MODELS OF SET THEORY: EXTENSIONS AND DEAD-ENDS

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Abstract. This article is a contribution to the study of extensions of arbitrary models of ZF (Zermelo–Fraenkel set theory), with no regard to countability or well-foundedness of the models involved. Our main results include the theorems below; in Theorems A and B, $\mathcal{N}$ is said to be a conservative elementary extension of $\mathcal{M}$ if $\mathcal{N}$ elementarily extends $\mathcal{M}$, and the intersection of every $\mathcal{N}$-definable set with the universe of $\mathcal{M}$ is $\mathcal{M}$-definable (parameters allowed). In Theorem B, ZFC is the result of augmenting ZF with the axiom of choice.

Theorem A. *Every model $\mathcal{M}$ of $\mathsf{ZF} + \exists p\,(\mathrm{V} = \mathrm{HOD}(p))$ has a conservative elementary extension $\mathcal{N}$ that contains an ordinal above all of the ordinals of $\mathcal{M}$.*

Theorem B. *If $\mathcal{N}$ is a conservative elementary extension of a model $\mathcal{M}$ of ZFC, and $\mathcal{N}$ has the same natural numbers as $\mathcal{M}$, then $\mathcal{M}$ is cofinal in $\mathcal{N}$.*

Theorem C. *Every consistent extension of ZF has a model $\mathcal{M}$ of power $\aleph_1$ such that $\mathcal{M}$ has no proper end extension to a model of ZF.*

§1. Introduction. This article concerns the model theory of Zermelo–Fraenkel (ZF) set theory. More specifically, it is about various types of extensions of *arbitrary* models of ZF, as opposed to countable or well-founded ones. This is a topic that I have worked on intermittently for over four decades. Indeed, Theorem C of the abstract answers the following question that was initially posed in my 1984 doctoral dissertation [9, Question 1.1.12]:

Question ♡. *Can every model of* ZF *be properly* ***end extended*** *to a model of* ZF?[1]

Note that if there is a proper class of strongly inaccessible cardinals, then every *well-founded* model of ZF can be properly end extended to a model of ZF. Furthermore, by a well-known result due to Keisler and Morley [30], every model of ZF whose class of ordinals has *countable cofinality* has a proper elementary end extension. The motivation for the above question emerges from the comparative study of the model theory of ZF and Peano Arithmetic (PA). Prima facie, ZF and PA are unrelated; after all PA axiomatizes basic intuitions concerning the familiar arithmetical structure of natural numbers, whereas ZF codifies the laws governing

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[1]The negative answer to the version of this question in which "end-extended" is strengthened to "rank extended" was established in the mid 1980s for models of ZFC [10]. Theorem 5.18 of this article provides a negative answer to Question ♡. This question was re-posed in [13]. A variant of the same question was posed recently by Noah Schweber on MathOverflow [43].

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Cantor's mysterious universe of sets, a vastly more complex structure. However, thanks to a clever coding idea, introduced first by Ackermann [1], ZF and PA turn out to be intimately connected at a formal level: PA is *bi-interpretable* with the theory $\mathsf{ZF}_{\mathrm{fin}} + \mathsf{TC}$, where $\mathsf{ZF}_{\mathrm{fin}}$ is obtained from ZF by replacing the axiom of infinity by its negation, and TC expresses "the transitive closure of every set exists."[2] In this light it is natural to compare and contrast the model theoretic behavior of models of ZF and PA in order to elucidate the role of the axiom of infinity in ZF set theory. A central result in the model theory of PA is the MacDowell–Specker Theorem [33], and its refinement (independently) by Phillips [39] and Gaifman [21], that states that every model of PA (of any cardinality) has a proper conservative elementary end extension. Using the aforementioned bi-interpretation between PA and ZF, the refinement of the MacDowell–Specker Theorem is equivalent to the following theorem.

THEOREM $\triangledown$. *Every model of* $\mathsf{ZF}_{\mathrm{fin}} + \mathsf{TC}$ (*of any cardinality*) *has a proper conservative elementary end extension*.

In the above, $\mathcal{N}$ is said to be a conservative elementary extension of $\mathcal{M}$ if $\mathcal{N}$ elementarily extends $\mathcal{M}$, and the intersection of every $\mathcal{N}$-definable set with M is $\mathcal{M}$-definable (parameters allowed). It is known that Theorem $\triangledown$ becomes false if $\mathsf{ZF}_{\mathrm{fin}} + \mathsf{TC}$ is replaced by ZF, but the answer to Question $\heartsuit$ (Theorem 5.18) remained open until now.[3]

The foremost motivation in writing this article was to present the rather intricate details of the solution to the above 40-year old question to my younger self. Another motivation was to publicize what can be described as a long overdue 'Additions and Corrections' to my 1987 paper [11], namely, Theorems A and B of the abstract (Theorem 3.1 and Corollary 4.2 of the article, respectively).

Section 2 is devoted to preliminaries needed for Sections 3–5 in which the main results are presented. Section 3 contains the proof of Theorem A of the abstract (and closely related material). Section 4 contains results on faithful extensions (a generalization of conservative extensions), including Theorem 4.1 that yields Theorem B of the abstract (as Corollary 4.2). A key result in Section 4 is Theorem 4.4 which is used in Section 5 in conjunction with other machinery to establish Theorem C of the abstract. Section 6 is an *Addendum and Corrigendum* to my aforementioned 1987 paper [11]. The Appendix presents the proof of a theorem due to the collective effort of Rubin, Shelah, and Schmerl that plays a key role in the proof of Theorem C of the abstract.

§2. Preliminaries. Here we collect the basic definitions, notations, conventions, and results that are relevant for the subsequent sections.

[2]The role of TC was elucidated in [17] by showing that TC cannot be dropped in this bi-interpretability result. The details of this bi-interpretability result are worked out by Kaye and Wong in [27]. Their work shows that indeed the two theories are definitionally equivalent (aka synonymous), i.e., they have a common definitional extension.

[3]As shown by Keisler and Silver [31], if κ is the first inaccessible cardinal, then $(V_\kappa, \in)$ does not have a proper elementary end extension. More generally, the results of Kaufmann [25] and the author [10] showed every consistent extension of ZFC has a model of power $\aleph_1$ that has no proper rank extension to another model of ZFC. For an exposition of the general framework for the analogues of the MacDowell–Specker Theorem for set theory, see [13].

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2.1. Definitions and basic facts. (Languages and theories of sets).

(a) $\mathcal{L}_{\text{set}} = \{=, \in\}$ is the language of ZF-set theory. We treat ZF as being axiomatized as usual, except that instead of including the scheme of replacement among the axioms of ZF, we include the schemes of separation and collection, as in [5, Appendix A]. Thus, in our setup the axioms of Zermelo set theory Z are obtained by removing the scheme of collection from the axioms of ZF. More generally, for $\mathcal{L} \supseteq \mathcal{L}_{\text{set}}$ we construe $\mathsf{ZF}(\mathcal{L})$ to be the natural extension of ZF in which the schemes of separation and collection are extended to $\mathcal{L}$-formulae; similarly we will use $\mathsf{Z}(\mathcal{L})$ to denote Zermelo set theory over $\mathcal{L}$, i.e., as the result of extending Z with the $\mathcal{L}$-separation scheme $\mathsf{Sep}(\mathcal{L})$. Officially speaking, $\mathsf{Sep}(\mathcal{L})$ consists of the universal closures of $\mathcal{L}$-formulae of the form

$$\forall v \exists w \forall x (x \in w \longleftrightarrow x \in v \wedge \varphi(x, \vec{y})).$$

Thus $\mathsf{ZF}(\mathcal{L})$ is the result of augmenting $\mathsf{Z}(\mathcal{L})$ with the $\mathcal{L}$-collection scheme $\mathsf{Coll}(\mathcal{L})$, which consists of the universal closures of $\mathcal{L}$-formulae of the form:

$$\big(\forall x \in v\ \exists y\ \varphi(x, y, \vec{z})\big) \to \big(\exists w\ \forall x \in v\ \exists y \in w\ \varphi(x, y, \vec{z})\big)\,.$$

When X is a new predicate, we write $\mathcal{L}_{\text{set}}(\mathrm{X})$ instead of $\mathcal{L}_{\text{set}} \cup \{\mathrm{X}\}$, and similarly we write $\mathsf{ZF}(\mathrm{X})$, $\mathsf{Sep}(\mathrm{X})$, $\mathsf{Coll}(\mathrm{X})$, etc. instead of $\mathsf{ZF}(\mathcal{L})$, $\mathsf{Sep}(\mathcal{L})$, $\mathsf{Coll}(\mathcal{L})$, etc. (respectively) when $\mathcal{L} = \mathcal{L}_{\text{set}}(\mathrm{X})$.

(b) For $n \in \omega$, we employ the common notation $(\Sigma_n, \Pi_n, \Delta_n)$ for the Levy hierarchy of $\mathcal{L}_{\text{set}}$-formulae, as in the standard references in modern set theory by Kunen [32], Jech [24], and Kanamori [25]. In particular, $\Delta_0 = \Sigma_0 = \Pi_0$ corresponds to the collections of $\mathcal{L}_{\text{set}}$-formulae all of whose quantifiers are bounded. In addition, we will also consider the Takahashi hierarchy of formulae $(\Sigma_n^{\mathcal{P}}, \Pi_n^{\mathcal{P}}, \Delta_n^{\mathcal{P}})$ (introduced in [46], and further studied by Mathias' [35]), where $\Delta_0^{\mathcal{P}}$ is the smallest class of $\mathcal{L}_{\text{set}}$-formulae that contains all atomic formulae, and is closed both under Boolean connectives and under quantification in the form $Qx \in y$ and $Qx \subseteq y$ where x and y are distinct variables, and Q is $\exists$ or $\forall$. The classes $\Sigma_1^{\mathcal{P}}, \Pi_1^{\mathcal{P}}$, etc. are defined inductively from the class $\Delta_0^{\mathcal{P}}$ in the same way that the formula-classes Σ_1, Π_1, etc. are defined from Δ_0-formulae.

(c) For $n \in \omega$ and $\mathcal{L} \supseteq \mathcal{L}_{\text{set}}$, the Levy hierarchy can be naturally extended to $\mathcal{L}$-formulae $(\Sigma_n(\mathcal{L}), \Pi_n(\mathcal{L}), \Delta_n(\mathcal{L}))$, where $\Delta_0(\mathcal{L})$ is the smallest family of $\mathcal{L}$-formulae that contains all atomic $\mathcal{L}$-formulae and is closed under Boolean operations and bounded quantification.

(d) KPR (Kripke–Platek with ranks) is the subtheory of ZF whose axioms consist of KP plus an axiom that asserts that for all ordinals $\alpha, \{x : \mathrm{rank}(x) < \alpha\}$ exists (where $\mathrm{rank}(x)$ is the usual rank function on sets). Following recent practice (initiated by Mathias [35]), the foundation scheme of KP is only limited to Π_1-formulae; thus in this formulation KP can prove the scheme of $\in$-induction for Σ_1-formulae. In contrast to Barwise's KP in [2], which includes the full scheme of foundation, our version of KP is finitely axiomatizable. Thus KPR is finitely axiomatizable.

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(e) ZBQC is the subtheory of Z, obtained by augmenting M_0 with Foundation (as a single axiom), AC (the axiom of choice), and the axiom of infinity, where M_0 is axiomatized by Extensionality, Pairing, Union, Powerset, and Δ_0-Separation.[4] Two distinguished strengthenings of ZBQC are Mac := ZBQC + TC (recall that TC asserts that every set has a transitive closure) , and Most := ZBQC + KP + Σ_1-Separation. ZBQC,Mac, and Most are finitely axiomatizable.[5]

(f) For $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{set}}$, the *dependent choice scheme,* denoted Π^1_∞ -DC($\mathcal{L}$), consists of the universal closure of $\mathcal{L}$-formulae of the following form:

$$\forall x \exists y\ \varphi(x, y, \vec{z}) \longrightarrow \forall x \exists f\ \left[f : \omega \to \mathrm{V},\ f(0) = x, \text{and } \forall n \in \omega\ \varphi(f(n), f(n+1), \vec{z})\right].$$

In the presence of ZF($\mathcal{L}$), thanks to the Reflection Theorem (Theorem 2.10), Π^1_∞-DC($\mathcal{L}$) is equivalent to the single sentence DC below:

$$\forall r\ \forall a \left[\begin{array}{c} (\forall x \in a\ \exists y \in a\ \langle x, y\rangle \in r) \longrightarrow \\ (\forall x \in a\ \exists f\ \ (f : \omega \to \mathrm{V},\ f(0) = x, \text{and } \forall n \in \omega\ \langle f(n), f(n+1)\rangle \in r)) \end{array} \right].$$

Note that DC is provable in ZFC.

2.2. Definitions and basic facts. (Model theoretic concepts). In what follows we make the blanket assumption that $\mathcal{M}, \mathcal{N}$, etc. are $\mathcal{L}$-structures, where $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{set}}$.

(a) We follow the convention of using M, M^*, M_0, etc. to denote (respectively) the universes of structures $\mathcal{M}, \mathcal{M}^*, \mathcal{M}_0$, etc. We denote the membership relation of $\mathcal{M}$ by $\in^{\mathcal{M}}$; thus an $\mathcal{L}_{\mathrm{set}}$-structure $\mathcal{M}$ is of the form $(M, \in^{\mathcal{M}})$.

(b) $\mathrm{Ord}^{\mathcal{M}}$ is the class of ordinals in the sense of $\mathcal{M}$, i.e.,

$$\mathrm{Ord}^{\mathcal{M}} := \{m \in M : \mathcal{M} \models \mathrm{Ord}(m)\},$$

where $\mathrm{Ord}(x)$ expresses "x is transitive and is well-ordered by $\in$." More generally, for a formula $\varphi(\vec{x})$ with parameters from M, where $\vec{m} = (x_1, \cdots, x_k)$, we write $\varphi^{\mathcal{M}}$ for

$$\left\{\vec{m} \in M^k : \mathcal{M} \models \varphi\,(m_1, \cdots, m_k)\right\}.$$

A subset D of M^k is $\mathcal{M}$-definable if it is of the form $\varphi^{\mathcal{M}}$ for some choice of φ. We write $\omega^{\mathcal{M}}$ for the set of finite ordinals (i.e., natural numbers) of $\mathcal{M}$, and ω for the set of finite ordinals in the real world, whose members we refer to as *metatheoretic natural numbers.* $\mathcal{M}$ is said to be *ω-standard* if $(\omega, \in)^{\mathcal{M}} \cong (\omega, \in)$. For $\alpha \in \mathrm{Ord}^{\mathcal{M}}$ we often use $\mathcal{M}_\alpha$ to denote the substructure of $\mathcal{M}$ whose universe is

$$M_\alpha := \{m \in M : \mathcal{M} \models m \in \mathrm{V}_\alpha\},$$

[4] ZBQC was championed by Mac Lane [34, p. 373] as a parsimonious foundation for mathematical practice. Mathias' [35] is an excellent source of information about ZBQC.

[5] In the presence of the other axioms of ZBQC, Δ_0-Separation is well-known to be equivalent to the closure of the universe under Gödel-operations (see [24, Theorem 13.4]). This makes it clear that ZBQC and Mac are finitely axiomatizable. Another way to see that Mac is finitely axiomatizable is to take advantage of the fact that it supports partial satisfaction classes (see footnote 14). The availability of partial satisfaction classes within Mac makes it clear that the extension Most of Mac is also finitely axiomatizable.

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where V_α is defined as usual as $\{x : \mathrm{rank}(x) < \alpha\}$, where $\mathrm{rank}(x) = \sup\{\mathrm{rank}(y) + 1 : y \in x\}$.

(c) For $c \in M$, $\mathrm{Ext}_{\mathcal{M}}(c)$ is the $\mathcal{M}$-extension of c, i.e., $\mathrm{Ext}_{\mathcal{M}}(c) := \{m \in M : m \in^{\mathcal{M}} c\}$.

(d) We say that $X \subseteq M$ *is coded* (*in* $\mathcal{M}$) if there is some $c \in M$ such that $\mathrm{Ext}_{\mathcal{M}}(c) = X$.

(e) Suppose $\mathcal{M} \models \mathsf{ZF}$. We say that $X \subseteq M$ is $\mathcal{M}$-*amenable* if $(\mathcal{M}, X) \models \mathsf{ZF(X)}$; here X is the interpretation of X.

2.3. Definitions. ('Geometric shapes' of extensions). Suppose $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{set}}$, $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures, and $\mathcal{M}$ a *submodel*[6] of $\mathcal{N}$ (written $\mathcal{M} \subseteq \mathcal{N}$).

(a) $\mathcal{M}^*$ *is the convex hull of* $\mathcal{M}$ *in* $\mathcal{N}$ if $M^* = \bigcup_{a \in M} \mathrm{Ext}_{\mathcal{N}}(a)$.

(b) Suppose $a \in M$.$\mathcal{N}$ *fixes* a if $\mathrm{Ext}_{\mathcal{M}}(a) = \mathrm{Ext}_{\mathcal{N}}(a)$, and $\mathcal{N}$ *enlarges* a if $\mathrm{Ext}_{\mathcal{M}}(a) \subsetneq \mathrm{Ext}_{\mathcal{N}}(a)$.

(c) $\mathcal{N}$ *end extends* $\mathcal{M}$ (written $\mathcal{M} \subseteq_{\mathrm{end}} \mathcal{N}$), if $\mathcal{N}$ fixes every $a \in M$. End extensions are also referred to in the literature as *transitive* extensions, and in the old days as *outer* extensions.

(d) $\mathcal{N}$ is a *powerset-preserving end extension* of $\mathcal{M}$ (written $\mathcal{M} \subseteq^{\mathcal{P}}_{\mathrm{end}} \mathcal{N}$) if $(i)\mathcal{M} \subseteq_{\mathrm{end}} \mathcal{N}$ and (ii) for if $a \in M$, $b \in N$, and $\mathcal{N} \models (b \subseteq a)$, then $b \in M$.

(e) $\mathcal{N}$ is a *rank extension of* $\mathcal{N}$ (written $\mathcal{M} \subseteq_{\mathrm{rank}} \mathcal{N}$) if $\mathcal{N}$ is an end extension of $\mathcal{M}$, and for all $a \in M$ and all $b \in N \backslash M$, $\mathcal{N} \models \mathrm{rank}(a) \in \mathrm{rank}(b)$. Here we assume that $\mathcal{M}$ and $\mathcal{N}$ are models of a sufficient fragment of ZF (such as KP) in which the rank function is well-defined. $\mathcal{N}$ is a *topped rank extension* of $\mathcal{M}$ if $\mathrm{Ord}^{\mathcal{N}} \backslash \mathrm{Ord}^{\mathcal{M}}$ has a least element.[7]

(f) $\mathcal{N}$ is a *cofinal extension of* $\mathcal{M}$ (written $\mathcal{M} \subseteq_{\mathrm{cof}} \mathcal{N}$) if for every $b \in N$ there is some $a \in M$ such that $b \in \mathrm{Ext}_{\mathcal{N}}(a)$, i.e., $\mathcal{N}$ is the convex hull of $\mathcal{M}$ in $\mathcal{N}$.

(g) $\mathcal{N}$ is *taller* than $\mathcal{M}$ (written $\mathcal{M} \subseteq_{\mathrm{taller}} \mathcal{N}$) if there is some $b \in M$ such that $M \subseteq \mathrm{Ext}_{\mathcal{N}}(b)$.

2.4. Definitions ('Logical shapes' of extensions). Suppose $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{set}}$, and $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures such that $\mathcal{M}$ is a submodel of $\mathcal{N}$.

(a) For a subset Γ of $\mathcal{L}$-formulae, $\mathcal{M}$ is a Γ-*elementary submodel of* $\mathcal{N}$ (written $\mathcal{M} \preceq_\Gamma \mathcal{N}$) if for all n-ary formulae $\gamma \in \Gamma$ and for all $a_1, \cdots, a_n$ in M, $\mathcal{M} \models \gamma(a_1, \cdots, a_n)$ iff $\mathcal{N} \models \gamma(a_1, \cdots, a_n)$.$\mathcal{M}$ is an *elementary submodel* of $\mathcal{N}$ (written $\mathcal{M} \preceq \mathcal{N}$) if $\mathcal{M} \preceq_\Gamma \mathcal{N}$ for $\Gamma = \mathcal{L}$-formulae. We say that $\mathcal{M}$ is an *elementary submodel* of $\mathcal{N}$ if $\mathcal{M} \preceq_\Gamma \mathcal{N}$ for the set Γ of all $\mathcal{L}$-formulae (written $\mathcal{M} \preceq \mathcal{N}$). For a given $\gamma \in \Gamma$ we say that γ is *absolute between* $\mathcal{M}$ *and* $\mathcal{N}$ if $\mathcal{M} \preceq_{\{\gamma\}} \mathcal{N}$. As usual, we write $\mathcal{M} \prec_\Gamma \mathcal{N}$ if $\mathcal{M} \preceq_\Gamma \mathcal{N}$ and $\mathcal{M} \subsetneq \mathcal{N}$ (similarly for $\mathcal{M} \prec \mathcal{N}$)

(b) $\mathcal{N}$ is a *self-extension* of $\mathcal{M}$ (written $\mathcal{M} \subseteq_{\mathrm{self}} \mathcal{N}$) if $\mathcal{N}$ is *interpretable* in $\mathcal{M}$ (i.e., $\mathcal{N}$ is isomorphic to a model $\mathcal{N}^*$ whose universe N^* and the interpretation

[6]The notion of a submodel here is the usual one in model theory; i.e., $\mathcal{M} \subseteq \mathcal{N}$ means $M \subseteq N$ and the interpretation of each nonlogical symbol of $\mathcal{L}$ in $\mathcal{M}$ is the restriction to M of the corresponding of interpretation in $\mathcal{N}$ (in particular, $\mathrm{Ext}_{\mathcal{M}}(a) \subseteq \mathrm{Ext}_{\mathcal{N}}(a)$ for all $a \in M$).

[7]Recently rank extensions are also referred to as *top extensions*, we will not use this terminology as it leads to the expression "a topped top extension."

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$R^{\mathcal{N}^*}$ of each $R \in \mathcal{L}$ are all $\mathcal{M}$-definable), and additionally, there is an $\mathcal{M}$-definable embedding j of $\mathcal{M}$ into $\mathcal{N}^*$. In this context, $\mathcal{N}$ is an *elementary self-extension* of $\mathcal{M}$ if the map j is an elementary embedding.

