x)<1$, so $0\leqslant\{y(x)\}^{\frac{1}{4}}<1$. Put $\{y(x)\}^{\frac{1}{4}}=1-\delta(x)$, where $0<\delta(x)\leqslant 1$.

Then $$y(x)=\{1-\delta(x)\}^4=1-4\delta(x)+6\delta^2(x)-4\delta^3(x)+\delta^4(x)$$
$$=1-\tfrac{1}{5}(1-x^2)(5-x^2).$$

Therefore
$$\delta(x)=\frac{(1-x^2)(5-x^2)}{5\{4-6\delta(x)+4\delta^2(x)-\delta^3(x)\}}>\frac{(1-x^2)(5-x^2)}{40} \quad \text{for} \quad -1<x<1.$$

Thus (2) becomes

$$\frac{\partial}{\partial t}A(x,t)=\tfrac{2}{5}(x^2-1)(5-x^2)-\frac{24t}{5}(1-x^2)+288t^{-\frac{1}{2}}[\exp(t^{\frac{1}{2}})-\{1-\delta(x)\}\exp(t^{\frac{1}{2}}\{1-\delta(x)\})]$$
$$\geqslant\tfrac{2}{5}(x^2-1)(5-x^2)-\frac{24t}{5}(1-x^2)+288t^{-\frac{1}{2}}\exp(t^{\frac{1}{2}})[\delta(x)]$$
$$>\tfrac{2}{5}(x^2-1)(5-x^2)-\frac{24t}{5}(1-x^2)+288t^{-\frac{1}{2}}\exp(t^{\frac{1}{2}})\left[\frac{(1-x^2)(5-x^2)}{40}\right]$$

from above. Further, for $t>0$, $\exp(t^{\frac{1}{2}})>t^{\frac{1}{2}}+(t^{\frac{3}{2}}/6)$. Thus, for $t\geqslant 0$ and $-1<x<1$,

$$\frac{\partial}{\partial t}A(x,t)>\tfrac{2}{5}(x^2-1)(5-x^2)-\frac{24t}{5}(1-x^2)+\frac{36}{5}\left(1+\frac{t}{6}\right)(1-x^2)(5-x^2)$$
$$=\tfrac{34}{5}(1-x^2)(5-x^2)+\frac{24t}{5}(1-x^2)\{-1+\tfrac{1}{4}(5-x^2)\}>0.$$

Thus for $t>0$ and $-1<x<1$, $A(x,t)>0$, and so

$$Lu_1-\frac{\partial u_1}{\partial t}+f(x,u_1)>Lu_2-\frac{\partial u_2}{\partial t}+f(x,u_2) \quad \text{for} \quad t>0,\ -1<x<1.$$

(c) For $x=\pm 1$, $Bu_1=t^2$, $Bu_2=t^2y(x)+xt^2y'(x)$
$$=t^2y(x)+\tfrac{4}{5}t^2x^2(3-x^2)$$
$$=t^2+\tfrac{8}{5}t^2 \quad \text{since} \quad x^2=1 \quad \text{and} \quad y(x)=1.$$

Therefore $$Bu_1<Bu_2 \quad \text{for} \quad x=\pm 1,\ t>0.$$

Thus all hypotheses of Theorem 2 except (f) are satisfied. However, it is false that $u_1(x,t)\leqslant u_2(x,t)$ for $t\geqslant 0$ and $-1\leqslant x\leqslant 1$, since $0\leqslant y(x)<1$ if $-1<x<1$.

(ii) Hypothesis (f) of Theorem 2 can, however, be replaced by the condition that $f(x,t,u)$ be monotone decreasing in u; this allows the proof to be slightly simplified, but

https://doi.org/10.1017/S0305004100053767 Published online by Cambridge University Press

is of little interest for the present discussion, as $f(x, t, u)$ is typically monotone increasing in u in the problems we are concerned with. A comparison theorem involving monotone decreasing f is proved by Chan(7).

4. *Upper and lower solutions.* We call ϕ an upper solution for (1) if ϕ is continuous for $x \in \bar{V}$, $t \geqslant 0$, has continuous second-order x_i-derivatives for $x \in V$, $t > 0$, continuous first-order x_i-derivatives for $x \in \bar{V}$, $t \geqslant 0$ and continuous first-order t-derivative for $x \in V$, $t > 0$, and satisfies:

$$L\phi - \frac{\partial \phi}{\partial t} + f(x, t, \phi) \leqslant 0 \quad \text{for} \quad x \in V, t > 0;$$

$$B\phi \geqslant 0 \quad \text{for} \quad x \in \partial V, t > 0; \quad \phi(x, 0) \geqslant u_0(x) \quad \text{for} \quad x \in \bar{V}.$$

A *lower solution* is defined analogously, by reversing the inequalities. Similarly the terms *strict upper solution* and *strict lower solution* are defined by replacing the inequalities in the above definitions by strict inequalities. It follows from Theorems 1 and 2 that, under the appropriate conditions, upper and lower (or strict upper and strict lower) solutions are in fact upper and lower bounds for the solution of (1) (note that from Theorem 2 we have at once that, under the hypotheses of that theorem, the solution of (1) is unique).

5. *Construction of upper and lower solutions.* We shall now examine, by means of the construction of upper and lower solutions, the behaviour of the solution of (1) under various assumptions as to the rate of growth of $f(x, t, u)$ with increasing u (the existence of the solution will be taken as a hypothesis in all that follows). We shall begin by working with a specific region

$$V_m = \left\{x: \sum_{i=1}^{n} x_i^{2m_i} < 1\right\},$$

where the m_i are arbitrarily chosen positive integers. We shall take the unit vector field $\boldsymbol{\nu}$ to be such that, for each $i = 1, \ldots, n$, $\nu_i(x) = \alpha_i(x)\, x_i^{2m_i - 1}$, where α_i is continuous on V_m. We denote the least upper and greatest lower bounds of $\sum_{i=1}^{n} \alpha_i(x)\, x_i^{2m_i}$ on ∂V_m by $\Theta(\alpha_1, \ldots, \alpha_n, m_1, \ldots, m_n)$ and $\theta(\alpha_1, \ldots, \alpha_n, m_1, \ldots, m_n)$ respectively, and we shall assume the field $\boldsymbol{\nu}$ to be such that θ is positive. We denote the (positive) least upper and greatest lower bounds of $\sum_{i=1}^{n} x_i^2$ on ∂V_m by $\Psi(m_1, \ldots, m_n)$ and $\psi(m_1, \ldots, m_n)$ respectively.

THEOREM 3. *Suppose that*

(*a*) *There exist constants* $A_i > 0$, $B_i \geqslant 0$ $(i = 1, \ldots, n)$ *and* $C \geqslant 0$ *such that, for all* $x \in V_m$ *and* $t > 0$, $0 < a_{ii}(x, t) \leqslant A_i$, $|b_i(x, t)| \leqslant B_i$ $(i = 1, \ldots, n)$ *and* $c(x, t) \geqslant -C$.

(*b*) *There exist constants* $M_1, M_2 > 0$ *such that, for all* $t > 0, x \in V_m$ *and* $u \geqslant 0$,

$$f(x, t, u) \geqslant M_1 u + M_2;$$

furthermore, f satisfies a uniform Lipschitz condition in u on any finite u-interval.

(*c*) *There exist constants* $D_0, \delta_1 > 0$ *such that, for all* $x \in \partial V_m$ *and* $t > 0$,

$$0 < d_0(x, t) \leqslant D_0 \quad \text{and} \quad \delta_1 \leqslant d_1(x, t);$$

also we require $\Psi < \psi + (2\delta_1 \theta / D_0)$.

https://doi.org/10.1017/S0305004100053767 Published online by Cambridge University Press

(*d*) $u_0(x) \geqslant 0$ *for all* $x \in \overline{V}_m$.

Then: (i) *A lower solution for* (1) *is given by*

$$w(x,t) = t\left(A - \sum_{i=1}^{n} x_i^2\right) \quad \text{for all} \quad x \in \overline{V}_m, t \geqslant 0$$

if A *is a constant chosen so that* $\Psi < A < \psi + (2\delta_1\theta/D_0)$, *and if*

$$\lambda > \max\left\{\frac{2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + CA}{M_1(A-\Psi)}, \frac{A}{M_2}\right\}.$$

(ii) *If* $\lambda > \max\left\{\dfrac{2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + C\left(\Psi + \dfrac{2\delta_1\theta}{D_0}\right)}{M_1\left(\psi + \dfrac{2\delta_1\theta}{D_0} - \Psi\right)}, \dfrac{\psi + \dfrac{2\delta_1\theta}{D_0}}{M_2}\right\}$,

and if $u(x,t)$ *is a solution of* (1), *then* $u(x,t) \to \infty$ *as* $t \to \infty$, *uniformly for* $x \in \overline{V}_m$.

Proof. (I) $w(x,0) = 0 \leqslant u_0(x)$ for all $x \in \overline{V}_m$.

(II)
$$\begin{aligned} Bw &= d_0(x,t)\,w + d_1(x,t)\sum_{i=1}^{n} \alpha_i(x)\,x_i^{2m_i-1}\frac{\partial w}{\partial x_i} \\ &= t\left\{d_0(x,t)\left(A - \sum_{i=1}^{n} x_i^2\right) - 2d_1(x,t)\sum_{i=1}^{n} \alpha_i(x)\,x_i^{2m_i}\right\} \\ &\leqslant t\{D_0(A-\psi) - 2\delta_1\theta\} < 0 \quad \text{for all} \quad t > 0, x \in \partial V_m. \end{aligned}$$

(III)
$$\begin{aligned} Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w) &= \sum_{i=1}^{n} a_{ii}(x,t)(-2t) + \sum_{i=1}^{n} b_i(x,t)(-2tx_i) \\ &\quad + c(x,t)\,t\left(A - \sum_{i=1}^{n} x_i^2\right) - \left(A - \sum_{i=1}^{n} x_i^2\right) + \lambda f(x,t,w) \\ &\geqslant -2t\sum_{i=1}^{n} A_i - 2t\sum_{i=1}^{n} B_i - CtA - A + \lambda[M_1 t(A-\Psi) + M_2] \\ &> 0 \quad \text{for all} \quad t > 0, x \in V_m. \end{aligned}$$

Thus part (i) is proved. Part (ii) follows at once by choosing A sufficiently close to $\psi + 2\delta_1\theta/D_0$ and applying Theorem 2.