(c) $\mathcal{N}$ is a *conservative* extension of $\mathcal{M}$ (written $\mathcal{M} \subseteq_{\text{cons}} \mathcal{N}$) if for every $\mathcal{N}$-definable D, $M \cap D$ is $\mathcal{M}$-definable.

(d) For $\mathcal{M} \models \mathsf{ZF}$, $\mathcal{N}$ is a *faithful* extension of $\mathcal{M}$ (written $\mathcal{M} \subseteq_{\text{faith}} \mathcal{N}$), if for every $\mathcal{N}$-definable D, $M \cap D$ is $\mathcal{M}$-amenable (as in Definition 2.2(e)).[8]

2.5. Remark. The following are readily verifiable:

(a) If $\langle \mathcal{N}_k : k \in \omega \rangle$ is a chain of models extending $\mathcal{M}$ whose union is $\mathcal{N}$, and $\mathcal{M} \preceq_{\Pi_k,\text{cons}} \mathcal{N}_k \preceq_{\Pi_k} \mathcal{N}_{k+1}$ for each $k \in \omega$, then $\mathcal{M} \preceq_{\text{cons}} \mathcal{N}$.[9] Here $\mathcal{M} \preceq_{\Pi_k,\text{cons}} \mathcal{N}_k$ is shorthand for: $\mathcal{M} \preceq_{\Pi_k} \mathcal{N}_k$ and $\mathcal{M} \preceq_{\text{cons}} \mathcal{N}_k$.

(b) $(\mathcal{M} \subseteq_{\text{self}} \mathcal{N}) \Rightarrow (\mathcal{M} \subseteq_{\text{cons}} \mathcal{N})$.

(c) Assuming $\mathcal{M} \models \mathsf{ZF}$, $(\mathcal{M} \subseteq_{\text{cons}} \mathcal{N}) \Rightarrow (\mathcal{M} \subseteq_{\text{faith}} \mathcal{N})$.

The implications in (b) and (c) are not reversible. More specifically, (b) is not reversible since, e.g., if $\mathcal{N}$ is a self-extension of $\mathcal{M}$, then $|M| = |N|$, but it is possible to build a conservative elementary extension $\mathcal{N}$ of a model $\mathcal{M}$ with $|M| < |N|$, e.g., if $\mathcal{U}$ is an $\mathcal{M}$-ultrafilter, then the ultrapower formation modulo $\mathcal{U}$ can be iterated along any linear order (of arbitrary cardinality) and results in a conservative elementary extension of $\mathcal{M}$. In particular, it is possible to arrange $\mathcal{M} \subseteq_{\text{cons}} \mathcal{N}$ such that M is countable but N is uncountable. Another way to see that (i) is not reversible is to note that by a theorem of Gaifman [20] every elementary self-extension of a model $\mathcal{M}$ of the ZF is a cofinal extension.[10] However, by Theorem 3.1 there are models of ZF that have taller conservative elementary extensions. To see that (c) is not reversible, note that if κ and λ are strongly inaccessible cardinals with $\kappa < \lambda$, then $(\mathrm{V}_\kappa, \in)$ is faithfully extended by $(\mathrm{V}_\lambda, \in)$, but since the truth predicate for $(\mathrm{V}_\kappa, \in)$ is parametrically definable in $(\mathrm{V}_\lambda, \in)$, by Tarski's undefinability of truth $(\mathrm{V}_\lambda, \in)$ is not a conservative extension of $(\mathrm{V}_\kappa, \in)$.

It is also noteworthy that the most common examples of definable extensions in the context of models of set theory are *elementary* extensions. However, the Boolean-valued approach to forcing provides a wealth of definable extensions that are not elementary. More specifically, if $\mathcal{M}$ is a model of ZFC, $\mathbb{B}$ is an element of $\mathcal{M}$ that is a complete Boolean algebra from the point of view of $\mathcal{M}$, and $\mathcal{U}$ is an ultrafilter on $\mathbb{B}$ that is in $\mathcal{M}$, then there is an $\mathcal{M}$-definable embedding j of $\mathcal{M}$ into the model $\mathcal{M}^{\mathbb{B}}/\mathcal{U}$ (where $\mathcal{M}^{\mathbb{B}}$ is the $\mathbb{B}$-valued forcing extension of the universe, as calculated in $\mathcal{M}$). However, in typical cases $\mathcal{M}$ and $\mathcal{M}^{\mathbb{B}}/\mathcal{U}$ are not elementarily equivalent, e.g., we can always arrange $\mathbb{B}$ so that the truth-value of the continuum hypothesis changes in the transition between the $\mathcal{M}$ and $\mathcal{M}^{\mathbb{B}}/\mathcal{U}$. Furthermore, even though $\mathcal{M}$ and $\mathcal{M}^{\mathbb{B}}/\mathcal{U}$ might be elementarily equivalent in some cases, in general the embedding j is not elementary if $\mathbb{B}$ is a nontrivial notion of forcing, since the statement "there is

[8] The notion of a faithful extension was first introduced and studied in the context of *elementary* extensions in [11].

[9] This fact readily follows from Theorem 2.11. It comes handy in the first proof of Theorem 3.1.

[10] Indeed Gaifman's theorem is stated for models $\mathcal{M}$ of the fragment ZR of ZF, where ZR is the result of extending Z (Zermelo set theory) with an axiom that every element is a member of some set of the form V_α.

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a filter over $\mathbb{B}$ that meets all the dense subsets of $\mathbb{B}$" holds in $\mathcal{M}^{\mathbb{B}}/\mathcal{U}$, but not in $\mathcal{M}$ (assuming $\mathbb{B}$ is a nontrivial notion of forcing).

2.6. Remark. Suppose $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}_{\text{set}}$-structures with $\mathcal{M} \subseteq \mathcal{N}$. The following are readily verifiable from the definitions involved:

(a) For each $n \in \omega$, $\mathcal{M} \preceq_{\Sigma_n} \mathcal{N}$ iff $\mathcal{M} \preceq_{\Pi_n} \mathcal{N}$.
(b) $\mathcal{M} \subseteq_{\text{end}} \mathcal{N}$, implies $\mathcal{M} \preceq_{\Delta_0} \mathcal{N}$; and $\mathcal{M} \preceq_{\Delta_0} \mathcal{N}$ implies $\mathcal{M} \preceq_{\Delta_1} \mathcal{N}$.
(c) Suppose $\mathcal{M} \preceq_{\Delta_0} \mathcal{N}$, $\mathcal{N}$ fixes $a \in M$, and there is bijection f in $\mathcal{M}$ between a and some $b \in M$. Then $\mathcal{N}$ fixes b as well (by Δ_0-elementarity, f remains a bijection between a and b as viewed from $\mathcal{N}$)
(d) Suppose $\mathcal{M}$ and $\mathcal{N}$ are models of KPR, $\mathcal{M} \preceq_{\Delta_0^{\mathcal{P}}} \mathcal{N}$, and $\alpha \in \text{Ord}^M$. Then $\text{V}_\alpha^{\mathcal{M}} = \text{V}_\alpha^{\mathcal{N}}$; this follows from the fact that the formula expressing that x is an ordinal and $y = \text{V}_x$ as a $\Sigma_1^{\mathcal{P}}$-formula in KPR (using the usual recursive definition of the V_α-hierarchy).
(e) If $\mathcal{M} \subseteq_{\text{end}}^{\mathcal{P}} \mathcal{N}$, then $\mathcal{M} \preceq_{\Delta_0^{\mathcal{P}}} \mathcal{N}$. On the other hand, since the formula expressing "x is an ordinal and $y = \text{V}_x$" can be written as a $\Sigma_1^{\mathcal{P}}$-formula, it is absolute between models $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M} \subseteq_{\text{end}}^{\mathcal{P}} \mathcal{N}$. Thus for models $\mathcal{M}$ and $\mathcal{N}$ of KPR:

$$\mathcal{M} \subseteq_{\text{end}}^{\mathcal{P}} \mathcal{N} \Rightarrow \mathcal{M} \subseteq_{\text{rank}} \mathcal{N}.$$

As pointed out by the referee, by [35, Proposition 6.17], the converse of the above implication also holds provided $\mathcal{M} \models \text{Most}$ and $\mathcal{N} \models \text{Mac}$.
(f) If $\mathcal{M} \preceq_{\Delta_0} \mathcal{N}$, and $\mathcal{M}^*$ is the convex hull of $\mathcal{M}$ in $\mathcal{N}$, then $\mathcal{M} \subseteq_{\text{cof}} \mathcal{M}^* \subseteq_{\text{end}} \mathcal{N}$. If $\mathcal{M}$ and $\mathcal{N}$ are furthermore assumed to be models of KPR, and $\mathcal{M} \preceq_{\Delta_0^{\mathcal{P}}} \mathcal{N}$, then (e) above implies that $\mathcal{M} \subseteq_{\text{cof}} \mathcal{M}^* \subseteq_{\text{rank}} \mathcal{N}$.
(g) If both $\mathcal{M}$ and $\mathcal{N}$ are models of KPR, and $\mathcal{M} \preceq_{\Delta_0} \mathcal{N}$, then the statement "$\mathcal{N}$ is taller than $\mathcal{M}$" is equivalent to "there is some $\gamma \in \text{Ord}^{\mathcal{N}}$ that exceeds each $\alpha \in \text{Ord}^{\mathcal{M}}$."
(h) Generic extensions of models of ZF are end extensions but not rank extensions since they have the same ordinals. However, by (f) above, powerset preserving end extensions of models of KPR are rank extensions, thus for models $\mathcal{M}$ and $\mathcal{N}$ of KPR we have

$$\mathcal{M} \subseteq_{\text{end}}^{\mathcal{P}} \mathcal{N} \Longrightarrow \mathcal{M} \subseteq_{\text{rank}} \mathcal{N}.$$

On the other hand, since $\mathcal{P}(x) = y$ is Π_1-statement, we have

$$\mathcal{M} \preceq_{\Sigma_1,\text{end}} \mathcal{N} \Longrightarrow \mathcal{M} \subseteq_{\text{end}}^{\mathcal{P}} \mathcal{N}.$$

The converse of the above implication need not be true for arbitrary models $\mathcal{M}$ and $\mathcal{N}$ of KPR. The referee has offered the following counterexample: let $\mathcal{N}$ be a model of ZFC that is ω-standard but whose ω_1 is nonstandard, and let $\mathcal{M}$ be the well-founded part of $\mathcal{N}$. Then $\mathcal{M} \models \text{KPR}$, and the property of being equinumerous to a von Neumann ordinal is a Σ_1-property of $\mathcal{P}^{\mathcal{N}}(\omega)$ that holds in $\mathcal{N}$ but fails in $\mathcal{M}$. However, if $\mathcal{M} \models \text{Most}$ and $\mathcal{N} \models \text{Mac}$, then by [35, Proposition 6.17] the converse of the above implication holds.
(i) As we shall see in Lemma 4.3, faithful end extensions of models of ZF are rank extensions.

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(j) Thanks to the provability of the induction scheme (over natural numbers) in ZF, if $\mathcal{N}$ is a faithful extension of a model $\mathcal{N}$ (where $\mathcal{M}$ is a model of ZF), then $\mathcal{N}$ fixes each element of $\omega^{\mathcal{M}}$ (but $\mathcal{N}$ need not fix $\omega^{\mathcal{M}}$, e.g., consider the case when $\mathcal{N}$ is an internal ultrapower of $\mathcal{M}$ modulo a nonprincipal ultrafilter over $\omega^{\mathcal{M}}$).

2.7. Definition. Reasoning within KP, for each object a in the universe of sets, let $\dot{a}$ be a constant symbol denoting a (where the map $a \mapsto \dot{a}$ is Δ_1, e.g., $\dot{a} = \langle 3, a\rangle$ as in Devlin's monograph [7]). Let $\mathrm{Sent}_{\mathcal{L}_{\mathrm{set}}}(x)$ be the $\mathcal{L}_{\mathrm{set}}$-formula that defines the proper class $\mathrm{Sent}_{\mathcal{L}_{\mathrm{set}}}$ of sentences in the language

$$\mathcal{L}^{+} = \mathcal{L} \cup \{\dot{a} : a \in \mathrm{V}\},$$

and let $\mathrm{Sent}_{\mathcal{L}_{\mathrm{set}}}(i, x)$ be the $\mathcal{L}_{\mathrm{set}}$-formula that expresses "$i \in \omega, x \in \mathrm{Sent}_{\mathcal{L}_{\mathrm{set}}}$, and x is a Σ_i-sentence." In (a) and (b) below, $\mathcal{M} \models \mathsf{KP}$, $S \subseteq M$, and $k \in \omega^{\mathcal{M}}$.[11]

(a) Given $k \in \omega$, S is a *Σ_k-satisfaction class for $\mathcal{M}$* if $(\mathcal{M}, S) \models \mathrm{Sat}(k, \mathrm{S})$, where $\mathrm{Sat}(k, \mathrm{S})$ is the universal generalization of the conjunction of the axioms (i) through (iv) below. We assume that first order logic is formulated using only the logical constants $\{\neg, \vee, \exists\}$.
  (i) $[(\mathrm{S}(\dot{x} = \dot{y}) \leftrightarrow x = y) \wedge (\mathrm{S}(\dot{x} \in \dot{y}) \leftrightarrow x \in y)]$.
  (ii) $\big[\mathrm{Sent}_{\mathcal{L}_{\mathrm{set}}}(k, \varphi) \wedge (\varphi = \neg\psi)\big] \rightarrow \big[\mathrm{S}(\varphi) \leftrightarrow \neg\mathrm{S}(\psi)\big]$.
  (iii) $\big[\mathrm{Sent}_{\mathcal{L}_{\mathrm{set}}}(k, \varphi) \wedge (\varphi = \psi_1 \vee \psi_2)\big] \rightarrow \big[\mathrm{S}(\varphi) \leftrightarrow \big(\mathrm{S}(\psi_1) \vee \mathrm{S}(\psi_2)\big)\big]$.
  (iv) $\big[\mathrm{Sent}_{\mathcal{L}_{\mathrm{set}}}(k, \varphi) \wedge \big(\varphi = \exists v\ \psi(v)\big)\big] \rightarrow \big[\mathrm{S}(\varphi) \leftrightarrow \exists x\ \mathrm{S}(\psi(\dot{x}))\big]$.
(b) S is a *full satisfaction class for $\mathcal{M}$* if $(\mathcal{M}, S) \models \mathrm{Sat}(k, \mathrm{S})$ for every $k \in \omega$.

2.8. Tarski's definability and undefinability of truth theorems. *Suppose $\mathcal{M}$is an $\mathcal{L}_{\mathrm{set}}$-structure.*

(a) (Definability/codability of truth). *If $\mathcal{M}$ is a structure coded as an element m of a model $\mathcal{N}$ of a sufficiently strong*[12] *fragment of* ZF, *then there is $s \in N$ such that $\mathcal{N}$ views s as the Tarskian satisfaction relation on m, written $\mathcal{N} \models \mathrm{sat}(s, m)$.*[13] *Moreover*, $\mathrm{sat}(x, y)$ *is Δ_1 in $\mathcal{N}$.*
(b) (Undefinability of truth). *If S is a full satisfaction class for $\mathcal{M}$, then S is not $\mathcal{M}$-definable.*

It is a well-known result of Levy that if $\mathcal{M} \models \mathsf{ZF}$, then there is a Δ_0-satisfaction class for $\mathcal{M}$ that is definable in $\mathcal{M}$ both by a Σ_1-formula and a Π_1-formula (see ([24], p. 186) for a proof). This makes it clear that for each $n \geq 1$, there is a Σ_n-satisfaction class for $\mathcal{M}$ that is definable in $\mathcal{M}$ by a Σ_n -formula. Levy's result extends to models of $\mathsf{ZF}(\mathcal{L})$ if $\mathcal{L}$ is finite as follows.

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[11] KP is chosen here only for convenience; much weaker theories of sets do the job here.

[12] There are two canonical fragments of ZF that are 'sufficiently strong' for this purpose, namely:
(1) KP (see Definition 2.1(d)), as shown by Friedman, Lu, and Wong in [19, Lemma 4.1].
(2) The fragment M_0 of Z (see Definition 2.1(e)), as shown by Mathias [35, Proposition 3.10].
Note, however, that much weaker systems suffice if one only wishes to have a set theory within which the Tarskian satisfaction relation of every internal set structure is *definable*, as opposed to: *coded as a set*. One such weak system is DS (for 'Devlin strengthened'), which is shown by Mathias ([36], Proposition 10.37) to be capable of defining the Tarskian satisfaction predicate for set structures.

[13] In other words, within $\mathcal{N}$, s is the set consisting of ordered pairs $(\varphi(\vec{x}), \vec{a})$ such that $m \models \varphi(\vec{x}/\vec{a})$.

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2.9. Levy's partial definability of truth theorem. *Let $\mathcal{L}$be a finite extension of $\mathcal{L}_{\text{set}}$. For each $n \in \omega$ there is an $\mathcal{L}$-formula $\text{Sat}_{\Sigma_n(\mathcal{L})}$ such that for all models $\mathcal{M}$of a sufficiently strong*[14] $\mathsf{ZF}(\mathcal{L})$, $\text{Sat}^{\mathcal{M}}_{\Sigma_n(\mathcal{L})}$ *is a $\Sigma_n(\mathcal{L})$-satisfaction class for $\mathcal{M}$. Furthermore, for $n \geq 1$, $\text{Sat}_{\Sigma_n(\mathcal{L})}$ is equivalent to a $\Sigma_n(\mathcal{L})$-formula (provably in $\mathsf{ZF}(\mathcal{L})$).*

The Reflection Theorem is often formulated as a theorem scheme of ZF (e.g., as in [24]), the proof strategy of the Reflection Theorem applies equally well to $\mathsf{ZF}(\mathcal{L})$ for finite extensions $\mathcal{L}$ of $\mathcal{L}_{\text{set}}$.

2.10. Montague–Vaught–Levy reflection theorem. *Let $\mathcal{L}$be a finite extension of $\mathcal{L}_{\text{set}}$. For each $n \in \omega$, $\mathsf{ZF}(\mathcal{L})$ proves that there are arbitrarily large ordinals γ such that the submodel of the universe determined byV_γ is a $\Sigma_n(\mathcal{L})$-elementary submodel of the universe.*[15]

2.11. Σ_n-elementary chains theorem. *Suppose $n \in \omega, \mathcal{L} \supseteq \mathcal{L}_{\text{set}}$, $(I, <_I)$ is a linear order, and $\{\mathcal{M}_i : i \in I\}$ is a collection of $\mathcal{L}$-structures such that $\mathcal{M}_i \preceq_{\Sigma_n(\mathcal{L})} \mathcal{M}_{i'}$ whenever $i <_I i'$. Then we have*

$$\mathcal{M}_i \preceq_{\Sigma_n(\mathcal{L})} \bigcup_{i \in I} \mathcal{M}_i.$$

The following is a special case of a result of Gaifman ([20], Theorem 1, p.54). In the statement and the proof, $\mathcal{M}^* \prec_{\Delta_0(\mathcal{L}),\text{end}} \mathcal{N}$ is shorthand for the conjunction of $\mathcal{M}^* \prec_{\Delta_0(\mathcal{L})} \mathcal{N}$ and $\mathcal{M}^* \prec_{\text{end}} \mathcal{N}$.

2.12. Gaifman splitting theorem. *Let $\mathcal{L} \supseteq \mathcal{L}_{\text{set}}$. Suppose $\mathcal{M} \models \mathsf{ZF}(\mathcal{L})$, $\mathcal{N}$ is an $\mathcal{L}$-structure with $\mathcal{M} \prec_{\Delta_0(\mathcal{L})} \mathcal{N}$, and $\mathcal{M}^*$ is the convex hull of $\mathcal{M}$ in $\mathcal{N}$. Then the following hold*:

(a) $\mathcal{M} \preceq_{\text{cof}} \mathcal{M}^* \preceq_{\Delta_0(\mathcal{L}),\text{end}} \mathcal{N}$.
(b) $\mathcal{M} \prec \mathcal{N} \Rightarrow \mathcal{M} \preceq_{\text{cof}} \mathcal{M}^* \preceq_{\text{end}} \mathcal{N}$.

Proof. To simplify the notation, we present the proof for $\mathcal{L} = \mathcal{L}_{\text{set}}$ since the same reasoning handles the general case. In our proof we use the notation φ^v, where φ is an $\mathcal{L}_{\text{set}}$-formula and v is a parameter, to refer to the Δ_0-formula obtained by restricting all the quantifiers of φ to the elements of v. A straightforward induction on the complexity of formulae shows that:

(1) If $\mathcal{K}$ is an $\mathcal{L}_{\text{set}}$-structure, and $m \in K$, then $\mathcal{K} \models \varphi^m$ iff $(m, \in)^{\mathcal{K}} \models \varphi$.

Note that in the above, φ is allowed to contain parameters in $\text{Ext}_{\mathcal{K}}(m)$. Suppose $\mathcal{M} \prec_{\Delta_0} \mathcal{N}$, where $\mathcal{M} \models \mathsf{ZF}$. Note that we are not assuming that $\mathcal{N} \models \mathsf{ZF}$. Putting (1) together with $\mathcal{M} \prec_{\Delta_0} \mathcal{N}$ implies that $(m, \in)^{\mathcal{M}} \preceq (m, \in)^{\mathcal{M}^*}$ for all $m \in M$. On the other hand, the definition of $\mathcal{M}^*$ makes it clear that $(m, \in)^{\mathcal{N}} = (m, \in)^{\mathcal{M}^*}$ if $m \in M$. Putting all this together, we have:

(2) For all $m \in M$, $(m, \in)^{\mathcal{M}} \preceq (m, \in)^{\mathcal{M}^*} = (m, \in)^{\mathcal{N}}$.

[14]There are two canonical fragments of ZF that are 'sufficiently strong' for this purpose: namely: KP, and $\mathsf{M}_0 + \mathsf{TC}$, see, e.g., Definitions 2.9 and 2.10 of McKenzie's [37] for the case of KP, a similar construction works for $\mathsf{M}_0 + \mathsf{TC}$. What is needed in both cases is TC, plus the ability of the theory to define the Tarskian satisfaction `

[15]This is expressible with the help of $\text{Sat}_{\Sigma_n(\mathcal{L})}$.

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It is easy to see, using the definition of $\mathcal{M}^*$, that $\mathcal{M} \subseteq_{\text{cof}} \mathcal{M}^* \subseteq_{\text{end}} \mathcal{N}$, and in particular $\mathcal{M}^* \preceq_{\Delta_0} \mathcal{N}$ by Remark 2.6(b). Thus the proof of (a) is complete once we show that $\mathcal{M} \preceq_{\Sigma_n} \mathcal{M}^*$ for each $n \in \omega$. Towards this aim, given $n \in \omega$, we can invoke the Reflection Theorem in $\mathcal{M}$ to get hold of some $U \subseteq \text{Ord}^{\mathcal{M}}$ such that:

(3) $(\mathcal{M}, U)$ satisfies "U is unbounded in Ord, and $\left[(\text{V}_\alpha, \in) \prec_{\Sigma_n} (\text{V}, \in)\right]$ for every $\alpha \in U$."

For each $\alpha \in U$ let $m_\alpha \in M$ such that $m_\alpha = \text{V}_\alpha^{\mathcal{M}}$. By (3), we have:

(4) If $\alpha, \beta \in U$ with $\alpha < \beta$, then $(m_\alpha, \in)^{\mathcal{M}} \prec_{\Sigma_n} (m_\beta, \in)^{\mathcal{M}}$.

Next, we observe that if a, b are in M with $(a, \in)^{\mathcal{M}} \prec_{\Sigma_n} (b, \in)^{\mathcal{M}}$, then for each k-ary Σ_n-formula $\sigma(\vec{x})$, $\mathcal{M}$ satisfies the Δ_0-statement $\delta_\sigma(a, b)$, where:

$$\delta_\sigma(a, b) := \forall \vec{x} \in a \left[\sigma^a(\vec{x}) \leftrightarrow \sigma^b(\vec{x})\right],$$

where the prefix $\forall \vec{x} \in a$ is shorthand for $\forall x_0 \in a \cdots \forall x_{k-1} \in a$. Therefore, by putting (4) and (2) together with the assumption $\mathcal{M} \prec_{\Delta_0} \mathcal{N}$ we can conclude:

(5) If $\alpha, \beta \in U$ with $\alpha < \beta$, then $(m_\alpha, \in)^{\mathcal{M}^*} \prec_{\Sigma_n} (m_\beta, \in)^{\mathcal{M}^*}$.

On the other hand, since U is unbounded in $\text{Ord}^{\mathcal{M}}$, the definition of m_α, together with the fact that $\mathcal{M} \subseteq_{\text{cof}} \mathcal{M}^*$, imply:

(6) $\mathcal{M} = \bigcup_{\alpha \in U} (m_\alpha, \in)^{\mathcal{M}}$ and $\mathcal{M}^* = \bigcup_{\alpha \in U} (m_\alpha, \in)^{\mathcal{M}^*}$.

Thanks to (2), (5), (6), and Theorem 2.11 (Σ_n -Elementary Chains Theorem) we can conclude:

(7) If $\alpha \in U$, then $(m_\alpha, \in)^{\mathcal{M}} \preceq (m_\alpha, \in)^{\mathcal{M}^*} \preceq_{\Sigma_n} \mathcal{M}^*$.

We can conclude that $\mathcal{M} \preceq_{\Sigma_n} \mathcal{M}^*$ by (4), (7), and Theorem 2.11. This concludes the proof of (a).

To prove (b), suppose $\mathcal{M} \prec \mathcal{N}$. With (a) at hand, we only need to show that $\mathcal{M}^* \preceq_{\Sigma_n} \mathcal{N}$ for each $n \in \omega$. We observe that if $\alpha \in U$, then by (5) $(m_\alpha, \in)^{\mathcal{M}^*} \prec_{\Sigma_n} \mathcal{M}^*$, and by (2), $(m_\alpha, \in)^{\mathcal{M}^*} = (m_\alpha, \in)^{\mathcal{N}}$. On the other hand, the assumption $\mathcal{M} \prec \mathcal{N}$ together with (3) makes it clear that if $\alpha \in U$, $(m_\alpha, \in)^{\mathcal{N}} \prec_{\Sigma_n} \mathcal{N}$. Thus $\mathcal{M}^* \preceq_{\Sigma_n} \mathcal{N}$ by Theorem 2.11. ⊣

2.13. Remark. In general, if $\mathcal{M} \subseteq \mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are both models of ZF, the condition "$\text{Ord}^{\mathcal{M}}$ is cofinal in $\text{Ord}^{\mathcal{N}}$" does not imply that $\mathcal{M}$ is cofinal in $\mathcal{N}$ (e.g., consider $\mathcal{M}$ and $\mathcal{N}$ where $\mathcal{N}$ is a set-generic extension of $\mathcal{M}$). However, the conjunction of Σ_1 -elementarity and "$\text{Ord}^{\mathcal{M}}$ is cofinal in $\text{Ord}^{\mathcal{N}}$," where $\mathcal{M} \models$ ZF (or even KPR) and $\mathcal{N}$ is an $\mathcal{L}_{\text{set}}$-structure implies that $\mathcal{M}$ is cofinal in $\mathcal{N}$, since the statement $(y \in \text{Ord} \wedge x = \text{V}_y)$ can be written as a Π_1-statement within KPR, and therefore it is absolute between $\mathcal{M}$ and $\mathcal{N}$ if $\mathcal{M} \prec_{\Sigma_1} \mathcal{N}$.[16] This is how elementary embeddings between inner models are usually formalized within ZF, as in Kanamori's monograph [25, Proposition 5.1].

Recall that for models of ZF, the $\mathcal{L}_{\text{set}}$-sentence $\exists p \left(\text{V} = \text{HOD}(p)\right)$ expresses: "there is some p such that every set is first order definable in some structure of the form $(\text{V}_\alpha, \in, p, \beta)_{\beta<\alpha}$ with $p \in \text{V}_\alpha$." The following result is classical.

[16] As noted in [25, Lemma 0.2, p. 5], the rank function is Δ_1^{ZF}, and this fact can be used to show that $(y \in \text{Ord} \wedge x = \text{V}_y)$ can be written as a Π_1-statement within ZF; an examination of the proof makes it clear that KPR suffices for this purpose.

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2.14. Myhill–Scott theorem [38]. *The following statements are equivalent for* $\mathcal{M} \models$ ZF:

(a) $\mathcal{M} \models \exists p \left(\mathrm{V} = \mathrm{HOD}(p)\right)$.
(b) $\mathcal{M}$ *carries a definable global well-ordering, i.e., for some* $p \in M$ *and some set-theoretic formula* $\varphi(x, y, z)$, $\mathcal{M}$ *satisfies "*$\varphi(x, y, p)$ *well-orders the universe*."
(c) $\mathcal{M}$ *carries a definable global choice function, i.e., for some* $p \in M$ *and some set-theoretic formula* $\psi(x, y, z)$, $\mathcal{M}$ *satisfies "*$\psi(x, y, p)$ *is the graph of a global choice function*."

§3. Taller conservative elementary extensions.

THEOREM 3.1. *Every model* $\mathcal{M}$ *of* ZF $+ \exists p \left(\mathrm{V} = \mathrm{HOD}(p)\right)$ *(of any cardinality) has a conservative elementary extension* $\mathcal{N}$ *such that* $\mathcal{N}$ *is taller than* $\mathcal{M}$.

We shall present two proofs of Theorem 3.1, the first one is based on a class-sized syntactic construction taking place within a model of set theory; it was inspired by Kaufmann's proof of the MacDowell–Specker theorem using the Arithmetized Completeness Theorem, as described by Schmerl [41].[17] The second one is based on a model-theoretic construction that is reminiscent of the original ultrapower proof[18] of the MacDowell–Specker theorem [33]. The first proof is short and devilish; the second proof is a bit longer but is more transparent due to its combinatorial flavor; it also lends itself to a refinement, as indicated in Remark 3.3 and Theorem 3.4.

FIRST PROOF OF THEOREM 3.1. *We need the following two Facts* 1 *and* 2.

FACT 1. *Suppose* $\mathcal{M}$ *is a model of* ZF *that carries an* $\mathcal{M}$*-definable global well-ordering, and* T *is an* $\mathcal{M}$*-definable class of first order sentences such that* $\mathcal{M}$ *satisfies "*T *is a consistent first order theory." Then there is a model* $\mathcal{N} \models T^{\mathrm{st}}$ *such that the elementary diagram of* $\mathcal{N}$ *is* $\mathcal{M}$*-definable, where* T^{st} *is the collection of sentences in* T *with standard shape.*[19]

PROOF OF FACT 1. Since $\mathcal{M}$ has a definable global well-ordering, the Henkin proof of the completeness theorem of first order logic can be applied within $\mathcal{M}$ to construct a Henkinized complete extension T^{Henkin} of T (in a language extending the language of T by class-many new constant symbols) such that T is definable in $\mathcal{M}$. This in turn allows $\mathcal{M}$ to define $\mathcal{N}$ by reading it off T^{Henkin}, as in the usual Henkin proof of the completeness theorem. ⊣

FACT 2. *If* $\mathcal{M} \models$ ZF, *then for each* $n \in \omega\mathcal{M} \models \mathrm{Con}(\mathrm{Th}_{\Pi_n}(V, \in, \dot{a})_{a \in \mathrm{V}})$; *here* Con(X) *expresses the formal consistency of* X, *and* $\mathrm{Th}_{\Pi_n}(V, \in, \dot{a})_{a \in \mathrm{V}}$ *is the* Π_n*-fragment of the elementary diagram of the universe*(*which is a definable class, using* $\neg\mathrm{Sat}_{\Sigma_n}$, *where* Sat_{Σ_n} *is as in Theorem* 2.9).

[17] As noted by Phillips [39] and Gaifman [21], the McDowell–Specker proof lends itself to a fine-tuning that ensures that the conservative elementary extension $\mathcal{N}$ of a given model $\mathcal{M}$ has the minimality property, i.e., there is no $\mathcal{K}$ such that $\mathcal{M} \prec \mathcal{K} \prec \mathcal{N}$. However, Kaufmann's slick proof does not seem to lend itself to this embellishment.

[18] The original proof by MacDowell and Specker of their result uses the concept of a "finitely additive 2-valued measure," which is an alternative formulation of the notion of an ultrafilter.

[19] More specifically, $\varphi \in T^{\mathrm{st}}$ iff $\varphi \in T$ and there is some standard formula φ^* such that $\mathcal{M}$ believes that φ is the result of substituting constants for the free variables of φ^*.

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PROOF OF FACT 2. This is an immediate consequence of the Reflection Theorem 2.10.

Starting with a model $\mathcal{M}$ of $\mathsf{ZF}+\exists p\,(\mathrm{V}=\mathrm{HOD}(p))$, by Theorem 2.14, $\mathcal{M}$ carries a global definable well-ordering. We will construct an increasing sequence of $\mathcal{L}_{\mathrm{set}}$-structures $\langle \mathcal{N}_k : k\in\omega\rangle$ that satisfies the following properties for each $k\in\omega$:

(1) $\mathcal{N}_0=\mathcal{M}$,
(2) $\mathcal{M}\preceq_{\Pi_{k+2}}\mathcal{N}_k\preceq_{\Pi_{k+1}}\mathcal{N}_{k+1}$.
(3) There is some $\alpha\in\mathrm{Ord}^{\mathcal{N}_1}$ that is above each $\beta\in\mathrm{Ord}^{\mathcal{M}}$, thus $\mathcal{M}\preceq_{\mathrm{taller}}\mathcal{N}_k$ for all $k\geq 1$.
(4) $\mathcal{N}_k$ is a self-extension of $\mathcal{M}$, and in particular $\mathcal{M}\preceq_{\mathrm{cons}}\mathcal{N}_k$.[20]

In other words:

$$\mathcal{M}=\mathcal{N}_0\preceq_{\Pi_3,\mathrm{taller}}\mathcal{N}_1\preceq_{\Pi_2}\mathcal{N}_2\preceq_{\Pi_3}\mathcal{N}_3\ \ldots,$$
$$\text{and for each } k\in\omega,\ \mathcal{M}\preceq_{\Pi_{k+2},\mathrm{cons},\mathrm{taller}}\mathcal{N}_k.$$

In light of part (a) of Remark 2.5, this shows that $\mathcal{M}\prec_{\mathrm{cons,taller}}\mathcal{N}$, where $\mathcal{N}:=\bigcup_{k\in\omega}\mathcal{N}_k$. So the proof will be complete once we explain how to recursively build the desired chain $\langle \mathcal{N}_k : k\in\omega\rangle$.

Using the notation of Definition 2.7, let $\mathcal{L}^{+}_{\mathrm{set}}=\mathcal{L}_{\mathrm{set}}\cup\{\dot{a}: a\in\mathrm{V}\}$ be the language $\mathcal{L}_{\mathrm{set}}$ of set theory with a constant $\dot{a}$ for each $a\in\mathrm{V}$. To build $\mathcal{N}_1$ we argue within $\mathcal{M}$: add a new constant c to $\mathcal{L}^{+}_{\mathrm{set}}$ and consider the theory T_1 defined as follows:

$$T_1=\{\mathrm{Ord}(c)\}\cup\{\dot{\alpha}\in c:\mathrm{Ord}(\alpha)\}\cup\mathrm{Th}_{\Pi_3}(\mathrm{V},\in,\dot{a})_{a\in\mathrm{V}}.$$

Thus T_1 is a proper class within $\mathcal{M}$. Arguing within $\mathcal{M}$, and using Fact 2, it is easy to see that $\mathrm{Con}(T_1)$ holds, and therefore by Fact 1 we can get hold of a proper class model $\mathcal{N}_1$ of T_1 such that $\mathcal{N}_1$ is a self-extension of $\mathcal{M}$. Thus $\mathcal{M}\prec_{\Pi_{3,\mathrm{cons}}}\mathcal{N}_1$, and there is an ordinal in $\mathcal{N}_1$ that is above all of the ordinals of $\mathcal{M}$.

Next suppose we have built $\langle \mathcal{N}_i : 1\leq i\leq k\rangle$ for some $k\in\omega$ while complying with (1)–(4). Consider the theory T_k defined in $\mathcal{M}$ as follows:

$$T_{k+1}=\mathrm{Th}_{\Pi_{k+1}}(\mathcal{N}_k,b)_{b\in N_k}\cup\mathrm{Th}_{\Pi_{k+2}}(\mathrm{V},\in,a)_{a\in\mathrm{V}}.$$

Fact 2 together with the inductive assumption that $\mathcal{M}\prec_{\Pi_{k+1}}\mathcal{N}_k$ makes it evident that $\mathrm{Con}(T_{k+1})$ holds in $\mathcal{M}$. Hence by Fact 1 we can get hold of a model $\mathcal{N}_{k+1}$ of T such that $\mathcal{N}_{k+1}$ is a self-extension of $\mathcal{M}$. This makes it clear that $\langle \mathcal{N}_i : 1\leq i\leq k+1\rangle$ meets (1)–(4). ⊣

SECOND PROOF OF THEOREM 3.1. Given $\mathcal{M}\models\mathsf{ZF}+\exists p\,(\mathrm{V}=\mathrm{HOD}(p))$, let $\mathbb{B}$ be the Boolean algebra consisting of $\mathcal{M}$-definable subsets of $\mathrm{Ord}^{\mathcal{M}}$ (ordered by $\subseteq$). In Stage 1 of the proof we will build an appropriate ultrafilter $\mathcal{U}$ on $\mathbb{B}$, and in Stage 2 we will verify that the definable ultrapower of $\mathcal{M}$ modulo $\mathcal{U}$, denoted $\mathcal{M}^{\mathrm{Ord}^{\mathcal{M}}}/\mathcal{U}$, is a conservative elementary extension of $\mathcal{M}$ that is taller than $\mathcal{M}$.

Stage 1: The following facts come handy in this stage:

FACT 1. *By Theorem* 2.9, *for each* $n\in\omega$ *there is a definable satisfaction predicate for* Σ_n*-formulae.*

[20] See Definition 2.4(b) and Remark 2.5(a).

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FACT 2. *By Theorem* 2.14, $\mathcal{M}$ *has a parametrically definable global well-ordering.*

To facilitate the construction of $\mathcal{U}$, let us introduce some conventions:

(i) Given $\mathfrak{X} \subseteq \mathcal{P}(M)$, we say that $\mathfrak{X}$ is $\mathcal{M}$-*listable*, if that there is a parametric formula $\psi(\alpha, \beta)$ such that for all $X \subseteq M$ we have:

$$X \in \mathfrak{X} \text{ iff } \exists \alpha \in \mathrm{Ord}^{\mathcal{M}} X = \{m \in M : \mathcal{M} \models \psi(\alpha, m)\}.$$

In the above context, if ψ is a Σ_n-formula, $\mathfrak{X}$ is said to be Σ_n-*listable in* $\mathcal{M}$. Moreover, we say that $\left\langle X_\alpha : \alpha \in \mathrm{Ord}^{\mathcal{M}} \right\rangle$ is an $\mathcal{M}$-*list* of $\mathfrak{X}$ if $X_\alpha = \{m \in M : \mathcal{M} \models \psi(\alpha, m)\}$ for all $\alpha \in \mathrm{Ord}^{\mathcal{M}}$.

(ii) Given an $\mathcal{M}$-listable $\mathfrak{X} \subseteq \mathbb{B}$, we say that $\mathfrak{X}$ is $\mathcal{M}$-*thick* if every $\mathcal{M}$-finite intersection of elements of $\mathfrak{X}$ is nonempty. More precisely, $\mathfrak{X}$ is $\mathcal{M}$ -thick if $\mathcal{M}$ satisfies the following, where ψ is a listing formula for $\mathfrak{X}$:

$$\forall k \in \omega \forall \langle \alpha_i : i < k \rangle \in \mathrm{Ord}^k \ \exists \gamma \in \mathrm{Ord} \ \forall i < k \ \psi(\alpha_i, \gamma).$$

(iii) For each $n \in \omega$, let $\mathbb{B}_n$ be the collection of $X \in \mathbb{B}$ such that X is Σ_n-definable in $\mathcal{M}$ (parameters allowed). Note that by Facts 1 and 2, $\mathbb{B}_n$ is $\mathcal{M}$-listable for each $n \in \omega$.

(iv) $\mathcal{F}$ is the collection of all $X \in \mathbb{B}$ of the form $\mathrm{Ord}^{\mathcal{M}} \backslash \mathrm{Ext}_{\mathcal{M}}(\alpha)$, where $\alpha \in \mathrm{Ord}^{\mathcal{M}}$.$\mathcal{F}$ is clearly $\mathcal{M}$-listable and $\mathcal{M}$-thick.

We wish to construct an ultrafilter $\mathcal{U} \subseteq \mathbb{B}$ that satisfies the following properties:

(1) $\mathcal{U}$ is an ultrafilter on $\mathbb{B}$.
(2) $\mathcal{F} \subseteq \mathcal{U}$.
(3) Given any $\mathcal{M}$-definable $f : \mathrm{Ord}^{\mathcal{M}} \to \mathcal{M}$, there is some $P \in \mathcal{U}$ such that either $f \restriction P$ is one-to-one, or the range of $f \restriction P$ is a set (as opposed to a class).
(4) Given any $n \in \omega$, $\mathcal{U} \cap \mathbb{B}_n$ is $\mathcal{M}$-listable. This condition is equivalent to: for any $\mathcal{M}$-list $\left\langle X_\alpha : \alpha \in \mathrm{Ord}^{\mathcal{M}} \right\rangle$, $\left\{\alpha \in \mathrm{Ord}^{\mathcal{M}} : X_\alpha \in \mathcal{U}\right\} \in \mathbb{B}$.

The desired ultrafilter $\mathcal{U}$ will be defined as $\bigcup_{n \in \omega} \mathcal{U}_n$, where $\mathcal{U}_{n+1}$ will be defined from $\mathcal{U}_n$ using an internal recursion within $\mathcal{M}$ of length $\mathrm{Ord}^{\mathcal{M}}$. Thus, intuitively speaking, $\mathcal{U}$ will be constructed in $\omega \times \mathrm{Ord}^{\mathcal{M}}$-stages.

- In what follows, for $n \in \omega$, we fix an $\mathcal{M}$-list $\left\langle f_{n,\alpha} : \alpha \in \mathrm{Ord}^{\mathcal{M}} \right\rangle$ of all parametrically definable Σ_n-functions[21] of $\mathcal{M}$ with domain $\mathrm{Ord}^{\mathcal{M}}$, and an $\mathcal{M}$-list $\left\langle S_{n,\alpha} : \alpha \in \mathrm{Ord}^{\mathcal{M}} \right\rangle$ of all elements of $\mathbb{B}_n$. Such $\mathcal{M}$-lists exist by Facts 1 and 2 listed at the beginning of Stage 1.

Let $\mathcal{U}_0 := \mathcal{F}$. Thus $\mathcal{U}_0$ is $\mathcal{M}$-listable and $\mathcal{M}$-thick. We now describe the inductive construction of $\mathcal{U}_{n+1}$ from $\mathcal{U}_n$. So suppose $n \in \omega$, and we have $\mathcal{U}_n \subseteq \mathbb{B}$ that satisfies the clauses $\mathbb{C}_1(n)$ through $\mathbb{C}_3(n)$ below. Note that conditions $\mathbb{C}_2(n)$ and $\mathbb{C}_3(n)$ only kick in for $n \geq 1$ and ensure that $\mathcal{U}_n$ abides by certain Σ_{n-1}-obligations for $n \geq 1$.

[21] The notion of $\mathcal{M}$-listability can be naturally extended to family $\mathcal{Y}$ of subsets of M^2, since $\mathcal{Y}$ can be coded as a family $\mathfrak{X}_{\mathcal{Y}}$ of subsets of M with the help of a canonical pairing function.

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$\mathbb{C}_1\ (n)$: $\mathcal{U}_n$ is $\mathcal{M}$-listable and $\mathcal{M}$-thick.
$\mathbb{C}_2\ (n)$: If $n \geq 1$, then $\forall \alpha \in \mathrm{Ord}^{\mathcal{M}}\ \exists X \in \mathcal{U}_n$ ($f_{n-1,\alpha} \restriction X$ is one-to-one, or the range of $f_{n-1,\alpha} \restriction X$ is a set).
$\mathbb{C}_3\ (n)$: If $n \geq 1$, then $\forall \alpha \in \mathrm{Ord}^{\mathcal{M}}$ ($S_{n-1,\alpha} \in \mathcal{U}_{n+1}$ or $\mathrm{Ord}^{\mathcal{M}} \backslash S_{n-1,\alpha} \in \mathcal{U}_n$).