A number of similar results concerning lower solutions can readily be proved under different hypotheses (*b*) and (*d*) concerning f and u_0, but with the remaining hypotheses unaltered (for details of the proofs of these and other results presented in summary form, see (6)). These results are presented in Table 1. Note that the Lipschitz conditions on f are required only to prove that the given lower solutions are lower bounds for the solution (in those cases where Theorem 2 is applied) and not to prove that the given functions are lower solutions.

THEOREM 4. *Suppose that*

(*a*) *There exist constants* $A_i > 0$, $B_i \geqslant 0$ $(i = 1, \ldots, n)$ *and* C *such that, for all* $x \in V_m$ *and* $t > 0$, $0 < A_i \leqslant a_{ii}(x,t)$, $|b_i(x,t)| \leqslant B_i (i = 1, \ldots, n)$ *and* $c(x,t) \leqslant C$.

https://doi.org/10.1017/S0305004100053767 Published online by Cambridge University Press

https://doi.org/10.1017/S0305004100053767 Published online by Cambridge University Press

Table 1. *Lower solutions for* (1) *on* V_m

Hypothesis concerning f	Hypothesis concerning u_0	Lower solution	Special conditions for validity
$f(x,t,0) \geqslant 0$ for all $x \in V_m$, $t > 0$; f satisfies a uniform Lipschitz condition in u on any finite u-interval	$u_0(x) \geqslant 0$ for all $x \in \bar{V}_m$	0	$\lambda > 0$
There exists $M > 0$ such that $f(x,t,u) \geqslant M$ for all $x \in V_m$, $t > 0$ and $u \geqslant 0$; f satisfies a uniform Lipschitz condition in u on any finite u-interval	$u_0(x) \geqslant 0$ for all $x \in \bar{V}_m$	$\lambda K\left(A - \sum_{i=1}^{n} x_i^2\right)(1-e^{-t})$	$\lambda > 0$ $0 < K < \dfrac{M}{2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + (C+1)A}$
There exist $M_1, M_2 > 0$ such that $f(x,t,u) \geqslant M_1 u + M_2$ for all $x \in V_m$, $t > 0$ and $u \geqslant 0$; f satisfies a uniform Lipschitz condition in u on any finite u-interval	$u_0(x) \geqslant 0$ for all $x \in \bar{V}_m$	$t\left(A - \sum_{i=1}^{n} x_i^2\right)$	$\lambda > \max\left\{\dfrac{2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + CA}{M_1(A-\Psi)}, \dfrac{A}{M_2}\right\}$
There exists $M > 0$ such that $f(x,t,u) \geqslant Mu$ for all $x \in V_m$, $t > 0$ and $u \geqslant 0$	$u_0(x) > \epsilon$ $(\epsilon > 0)$ for all $x \in \bar{V}_m$	$\dfrac{\epsilon}{A} e^t \left(A - \sum_{i=1}^{n} x_i^2\right)$	$\lambda > \dfrac{2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + (C+1)A}{M(A-\Psi)}$
There exist $M, \alpha > 0$ such that $f(x,t,u) \geqslant Mu^{1+(1/\alpha)}$ for all $x \in V_m$, $t > 0$ and $u \geqslant 0$	$u_0(x) > \epsilon$ $(\epsilon > 0)$ for all $x \in \bar{V}_m$	$\dfrac{(t^*)^\alpha \epsilon}{A(t^*-t)^\alpha}\left(A - \sum_{i=1}^{n} x_i^2\right)$ $(0 < t < t^*)$	$\lambda > \dfrac{A^{1/\alpha}}{Me^{1/\alpha}(A-\Psi)^{1+(1/\alpha)}}\left(\dfrac{\alpha A}{t^*} + 2\sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n} B_i + CA\right)$
There exist $M_1, M_2, \alpha > 0$ such that $f(x,t,u) \geqslant M_1 u^{1+(1/\alpha)} + M_2$ for all $x \in V_m$, $t > 0$ and $u \geqslant 0$; f satisfies a uniform Lipschitz condition in u on any finite u-interval	$u_0(x) \geqslant 0$ for all $x \in \bar{V}_m$	$\dfrac{t}{(t^*-t)^\alpha}\left(A - \sum_{i=1}^{n} x_i^2\right)$ $(0 < t < t^*)$	λ sufficiently large, depending on A_i, $B_i (i = 1. \ldots, n)$, C, M_1, M_2, α, A, Ψ and t^*

(*b*) *For any bounded positive u-interval* I, *there exists* $M > 0$, *dependent only on* I, *such that* $f(x,t,u) \leqslant M$ *for all* $x \in V_m$, $t > 0$ *and* $u \in I$.

(*c*) *There exist constants* $\delta_0 > 0$, $D_1 \geqslant 0$ *such that, for all* $x \in \partial V_m$ *and* $t > 0$, $d_0(x,t) \geqslant \delta_0$ *and* $0 \leqslant d_1(x,t) \leqslant D_1$.

(*d*) *In the case* $C \geqslant 0$, *we have* $2\sum_{i=1}^{n} B_i + C\left(\Psi + \frac{2D_1\Theta}{\delta_0}\right) < 2\sum_{i=1}^{n} A_i$.