The following lemma is crucial for the construction of $\mathcal{U}_{n+1} \supseteq \mathcal{U}_n$ such that $\mathcal{U}_{n+1}$ satisfies $\mathbb{C}_1(n+1)$ through $\mathbb{C}_3(n+1)$.

LEMMA 3.1.1. *Suppose $n \in \omega$, $\mathcal{F} \subseteq \mathfrak{X} \subseteq \mathbb{B}$, $\mathfrak{X}$ is Σ_n-listable in $\mathcal{M}$, and $\mathfrak{X}$ is $\mathcal{M}$-thick. Then there is a large enough $k \in \omega$ that satisfies the following two conditions*:

(a) *Given any $A \subseteq \mathrm{Ord}^{\mathcal{M}}$ that is Σ_m-definable in $\mathcal{M}$, let $\widehat{A} \in \mathbb{B}$ be defined by: $\widehat{A} = A$ if $\mathfrak{X} \cup \{A\}$ is $\mathcal{M}$-thick, and otherwise $\widehat{A} := \mathrm{Ord}^{\mathcal{M}} \backslash A$. Then $\mathfrak{X} \cup \{\widehat{A}\}$ is $\mathcal{M}$-thick, and $\widehat{A}$ is Σ_p-definable in $\mathcal{M}$ for $p = \max\{m, n\} + k$.*
(b) *Given any $f : \mathrm{Ord}^{\mathcal{M}} \to M$ that is Σ_m-definable in $\mathcal{M}$, there is some $P \in \mathbb{B}$ that is Σ_p-definable in $\mathcal{M}$ for $p = \max\{m, n\} + k$ such that $\mathfrak{X} \cup \{P\}$ is $\mathcal{M}$-thick, and $f \restriction P$ is either one-to-one, or the range of $f \restriction P$ is coded as an element of $\mathcal{M}$.*

PROOF. (a) is easy to see, so we leave the proof to the reader. Our proof of (b) will not explicitly specify the bound k, instead we focus on the combinatorics, which is uniform enough to yield the existence of such a bound. Given an $\mathcal{M}$-definable $f : \mathrm{Ord}^{\mathcal{M}} \to M$, we distinguish two cases:

Case 1: There is some $\gamma \in \mathrm{Ord}^{\mathcal{M}}$ such that $\mathfrak{X} \cup \{Y_\gamma\}$ is $\mathcal{M}$-thick, where

$$Y_\gamma = \{\alpha \in \mathrm{Ord}^{\mathcal{M}} : \mathcal{M} \models f(\alpha) < \gamma\}.$$

Case 2: Not case 1.

If case 1 holds, then we let $P := Y_{\gamma_0}$, where γ_0 is the first ordinal in $\mathrm{Ord}^{\mathcal{M}}$ witnessing the veracity of case 1. Clearly the range of $f \restriction P$ is coded as an element of $\mathcal{M}$.

If case 2 holds, then let $\mathfrak{X}^* :=$ the set of all $\mathcal{M}$-finite intersections of $\mathfrak{X}$. Note that $\mathfrak{X}^*$ is $\mathcal{M}$-listable, so we can let $\left\langle D_\eta : \eta \in \mathrm{Ord}^{\mathcal{M}} \right\rangle$ be an $\mathcal{M}$ -list of $\mathfrak{X}^*$. It is also clear that the $\mathcal{M}$ -thickness of $\mathfrak{X}$ is inherited by $\mathfrak{X}^*$. We will construct an $\mathcal{M}$-definable set $P = \{p_\eta : \eta \in \mathrm{Ord}^{\mathcal{M}}\}$ by recursion (within $\mathcal{M}$) such that the restriction of f to P is one-to-one, and such that $\mathfrak{X} \cup \{P\}$ is $\mathcal{M}$-thick. Suppose we have already built P up to some $\gamma \in \mathrm{Ord}^{\mathcal{M}}$ as $\{p_\eta : \eta < \gamma\}$ such that the following two conditions hold in $\mathcal{M}$:

$(*)$ $p_\eta < p_{\eta'}$ and $f(p_\eta) < f(p_{\eta'})$ whenever $\eta < \eta' < \gamma$.
$(**)$ $p_\eta \in D_\eta$ for all $\eta < \gamma$.

Within $\mathcal{M}$, let $\theta = \sup\{f(p_\eta) : \eta < \gamma\}$. Note that θ is well-defined because $\mathcal{M}$ satisfies the replacement scheme. Since we know that case 2 holds, there must exist some $D_\delta \in \mathfrak{X}^*$ such that:

$$D_\delta \cap Y_{\theta+1} = \varnothing, \text{ where } Y_{\theta+1} := \{\alpha \in \mathrm{Ord}^{\mathcal{M}} : \mathcal{M} \models f(\alpha) < \theta + 1\}.$$

Therefore within $\mathcal{M}$ can choose p_γ to be the first member of $D_\delta \cap D_\gamma$ that is above $\{s_\eta : \eta < \gamma\}$. Since $\mathfrak{X}$ is $\mathcal{M}$-thick and $\mathcal{F} \subseteq \mathfrak{X}$, $D_\delta \cap D_\gamma$ is unbounded in $\mathrm{Ord}^{\mathcal{M}}$, hence s_γ is well-defined. Thus we have constructed an $\mathcal{M}$-definable set $P = \{p_\eta : \eta \in \mathrm{Ord}^{\mathcal{M}}\}$ such that $p_\eta \in D_\eta$ for all $\eta \in \mathrm{Ord}^{\mathcal{M}}$ (so $\mathfrak{X} \cup \{P\}$ is $\mathcal{M}$-thick), and f is

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strictly monotone increasing on P. This concludes the proof of part (b) of Lemma 3.1.1. ⊣

Using Lemma 3.1.1, it is now straightforward to use transfinite recursion within $\mathcal{M}$ of length $\mathrm{Ord}^{\mathcal{M}}$ to construct a family $\mathcal{U}_{n+1} \subseteq \mathbb{B}$ that extends $\mathcal{U}_n$ and which satisfies clauses $\mathbb{C}_1(n+1)$ through $\mathbb{C}_3(n+1)$. $\mathcal{U}_{n+1}$ will be of the form:

$$\mathcal{U}_n \cup \bigcup_{\alpha \in \mathrm{Ord}^{\mathcal{M}}} \{A_\alpha, P_\alpha\},$$

where A_α and P_α are defined within $\mathcal{M}$ by transfinite recursion on α. Suppose for some $\alpha \in \mathrm{Ord}^{\mathcal{M}}$ we have built subsets A_β and P_β of $\mathbb{B}$ for each $\beta < \alpha$, and $\mathcal{U}_n \cup \bigcup_{\beta<\alpha} \{A_\beta, P_\beta\}$ is $\mathcal{M}$ -thick. Now we can use part (a) of Lemma 3.1.1 to construct $A_\alpha \in \mathbb{B}$ such that $\mathcal{U}_n \cup \bigcup_{\beta<\alpha} \{A_\beta, P_\beta\} \cup \{A_\alpha\}$ is $\mathcal{M}$-thick and $A_\alpha = S_{n,\alpha}$ or $A_\alpha = \mathrm{Ord}^{\mathcal{M}} \backslash S_{n,\alpha}$. In the next step, using part (b) of Lemma 3.1.1, we can construct $P_\alpha \in \mathbb{B}$ which 'takes care of' $f_{n,\alpha}$ (i.e., the restriction of $f_{n,\alpha}$ to P_α is either one-to-one, or its range is a set) and which has the property that $\mathcal{U}_n \cup \bigcup_{\beta<\alpha} \{A_\beta, P_\beta\} \cup \{A_\alpha, P_\alpha\}$ is $\mathcal{M}$-thick. This concludes the recursive construction of A_α and P_α for $\alpha \in \mathrm{Ord}^{\mathcal{M}}$, thus we can define

$$\mathcal{U}_{n+1} := \mathcal{U}_n \cup \bigcup_{\alpha \in \mathrm{Ord}^{\mathcal{M}}} \{A_\alpha, P_\alpha\}.$$

The construction makes it clear that $\mathcal{U}_{n+1}$ satisfies $\mathbb{C}_1(n+1)$ through $\mathbb{C}_3(n+1)$. This, in turn, makes evident that for

$$\mathcal{U} := \bigcup_{n \in \omega} \mathcal{U}_n,$$

$\mathcal{U}$ has properties (1)–(4) promised at the beginning of Stage 1. Note that $\mathcal{U}$ satisfies condition (4) since $\mathcal{U}_{n+1}$ and $\mathbb{B}_n$ are both $\mathcal{M}$-listable, and by clause $\mathbb{C}_3(n+1)$, $\mathcal{U} \cap \mathbb{B}_n \subseteq \mathcal{U}_{n+1}$. This concludes Stage 1.

Stage 2: Let $\mathcal{N} := \mathcal{M}^{\mathrm{Ord}^{\mathcal{M}}} / \mathcal{U}$ be the definable ultrapower of $\mathcal{M}$ modulo $\mathcal{U}$ whose universe consists of $\mathcal{U}$-equivalence classes $[f]_{\mathcal{U}}$ of $\mathcal{M}$-definable functions $f : \mathrm{Ord}^{\mathcal{M}} \to M$. Thanks to the availability of an $\mathcal{M}$ -definable global well-ordering, the following Łoś-style result can be readily verified.

THEOREM 3.1.2. *For all k-ary formulae $\varphi(x_0, \cdots, x_{k-1})$ of $\mathcal{L}_{\mathrm{set}}$ and all k-tuples of $\mathcal{M}$-definable functions $\langle f_i : i < k \rangle$ from $\mathrm{Ord}^{\mathcal{M}}$ to M, we have*

$$\mathcal{N} \models \varphi([f_0]_{\mathcal{U}}, \cdots, [f_{k-1}]_{\mathcal{U}}) \text{ iff } \left\{\alpha \in \mathrm{Ord}^{\mathcal{M}} : \mathcal{M} \models \varphi\left(f_0(\alpha), \cdots, f_{k-1}(\alpha)\right)\right\} \in \mathcal{U}.$$

For each $m \in M$ let $\widetilde{m} : \mathrm{Ord}^{\mathcal{M}} \to \{m\}$. Theorem 3.1.2 assures us that the map $j : \mathcal{M} \to \mathcal{N}$ given by $j(m) = [\widetilde{m}]_{\mathcal{U}}$ is an elementary embedding, thus by identifying $\mathcal{M}$ with the image of j, we can construe $\mathcal{N}$ as an elementary extension of $\mathcal{M}$. Let $\mathrm{id} : \mathrm{Ord}^{\mathcal{M}} \to \mathrm{Ord}^{\mathcal{M}}$ be the identity map. By property (2) of $\mathcal{U}$ specified in Stage 1, every element of $\mathcal{U}$ is unbounded in $\mathrm{Ord}^{\mathcal{M}}$. So by Theorem 3.1.2, $\mathcal{N}$ is taller than $\mathcal{M}$ since $[\mathrm{id}]_{\mathcal{U}}$ exceeds all ordinals of $\mathcal{M}$.

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It remains to verify that $\mathcal{N}$ is a conservative extension of $\mathcal{M}$. Given a formula $\psi(x, y_0, \cdots, y_{k-1})$ and parameters $s_0, \cdots, s_{k-1}$ in N, let

$$Y = \{m \in M : \mathcal{N} \models \varphi([\widetilde{m}]_{\mathcal{U}}, s_0, \cdots, s_{k-1})\}.$$

We wish to show that Y is $\mathcal{M}$-definable. Choose $f_0, \cdots, f_{k-1}$ so that $s_i = [f_i]_{\mathcal{U}}$ for all $i < p$ and consider the formula $\theta(x, \alpha)$ defined below:

$$\theta(x, \alpha) := \alpha \in \mathrm{Ord} \wedge \varphi(x, f_0(\alpha), \cdots, f_{k-1}(\alpha)).$$

Pick $n \in \omega$ large enough so that $\theta(x, \alpha)$ is a Σ_n-formula. For each $m \in M$, let

$$S_m := \{\alpha \in \mathrm{Ord}^{\mathcal{M}} : \mathcal{M} \models \theta(m, \alpha)\}.$$

Note that the choice of n, $S_m \in \mathbb{B}_n$ for each $m \in M$, so $S_m \in \mathcal{U}$ iff $S_m \in \mathcal{U} \cap \mathbb{B}_n$ for each $m \in M$. On the other hand, by Theorem 3.1.2 we have

$$m \in Y \text{ iff } S_m \in \mathcal{U} \text{ for each } m \in M.$$

So we can conclude that $m \in Y$ iff $S_m \in \mathcal{U} \cap \mathbb{B}_n$ for each $m \in M$. Since $\mathcal{U} \cap \mathbb{B}_n$ is $\mathcal{M}$-listable by property (4) of $\mathcal{U}$ specified in Stage 1, we can write $\mathcal{U} \cap \mathbb{B}_n$ as an $\mathcal{M}$-list $\left\langle X_\eta : \eta \in \mathrm{Ord}^{\mathcal{M}} \right\rangle$ given by some formula $\psi(x, y)$. Thus the following holds for all $m \in M$, which shows that Y is $\mathcal{M}$-definable:

$$m \in Y \text{ iff } \exists \eta \, (S_m = X_\eta) \text{ iff } \mathcal{M} \models \exists \eta \, \forall \alpha \, \big(\theta(m, \alpha) \leftrightarrow \psi(\eta, m)\big).$$

COROLLARY 3.2. *Work in* $\mathsf{ZF} + \exists p \, \big(\mathrm{V} = \mathrm{HOD}(p)\big)$ *in the metatheory, and let* $\mathcal{M}$ *be an arbitrary model of* $\mathsf{ZF} + \exists p \, \big(\mathrm{V} = \mathrm{HOD}(p)\big)$.

(a) *For every regular cardinal* κ *there is a conservative elementary extension* $\mathcal{N}_\kappa$ *of* $\mathcal{M}$ *that is taller than* $\mathcal{M}$ *and* $\mathrm{cf}(\mathrm{Ord}^{\mathcal{N}}) = \kappa$.[22]

(b) *There is a proper class model* $\mathcal{N}_{\mathrm{Ord}}$ *whose elementary diagram is a definable class such that* (1) $\mathcal{M} \prec_{\mathrm{cons,taller}} \mathcal{N}_{\mathrm{Ord}}$ *and* (2) $\mathrm{Ord}^{\mathcal{N}_{\mathrm{Ord}}}$ *contains a cofinal class of order-type* Ord.

PROOF. To establish (a), we observe that given a regular cardinal κ, Theorem 3.1 allows us to build an elementary chain of models $\langle \mathcal{M}_\alpha : \alpha \in \kappa \rangle$ with the following properties:

(1) $\mathcal{M}_0 = \mathcal{M}$.

(2) For each $\alpha \in \kappa \mathcal{M}_{\alpha+1}$ is a conservative elementary extension of $\mathcal{M}$ that is taller than $\mathcal{M}$.

(3) For limit $\alpha \in \kappa$, $\mathcal{M}_\alpha = \bigcup_{\beta \in \alpha} \mathcal{M}_\beta$.

The desired model $\mathcal{N}$ is clearly $\mathcal{N}_\kappa = \bigcup_{\alpha \in \kappa} \mathcal{M}_\alpha$. This concludes the proof of part (a).

With part (a) at hand, it is evident that in the presence of a global definable well-ordering (available thanks to the veracity of $\exists p \, \big(\mathrm{V} = \mathrm{HOD}(p)\big)$ in the metatheory), we can construct the desired model $\mathcal{N}_{\mathrm{Ord}}$ as the union of an elementary chain of models $\langle \mathcal{M}_\alpha : \alpha \in \mathrm{Ord} \rangle$ that satisfies (1) and it also satisfies the result of replacing κ with Ord in (2) and (3). ⊣

[22] As indicated by the proof of this part, ZFC suffices as the metatheory for this part.

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REMARK 3.3. We should point out that condition (3) of the proof of Theorem 3.1 was inserted so as to obtain the following corollary which is of interest in light of a result of [12] that states that every model of ZFC has a cofinal conservative extension that possesses a minimal *cofinal* elementary extension.[23]

THEOREM 3.4. *Every model of* $\mathsf{ZF} + \exists p\,(\mathrm{V} = \mathrm{HOD}(p))$ *has a cofinal conservative elementary extension that possesses a minimal elementary end extension.*

PROOF. Let $\mathcal{M}$ be a model of $\mathsf{ZF} + \exists p\,(\mathrm{V} = \mathrm{HOD}(p))$, $\mathcal{N}$ be the model constructed in the second proof of Theorem 3.1, and let $\mathcal{M}^*$ be the convex hull of $\mathcal{M}$ in $\mathcal{N}$. Note that by Gaifman Splitting Theorem we have

$$\mathcal{M} \preceq_{\mathrm{cof}} \mathcal{M}^* \prec_{\mathrm{end}} \mathcal{N}.$$

The above together with the fact that $\mathcal{N}$ is conservative extension of $\mathcal{M}$ makes it clear that $\mathcal{M}^*$ is a conservative extension of $\mathcal{M}$.[24] We claim that $\mathcal{N}$ is a minimal elementary end extension of $\mathcal{M}^*$. First note that $\mathcal{N}$ is generated by members of $\mathcal{M}$ and the $\mathcal{U}$-equivalence class of the identity function $[\mathrm{id}]_{\mathcal{U}}$ via the definable terms of $\mathcal{M}$, i.e., the following holds:

For any $a \in N$, there exists some $\mathcal{M}$-definable function f such that $\mathcal{N} \models f([\mathrm{id}]_{\mathcal{U}}) = a$.

To verify that $\mathcal{N}$ is a minimal elementary end extension of $\mathcal{M}^*$ it is sufficient to show that if $s \in N \setminus M^*$ then $[\mathrm{id}]_{\mathcal{U}}$ is definable in $\mathcal{N}$ using parameters from $M \cup \{s\}$. Choose h such that $s = [h]_{\mathcal{U}}$ and recall that by our construction of $\mathcal{U}$ we know that there exists $X \subseteq \mathrm{Ord}^{\mathcal{M}}$ with $X \in \mathcal{U}$ such that either $h \restriction X$ is one-to-one, or the image of X under h is a set (and thus of bounded $\mathcal{M}$-rank). However, since s was chosen to be in $N \setminus M^*$ we can rule out the latter possibility. Note that since $X \in \mathcal{U}$ the extension of X in $\mathcal{N}$ will contain $[\mathrm{id}]_{\mathcal{U}}$, i.e., if X is defined in $\mathcal{M}$ by $\varphi(x)$, then $\mathcal{N} \models \varphi([\mathrm{id}]_{\mathcal{U}})$. Moreover, by elementarity, since h is one-to-one on X in $\mathcal{M}$, it must remain so in $\mathcal{N}$. Hence, in light of the fact that $\mathcal{N} \models h([\mathrm{id}]_{\mathcal{U}}) = s$, we can define $[\mathrm{id}]_{\mathcal{U}}$ within $\mathcal{N}$ to be the unique element of $h^{-1}(s) \cap X$. This completes the proof. ⊣

REMARK 3.5. As Theorem 4.1 of the next section shows, in Theorem 3.1 the model $\mathcal{N}$ cannot be required to fix $\omega^{\mathcal{M}}$ and in particular $\mathcal{N}$ cannot be arranged to end extend $\mathcal{M}$. By putting this observation together with Remark 2.6(j) we can conclude that in the proof of Corollary 3.2, $\omega^{\mathcal{N}_\kappa}$ is κ-like (i.e., its cardinality is κ but each proper initial segment of it is of cardinality less than κ), and $\omega^{\mathcal{N}_{\mathrm{Ord}}}$ is Ord-like (i.e., it is a proper class every proper initial segment of which forms a set).

We next address the question of the extent to which the hypothesis in Theorem 3.1 that $\mathcal{M}$ is a model of $\exists p\,(\mathrm{V} = \mathrm{HOD}(p))$ can be weakened. In what follows AC stands for the axiom of choice, and DC stands for the axiom of dependent choice.

PROPOSITION 3.6. *Suppose* $\mathcal{M}$ *and* $\mathcal{N}$ *are models of* ZF, $\mathcal{M} \prec_{\mathrm{cons}} \mathcal{N}$, *and* $\mathcal{N}$ *is taller than* $\mathcal{M}$. *If every set can be linearly ordered in* $\mathcal{N}$, *then there is an* $\mathcal{M}$*- definable global linear ordering.*

[23] $\mathcal{N}$ is a *minimal* elementary extension of $\mathcal{M}$ if $\mathcal{M} \prec \mathcal{N}$ and there is no $\mathcal{K}$ such that $\mathcal{M} \prec \mathcal{K} \prec \mathcal{N}$.
[24] By Theorem 4.1, $\mathcal{M}^*$ is a proper elementary extension of $\mathcal{M}$.

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PROOF. Suppose $\mathcal{M}$ and $\mathcal{N}$ are as in the assumptions of the proposition. Fix $\alpha \in \mathrm{Ord}^{\mathcal{N}}$ such that α is above all the ordinals of $\mathcal{M}$ and let R denote a linear ordering of $\mathrm{V}_\alpha^{\mathcal{N}}$ in $\mathcal{N}$. By conservativity, there is a formula $r(x, y)$ (possibly with parameters from M) such that $r(x, y)$ that defines $R \cap M^2$ in $\mathcal{M}$. It is clear that $r(x, y)$ defines a global linear ordering in $\mathcal{M}$. ⊣

COROLLARY 3.7. *There are models of* ZFC *that have no taller conservative elementary extensions to a model of* ZFC.

PROOF. This follows from Proposition 3.6, and the well-known fact that there are models of ZFC in which the universe is not definably linearly ordered.[25] ⊣

QUESTION 3.8. *Can the conclusion of Proposition 3.6 be improved to* $\mathcal{M} \models \exists p\,(\mathrm{V} = \mathrm{HOD}(p))$?

REMARK 3.9. If $\mathcal{M} \models \mathsf{ZF}$ and $\mathcal{M} \prec_{\Delta_0,\,\mathrm{cons}} \mathcal{N}$, then each $k \in \omega^{\mathcal{M}}$ is fixed. To see this, suppose to the contrary that $r \in \mathrm{Ext}_{\mathcal{N}}(k) \backslash \mathrm{Ext}_{\mathcal{M}}(k)$. Consider the set $S = \{x \in \mathrm{Ext}_{\mathcal{M}}(k) : \mathcal{N} \models x < r\}$. It is easy to see that S has no last element. On the other hand, by the assumption $\mathcal{M} \prec_{\mathrm{cons}} \mathcal{N}$, the set $S = \{x \in \mathrm{Ext}_{\mathcal{M}}(k) : \mathcal{N} \models x < r\}$ is a definable subset of the predecessors of r and thus by the veracity of the separation scheme in $\mathcal{M}$, S is coded in $\mathcal{M}$ by some element, which makes it clear that S must have a last element, contradiction. This shows that in Theorem 4.1, $\mathcal{N}$ fixes each $k \in \omega^{\mathcal{M}}$. A natural question is whether a taller conservative elementary extension of $\mathcal{M}$ can be arranged to fix $\omega^{\mathcal{M}}$. As we shall see in the next section, this question has a negative answer for models of ZFC (see Corollary 4.2).

REMARK 3.10. Theorem 4.1 should also be contrasted with Gaifman's theorem mentioned in Remark 2.5 which bars taller elementary self-extensions of models of (a fragment of) ZF. Thus in Gaifman's theorem, (a) the condition "$\mathcal{N}$ is an elementary self extension of $\mathcal{M}$" cannot be weakened to "$\mathcal{N}$ is an elementary conservative extension of $\mathcal{M}$." Also note that in the first proof of Theorem 3.1, each $\mathcal{N}_k$ is a self-extension of $\mathcal{M}$ whose associated embedding j is a Σ_{k+2}-elementary embedding whose image is *not* cofinal in $\mathcal{M}$.

§4. Faithful extensions. This section contains the proof of Theorem B of the abstract (see Corollary 4.2). Recall from Definition 2.4 and Remark 2.5 that the notion of a faithful extension is a generalization of the notion of a conservative extension. The notion of a full satisfaction class was defined in Definition 2.7(b).

THEOREM 4.1. *Suppose* $\mathcal{M}$ *and* $\mathcal{N}$ *are both models of* ZFC *such that* $\mathcal{M} \prec_{\Delta_0^{\mathcal{P}},\mathrm{faith}} \mathcal{N}$, $\mathcal{N}$ *fixes* $\omega^{\mathcal{M}}$, *and* $\mathcal{N}$ *is taller than* $\mathcal{M}$. *Then*:

(a) *There is some* $\gamma \in \mathrm{Ord}^{\mathcal{N}}$ *such that* $\mathcal{M} \preceq \mathcal{N}_\gamma$. *Thus either* $\mathcal{M} = \mathcal{N}_\gamma$, *in which case* $\mathcal{N}$ *is a topped rank extension of* $\mathcal{M}$, *or* $\mathcal{M} \prec \mathcal{N}_\gamma$.[26]

[25] Easton proved (in his unpublished dissertation [8]) that assuming Con(ZF) there is a model $\mathcal{M}$ of ZFC that carries no $\mathcal{M}$-definable global choice function for the class of pairs in $\mathcal{M}$; and in particular the universe cannot be definably linearly ordered in $\mathcal{M}$. Easton's theorem was exposited by Felgner [18, p. 231]; for a more recent and streamlined account, see Hamkins' MathOverflow answer [23].

[26] As pointed out in Remark 4.5, both cases of the dichotomy can be realized.

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(b) *In particular, there is a full satisfaction class S for $\mathcal{M}$ such that S can be written as $D \cap M$, where D is $\mathcal{N}$ -definable.*

PROOF. Fix[27] some $\mathcal{N}$-ordinal $\lambda \in \mathrm{Ord}^{\mathcal{N}}$ that dominates each $\mathcal{M}$-ordinal, and some $\mathcal{N}$-ordinal $\beta > \lambda$. Note that:

(1) $\mathcal{N}_\beta \prec_{\Delta_0^{\mathcal{P}}} \mathcal{N}$.

Thanks to the availability of AC in $\mathcal{N}$, we can choose an ordering $\lhd$ in $\mathcal{N}$ such that $\mathcal{N}$ satisfies "$\lhd$ is a well-ordering of V_β." Then for each $m \in M$, we define the following set K_m (again within $\mathcal{N}$) as:

$$K_m := \{a \in \mathrm{V} : \mathcal{N} \models a \in \mathrm{Def}(\mathrm{V}_\beta, \in, \lhd, \lambda, m)\},$$

where $x \in \mathrm{Def}(\mathrm{V}_\beta, \in, \lhd, \lambda, m)$ is shorthand for the formula of set theory that expresses:

$$x \text{ is definable in the structure } (\mathrm{V}_\beta, \in, \lhd, \lambda, m).$$

Note that the only allowable parameters used in a definition of x are λ and m. Thus, intuitively speaking, within $\mathcal{N}$ the set K_m consists of elements a of V_β such that a is first order definable in $\left(\mathrm{V}_\beta, \in, \lhd, \lambda, m\right)$. Also note that since we are not assuming that $\mathcal{M}$ is ω-standard, an element $a \in K_m$ need not be definable in the structure $\left(\mathcal{N}_\beta, \lhd, \lambda, m\right)$ in the real world. Next we move outside of $\mathcal{N}$ and define K as follows:

$$K := \bigcup_{m \in M} K_m.$$

We observe that for each $m \in M$, K_m is coded in $\mathcal{N}$, but there is no reason to expect that K is coded in $\mathcal{N}$. Thus every element of K is definable, in the sense of $\mathcal{N}$, in $(\mathrm{V}_\beta, \in, \lhd)$ with some appropriate choice of parameters in $\{\lambda\} \cup M$. Let $\mathcal{K}$ be the submodel of $\mathcal{N}$ whose universe is K. Using the Tarski test for elementarity and the fact that, as viewed from $\mathcal{N}$, $\lhd$ well-orders V_β we have

(2) $\mathcal{M} \subsetneq \mathcal{K} \preceq \mathcal{N}_\beta$.

By putting (1) and (2) together we conclude

(3) $\mathcal{M} \prec_{\Delta_0^{\mathcal{P}}} \mathcal{K} \preceq \mathcal{N}_\beta \prec_{\Delta_0^{\mathcal{P}}} \mathcal{N}$.

Let O^* be the collection of 'ordinals' of $\mathcal{K}$ that are above the 'ordinals' of $\mathcal{M}$, i.e.,

$$O^* = \left\{\gamma \in \mathrm{Ord}^{\mathcal{K}} : \forall \alpha \in \mathrm{Ord}^{\mathcal{M}}\ \mathcal{N} \models \alpha \in \gamma\right\}.$$

Clearly O^* is nonempty since $\lambda \in O$. We now consider the following two cases. As we shall see, Case I leads to the conclusion of the theorem, and Case II is impossible.

Case I. O^* has a least ordinal (under $\in^{\mathcal{K}}$).

Case II. O^* has no least ordinal.

Suppose Case I holds and let $\gamma_0 = \min(O^*)$. Clearly γ_0 is a limit ordinal of $\mathcal{N}$. By the choice of γ_0, $\mathrm{Ord}^{\mathcal{M}}$ is cofinal in $\mathrm{Ord}^{\mathcal{K}_{\gamma_0}}$. Since by (3) and part (d) of Remark 2.6, $\mathrm{V}_\alpha^{\mathcal{M}} = \mathrm{V}_\alpha^{\mathcal{K}}$ for each $\alpha \in \mathrm{Ord}^{\mathcal{M}}$, this makes it clear that:

[27] The proof strategy of Theorem 4.1 is a variant of the proofs of the following results: [10, Theorem 1.5], ([11], Theorem 3.3), [16, Theorem 2.1], ([14, Theorem 2.1.3), and [15, Lemma 2.19]. A variant of the same strategy is used in the proof of Theorem 4.5. The origins of the strategy can be traced to Kaufmann's refinement [26, Lemma 1.4] of a Skolem hull argument due to Keisler and Silver ([31, Theorem 2.1).

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(4) $\mathcal{M} \preceq_{\Delta_0^{\mathcal{P}},\, \mathrm{cof}} \mathcal{K}_{\gamma_0}$.

Since $\mathcal{K}_{\gamma_0}$ is the convex hull of $\mathcal{M}$ in $\mathcal{K}$, by Gaifman Splitting Theorem, (4) shows:

(5) $\mathcal{M} \preceq \mathcal{K}_{\gamma_0}$.

On the other hand since by (3) $\mathcal{K} \preceq \mathcal{N}_\beta$, we have

(6) $\mathcal{K}_{\gamma_0} = \mathrm{V}_{\gamma_0}^{\mathcal{K}} \preceq \mathrm{V}_{\gamma_0}^{\mathcal{N}_\beta} = \mathrm{V}_{\gamma_0}^{\mathcal{N}} = \mathcal{N}_{\gamma_0}$.

By (5) and (6), $\mathcal{M} \preceq \mathcal{N}_{\gamma_0}$. Thus the proof of the theorem will be complete once we show that Case II leads to a contradiction. Within $\mathcal{N}$ let s_β be the elementary diagram of $\left(\mathrm{V}_\beta, \in\right)$, i.e., $\mathcal{N}$ views s_β is the Tarskian satisfaction class for $\left(\mathrm{V}_\beta, \in\right)$, and let $\Phi := \bigcup_{m \in M} \Phi_m$, where:

$$\Phi_m := \{x \in M : \mathcal{N} \models x \text{ is (the code of) a formula } \varphi(c, \dot{m}) \text{ such that } \varphi(\dot{\lambda}, \dot{m}) \in s_\beta\}.$$

So intuitively speaking, Φ is the result of replacing $\dot{\lambda}$ by c (where c is a fresh constant) in the sentences in the elementary diagram of $\mathcal{N}_\beta$ (as computed in $\mathcal{N}$) whose constants are in $\{\dot{\lambda}\} \cup \{\dot{m} : m \in M\}$. Since $\mathcal{N}$ need not be ω-standard, the elements of Φ might be nonstandard formulae.

By the assumption that $\mathcal{N}$ is a faithful extension of $\mathcal{M}$, Φ is $\mathcal{M}$-amenable. Next let

$$\Gamma := \left\{ \begin{array}{c} t(c, \dot{m}) \in M : t(c, \dot{m}) \in \Phi, \text{ and } \forall \theta \in \mathrm{Ord}\ \left(t(c, \dot{m}) > \dot{\theta}\right) \in \Phi, \text{ and} \\ t \text{ is a definable term in the language } \mathcal{L}_{\mathrm{set}} \cup \{c\} \cup \{\dot{m} : m \in M\} \end{array} \right\}.$$

Officially speaking, Γ consists of *syntactic objects* $\varphi(c, \dot{m}, x)$ in $\mathcal{M}$ that satisfy the following three conditions in $(\mathcal{M}, \Phi)$:

(i) $\left[\exists!\, x \varphi(c, \dot{m}, x)\right] \in \Phi$.

(ii) $\left[\forall x \left(\varphi(c, \dot{m}, x) \to x \in \mathrm{Ord}\right)\right] \in \Phi$.

(iii) $\forall \theta \in \mathrm{Ord}\ \left[\forall x \left(\varphi(c, \dot{m}, x) \to \dot{\theta} \in x\right)\right] \in \Phi$.

Note that Γ is definable in $(\mathcal{M}, \Phi)$. Since we are considering Case II, $(\mathcal{M}, \Phi) \models \psi$, where:

$$\psi := \forall t \left[t \in \Gamma \longrightarrow \exists t' \in \Gamma\ (t' \in t) \in \Phi\right].$$

The veracity of the dependent choice scheme in $\mathcal{M}$ (see part (f) of Definition 2.1), together with the facts that Φ is $\mathcal{M}$-amenable and Γ is definable in $(\mathcal{M}, \Phi)$ make it clear that there is a sequence $s = \left\langle t_n : n \in \omega^{\mathcal{M}} \right\rangle$ in $\mathcal{M}$ such that:

$$(\mathcal{M}, \Phi) \models \forall n \in \omega\ \left[t_n \in \Gamma \wedge \left((t_{n+1} \in t_n) \in \Phi\right)\right].$$

Since s is a countable object in $\mathcal{M}$, and $\mathcal{N}$ fixes $\omega^{\mathcal{M}}$ by assumption, s is fixed in the passage from $\mathcal{M}$ to $\mathcal{N}$ by Remark 2.6(c). On the other hand, since $\mathcal{N}$ has a satisfaction predicate s_β for $\mathcal{N}_\beta$, this leads to a contradiction because we have

$$\mathcal{N} \models \left\langle t_n^{(\mathrm{V}_\beta, \in)}(\dot{\lambda}) : n \in \omega \right\rangle \text{ is an infinite decreasing sequence of ordinals.}$$

In the above $t_n(\dot{\lambda})$ is the term obtained by replacing c with $\dot{\lambda}$ in t_n, and $t_n^{(\mathrm{V}_\beta, \in)}(\dot{\lambda})$ is the interpretation of $t_n(\dot{\lambda})$ in $(\mathrm{V}, \in)$ using the Tarskian satisfaction class for $(\mathrm{V}_\beta, \in)$. $\dashv$

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COROLLARY 4.2. *If* $\mathcal{M} \models \mathsf{ZFC}$, $\mathcal{M} \prec_{\Delta_0^{\mathcal{P}},\,\mathrm{cons}} \mathcal{N}$, *and* $\mathcal{M}$ *fixes* $\omega^{\mathcal{M}}$, *then* $\mathcal{M}$ *is cofinal in* $\mathcal{N}$. *Thus a conservative elementary extension of* $\mathcal{M}$ *that fixes* $\omega^{\mathcal{M}}$ *is a cofinal extension*.

PROOF. If $\mathcal{M} \models \mathsf{ZFC}$, $\mathcal{M} \prec_{\mathrm{cons}} \mathcal{N}$, and $\mathcal{M}$ is not cofinal in $\mathcal{N}$, then $\mathcal{N}$ is taller than $\mathcal{M}$. But then by part (b) of Theorem 4.1, there is an $\mathcal{M}$-definable full satisfaction class over $\mathcal{M}$, which contradicts Tarski's Undefinability of Truth Theorem. ⊣

In Theorem 4.4 below we show that the conclusion of Theorem 4.1 holds if ZFC is weakened to ZF, but the assumption that $\mathcal{N}$ fixes $\omega^{\mathcal{M}}$ is strengthened to $\mathcal{M} \subsetneq_{\mathrm{end}} \mathcal{N}$. We first establish a lemma that will come handy in the proof of Theorem 4.4.

LEMMA 4.3. *Suppose* $\mathcal{M}$ *and* $\mathcal{N}$ *are models of* ZF, *and* $\mathcal{M} \subsetneq_{\mathrm{end,faithful}} \mathcal{N}$. *Then* $\mathcal{M} \subsetneq_{\mathrm{rank}} \mathcal{N}$.

PROOF. Suppose $\mathcal{M}$ and $\mathcal{N}$ are as in the assumptions of the lemma. Since $\mathcal{N}$ end extends $\mathcal{M}$, in order to show that $\mathcal{N}$ rank extends $\mathcal{M}$ it is sufficient that if $\alpha \in \mathrm{Ord}^{\mathcal{M}}$ and $m \in M$ with $\mathcal{M} \models m = \mathrm{V}_\alpha$, then $\mathcal{N} \models m = \mathrm{V}_\alpha$. For this purpose, it is sufficient to verify that $\mathcal{N}$ is a powerset-preserving end extension of $\mathcal{M}$, since the formula expressing $x = \mathrm{V}_\alpha$ is $\Sigma_1^{\mathcal{P}}$ and is therefore absolute for powerset-preserving end extensions (as noted in Remark 2.6(e)). Towards this goal, let $a \in M$ and suppose $b \in N$ such that:

(1) $\mathcal{N} \models b \subseteq a$.

Let $X_b = \mathrm{Ext}_{\mathcal{N}}(b) \cap \mathrm{Ext}_{\mathcal{M}}(a) = \mathrm{Ext}_{\mathcal{N}}(b)$. Thanks to the assumption that $\mathcal{N}$ is a faithful extension of $\mathcal{M}$, the expansion $(\mathcal{M}, X_b)$ satisfies Separation in the extended language and therefore there is an element $b' \in M$ that codes X_b. Thus

(2) $\mathrm{Ext}_{\mathcal{M}}(b') = \mathrm{Ext}_{\mathcal{N}}(b) \cap \mathrm{Ext}_{\mathcal{M}}(a)$, and

Since $\mathcal{N}$ end extends $\mathcal{M}$, we have

(3) $\mathrm{Ext}_{\mathcal{M}}(a) = \mathrm{Ext}_{\mathcal{N}}(a)$ and $\mathrm{Ext}_{\mathcal{M}}(b') = \mathrm{Ext}_{\mathcal{N}}(b')$.

So by (1) and (2),

(4) $\mathrm{Ext}_{\mathcal{N}}(b') = \mathrm{Ext}_{\mathcal{N}}(b) \cap \mathrm{Ext}_{\mathcal{N}}(a)$.

By (1), $\mathrm{Ext}_{\mathcal{N}}(b) \subseteq \mathrm{Ext}_{\mathcal{N}}(a)$, so combined with (4) this yields:

(5) $\mathrm{Ext}_{\mathcal{N}}(b') = \mathrm{Ext}_{\mathcal{N}}(b)$,

which by Extensionality makes it clear that $b' = b$. ⊣

- The proof of Theorem 4.4 below is a variant of the proof of Theorem 4.1, and will strike the reader who has worked through the proof of Theorem 4.1 as repetitious, but the proof is a bit more involved here since we do not have access to the axiom of choice in $\mathcal{M}$ or in $\mathcal{N}$. Note that in Theorem 4.4 $\mathcal{N}$ is assumed to end extend $\mathcal{M}$, whereas in Theorem 4.1 $\mathcal{N}$ is only assumed to fix $\omega^{\mathcal{M}}$. Of course Theorem 4.4 follows from Theorem 4.1 if $\mathcal{M}$ and $\mathcal{N}$ are models of ZFC.

THEOREM 4.4. *Suppose* $\mathcal{M}$ *and* $\mathcal{N}$ *are models of* ZF, *and* $\mathcal{M} \subsetneq_{\mathrm{end,faithful}} \mathcal{N}$. *Then*:

(a) *There is some* $\gamma \in \mathrm{Ord}^{\mathcal{N}} \backslash \mathrm{Ord}^{\mathcal{M}}$ *such that* $\mathcal{M} \preceq \mathcal{N}_\gamma$. *Thus either* $\mathcal{M} = \mathcal{N}_\gamma$, *in which case* $\mathcal{N}$ *is a topped rank extension of* $\mathcal{M}$, *or* $\mathcal{M} \prec \mathcal{N}_\gamma$.

(b) *In particular, there is a full satisfaction class* S *for* $\mathcal{M}$ *such that* S *can be written as* $D \cap M$, *where* D *is* $\mathcal{N}$ *-definable.*

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PROOF. Assume $\mathcal{M}$ and $\mathcal{N}$ are as in the assumptions of the theorem. By Lemma 4.3 we can assume that $\mathcal{N}$ rank extends $\mathcal{M}$. Fix some $\lambda \in \mathrm{Ord}^{\mathcal{N}} \backslash \mathrm{Ord}^{\mathcal{M}}$ and some ordinal $\beta > \lambda$. For each $m \in M$, we can define the following set within $\mathcal{N}$:

$$O_m := \left\{\gamma \in \beta : \mathcal{N} \models \gamma \in \mathrm{Def}(\mathrm{V}_\beta, \in, \lambda, m)\right\},$$

where $x \in \mathrm{Def}(\mathrm{V}_\beta, \in, \lambda, m)$ is shorthand for the formula of set theory that expresses:

$$x \text{ is definable in } (\mathrm{V}_\beta, \in) \text{ using at most the parameters } \lambda \text{ and } m.$$

Clearly $\lambda \in O_m$ and O_m is coded in $\mathcal{N}$. Next let

$$O := \bigcup_{m \in M} O_m.$$

Let $O^* = O \backslash \mathrm{Ord}^{\mathcal{M}}$. Clearly O^* is nonempty since $\lambda \in O^*$. We now consider two cases:

(I) O^* has a least element (under $\in^{\mathcal{N}}$).
(II) O^* has no least element.

Case I. Let γ_0 be the least element of O^*. It is easy to see that γ_0 is a limit ordinal in $\mathcal{N}$. We claim that:

$$\mathcal{M} \preceq \mathcal{N}_{\gamma_0}.$$

To verify the claim, by the Tarski criterion of elementarity, it suffices to show that if $\mathcal{N}_{\gamma_0} \models \exists y \varphi(y, m)$ for some $m \in M$, then there is some $m_0 \in M$ such that $\mathcal{N}_{\gamma_0} \models \varphi(m_0, m)$. Let γ_m be the first ordinal below γ_0 such that:

$$\mathcal{N}_{\gamma_0} \models \exists y \in \mathrm{V}_{\gamma_m}\ \varphi(y, m).$$

So γ_m is definable in $\mathcal{N}_\beta$ with parameters γ_0 and m, and since $\gamma_0 \in O$ by assumption, this shows that γ_m is definable in $\mathcal{N}_\beta$ with parameter m. So $\gamma_m \in O$. By the fact that γ_0 was chosen to be the least element of O^*, this shows that $\gamma_m \in \mathrm{Ord}^{\mathcal{M}}$, which makes it clear that there is some $m_0 \in M$ such that:

$$\mathcal{N}_{\gamma_0} \models \varphi(m_0, m).$$

This concludes the verification of $\mathcal{M} \preceq \mathcal{N}_{\gamma_0}$. To finish the proof of Theorem 4.4 it suffices to show that Case II is impossible. Suppose to the contrary that O^* has no least element. Within $\mathcal{N}$ let s_β be the Tarskian satisfaction class for $\left(\mathrm{V}_\beta, \in\right)$. For each $m \in M$ we define

$$\Phi_m := \{\varphi(c, \dot{m}) \in M : \mathcal{N} \models \varphi(\dot{\lambda}, \dot{m}) \in s_\beta\}.$$

Note the constant c is interpreted as λ in the right-hand side of the above definition of Φ_m. Next we define

$$\Phi := \bigcup_{m \in M} \Phi_m.$$

Observe that $\Phi \subseteq M$ and Φ is $\mathcal{M}$ -amenable since $\mathcal{N}$ is a faithful rank extension of $\mathcal{M}$. So intuitively speaking, Φ is the subset of the elementary diagram of $\mathcal{N}_\beta$ (as computed in $\mathcal{N}$) that consists of sentences whose constants are in $\{c\} \cup \{\dot{m} :$

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$m \in M\}$. Since $\mathcal{N}$ need not be ω-standard, the elements of Φ need not be standard formulae. Also note that the constant c is interpreted as λ in the right-hand side of the above definition of Φ_m. Now let

$$\Gamma := \left\{ \begin{array}{c} t(c, \dot{m}) \in M : t(c, \dot{m}) \in \Phi, \text{ and } \forall \theta \in \mathrm{Ord}\ \left(t(c, \dot{m}) > \dot{\theta}\right) \in \Phi, \text{ and} \\ t \text{ is a definable term in the language } \mathcal{L}_{\mathrm{set}} \cup \{c\} \cup \{\dot{m} : m \in M\} \end{array} \right\},$$

So, officially speaking, Γ consists of *syntactic objects* $\varphi(c, \dot{m}, x)$ in $\mathcal{M}$ that satisfy the following three conditions in $(\mathcal{M}, \Phi)$:

(i) $\left[\exists! x\ \varphi(c, \dot{m}, x)\right] \in \Phi$.
(ii) $\left[\forall x\ \left(\varphi(c, \dot{m}, x) \to x \in \mathrm{Ord}\right)\right] \in \Phi$.
(iii) $\forall \theta \in \mathrm{Ord}\ \left[\forall x\ \left(\varphi(c, \dot{m}, x) \to \dot{\theta} \in x\right)\right] \in \Phi$.