(*e*) *There exists a constant* $\epsilon > 0$ *such that* $u_0(x) < \epsilon$ *for all* $x \in \overline{V}_m$.

Then: (i) *A strict upper solution for* (1) *is given by*

$$w(x,t) = \frac{\epsilon}{A-\Psi}\left(A - \sum_{i=1}^{n} x_i^2\right) \text{ for all } x \in \overline{V}_m,\ t \geqslant 0$$

if A *is a constant chosen so that*

$$\Psi + \frac{2D_1\Theta}{\delta_0} < A < \frac{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i}{C} \quad \text{if} \quad C > 0$$

$$A > \Psi + \frac{2D_1\Theta}{\delta_0} \quad \text{if} \quad C = 0$$

$$A > \max\left\{\Psi + \frac{2D_1\Theta}{\delta_0},\ \Psi + \frac{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i}{C}\right\} \quad \text{if} \quad C < 0$$

and if

$$0 < \lambda < m_0(\epsilon, A),$$

where $m_0(\epsilon, A)$, *apart from depending on* ϵ *and* A, *depends on the coefficients in* L *and* B, *the quantities* Ψ *and* Θ *and the function* f.

(ii) *If* $0 < \lambda < m_0(\epsilon, A)$, *and if* $u(x,t)$ *is a solution of* (1), *then for all* $t \geqslant 0$ *and* $x \in \overline{V}_m$, $u(x,t) \leqslant \frac{\epsilon A}{A-\Psi}$.

Proof. (I) $\quad w(x,0) \geqslant \epsilon > u_0(x) \quad \text{for all} \quad x \in \overline{V}_m.$

(II) $$Bw = \frac{\epsilon}{A-\Psi}\left\{d_0(x,t)\left(A - \sum_{i=1}^{n} x_i^2\right) - 2d_1(x,t)\sum_{i=1}^{n} \alpha_i(x)\, x_i^{2m_i}\right\}$$

$$\geqslant \frac{\epsilon}{A-\Psi}\{\delta_0(A-\Psi) - 2D_1\Theta\} > 0 \quad \text{for all} \quad t > 0, \quad x \in \partial V_m.$$

(III)

$$Lw - \frac{\partial w}{\partial t} + \lambda f(x,t,w)$$

$$= \sum_{i=1}^{n} a_{ii}(x,t)(-2)\left(\frac{\epsilon}{A-\Psi}\right) + \sum_{i=1}^{n} b_i(x,t)(-2x_i)\left(\frac{\epsilon}{A-\Psi}\right) + c(x,t)\,w + \lambda f(x,t,w).$$

Now for $t > 0$ and $x \in V_m$, $\epsilon \leqslant w(x,t) \leqslant \epsilon A/(A-\Psi)$, so by (*b*) we have

$$f(x,t,w) \leqslant M(\epsilon, A, \Psi).$$

https://doi.org/10.1017/S0305004100053767 Published online by Cambridge University Press

Table 2. *Upper solutions for* (1) *on* V_m

Hypothesis concerning f	Hypothesis concerning u_0	Upper solution	Special conditions for validity
For any bounded positive u-interval I, there exists $M > 0$ depending only on I, such that $f(x,t,u) \leqslant M$ for all $x \in V_m, t > 0, u \in I$	$u_0(x) < \epsilon$ $(\epsilon > 0)$ for all $x \in \bar{V}_m$	$\frac{\epsilon}{A-\Psi}\left(A - \sum_{i=1}^{n} x_i^2\right)$	$0 < \lambda < m_0(\epsilon, A)$ (see Theorem 4 for the definition of $m_0(\epsilon, A)$)
There exists $M > 0$ such that $f(x,t,u) \leqslant Mu$ for all $x \in V_m, t > 0$ and $u \geqslant 0$	$u_0(x) < \epsilon$ $(\epsilon > 0)$ for all $x \in \bar{V}_m$	$\frac{\epsilon}{A-\Psi} e^{-\lambda t}\left(A - \sum_{i=1}^{n} x_i^2\right)$	$0 < \lambda < \frac{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i - CA}{M(A+1)}$ if $C \geqslant 0$ $0 < \lambda < \frac{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i - C(A-\Psi)}{M(A+1)}$ if $C < 0$
There exist $M_1 \geqslant 0$, $M_2 > 0$ and α such that $0 < \alpha \leqslant 1$, such that $f(x,t,u) \leqslant M_1 + M_2 u^{1-\alpha}$ for all $x \in V_m, t > 0$ and $u \geqslant 0$	$u_0(x) < \epsilon$ $(\epsilon > 0)$ for all $x \in \bar{V}_m$	$K\lambda^{1/\alpha} M_2^{1/\alpha}\left(A - \sum_{i=1}^{n} x_i^2\right)$	$\lambda > 0$ $K > \max\left\{1, \left[\frac{\lambda^{1-(1/\alpha)} M_2^{-1/\alpha} M_1 + A^{1-\alpha}}{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i - CA}\right]^{1/\alpha}, \frac{\epsilon}{\lambda^{1/\alpha} M_2^{1/\alpha}(A-\Psi)}\right\}$ if $C \geqslant 0$ $K > \max\left\{1, \left[\frac{\lambda^{1-(1/\alpha)} M_2^{-1/\alpha} M_1 + A^{1-\alpha}}{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i - C(A-\Psi)}\right]^{1/\alpha}, \frac{\epsilon}{\lambda^{1/\alpha} M_2^{1/\alpha}(A-\Psi)}\right\}$ if $C < 0$

https://doi.org/10.1017/S0305004100053767 Published online by Cambridge University Press