Clearly Γ is definable in $(\mathcal{M}, \Phi)$. Since we are in Case II, $(\mathcal{M}, \Phi) \models \psi$, where ψ is the sentence that expresses:

$$\Gamma \neq \varnothing \wedge \forall t \left[t \in \Gamma \longrightarrow \exists t' \in \Gamma\ (t' \in t) \in \Phi\right].$$

In contrast to the proof of Theorem 4.1 we cannot at this point conclude that there is a countable descending chain $\langle t_n : n \in \omega \rangle$ in $\mathcal{M}$ since DC need not hold in $\mathcal{M}$. Instead, we will use the following argument that takes advantage of the Reflection Theorem. Choose $k \in \omega$ such that ψ is a $\Sigma_k(\Phi)$-statement, and use the Reflection Theorem in $(\mathcal{M}, \Phi)$ to pick $\mu \in \mathrm{Ord}^{\mathcal{M}}$ such that:

$$\left(\mathcal{M}_\mu, \Phi \cap M_\mu\right) \prec_{\Sigma_k(\mathrm{X})} (\mathcal{M}, \Phi).^{28}$$

Then ψ holds in $\left(\mathcal{M}_\mu, \Phi \cap M_\mu\right)$. Within $\mathcal{M}$, let

$$w = \{v \in \mathrm{V}_\mu : v \in \Gamma\}.$$

Thus $(\mathcal{M}, \Phi) \models w \neq \varnothing \wedge \exists t' \in \Gamma\ (t' \in t) \in \Phi$. Observe that since $\mathcal{N}$ has access to the Tarskian satisfaction class for $\mathcal{N}_\beta$, $\mathcal{N}$ can evaluate each term $t(c)$ in w as an ordinal $\delta < \beta$, where $\mathcal{N}_\beta \models t(\dot{\lambda}) = \delta$, where $t(\dot{\lambda})$ is the term obtained by replacing all occurrences of the constant c with $\dot{\lambda}$ in $t(c)$. So we can consider $s \in N$, where:

$$\mathcal{N} \models s = \left\{\delta : \exists t(c) \in w\ (\mathrm{V}_\beta, \in) \models \delta = t(\dot{\lambda})\right\}.$$

Within $\mathcal{N}$, s is a nonempty set of ordinals that has no least element, which of course is a contradiction. This shows that Case II is impossible, thus concluding the proof. ⊣

Remark 4.5. Arguing in ZFC, suppose κ is the first strongly inaccessible cardinal, λ is the second strongly inaccessible cardinal, $\mathcal{M} = (\mathrm{V}_\kappa, \in)$ and $\mathcal{N} = (\mathrm{V}_\lambda, \in)$. Then $\mathcal{N}$ is a proper faithful rank extension of $\mathcal{M}$ in which the first clause of the dichotomy of the conclusion of part (a) of Theorem 4.1 (and 4.4) holds, but not the second one. On the other hand, if κ is a weakly compact cardinal, then as noted by Kaufmann [26, Proposition 2.3], $(\mathrm{V}_\kappa, \in)$ has a faithful topless elementary end extension $\mathcal{N}$; in

[28] Here X is interpreted by the left-hand side structure as $\Phi \cap M_\mu$ and by the right-hand side structure as Φ

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this scenario the second clause of the dichotomy of the conclusion of part (a) of Theorem 4.1 (and 4.4) holds, but not the first one.

REMARK 4.6. If the assumptions of Theorem 4.4 are strengthened by adding the assumption that $\mathcal{M} \prec \mathcal{N}$, then as in the proof of Theorem 3.3 of [11] we can use 'Kaufmann's trick' to conclude that there is some $\mathcal{K}$ such that $\mathcal{M} \prec \mathcal{K} \preceq \mathcal{N}$ and $\mathcal{K}$ is a topped rank extension of $\mathcal{M}$. This yields a strengthening of Theorem 3.3 of [11] by eliminating the assumption that the axiom of choice holds in $\mathcal{M}$ and $\mathcal{N}$.

REMARK 4.7. An inspection of the proof of Theorem 4.1 makes it clear that the assumption that $\mathcal{M} \models \mathsf{ZFC}$ can be reduced to $\mathcal{M} \models \mathsf{ZF} + \mathsf{DC}$, and the assumption that $\mathcal{N} \models \mathsf{ZFC}$ can be reduced to $\mathcal{N} \models \mathsf{KPR}$ + "every set can be well-ordered." Here KPR is as in part (d) of Definition 2.1. Similarly, an inspection of the proof of Theorem 4.4 shows that the assumption that $\mathcal{N} \models \mathsf{ZF}$ can be reduced to $\mathcal{N} \models \mathsf{KPR}$.

QUESTION 4.8. *Can Theorem 4.1 and Corollary 4.2 be strengthened by assuming that $\mathcal{M}$ and $\mathcal{N}$ are models of* ZF?

§5. Dead-end models. In this section we will establish Theorem C of the abstract, the proof uses many ingredients, including the following one that refines a result obtained independently by Kaufmann and the author who demonstrated Theorem 5.1 for models of ZF in which the Axiom of Choice holds (see "Added in Proof" of [26] and Remark 1.6 of [10]).

THEOREM 5.1. *No model of* ZF *has a conservative proper end extension satisfying* ZF.

PROOF. This follows from putting part (b) of Theorem 4.4 together with Tarski's Undefinability of Truth Theorem (as in Theorem 2.8). ⊣

The rest of the section is devoted to presenting results that in conjunction with Theorem 5.1 will allow us to establish Theorem C of the abstract (as Theorem 5.18).

DEFINITION 5.2. Suppose $\mathcal{M}$ is an $\mathcal{L}_{\text{set}}$-structure.

(a) $X \subseteq M$ is *a class*[29] in $\mathcal{M}$ if $\forall a \in M\ \exists b \in M\ X \cap \mathrm{Ext}_{\mathcal{M}}(a) = \mathrm{Ext}_{\mathcal{M}}(b)$.
(b) $\mathcal{M}$ is *rather classless* if every class of M is $\mathcal{M}$-definable.
(c) $\mathcal{M}$ is $\aleph_1$-like if $|M| = \aleph_1$ but $|\mathrm{Ext}_{\mathcal{M}}(a)| \leq \aleph_0$ for each $a \in M$.

THEOREM 5.3 (Keisler–Kunen [29] and Shelah [44]). *Every countable model of* ZF *has an elementary end extension to an $\aleph_1$-like rather classless model.*

PROPOSITION 5.4. *No rather classless model of* ZF *has a proper rank extension to a model of* ZF.

PROOF. This follows from putting Theorem 4.4 together with the observation that a rank extension of a rather classless model of ZF is a conservative extension, and therefore a faithful extension. ⊣

REMARK 5.5. As pointed out in Remark 1.6 of [11] it is possible for a rather classless model to have a proper end extension satisfying ZF, since $\aleph_1$-like rather

[29] Classes of $\mathcal{M}$ are sometimes referred to as *piecewise coded* subsets of $\mathcal{M}$.

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classless models exist by Theorem 5.3, and one can use the Boolean-valued approach to forcing to construct set generic extensions of such models.

DEFINITION 5.6. A *ranked tree* τ is a two-sorted structure $\tau = (\mathbb{T}, \leq_{\mathbb{T}}, \mathbb{L}, \leq_{\mathbb{L}}, \rho)$ satisfying the following three properties:

(1) $(\mathbb{T}, \leq_{\mathbb{T}})$ is a tree, i.e., a partial order such that any two elements below a given element are comparable.
(2) $(\mathbb{L}, \leq_{\mathbb{L}})$ is a linear order with no last element.
(3) ρ is an order preserving map from $(\mathbb{T}, \leq_{\mathbb{T}})$ onto $(\mathbb{L}, \leq_{\mathbb{L}})$ with the property that for each $t \in \mathbb{T}$, ρ maps the set of predecessors of t onto the initial segment of $(\mathbb{L}, \leq_{\mathbb{L}})$ consisting of elements of L less than $\rho(t)$.

DEFINITION 5.7. Suppose $\tau = (\mathbb{T}, \leq_{\mathbb{T}}, \mathbb{L}, \leq_{\mathbb{L}}, \rho)$ is a ranked tree. A linearly ordered subset B of $\mathbb{T}$ is said to be a *branch* of τ if the image of B under ρ is $\mathbb{L}$. The *cofinality* of τ is the cofinality of $(\mathbb{L}, \leq_{\mathbb{L}})$.

DEFINITION 5.8. Given a structure $\mathcal{M}$ in a language $\mathcal{L}$, we say that a ranked tree τ is $\mathcal{M}$ -definable if $\tau = \mathbf{t}^{\mathcal{M}}$, where $\mathbf{t}$ is an appropriate sequence of $\mathcal{L}$-formulae whose components define the corresponding components of τ in $\mathcal{M}$.$\mathcal{M}$ is *rather branchless* if for each $\mathcal{M}$-definable ranked tree τ, all branches of τ (if any) are $\mathcal{M}$-definable.

THEOREM 5.9. *Suppose $\mathcal{M}$ is a countable structure in a countable language.*

(a) (*Keisler–Kunen* [29], *essentially*). *It is a theorem of* $\mathsf{ZFC} + \diamondsuit_{\omega_1}$ *that $\mathcal{M}$ can be elementarily extended to a rather branchless model.*
(b) (*Shelah* [44]). *It is a theorem of* ZFC *that $\mathcal{M}$ can be elementarily extended to a rather branchless model.*

DEFINITION 5.10. Suppose $(\mathbb{P}, \leq_{\mathbb{P}})$ is a poset (partially ordered set).

(a) $(\mathbb{P}, \leq_{\mathbb{P}})$ is *directed* if any given pair of elements of $\mathbb{P}$ has an upper bound.
(b) A subset F of $\mathbb{P}$ is a *filter* over $(\mathbb{P}, \leq_{\mathbb{P}})$ if the sub-poset $(F, \leq_{\mathbb{P}})$ is directed.
(c) A filter over $(\mathbb{P}, \leq_{\mathbb{P}})$ is *maximal* if it cannot be properly extended to a filter over $(\mathbb{P}, \leq_{\mathbb{P}})$.[30]
(d) A subset C of $\mathbb{P}$ is *cofinal* in $(\mathbb{P}, \leq_{\mathbb{P}})$ if $\forall x \in \mathbb{P}\ \exists y \in C\ x \leq_{\mathbb{P}} y$.

DEFINITION 5.11. Suppose s is an infinite set.

(a) $[s]^{<\omega}$ is the directed poset of finite subsets of s, ordered by containment. Note that $[s]^{<\omega}$ is a directed set with no maximum element.
(b) $\mathrm{Fin}(s, 2)$ is the poset of finite functions from s into $\{0, 1\}$, ordered by containment (where a function is viewed as a set of ordered pairs).

EXAMPLE 5.12. Given an infinite set a, and $s \subseteq a$, let $\chi_s : s \to 2$ be the characteristic function of s, i.e., $\chi_s(x) = 1$ iff $x \in s$. Clearly $[\chi_s]^{<\omega}$ is a maximal filter of $\mathrm{Fin}(a, 2)$. More generally, if $\mathcal{N} \models \mathsf{ZF}$, and $\mathcal{N} \models$ "$s \subseteq a$ and a is infinite," consider $(\mathbb{P}, \leq_{\mathbb{P}})$, where:

$$\mathbb{P} := \mathrm{Ext}_{\mathcal{N}}(\mathrm{Fin}^{\mathcal{N}}(a, 2)),$$

[30] Thus a maximal filter can be described as a *maximally compatible* subset of $(\mathbb{P}, \leq_{\mathbb{P}})$.

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and $\leq_{\mathbb{P}}$ is set-inclusion (among members of $\mathbb{P}$) in the sense of $\mathcal{N}$. Then F_s is a maximal filter over $(\mathbb{P}, \leq_{\mathbb{P}})$, where:

$$F_s := \mathrm{Ext}_{\mathcal{N}}\left([\chi_s]^{<\omega}\right)^{\mathcal{N}}.$$

DEFINITION 5.13. A structure $\mathcal{M}$ is a *Rubin* model if it has the following two properties:

(a) Every $\mathcal{M}$-definable directed poset $(\mathbb{D}, \leq_{\mathbb{D}})$ with no maximum element has a cofinal chain of length ω_1.
(b) Given any $\mathcal{M}$-definable poset $\mathbb{P}$, and any maximal filter F of $\mathbb{P}$, if F has a cofinal chain of length ω_1, then F is coded in $\mathcal{M}$.

REMARK 5.14. If $\tau = (\mathbb{T}, \leq_{\mathbb{T}}, \mathbb{L}, \leq_{\mathbb{L}}, \rho)$ is a ranked tree, then each branch of τ is a maximal filter over $(\mathbb{T}, \leq_{\mathbb{T}})$. This makes it clear that every Rubin model is rather branchless. Also, a rather branchless model of ZF is rather classless. To see this consider the ranked tree τ_{class} defined within a model $\mathcal{M}$ of ZF as follows: The nodes of τ_{class} are ordered pairs (s, α), where $s \subseteq \mathrm{V}_\alpha$, the rank of (s, α) is α and $(s, \alpha) < (t, \beta)$ if $\alpha \in \beta$ and $s = t \cap \mathrm{V}_\alpha$. It is easy to see that $\mathcal{M}$ is rather classless iff every branch of $\tau_{\mathrm{class}}^{\mathcal{M}}$ is $\mathcal{M}$-definable.[31] Hence we have the following chain of implications:

$$\text{Rubin} \;\Rightarrow\; \text{rather branchless} \;\Rightarrow\; \text{rather classless}.$$

THEOREM 5.15 (Rubin[32] [40, Corollary 2.4]). *It is a theorem of* $\mathsf{ZFC} + \diamondsuit_{\omega_1}$ *that if* $\mathcal{M}$ *is a countable structure in a countable language, then* $\mathcal{M}$ *has an elementary extension of cardinality* $\aleph_1$ *that is a Rubin model*.[33]

DEFINITION 5.16. Suppose $\mathcal{M}$ is a model of ZF. $\mathcal{M}$ is *weakly Rubin* if (a) and (b) below hold:

(a) $\mathcal{M}$ is rather classless (this is the asymptotic case of (ii) below, if one could use $a = \mathrm{V}$).
(b) For every element a of $\mathcal{M}$ that is infinite in the sense of $\mathcal{M}$ the following two statements hold:
  (1) $([a]^{<\omega})^{\mathcal{M}}$ has a cofinal chain of length ω_1.
  (2) If F is a maximal filter of $\mathrm{Fin}^{\mathcal{M}}(a, 2)$ and F has a cofinal chain of length ω_1, then F is coded in $\mathcal{M}$.

As demonstrated in the Appendix, Schmerl's strategy of $\diamondsuit_{\omega_1}$-elimination in [42] (which is based on an absoluteness argument first presented by Shelah [44]) can be

[31] The ranked tree τ_{class} was first introduced by Keisler [29, Example 2.1], who noted that $\mathcal{M}$ is rather classless iff every branch of $\tau_{\mathrm{class}}^{\mathcal{M}}$ is $\mathcal{M}$-definable, which is the key property that we need here. As noted by the referee, this key property is also satisfied by the subtree σ_{class} of τ_{class} whose tree-nodes are of the form (α, s) where the ordinal rank of s is α. Note that the branches of $\tau_{\mathrm{class}}^{\mathcal{M}}$ correspond to all classes of $\mathcal{M}$ (including improper ones, i.e., those that form a set), whereas the branches of $\sigma_{\mathrm{class}}^{\mathcal{M}}$ correspond to all proper classes of $\mathcal{M}$.

[32] The attribution to Rubin is informed by fact that the credit for [40, Lemma 2.3] is explicitly given to Rubin (at the end of the introduction of the article).

[33] The proof of Theorem 5.15 in [40, Lemma 2.3] has a small gap (it is assumed that $T_0 \vdash \exists \vec{x}\, \psi(\vec{x})$ instead of assuming the consistency of $T_0 + \exists \vec{x}\, \psi(\vec{x})$). The improved proof was presented in [9, Theorem 2.1.3] and is reproduced here in the Appendix.

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employed to build weakly Rubin models within ZFC. Thus we have the following theorem.

THEOREM 5.17 (Rubin–Shelah–Schmerl). *It is a theorem of* ZFC *that every countable model of* ZF *has an elementary extension to a weakly Rubin model of cardinality* $\aleph_1$.

We are finally ready to establish the main result of this section on the existence of 'dead-end' models. Note that since every generic extension is an end extension (even for ill-founded models), the model $\mathcal{M}$ in Theorem 5.18 has no proper generic extension.

THEOREM 5.18. *Every countable model* $\mathcal{M}_0 \models \mathsf{ZF}$ *has an elementary extension* $\mathcal{M}$ *of power* $\aleph_1$ *that has no proper end extension to a model* $\mathcal{N} \models \mathsf{ZF}$. *Thus every consistent extension of* ZF *has a model of power* $\aleph_1$ *that has no proper end extension to a model of* ZF.

PROOF. We begin[34] with a basic fact that will be called upon in the proof. ⊣

FACT (∇). *Suppose* $\mathcal{M} \models \mathsf{ZF}$ *and* $\mathcal{N} \models \mathsf{ZF}$ *with* $\mathcal{M} \subseteq_{\text{end}} \mathcal{N}$, *and* $a \in M$. *If* $s \in N$, s *is finite as viewed in* $\mathcal{N}$, *and* $\mathcal{N} \models s \subseteq a$, *then* $s \in M$. *Thus for all* $a \in M$, *we have*

$$\left([a]^{<\omega}\right)^{\mathcal{M}} = \left([a]^{<\omega}\right)^{\mathcal{N}}.$$

PROOF. The assumptions on $\mathcal{M}$ and $\mathcal{N}$ readily imply:
(1) $\omega^{\mathcal{M}} = \omega^{\mathcal{N}}$, and
(2) $\mathcal{M} \preceq_{\Delta_1} \mathcal{N}$.
Let $x = \text{Fn}(y, a)$ be the usual formula expressing "x is the set of all functions $f : y \to a$." It is easy to see that:
(3) The predicate $\left(x = \text{Fn}(y, a) \wedge y \in \omega\right)$ is Δ_1 within ZF.
Observe that $x = \text{Fn}(y, a)$ is clearly expressible by a Π_1-formula within ZF (with no restriction on y). With the added condition that $y \in \omega$, we can take advantage of recursion to express $x = \text{Fn}(y, a)$ by the following Σ_1-formula:

$$\exists \langle s_0, \cdots, s_y \rangle \left[s_y = x \wedge s_0 = \{\varnothing\} \wedge \forall z < y \, \left(s_{z+1} = \left\{ f \cup \{(z, v)\} : f \in s_z \wedge v \in a \right\}\right)\right].$$

Now assume $s \in N$, $\mathcal{N} \models s \subseteq a$, and for some $k \in N, \mathcal{N} \models k \in \omega \wedge |s| = k$. Thanks to (1) and the assumption $\mathcal{M} \subseteq_{\text{end}} \mathcal{N}$, $k \in M$. Therefore by (2) and (3), if $b \in M$ such that $\mathcal{M} \models b = \text{Fn}(k, a)$, then $\mathcal{N} \models b = \text{Fn}(k, a)$. Thus if $f \in N$ such that $f : k \to a$ is an injective function such that $\text{range}(f) = s$, then $f \in b$, and since $\mathcal{N}$ is an end extension of $\mathcal{M}$, $f \in M$. Hence $\text{range}(f) = s \in M$. □ Fact (∇))

Given a countable model $\mathcal{M}_0$ of ZF, by Theorem 5.17 there is a weakly Rubin model $\mathcal{M}$ that elementarily extends $\mathcal{M}_0$. By Theorem 5.1, to prove Theorem 5.18 it is sufficient to verify that every end extension $\mathcal{N}$ of $\mathcal{M}$ that satisfies ZF is a conservative extension. Towards this goal, suppose $\mathcal{M} \subsetneq_{\text{end}} \mathcal{N} \models \mathsf{ZF}$. In light of Remark 2.6(e), it suffices to show that $\mathcal{M} \subsetneq^{\mathcal{P}}_{\text{end}} \mathcal{N}$.

To show $\mathcal{M} \subsetneq^{\mathcal{P}}_{\text{end}} \mathcal{N}$, suppose that for $a \in M$ and $s \in N$, $\mathcal{N} \models s \subseteq a$. We will show that $s \in M$. By Fact (∇) we may assume that a is infinite as viewed from

[34] The proof of Fact (∇) presented here is a reformulation of a proof that was suggested by the referee. Our original proof admittedly used too much machinery.