By (d) and the choice of A, we have that if $C \geqslant 0$, then

$$2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i - CA > 0,$$

while if $C < 0$ then $\quad 2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i - C(A-\Psi) > 0.$

Thus we define the number $m_0(\epsilon, A) > 0$ as follows:

$$\text{If } C \geqslant 0, \qquad m_0(\epsilon, A) = \frac{\epsilon\left\{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i - CA\right\}}{(A-\Psi)M(\epsilon, A, \Psi)}.$$

$$\text{If } C < 0, \qquad m_0(\epsilon, A) = \frac{\epsilon\left\{2\sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n} B_i - C(A-\Psi)\right\}}{(A-\Psi)M(\epsilon, A, \Psi)}.$$

It then follows easily that if $0 < \lambda < m_0(\epsilon, A)$, then for all $t > 0$ and $x \in V_m$,

$$Lw - (\partial w/\partial t) + \lambda f(x, t, w) < 0.$$

Thus part (i) of the theorem is proved. Part (ii) follows at once using Theorem 1.

Again a number of similar results can be proved and are presented in summary form in Table 2.

If one is prepared to abandon the explicitly constructive approach used above, one may prove a collection of results qualitatively similar to the above, but applying to an arbitrary region V. The method of proof proposed here requires that the coefficients a_{ij}, $b_i (i, j = 1, \ldots, n)$ in L be independent of t. We therefore write

$$L_1 u \equiv \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i},$$

where $a_{ij} = a_{ji} \in C^{\alpha}(\overline{V}), b_i \in C^{\alpha}(\overline{V})$ for $i, j = 1, \ldots, n$ and denote by (3) the problem (1) with Lu replaced by $L_1 u + c(x, t)u$. Now consider the eigenvalue problem

$$\left.\begin{aligned} L_1\phi + \mu\phi &= 0 \quad \text{for} \quad x \in V, \\ a\phi + b\frac{\partial \phi}{\partial \nu} &= 0 \quad \text{for} \quad x \in \partial V, \end{aligned}\right\} \tag{4}$$

where a and b are positive constants. The inverse of the operator L_1 with boundary condition as in (4) is a compact operator on the Banach space $C(\overline{V})$ (see (5)). It follows from the maximum principle and the results of Krein and Rutman (12) on operators which are strongly positive with respect to a cone in a Banach space, that (4) has a unique normalized eigenfunction $\phi_1(x; a, b)$ which is strictly positive on $\overline{V}$, and the corresponding eigenvalue $\mu_1(a, b)$ is real, positive and simple. Obviously, there exist positive constants $\alpha_1(a, b)$, $\alpha_2(a, b)$ such that $\alpha_1(a, b) \leqslant \phi_1(x; a, b) \leqslant \alpha_2(a, b)$ for all $x \in \overline{V}$.

We are now in a position to state and prove results analogous to those summarized in Tables 1 and 2, but applying to the problem (3) and the arbitrary region V. The

https://doi.org/10.1017/S0305004100053767 Published online by Cambridge University Press

nature of the modifications required will be sufficiently indicated by stating the analogues of Theorems 3 and 4, omitting the proofs, which are quite straightforward.

THEOREM 5. *Suppose that*

(*a*) *There exists a constant* $C \geqslant 0$ *such that* $c(x,t) \geqslant -C$ *for all* $x \in V$, $t > 0$.

(*b*) *There exist constants* $M_1, M_2 > 0$ *such that, for all* $t > 0$, $x \in V$ *and* $u \geqslant 0$,

$$f(x,t,u) \geqslant M_1 u + M_2;$$

furthermore, f *satisfies a uniform Lipschitz condition in* u *on any finite* u*-interval.*

(*c*) *There exist constants* $D_0, \delta_1 > 0$ *such that, for all* $x \in \partial V$ *and* $t > 0$, $0 < d_0(x,t) \leqslant D_0$ *and* $\delta_1 \leqslant d_1(x,t)$.

(*d*) $u_0(x) \geqslant 0$ *for all* $x \in \overline{V}$.

Then if
$$\lambda > \max\left\{\frac{\mu_1(D_0, \frac{1}{2}\delta_1) + C\alpha_2(D_0, \frac{1}{2}\delta_1)}{M_1 \alpha_1(D_0, \frac{1}{2}\delta_1)}, \frac{\alpha_2(D_0, \frac{1}{2}\delta_1)}{M_2}\right\},$$

a lower solution for (3) *is given by* $w(x,t) = t\phi_1(x; D_0, \frac{1}{2}\delta_1)$ *for all* $x \in \overline{V}$, $t \geqslant 0$, *and if a solution* $u(x,t)$ *of* (3) *exists for all* $t \geqslant 0$, *then* $u(x,t) \to \infty$, *uniformly for* $x \in \overline{V}$.

THEOREM 6. *Suppose that*

(*a*) *There exist constants* $\delta_0, D_1 > 0$ *such that* $d_0(x,t) \geqslant \delta_0$ *and* $0 \leqslant d_1(x,t) \leqslant D_1$ *for all* $x \in \partial V$ *and* $t > 0$.