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$\mathcal{M}$. Note that this implies that $\mathcal{M}$ views $[a]^{<\omega}$ as a directed set with no maximum element. Fact (∇) assures us that:

$(*)$ $\forall m \in M\ ([m]^{<\omega})^{\mathcal{M}} = ([m]^{<\omega})^{\mathcal{N}}$, and

Since $\mathrm{Fin}(m,2) \subseteq [m \times \{0,1\}]^{<\omega}$, $(*)$ implies:

$(**)$ $\forall m \in M\, \mathrm{Fin}^{\mathcal{N}}(m,2) = \mathrm{Fin}^{\mathcal{M}}(m,2)$.

Let $\chi_s \in N$ be the characteristic function of s in $\mathcal{N}$, i.e., as viewed from $\mathcal{N}$, $\chi_s : a \to \{0,1\}$ and $\forall x \in a (x \in s \leftrightarrow \chi_s(x) = 1)$. Let

$$F_s := \mathrm{Ext}_{\mathcal{N}}\left([\chi_s]^{<\omega}\right)^{\mathcal{N}}.$$

As noted in Example 5.12, F_s is a maximal filter over $\mathrm{Ext}_{\mathcal{N}}(\mathrm{Fin}^{\mathcal{N}}(a,2))$, so by $(**)$ and the assumption that $\mathcal{N}$ end extends $\mathcal{M}$, F_s is a maximal filter over $\mathrm{Ext}_{\mathcal{M}}(\mathrm{Fin}^{\mathcal{M}}(a,2))$. Note that directed set $\mathrm{Ext}_{\mathcal{M}}\left([a]^{<\omega}\right)^{\mathcal{M}}$ has a cofinal chain $\langle p_\alpha : \alpha \in \omega_1 \rangle$ thanks to the assumption that $\mathcal{M}$ is weakly Rubin. Together with $(*)$, this readily implies that F_s has a cofinal chain $\langle q_\alpha : \alpha \in \omega_1 \rangle$, where:

$$q_\alpha := (\chi_s \restriction p_\alpha)^{\mathcal{N}} = (\chi_s \restriction p_\alpha)^{\mathcal{M}}.$$

Therefore, by the assumption that $\mathcal{M}$ is weakly Rubin, F_s is $\mathcal{M}$-definable and thus coded in $\mathcal{M}$ by some $m \in M$. This makes it clear that $\mathcal{M} \models \chi_s = \cup m$, and thus $s \in M$ since s is Δ_0-definable from χ_s. This concludes the verification that $\mathcal{M} \subsetneq^{\mathcal{P}}_{\mathrm{end}} \mathcal{N}$, which as explained earlier, is sufficient to establish Theorem 5.18. ⊣

An examination of the proof of Theorem 5.1, together with Remark 4.7, makes it clear that Theorem 5.1 can be refined as follows.

THEOREM 5.19. *No model of* $\mathsf{KPR} + \Pi^1_\infty$-$\mathsf{DC}$ *has a conservative proper end extension to a model of* KPR.

The proof of Theorem 5.18, together with Theorem 5.19 allows us to refine Theorem 5.18 as follows.

THEOREM 5.20. *Every countable model* $\mathcal{M}_0 \models \mathsf{KPR} + \Pi^1_\infty$-$\mathsf{DC}$ *has an elementary extension* $\mathcal{M}$ *of power* $\aleph_1$ *that has no proper end extension to a model* $\mathcal{N} \models \mathsf{KPR}$. *Thus every consistent extension of* $\mathsf{KPR} + \Pi^1_\infty$-$\mathsf{DC}$ *has a model of power* $\aleph_1$ *that has no proper end extension to model of* KPR.

QUESTION 5.21. *Is there an* ω*-standard model of* $\mathsf{ZF(C)}$ *that has no proper end extension to a model of* $\mathsf{ZF(C)}$?

REMARK 5.22. Weakly Rubin models are never ω -standard (even though ω-standard rather classless models exist in abundance). However, a slight variation of the existence proof of Rubin models shows that a Rubin model $\mathcal{M}$ can be arranged to have an $\aleph_1$-like $\omega^{\mathcal{M}}$. Note, however, that the answer to Question 5.21 is known to be in the negative if "end extension" is strengthened to "rank extension"; this follows from ([10], Theorem 1.5(b)), which asserts that no $\aleph_1$-like model of ZFC that elementarily end extends a model all of whose ordinals are pointwise definable has a proper rank extension to a model of ZFC. The proof of [10, Theorem 1.5(b)] together with part (b) of Theorem 4.4 makes it clear that the same goes for ZF, i.e., there are ω-standard models of ZF that have no proper rank extension to a model of ZF.

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§6. Addendum and corrigendum to [11]. Theorem 2.2 of [11] incorrectly stated (in the notation of the same paper, where Gothic letters are used for denoting structures) that if $\mathfrak{A} \models \mathsf{ZF}$, $\mathfrak{A} \prec_{\text{cons}} \mathfrak{B}$ and $\mathfrak{B}$ fixes each $a \in \omega^{\mathfrak{A}}$, then $\mathfrak{A}$ is cofinal in $\mathfrak{B}$. As shown in Theorem 3.1 of this article, it is possible to arrange taller conservative elementary extensions of models of ZFC, thus Theorem 2.2 of [11] is false as stated. However, by assuming that 'slightly' strengthening the conditions:

$$\mathfrak{A} \models \mathsf{ZF}, \text{ and } \mathfrak{B} \text{ fixes each } a \in \omega^{\mathfrak{A}},$$

to:

$$\mathfrak{A} \models \mathsf{ZFC}, \text{ and } \mathfrak{B} \text{ fixes } \omega^{\mathfrak{A}},$$

one obtains a true statement, as indicated in Corollary 4.2 of this article.[35] As pointed out in Remark 3.9, $\mathfrak{B}$ fixes each $a \in \omega^{\mathfrak{A}}$ whenever $\mathfrak{A} \prec_{\text{cons}} \mathfrak{B} \models \mathsf{ZF}$.

Several corollaries were drawn from Theorem 2.2 of [11]; what follows are the modifications in them, necessitated by the above correction. We assume that the reader has [11] handy for ready reference.

- In Corollary 2.3(a) of [11], the assumption that $\mathfrak{B}$ fixes every integer of $\mathfrak{A}$ should be strengthened to: $\mathfrak{B}$ fixes $\omega^{\mathfrak{A}}$.
- Part (b) of Corollary 2.3 of [11] is correct as stated in [11]: if $\mathfrak{A}$ is a Rubin model of ZFC and $\mathfrak{A} \prec \mathfrak{B}$ that fixes every $a \in \omega^{\mathfrak{A}}$, then $\mathfrak{B}$ is a conservative extension of $\mathfrak{A}$. This is really what the proof of part (a) of Corollary 2.3 of [11] shows. However, the following correction is needed: towards the end of the proof of part (a) of Corollary 2.3 of [11], after defining the set X, it must simply be stated as follows: But $X = Y \cap A$, where Y is the collection of members $a \in A$ satisfying $[\varphi(b, a) \wedge a \in R_\alpha]$. Since $\mathfrak{B}$ is an end extension of $\mathfrak{C}$, $Y \in C$, so by claim $(*)$, X is definable in $\mathfrak{A}$.
- Corollary 2.4 of [11] is correct as stated since it does not rely on the full force of Theorem 2.2 of [11] and only relies on the fact that no elementary conservative extension of a model of ZF is an end extension. The latter follows from Corollary 4.2 of this article.
- The condition stipulating that $\mathfrak{B}$ fixes $\omega^{\mathfrak{A}}$ must be added to condition (i) of Corollary 2.5 of [11].

Appendix: Proofs of Theorems 5.15 and 5.17

THEOREM A.1 (Rubin–Shelah–Schmerl). *It is a theorem of* ZFC *that every countable model of* ZF *has an elementary extension to a weakly Rubin model of cardinality* $\aleph_1$.

[35] To my knowledge, there is only one place in the published literature in which Theorem 2.2 of [11] has been used, namely, in a recent paper of Goldberg and Steel [22]. In the discussion after Lemma 3.5 of the aforementioned paper, it is noted that Theorem 2.2 of [11] implies that an elementary embedding $\pi : \mathcal{M} \to \mathcal{N}$ (where $\mathcal{M}$ and $\mathcal{N}$ are inner models of ZFC) has the property that $\pi^{-1}[S] \in M$ for each $S \in N$ (paraphrased in [22] as 'π is a close embedding of $\mathcal{M}$ into $\mathcal{N}$') iff for each parametrically definable subset D of $\mathcal{N}$, $\pi^{-1}[D]$ is parametrically definable in $\mathcal{M}$. In the context of [22], the relevant models $\mathcal{M}$ and $\mathcal{N}$ are both ω-models (they are inner models, thus well-founded), so Corollary 4.2 of this article applies.

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The proof of Theorem A.1 has two distinct stages: in the first stage we prove within ZFC + $\diamondsuit_{\omega_1}$ that every countably infinite structure in a countable language has an elementary extension to a *Rubin* model of cardinality $\aleph_1$ (Theorem 5.15), and then in the second stage we use the result of the first stage together with a forcing-and-absoluteness argument to establish the theorem.[36]

Stage 1 of the proof of Theorem A.1. Fix a $\diamondsuit_{\omega_1}$ sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$. Given a countable language $\mathcal{L}$, and countably infinite $\mathcal{L}$-structure model $\mathcal{M}_0$, assume without loss of generality that $M_0 = \omega$. We plan to inductively build two sequences $\langle \mathcal{M}_\alpha : \alpha < \omega_1 \rangle$, and $\langle \mathcal{O}_\alpha : \alpha < \omega_1 \rangle$. The first is a sequence of approximations to our final model $\mathcal{M}$. The second sequence, on the other hand, keeps track of the increasing list of 'obligations' we need to abide by throughout the construction of the first sequence. More specifically, each $\mathcal{O}_\alpha$ will be of the form $\{\{V_n^\alpha,\ W_n^\alpha\} : n \in \omega\}$, where $\{V_n^\alpha,\ W_n^\alpha\}$ is pair of disjoint subsets of M_α that are inseparable in $\mathcal{M}_\alpha$ and should be kept inseparable in each $\mathcal{M}_\beta$, for all $\beta > \alpha$. Here we say that two disjoint subsets V and W of a model $\mathcal{N}$ are *inseparable* in $\mathcal{N}$ if there is no $\mathcal{N}$ -definable $X \subseteq N$ such that $V \subseteq X$, and $W \cap X = \varnothing$.

We only need to describe the construction of these two sequences for stages $\alpha + 1$ for limit ordinals α since:

- $\mathcal{O}_0 := \varnothing$.
- For limit α, $\mathcal{M}_\alpha := \bigcup_{\beta<\alpha} \mathcal{M}_\beta$ and $\mathcal{O}_\alpha := \bigcup_{\beta<\alpha} \mathcal{O}_\beta$.
- For nonlimit α, $\mathcal{M}_{\alpha+1} := \mathcal{M}_\alpha$ and $\mathcal{O}_{\alpha+1} := \mathcal{O}_\alpha$.

At stage $\alpha + 1$, where α is a limit ordinal, we have access to a model $\mathcal{M}_\alpha$ (where $M_\alpha \in \omega_1$), and a collection $\mathcal{O}_\alpha$ of inseparable pairs of subsets of M_α. We now look at S_α, and consider two cases: either S_α is parametrically undefinable in $\mathcal{M}_\alpha$, or not. In the latter case we 'do nothing', and define $\mathcal{M}_{\alpha+1} := \mathcal{M}_\alpha$ and $\mathcal{O}_{\alpha+1} := \mathcal{O}_\alpha$. But if the former is true, we augment our list of obligations via:

$$\mathcal{O}_{\alpha+1} := \mathcal{O}_\alpha \cup \{\{S_\alpha,\ M_\alpha \backslash S_\alpha\}\}.$$

Notice that if S_α is parametrically undefinable in $\mathcal{M}_\alpha$, then $\{S_\alpha,\ M_\alpha \backslash S_\alpha\}$ is inseparable in $\mathcal{M}_\alpha$. Then we use Lemma A.2 below to build an elementary extension $\mathcal{M}_{\alpha+1}$ of $\mathcal{M}_\alpha$ that satisfies the following three conditions:

(1) For each $\{V,\ W\} \in \mathcal{O}_{\alpha+1}$, V and W are inseparable in $\mathcal{M}_{\alpha+1}$.

(2) For every $\mathcal{M}_\alpha$-definable directed set $\mathbb{D}$ with no last element, there is some $d^* \in \mathbb{D}^{\mathcal{M}_{\alpha+1}}$ such that for each $d \in \mathbb{D}^{\mathcal{M}_\alpha}$, $\mathcal{M}_{\alpha+1} \models (d <_{\mathbb{D}} d^*)$.

(3) $M_{\alpha+1} = M_\alpha + \omega$ (ordinal addition).

This concludes the description of the sequences $\langle \mathcal{M}_{\alpha+1} : \alpha < \omega_1 \rangle$, and $\langle \mathcal{O}_\alpha : \alpha < \omega_1 \rangle$.

Let $\mathcal{M} := \bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha$, and $\mathcal{O} := \bigcup_{\alpha<\omega_1} \mathcal{O}_\alpha$; note that:

(4) For each $\{V,\ W\} \in \mathcal{O}$, V and W are inseparable in $\mathcal{M}$.

We now verify that $\mathcal{M}$ is a Rubin model. Suppose, on the contrary, that for some $\mathcal{M}$-definable partial order $\mathbb{P}$, there is a maximal filter $F \subseteq \mathbb{P}$ that is not $\mathcal{M}$ -definable,

[36] It appears to be unknown whether Theorem 5.15 (existence of Rubin elementary extensions) can be proved in ZFC alone. However, as shown here, ZFC can prove that every countable model of ZF has an elementary extension to a *weakly Rubin model* (see also Remark A.7).

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and F contains a cofinal ω_1-chain E. By a standard Löwenheim–Skolem argument there is some limit $\alpha < \omega_1$ such that:

(5) $(\mathcal{M}_\alpha, S_\alpha) \prec (\mathcal{M}, F)$, where $S_\alpha = F \cap \alpha$.

In particular,

(6) S_α is an undefinable subset of $\mathcal{M}_\alpha$.

Since S_α is a countable subset of some $\mathcal{M}$ -definable partial order $\mathbb{P}$, and E is chain of length ω_1 that is cofinal in $\mathbb{P}$, there is some $e \in \mathbb{P}$ such that $p <_{\mathbb{P}} e$ for each $p \in S_\alpha$. Fix some $\beta < \omega_1$ such that $e \in M_\beta$. We claim that the formula $\varphi(x) = (x \in \mathbb{P}) \to (x <_{\mathbb{P}} e)$ separates S_α and $\mathbb{P}^{M_\alpha} \backslash S_\alpha$ in $\mathcal{M}$. This is easy to see since if $q \in \mathbb{P}^{M_\alpha} \backslash S_\alpha$, then by (5), S_α is a maximal filter on $\mathbb{P}^{M_\alpha}$, and therefore there is some $p \in S_\alpha$ such that $\mathcal{M}_\alpha \models$ "p and q have no common upper bound," which by (5) implies the same to be true in $\mathcal{M}$, in turn implying that $\neg(q <_{\mathbb{P}} e)$, as desired. We have arrived at a contradiction since on the one hand, based on (2) and (6), the formula $\varphi(x)$ witnesses the separability of S_α and $M_\alpha \backslash S_\alpha$ within $\mathcal{M}$, and on the other hand $\{S_\alpha,\ M_\alpha \backslash S_\alpha\} \in \mathcal{O}$ by (5), and therefore (4) dictates that S_α and $M_\alpha \backslash S_\alpha$ are inseparable in $\mathcal{M}$. Thus the proof of Theorem A.1 will be complete once we establish Lemma A.2 below.

LEMMA A.2. *Suppose $\mathcal{M}$ is a countable $\mathcal{L}$-structure, where $\mathcal{L}$ is countable. Let $\{\{V_n, W_n\} : n \in \omega\}$ is a countable list of $\mathcal{M}$-inseparable pairs of subsets of M, and $\{(\mathbb{D}_n, \leq_{\mathbb{D}_n}) : n \in \omega\}$ be an enumeration of $\mathcal{M}$-definable directed sets with no last element. Then there exists an elementary extension $\mathcal{N}$ of $\mathcal{M}$ that satisfies the following two properties*:

(a) *V_n and W_n remain inseparable in $\mathcal{N}$ for all $n \in \omega$.*

(b) *For each $n \in \omega$ there is some $d_n \in N$ such that $\mathcal{N} \models d_n \in \mathbb{D}_n$ (i.e., d_n satisfies the formula that defines $\mathbb{D}_n$ in $\mathcal{M}$), and for each $m \in \mathbb{D}_n, \mathcal{N} \models (m <_{\mathbb{D}_n} d_n)$.*

Let $\mathcal{L}^+$ be the language $\mathcal{L}$ augmented with constants from $C_1 = \{\dot{m} : m \in M\}$ and $C_2 = \{d_n : n \in \omega\}$ (where C_1 and C_2 are disjoint), and consider the $\mathcal{L}^+$-theory T below:

$$T := \mathrm{Th}(\mathcal{M}, m)_{m \in M} + \{d_n \in \mathbb{D}_n : n \in \omega\} \cup \{\dot{m} <_{\mathbb{D}_n} d_n : \mathcal{N} \models (m \in \mathbb{D}_n), n \in \omega\}.$$

T is readily seen to be consistent since it is finitely satisfiable in $\mathcal{M}$. Moreover, it is clear that if $\mathcal{N} \models T$, then $\mathcal{M} \prec \mathcal{N}$ and $\mathcal{N}$ satisfies condition (b) of the theorem. To arrange a model of T in which condition (a) also holds requires a delicate omitting types argument. First, we need a pair of preliminary lemmas. The proofs of Lemma A.3 and A.4 are routine and left to the reader. Note that they are each other's contrapositive, so it is sufficient to only verify one of them.

LEMMA A.3. *The following two conditions are equivalent for a sentence $\varphi(d_{k_1}, \cdots, d_{k_p})$ of $\mathcal{L}^+$*:

(i) $T \vdash \varphi(d_{k_1}, \cdots, d_{k_p})$.

(ii) $\mathcal{M} \models \exists r_1 \in \mathbb{D}_{k_1} \forall s_1 \in \mathbb{D}_{k_1} \cdots \exists r_p \in \mathbb{D}_{k_p} \forall s_p \in \mathbb{D}_{k_p} \left[\left(\bigwedge_{1 \leq i \leq p} r_i <_{\mathbb{D}_{k_i}} s_i \right) \to \varphi(s_1, \cdots, s_p) \right]$.

LEMMA A.4. *The following two conditions are equivalent for a sentence $\varphi(d_{k_1}, \cdots, d_{k_p})$ of $\mathcal{L}^+$*:

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(i) $T + \varphi(d_{k_1}, \cdots, d_{k_p})$ *is consistent*.

(ii) $\mathcal{M} \models \forall r_1 \in \mathbb{D}_{k_1} \exists s_1 \in \mathbb{D}_{k_1} \cdots \forall r_p \in \mathbb{D}_{k_p} \exists s_p \in \mathbb{D}_{k_p} \left[\left(\bigwedge_{1 \leq i \leq p} r_i <_{\mathbb{D}_{k_i}} s_i \right) \wedge \varphi(s_1, \cdots, s_p) \right]$.

We are now ready to carry out an omitting types argument to complete the proof of Lemma A.2. For each formula $\psi(y, \vec{x})$ of $\mathcal{L}^+$, and each $n \in \omega$, consider the following q-type formulated in the language $\mathcal{L}^+$, where $\vec{x} = \langle x_1, \cdots, x_q \rangle$ is a q-tuple of variables:

$$\Sigma_n^{\psi}(\vec{x}) := \{\psi(\dot{a}, \vec{x}) : a \in V_n\} \cup \{\neg\psi(\dot{b}, \vec{x}) : b \in W_n\}.$$

Note that Σ_n^{ψ} expresses that $\psi(y, \vec{x})$ separates V_n and W_n.

LEMMA A.5. *Σ_n^{ψ} is locally omitted by T for each formula $\psi(y, \vec{x})$, and each $n \in \omega$.*

PROOF. Suppose to the contrary that there is a formula $\theta(\vec{x}, d_{k_1}, \cdots, d_{k_p})$ of $\mathcal{L}^+$ and some $n \in \omega$ such that (1)–(3) below hold:

(1) $T + \exists \vec{x}\, \theta(\vec{x}, d_{k_1}, \cdots, d_{k_p})$ is consistent.

(2) For all $a \in V_n$, $T \vdash \forall \vec{x} \left[\theta(\vec{x}, d_{k_1}, \cdots, d_{k_p}) \rightarrow \psi(\dot{a}, \vec{x}) \right]$.

(3) For all $b \in W_n$, $T \vdash \forall \vec{x} \left[\theta(\vec{x}, d_{k_1}, \cdots, d_{k_p}) \rightarrow \neg\psi(\dot{b}, \vec{x}) \right]$.

Invoking Lemmas A.3 and A.4, (1)–(3) translate to (1′)–(3′) below:

(1′) $\mathcal{M} \models \forall r_1 \in \mathbb{D}_{k_1} \exists s_1 \in \mathbb{D}_{k_1} \cdots \forall r_p \in \mathbb{D}_{k_p} \exists s_p \in \mathbb{D}_{k_p} \left[\left(\bigwedge_{1 \leq i \leq p} r_i <_{\mathbb{D}_{k_i}} s_i \right) \wedge \exists \vec{x}\, \theta(\vec{x}, \vec{s}) \right]$, where $\vec{s} = \langle s_1, \cdots, s_p \rangle$.