(*b*) *There exists a constant* $C < \mu_1(\frac{1}{2}\delta_0, D_1)$ *such that* $c(x,t) \leqslant C$ *for all* $x \in V$ *and* $t > 0$.

(*c*) *For any bounded positive* u*-interval* I, *there exists* $M > 0$, *dependent only on* I, *such that* $f(x,t,u) \leqslant M$ *for all* $x \in V$, $t > 0$ *and* $u \in I$.

(*d*) *There exists a constant* $\epsilon > 0$ *such that* $u_0(x) < \epsilon$ *for all* $x \in \overline{V}$.

Then if
$$0 < \lambda < \frac{\epsilon[\mu_1(\frac{1}{2}\delta_0, D_1) - C]}{M[\epsilon, \alpha_1(\frac{1}{2}\delta_0, D_1), \alpha_2(\frac{1}{2}\delta_0, D_1)]}$$
$\left(\text{where } M[\epsilon, \alpha_1, \alpha_2] \text{ is the value of } M \text{ in (c) corresponding to } I = \left[\epsilon, \frac{\epsilon\alpha_2}{\alpha_1}\right]\right)$, *a strict upper solution for* (3) *is given by*

$$v(x,t) = \frac{\epsilon}{\alpha_1(\frac{1}{2}\delta_0, D_1)}\phi_1(x; \tfrac{1}{2}\delta_0, D_1) \quad \text{for all} \quad x \in \overline{V}, t \geqslant 0,$$

and if $u(x,t)$ *is a solution of* (3), *then* $u(x,t) < [\epsilon\alpha_2(\frac{1}{2}\delta_0, D_1)]/[\alpha_1(\frac{1}{2}\delta_0, D_1)]$ *for all* $t > 0$, $x \in \overline{V}$.

6. *Discussion.* In order to relate the results obtained above to those obtained for the steady-state problem by, in particular, Keller and Cohen (11) and Amann (1, 3), we shall suppose that we are dealing with the time-dependent problem

$$\left.\begin{aligned} L_1 u + c(x)u - \frac{\partial u}{\partial t} + \lambda f(x,u) = 0 \quad \text{for} \quad x \in V, t > 0, \\ d_0(x)u + d_1(x)\frac{\partial u}{\partial \nu} = 0 \quad \text{for} \quad x \in \partial V, t > 0, \\ u(x,0) = u_0(x) \quad \text{for} \quad x \in \overline{V} \end{aligned}\right\} \tag{5}$$

https://doi.org/10.1017/S0305004100053767 Published online by Cambridge University Press

and its related steady-state problem

$$\left.\begin{aligned} L_1 u + c(x)\, u + \lambda f(x,u) &= 0 \quad \text{for} \quad x \in V, \\ d_0(x)\, u + d_1(x) \frac{\partial u}{\partial \nu} &= 0 \quad \text{for} \quad x \in \partial V, \end{aligned}\right\} \tag{6}$$

and we suppose that the results of Keller and Cohen, and Amann, can be applied, which requires in particular that (6) be self-adjoint, also that for all $x \in V$, $f(x,u) \geqslant 0$ for all $u > 0$ and $f(x,0) > 0$. Note, however, that Amann (3) has shown that the requirement of self-adjointness can to a large extent be dispensed with.

As remarked earlier, the existence of a positive solution of (6) may be shown to be equivalent to the existence and uniform boundedness over all time of the solution of (5) with $u_0 \equiv 0$. The proof of this requires that both f and f_u satisfy uniform Lipschitz conditions in u and in x, for $x \in \overline{V}$ and any bounded non-negative u-interval (see (6)).

In reviewing the results obtained on the steady-state problem, it is convenient to consider three categories of functions f.

(1) *f monotone increasing and concave in u (but not asymptotically linear)*. In this case Keller and Cohen show that either there exists $\lambda^* > 0$ such that a positive solution of (6) exists for $0 < \lambda < \lambda^*$ but not for $\lambda \geqslant \lambda^*$, or else a positive solution of (6) exists for all $\lambda > 0$.

(2) *f monotone increasing and asymptotically linear in u*. In this case, Amann has shown that there exists a finite $\lambda^* > 0$ such that positive solutions of (6) exist for $0 < \lambda < \lambda^*$ but not for $\lambda > \lambda^*$; a positive solution of (6) may or may not exist for $\lambda = \lambda^*$.

(3) *f monotone increasing and convex in u (but not asymptotically linear)*. In this case Keller and Cohen show that there exists a finite $\lambda^* \geqslant 0$ such that positive solutions of (6) exist for $0 < \lambda < \lambda^*$ but not for $\lambda > \lambda^*$; a positive solution of (6) may or may not exist for $\lambda = \lambda^*$. Amann ((3), theorem 21·2) has since shown that $\lambda^* > 0$, and also ((3), theorem 21·4) that under certain conditions a positive solution of (6) does exist for $\lambda = \lambda^*$.

Amann (3) has also now shown that the requirement that f be monotone increasing is not essential to the above results, and can be replaced by the weaker condition that f_u is bounded below, uniformly with respect to $x \in \overline{V}$.

Now suppose that (5) is such that the results of the present paper apply. In considering the relation between the nature of the function f and the behaviour of the solution of (5) as $t \to \infty$, we again find three categories of functions f emerging, which while not identical to the above, clearly correspond closely to them.

(1 A) *$f(x,u) \leqslant M_1 + M_2 u^{1-\alpha}$ for some M_1, $M_2 > 0$, $0 < \alpha \leqslant 1$*. For any $\lambda > 0$, the solution of (5) is in this case bounded above as $t \to \infty$. Thus positive solutions of (6) will exist for all $\lambda > 0$, and one of the two alternatives which are given by Keller and Cohen for functions in category (1) can be ruled out (partial results in this direction have been obtained by Keller and Cohen ((11), theorem 4·2) and by Hudjaev ((9), theorem 2)).

https://doi.org/10.1017/S0305004100053767 Published online by Cambridge University Press

(2A) $M_1 u + M_2 \leqslant f(x,u) \leqslant M_1^* u + M_2^*$ *for some positive* M_1, M_2, M_1^*, M_2^*; *and* f *satisfies a uniform Lipschitz condition in* u *on any bounded* u*-interval*:

(i) If λ is sufficiently small, then the solution of (5) is bounded above as $t \to \infty$ (Theorem 4), and so (6) will have a positive solution.

(ii) If λ is sufficiently large, then the solution of (5) either tends to ∞ as $t \to \infty$ or ceases to exist for t greater than some finite value, and so (6) will have no positive solution.

These facts are in general accord with those proved by Amann for asymptotically linear f, but deal with a much wider class of functions.

(3A) $f(x,u) \geqslant Mu^{1+\alpha}$ *for some* $M, \alpha > 0$; *and* f *satisfies a uniform Lipschitz condition in* u *on any bounded* u*-interval*:

(i) If λ is sufficiently small (depending on u_0), then the solution of (5) is bounded above as $t \to \infty$ (Theorem 4), and so (6) will have a positive solution.

(ii) For any $\lambda > 0$, if u_0 is sufficiently large (depending on λ), then the solution of (5) ceases to exist for t greater than some finite value.

(iii) For any positive u_0, if λ is sufficiently large (depending on u_0), then the solution of (5) ceases to exist for t greater than some finite value.

(iv) If we impose the stronger condition that $f(x,u) \geqslant M_1 u^{1+\alpha} + M_2$ for some $M_1, M_2, \alpha > 0$ (as well as satisfying the Lipschitz condition), then for any non-negative u_0, if λ is sufficiently large, the solution of (5) ceases to exist for t greater than some finite value, so (6) has no positive solution.

Thus the information obtained by studying the time-dependent problem parallels in many respects that obtained by studying the steady-state problem; however, the conditions which need to be imposed on f are less stringent and there is no requirement that the problem be self-adjoint.

7. *Note.* For the heat-generation problem

$$\left.\begin{aligned} \sum_{i=1}^{3} \frac{\partial^2 u}{\partial y_i^2} - \frac{\partial u}{\partial t} + \lambda e^u = 0 \quad &\text{for} \quad y \in V, t > 0, \\ Hu + K\frac{\partial u}{\partial \mathbf{n}} = 0 \quad &\text{for} \quad y \in \partial V, t > 0, \\ u(y,0) = 0 \quad &\text{for} \quad y \in V, \end{aligned}\right\} \tag{7}$$

where H and K are positive constants and $\mathbf{n}$ is the outward normal to ∂V, Boddington, Gray and Harvey (4) have obtained empirically verified estimates for the critical parameter value λ^* (introduced in the above discussion), for various specific regions V. It is of interest to compare these estimates with the lower bound for λ^* obtainable from Theorem 4 of this paper.

If we take $f(x,t,u)$ to be e^u, $B_i = 0$ $(i = 1, \ldots, n)$ and $C = 0$ in Theorem 4, then it is easily verified that the solution of (1) is bounded as $t \to \infty$ if

$$\lambda < \frac{2\epsilon \sum_{i=1}^{n} A_i}{A - \Psi} \exp\left(-\frac{\epsilon A}{A - \Psi}\right)$$

https://doi.org/10.1017/S0305004100053767 Published online by Cambridge University Press

https://doi.org/10.1017/S0305004100053767 Published online by Cambridge University Press

Table 3. *Estimates and bounds for* λ^*

Region	Estimate for λ^* (see (4))	Lower bound for λ^* (see this paper)
Sphere (radius a)	$\frac{H}{a[(0{\cdot}9061)\,K+(0{\cdot}3000)\,aH]}$	$\frac{H}{a[(0{\cdot}9061)\,K+(0{\cdot}4530)\,aH]}$
Infinite cylinder (radius a)	$\frac{H}{a[(1{\cdot}3591)\,K+(0{\cdot}5000)\,aH]}$	$\frac{H}{a[(1{\cdot}3591)\,K+(0{\cdot}6796)\,aH]}$
Infinite slab (thickness $2a$)	$\frac{H}{a[(2{\cdot}7183)\,K+(1{\cdot}1669)\,aH]}$	$\frac{H}{a[(2{\cdot}7183)\,K+(1{\cdot}3591)\,aH]}$
Equicylinder $\left(\begin{matrix}\text{height } 2a\\ \text{radius } a\end{matrix}\right)$	$\frac{H}{a[(0{\cdot}9061)\,K+(0{\cdot}3516)\,aH]}$	$\frac{H}{a[(0{\cdot}9061)\,K+(0{\cdot}9061)\,aH]}$
Thin circular disc $\left(\begin{matrix}\text{thickness } 2a\\ \text{radius } 10a\end{matrix}\right)$	$\frac{H}{a[(2{\cdot}2654)\,K+(1{\cdot}0797)\,aH]}$	$\frac{H}{a[(2{\cdot}6650)\,K+(2{\cdot}6650)\,aH]}$
Long circular cylinder $\left(\begin{matrix}\text{height } 10a\\ \text{radius } a\end{matrix}\right)$	$\frac{H}{a[(1{\cdot}2359)\,K+(0{\cdot}4759)\,aH]}$	$\frac{H}{a[(1{\cdot}3325)\,K+(1{\cdot}3325)\,aH]}$
Cube (side $2a$)	$\frac{H}{a[(0{\cdot}9061)\,K+(0{\cdot}3892)\,aH]}$	$\frac{H}{a[(1{\cdot}5694)\,K+(1{\cdot}3591)\,aH]}$
Infinite square rod (side $2a$)	$\frac{H}{a[(1{\cdot}3597)\,K+(0{\cdot}5814)\,aH]}$	$\frac{H}{a[(1{\cdot}9221)\,K+(2{\cdot}0387)\,aH]}$

so that
$$\frac{2\epsilon \sum_{i=1}^{n} A_i}{A-\Psi} \exp\left(-\frac{\epsilon A}{A-\Psi}\right)$$
is a lower bound for λ^*. If $u_0(x) = 0$ for all $x \in \overline{V}_m$, we can obtain an optimal lower bound
$$\frac{2}{Ae} \sum_{i=1}^{n} A_i \quad \text{by taking} \quad \epsilon = \frac{A-\Psi}{A}.$$

If we take V in (7) to be $\{y: (y_1/a_1)^{2m_1} + (y_2/a_2)^{2m_2} + (y_3/a_3)^{2m_3} < 1\}$, a change of variables transforms (7) into a problem for which this lower bound may be used, with $A_i = 1/a_i^2$ $(i = 1, 2, 3)$. By appropriately choosing a_1, a_2, a_3, m_1, m_2 and m_3 and letting some of these tend to ∞, the various regions V discussed by Boddington, Gray and Harvey can be dealt with. The results are summarized in Table 3, and show that the lower bound for λ^* obtained from Theorem 4 is often remarkably close to the empirical estimate of Boddington, Gray and Harvey.

The material in this paper formed part of a Ph.D. thesis submitted to Massey University, Palmerston North, New Zealand; the author wishes to thank his supervisor, Dr G. C. Wake, for his advice and encouragement.

REFERENCES

(1) Amann, H. Multiple positive fixed points of asymptotically linear maps. *J. Functional Analysis* **17** (1974), 174–213.
(2) Amann, H. Supersolutions, monotone iterations and stability. *J. Differential Equations* **21** (1976), 363–377.
(3) Amann, H. Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. *SIAM Review* **18** (1976), 620–709.
(4) Boddington, T., Gray, P. and Harvey, D. I. Thermal theory of spontaneous ignition: criticality in bodies of arbitrary shape. *Philos. Trans. Roy. Soc. London* A **270** (1971), 467–506.
(5) Browder, F. E. Linear parabolic differential equations of arbitrary order; general boundary-value problems for elliptic equations. *Proc. Nat. Acad. Sci. U.S.A.* **39** (1953), 185–190. Also errata, p. 1298.
(6) Carter, M. R. A non-linear boundary value problem arising in the theory of thermal explosions. Ph.D. thesis, Massey University, Palmerston North, New Zealand, 1976.
(7) Chan, C. Y. Positive solutions for nonlinear parabolic second initial boundary value problems. *Quart. Appl. Math.* **31** (1974), 443–454.
(8) Cohen, D. S. Multiple stable solutions of non-linear boundary value problems arising in chemical reactor theory. *SIAM J. Appl. Math.* **20** (1971), 1–13.
(9) Hudjaev, S. I. Boundary problems for certain quasi-linear elliptic equations. *Soviet Math. Dokl.* **5** (1964), 188–192.
(10) Keener, J. P. and Keller, H. B. Positive solutions of convex non-linear eigenvalue problems. *J. Differential Equations* **16** (1974), 103–125.
(11) Keller, H. B. and Cohen, D. S. Some positone problems suggested by non-linear heat generation. *J. Math. Mech.* **16** (1967), 1361–1376.
(12) Krein, M. G. and Rutman, M. A. Linear operators leaving invariant a cone in a Banach space. *Amer. Math. Soc. Transl.*, Ser. 1, **10** (1962), 199–325.
(13) McNabb, A. Comparison and existence theorems for multicomponent diffusion systems. *J. Math. Anal. Appl.* **3** (1961), 133–144.
(14) Parter, S. V. Solutions of a differential equation arising in chemical reactor processes. *SIAM J. Appl. Math.* **26** (1974), 687–716.
(15) Sattinger, D. H. Monotone methods in non-linear elliptic and parabolic boundary value problems. *Indiana Univ. Math. J.* **21** (1972), 979–1000.

https://doi.org/10.1017/S0305004100053767 Published online by Cambridge University Press