(2′) For all $a \in V_n$, $\mathcal{M} \models \lambda(a)$, where:

$$\lambda(\mathbf{v}) := \forall \vec{x}\ \exists r_1 \in \mathbb{D}_{k_1} \forall s_1 \in \mathbb{D}_{k_1} \cdots \exists r_p \in \mathbb{D}_{k_p} \forall s_p \in \mathbb{D}_{k_p} \left[\left(\bigwedge_{1 \leq i \leq p} r_i <_{\mathbb{D}_{k_i}} s_i \right) \rightarrow \left(\theta(\vec{x}, \vec{s}) \rightarrow \psi(\mathbf{v}, \vec{x}\ ) \right) \right].$$

(3′) For all $b \in W_n$, $\mathcal{M} \models \gamma(b)$, where:

$$\gamma(\mathbf{w}) := \forall \vec{x}\ \exists r_1 \in \mathbb{D}_{k_1} \forall s_1 \in \mathbb{D}_{k_1} \cdots \exists r_n \in \mathbb{D}_{k_p} \forall s_p \in \mathbb{D}_{k_p} \left[\left(\bigwedge_{1 \leq i \leq p} r_i <_{\mathbb{D}_{k_i}} s_i \right) \rightarrow \left(\theta(\vec{x}, \vec{s}) \rightarrow \neg\psi(\mathbf{w}, \vec{x}) \right) \right].$$

Let $\Lambda := \{m \in M : \mathcal{M} \models \lambda(m)\}$, and $\Gamma := \{m \in M : \mathcal{M} \models \gamma(m)\}$, and observe that $V_n \subseteq \Lambda$ by (2′) and $W_n \subseteq \Gamma$ by (3′). To arrive at a contradiction, we will show that $\Lambda \cap \Gamma = \emptyset$, which implies that V_n and W_n are separable in $\mathcal{M}$. To this end, suppose to the contrary that for some $m \in M$,

(4) $\mathcal{M} \models \lambda(m) \wedge \gamma(m)$.

Since each $(\mathbb{D}_n, \leq_{\mathbb{D}_n})$ is a directed set, (4) implies:

(5) $\mathcal{M} \models \forall \vec{x}\, \exists r_1 \in \mathbb{D}_{k_1} \forall s_1 \in \mathbb{D}_{k_1} \cdots \exists r_p \in \mathbb{D}_{k_p} \forall s_p \in \mathbb{D}_{k_p} \left[\left(\bigwedge_{1 \leq i \leq p} r_i <_{\mathbb{D}_{k_i}} s_i \right) \rightarrow \left(\psi(m, \vec{x}) \wedge \neg\psi(m, \vec{x}) \right) \right]$.

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Recall that no $(\mathbb{D}_n, \leq_{\mathbb{D}_n})$ has a last element. This shows that (5) yields a contradiction, so the proof is complete. (Lemma A.5) ⊣

PROOF OF LEMMA A.2. Put Lemma A.5 together with the Henkin–Orey omitting types theorem [5, Theorem 2.2.9] to conclude that there exists a model $\mathcal{N}$ of T that satisfies properties (a) and (b). ⊣

Stage 2 of the proof of Theorem A.1. In this stage we employ the result of the first stage together with some set-theoretical considerations to establish the Rubin–Shelah–Schmerl Theorem. The main idea here is a variant of the one used by Schmerl [42], which itself based on a method of $\diamondsuit_{\omega_1}$-elimination introduced by Shelah [44]. This stage has three steps.

Step 1 of Stage 2. It is well-known that there is an ω_1-closed notion $\mathbb{Q}_1$ in the universe V of ZFC such that the $\mathbb{Q}_1$-generic extension $\mathrm{V}^{\mathbb{Q}_1}$ of V satisfies $\mathsf{ZFC} + \diamondsuit_{\omega_1}$ [32, Theorem 8.3] (note that $\aleph_1$ is absolute between V and $\mathrm{V}^{\mathbb{Q}_1}$ since $\mathbb{Q}_1$ is ω_1-closed). Thus by Step 1, given a model $\mathcal{M}$ in the universe V of ZF, the forcing extension $\mathrm{V}^{\mathbb{Q}_1}$ satisfies "there is an elementary extension $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{N}$ is a Rubin model." In $\mathrm{V}^{\mathbb{Q}_1}$, since $\mathcal{N}$ is a Rubin model, for each $s \in N$ that is infinite in the sense of $\mathcal{N}$, there is a subset $C_s = \{c^s_\alpha : \alpha \in \omega_1\}$ of $\mathcal{N}$ such that $\langle c^s_\alpha : \alpha \in \omega_1\rangle$ is an ω_1-chain that is cofinal in the directed set $([s]^{<\omega})^{\mathcal{M}}$.

- Let $\mathbb{P}_s$ be the suborder of $\big(\mathrm{Fin}(s,2)\big)^{\mathcal{M}}$ consisting of partial $\mathcal{M}$-finite functions f from s into 2 such that $\mathrm{dom}(f) \in C_s$. Note that $\mathbb{P}_s$ is a tree-order (the predecessors of any given element are comparable).
- $\mathbb{P}_s$ can be turned into a ranked-tree τ_s by defining $\rho(f) = \alpha$ if $\mathrm{dom}(f) = c_\alpha$. The ranked tree τ_s has the key property that each maximal filter of $\big(\mathrm{Fin}(s,2)\big)^{\mathcal{M}}$ is uniquely determined by a branch of τ_s.
- Let C be the binary predicate on $\mathcal{N}$ defined by $C(s,x)$ iff $x \in C_s$, and let

$$\mathcal{N}^+ = (\mathcal{N}, C)\,.$$

Note that the $\mathcal{N}^+$-definable ranked trees include trees of the form τ_s, as well as the ranked tree $\tau^{\mathcal{N}}_{\mathrm{class}}$ (whose branches uniquely determine classes of $\mathcal{N}$, as noted in Remark 5.14).

- This gives us $\aleph_1$-many ranked trees 'of interest' in $\mathrm{V}^{\mathbb{Q}_1}$, namely, $\tau^{\mathcal{N}}_{\mathrm{class}}$ together with τ_s for each infinite set s of $\mathcal{N}$.

Step 2 of Stage 2. The concept of a weakly specializing function plays a key role in this step. Given a ranked tree $(\mathbb{T},\ \leq_{\mathbb{T}},\ \mathbb{L},\ \leq_{\mathbb{L}},\ \rho)$, we say that $f : \mathbb{T} \to \omega$ *weakly specializes* $(\mathbb{T},\ \leq_{\mathbb{T}})$ if f has the following property:

If $x \leq_{\mathbb{T}} y$ and $x \leq_{\mathbb{T}} z$ and $f(x) = f(y) = f(z)$, then y and z are $\leq_{\mathbb{T}}$-comparable.

The following lemma captures the essence of a weakly specializing function for our purposes.

LEMMA A.6. *Suppose* $\tau = (\mathbb{T},\ \leq_{\mathbb{T}},\ \mathbb{L},\ \leq_{\mathbb{L}},\ \rho)$ *is a ranked tree, where* $(\mathbb{L},\ \leq_{\mathbb{L}})$ *has cofinality* ω_1, *and* f *weakly specializes* $(\mathbb{T},\ \leq_{\mathbb{T}})$. *Then every branch of* τ *is parametrically definable in the expansion* (τ, f) *of* τ.

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PROOF. Suppose B is a branch of τ. Then since $(\mathbb{L}, \leq_{\mathbb{L}})$ has cofinality ω_1, there is a subset S of B of order type ω_1 that is unbounded in B. Therefore there is some subset B_0 of B that is unbounded in B such that f is constant on B_0. Fix some element $b_0 \in B_0$ and consider subset X of $\mathbb{T}$ that is defined in (τ, f) by the formula:

$$\varphi(x, b_0) := [x \leq_{\mathbb{T}} b_0 \vee (b_0 \leq_{\mathbb{T}} x \wedge f(b_0) = f(x))].$$

By the defining property of weakly specializing functions, X is linearly ordered and is unbounded in B. This makes it clear that B is definable in (τ, f) as the downward closure of X, i.e., by the formula $\psi(x, b_0) := \exists y \left[\varphi(y, b_0) \wedge x \leq_{\mathbb{T}} y\right]$. ⊣

By a remarkable theorem, due independently to Baumgartner [3] and Shelah [44], given any ranked tree τ of cardinality $\aleph_1$ and of cofinality ω_1, there is a c.c.c. notion of forcing $\mathbb{Q}_\tau$ such that $\mathrm{V}^{\mathbb{Q}_1 * \mathbb{Q}_\tau} := \left(\mathrm{V}^{\mathbb{Q}_1}\right)^{\mathbb{Q}_\tau}$ contains a function f_τ such that weakly specializes τ.[37] Here $\mathbb{Q}_\tau$ consists of all finite attempts for building a function that weakly specializes τ (ordered by inclusion). Moreover, given any collection $\{\tau_\alpha : \alpha \in \kappa\}$ of ranked trees of cofinality ω_1, the *finite support product* $\mathbb{Q}$ of $\{\mathbb{Q}_{\tau_\alpha} : \alpha \in \kappa\}$ is also known to have the c.c.c. property,[38] as indicated in [6, Corollary 3.3].[39] Note that for each $\alpha \in \kappa$, $\mathbb{Q}$ forces the existence of a function f_{τ_α} that weakly specializes τ_α.

- Thus by choosing $\{\tau_\alpha : \alpha \in \omega_1\}$ to be the collection of ranked trees that are parametrically definable in $\mathcal{N}^+$, there is a function $g(s, x)$ in $\mathrm{V}^{\mathbb{Q}_1 * \mathbb{Q}} := \left(\mathrm{V}^{\mathbb{Q}_1}\right)^{\mathbb{Q}}$ such that for each infinite set s of $\mathcal{N}$, $g(s, x)$ weakly specializes τ_s (τ_s was defined in Step 1 of Stage 2), and a function f that weakly specializes $\tau^{\mathcal{N}}_{\mathrm{class}}$.

The next lemma shows that forcing with $\mathbb{Q}$ does not add new branches to ranked trees in the ground model that have cofinality ω_1. Note that when the lemma below is applied to the case when V is replaced by $\mathrm{V}^{\mathbb{Q}_1}$, it shows that the ample supply of rather branchless models in $\mathrm{V}^{\mathbb{Q}_1}$ remain rather branchless after forcing with $\mathbb{Q}$, i.e., in $\mathrm{V}^{\mathbb{Q}_1 * \mathbb{Q}}$.

LEMMA A.7. *Suppose $\tau = (\mathbb{T}, \leq_{\mathbb{T}}, \mathbb{L}, \leq_{\mathbb{L}}, \rho)$ is a ranked tree, where $(\mathbb{L}, \leq_{\mathbb{L}})$ has cofinality ω_1. If B is a branch of τ in $\mathrm{V}^{\mathbb{Q}}$, then $B \in \mathrm{V}$.*

PROOF. Suppose B is a branch of τ *in* $\mathrm{V}^{\mathbb{Q}}$, and let f be the $\mathbb{Q}$-generic function in $\mathrm{V}^{\mathbb{Q}}$ that specializes τ. Since $\mathbb{Q}$ does not collapse ω_1 and the rank order of τ is assumed to have cofinality ω_1 in V, there is for some $k \in \omega$ such that:

(1) $\mathrm{V}^{\mathbb{Q}} \models$ "$f^{-1}\{k\} \cap B$ is unbounded in B."

In $\mathrm{V}^{\mathbb{Q}}$, let $x_0 \in B$ with $f(x_0) = k$. Coupled with (1) this shows that there is some $q_0 \in \mathbb{Q}$ such that:

(2) $q_0 \Vdash \left[f(x_0) = k \wedge \exists y >_{\mathbb{T}} x_0 (y \in B \wedge f(y) = k)\right]$.

[37] This result generalizes the fact, established by Baumgartner, Malitz, and Reinhardt [4] that every ω_1-Aronszajn tree can be turned into a special Aronszajn tree in a c.c.c. forcing extension of the universe.

[38] It is well-known that the finite support product of a given collection of forcing notions $\{\mathbb{P}_\alpha : \alpha \in \kappa\}$ has the c.c.c. property iff every finite sub-product of $\{\mathbb{P}_\alpha : \alpha \in \kappa\}$ has the c.c.c. property.

[39] Corollary 3.3 of [6] is overtly about specializing ω_1-Aronszajn trees, but as shown by Baumgartner ([3], Section 7), given a tree τ of height ω_1 with at most $\aleph_1$ many branches, there is an Aronszajn subtree τ_0 of τ of height ω_1, and moreover, any function f_0 that specializes τ_0 can be canonically extended to a function f that specializes τ. For more details, see the proof of ([45], Lemma 3.5).

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Let B_0 be the subset of $\mathbb{T}$ defined in V as:

$$B_0 = \left\{ y \in \mathbb{T} : \exists q \in \mathbb{Q}\, (q \supseteq q_0 \wedge q \Vdash \left[y >_{\mathbb{T}} x_0 \wedge y \in B \wedge f(y) = k \right] \right\}.$$

We wish to show:

$(*)$ $\mathrm{V}^{\mathbb{Q}} \models B_0$ is a cofinal subset of B.

To verify $(*)$, suppose $b_0 \in B$. By (1), for some $y \in \mathbb{T}$, $\mathrm{V}^{\mathbb{Q}}$ satisfies "$y >_{\mathbb{T}} b_0$, $y \in B$, and $f(y) = k$." Therefore there is some $\mathbb{Q}$-condition $q \supseteq q_0$ such that:

$q \Vdash \left[y >_{\mathbb{T}} x_0 \wedge y \in B \wedge f(y) = k \right]$.

Since $B_0 \in \mathrm{V}$, $(*)$ makes it clear that $B \in \mathrm{V}$ as well, since B can be defined in V as the downward closure of B_0. ⊣

- Let Ω be the collection consisting of the ranked tree $\tau^{\mathcal{N}}_{\text{class}}$ together with the family of ranked trees $\left\{ \tau^{\mathcal{N}}_s : \mathcal{N} \models |s| \geq \aleph_0 \right\}$. Since $\mathcal{N}$ is a Rubin model in V, each of the trees in Ω are rather branchless in V, and Lemma A.7 assures us the ranked trees in Ω remain rather branchless in $\mathrm{V}^{\mathbb{Q}}$. Lemma A.6, on the other hand, assures us that each branch of the ranked trees in Ω has a simple definition in $\mathrm{V}^{\mathbb{Q}}$.
- Let $\mathcal{L}^+_{\text{set}}$ be the finite language obtained by augmenting the usual language $\mathcal{L}_{\text{set}}$ of set theory with extra symbols for denoting the binary relation C, the binary function g, and the unary function f. Then let $\mathcal{L}$ be the countable language that is the result of augmenting $\mathcal{L}^+_{\text{set}}$ with constants for each element of $\mathcal{M}$. The content of the above bullet item allows us to write a single sentence ψ in $\mathcal{L}_{\omega_1,\omega}(\mathrm{Q})$ (where Q is the quantifier "there exist uncountably many") that captures the following salient features of the structure $\left(\mathcal{N}^+, C, g, f\right)$ that hold in $\mathrm{V}^{\mathbb{Q}}$:

(1) The $\mathcal{L}_{\text{set}}$-reduct of $(\mathcal{N}^+, C, g, f)$ elementarily extends $\mathcal{M}$.
(2) For each infinite s, $\tau^{\mathcal{N}}_s$ has uncountable cofinality, is rather branchless, and $g(s, x)$ weakly specializes $\tau^{\mathcal{N}}_s$.
(3) $\tau^{\mathcal{N}}_{\text{class}}$ has uncountable cofinality, is rather branchless, and f weakly specializes $\tau^{\mathcal{N}}_{\text{class}}$.

More specifically, ψ is the conjunction of the following sentences:

- $\psi_1 =$ the conjunction of the countably many sentences in the elementary diagram of $\mathcal{M}$.
- $\psi_2 =$ the sentence expressing (2) above.
- $\psi_3 =$ the sentence expressing (3) above.

We elaborate on how to write ψ_2, a similar idea is used to write ψ_3. The quantifier Q allows us to express that τ_s has uncountable cofinality, and the range of the specializing function $g(s, x)$ is countable. To express the rather branchless feature of τ_s, let $\psi(x, y)$ be as in the proof of Lemma A.6. Note that in the notation of Lemma A.6, for any $b \in \mathbb{T}\{x \in \mathbb{T}_s : \psi(x, b)\}$ is linearly ordered by $\leq_{\mathbb{T}}$. By Lemma A.6, in $\mathrm{V}^{\mathbb{Q}}$, if s is infinite and B is a branch of τ_s, then there is some $b_0 \in B$ such that $\psi(x, b_0)$ defines B. Together with the fact that τ_s is rather branchless in $\mathrm{V}^{\mathbb{Q}}$ (thanks to Lemma A.7) the $\mathcal{L}_{\omega_1,\omega}$-formula below (no need for the quantifier Q here) expresses the rather branchless feature of τ_s. In the formula below $\mathbb{T}_s$ is the set of the nodes of the ranked tree τ_s, and "unbounded rank" is short for "unbounded

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τ_s-rank":

$$\forall b \in \mathbb{T}_s \left[\{x \in \mathbb{T}_s : \psi(x,b)\} \text{ has unbounded rank } \rightarrow \bigvee_{\theta(x,y)\in\mathcal{L}^+_{\text{set}}} \exists y\, \forall x\ \big(\psi(x,b) \leftrightarrow \theta(x,y)\big) \right].$$

Step 3 of Stage 2. Let ψ be the sentence described above. Since ψ has a model in $V^{\mathbb{Q}_1 * \mathbb{Q}}$, and by Keisler's completeness theorem for $\mathcal{L}_{\omega_1,\omega}(\mathsf{Q})$ [28], the consistency of sentences in $\mathcal{L}_{\omega_1,\omega}(\mathsf{Q})$ is absolute for extensions of V that do not collapse $\aleph_1$, we can conclude that ψ has a model in V since $\aleph_1$ is absolute between V and $V^{\mathbb{Q}_1 * \mathbb{Q}}$ (recall that $\mathbb{Q}_1$ is ω_1-closed and $\mathbb{Q}$ is c.c.c.). Thanks to Lemmas A.6 and A.7, the $\mathcal{L}_{\text{set}}$-reduct of any model of ψ is a weakly Rubin model that elementarily extends $\mathcal{M}$ (and the cardinality of the model can be arranged to be $\aleph_1$ by a Löwenheim–Skolem argument). This concludes the proof of Theorem A.1. ⊣

Remark A.7. Schmerl [42] used a similar argument as the one used in the proof of Theorem A.1 to show that every countable model $\mathcal{M}$ of ZFC has an elementary extension $\mathcal{N}$ of cardinality $\aleph_1$ such that every $\omega^{\mathcal{N}}$-complete ultrafilter over $\mathcal{N}$ is $\mathcal{N}$-definable (and thus coded in $\mathcal{N}$). The proof of Theorem A.1 can be dovetailed with Schmerl's proof so as to show that every countable model $\mathcal{M}$ of ZFC has an elementary extension $\mathcal{N}$ of cardinality $\aleph_1$ that is weakly Rubin and also has the additional property that every $\omega^{\mathcal{N}}$-complete ultrafilter over $\mathcal{N}$ is $\mathcal{N}$-definable.

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REFERENCES

[1] W. Ackermann, *Zur widerspruchsfreiheit der zahlentheorie*. ***Mathematische Annalen***, vol. 117 (1940), pp. 162–194.

[2] J. Barwise, ***Admissible Sets and Structures***, Springer-Verlag, Berlin, 1975.

[3] J. Baumgartner, *Applications of the proper forcing axiom*, ***Handbook of Set-Theoretic Topology*** (K. Kunen and Jerry E. Vaughan, editors), North-Holland, Amsterdam, 1984, pp. 913–959.

[4] J. Baumgartner, J. Malitz, and W. Reinhardt, *Embedding trees in the rationals*. ***Proceedings of the National Academy of Sciences of the United States of America***, vol. 67 (1970), pp. 1748–1753.

[5] C. C. Chang and H. J. Keisler, ***Model Theory***, North Holland, Amsterdam, 1973.

[6] D. Chodounský and J. Zapletal, *Why Y-cc?* ***Annals of Pure and Applied Logic***, vol. 166 (2015), pp. 1123–1149.

[7] K. Devlin, ***Constructibility***, Springer-Verlag, Berlin, 1984.

[8] W. Easton, ***Powers of regular cardinals***, Doctoral dissertation, Princeton University, 1964.

[9] A. Enayat, ***Topics in the model theory of set theory***, Doctoral dissertation, University of Wisconsin (Madison), 1984.

[10] ———, *On certain elementary extensions of models of set theory*. ***Transaction of American Mathematical Society***, vol. 283 (1984), pp. 705–715.

[11] ———, *Conservative extensions of models of set theory and generalizations*. ***Journal of Symbolic Logic***, vol. 51 (1986), pp. 1005–1021.

https://doi.org/10.1017/jsl.2026.10199 Published online by Cambridge University Press

[12] ———, *Minimal elementary extensions of models of arithmetic and set theory*. ***Archive for Mathematical Logic***, vol. 30 (1990), pp. 181–192.

[13] ———, *Analogues of the MacDowell-Specker theorem in set theory*, ***Models, Algebras and Proofs*** (X. Caicedo and C. H. Montenegro, editors), Marcel Dekker Inc., New York, 1998, pp. 25–50.

[14] ———, *Automorphisms, Mahlo cardinals, and NFU, automorphisms*, ***Nonstandard Models of Arithmetic and Set Theory*** (A. Enayat and R. Kossak, editors), Contemporary Mathematics, 361, AMS Publications, Providence, 2004, pp. 37–59.

[15] ———, *Set theory with a proper class of indiscernibles*. ***Fundamenta Mathematicae***, vol. 259 (2022), pp. 33–76.

[16] A. Enayat and J. D. Hamkins, *ZFC proves that Ord is not weakly compact for definable classes*. ***Journal of Symbolic Logic***, vol. 83 (2018), pp. 146–164.

[17] A. Enayat, J. Schmerl, and A. Visser, *ω-models of finite set theory*, ***Set Theory, Arithmetic, and Foundations of Mathematics: Theorems, Philosophies*** (J. Kennedy and R. Kossak, editors), Cambridge University Press, Cambridge, 2011, pp. 43–65.

[18] U. Felgner, *Choice functions on sets and classes*, ***Sets and Classes (on the Work by Paul Bernays)*** (Gert H. Müller, editor), Studies in Logic and the Foundations of Mathematics, 84, North-Holland, Amsterdam, 1976, pp. 217–255.

[19] S. Friedman, W. Li, and T. L. Wong, *Fragments of Kripke-Platek, set theory and the metamathematics of α-recursion theory*. ***Archive for Mathematical Logic***, vol. 55 (2016), pp. 899–924.

[20] H. Gaifman, *Elementary embeddings of models of set theory and certain subtheories*, ***Proceedings of Symposia in Pure Mathematics 13 (Part 2)*** (Thomas J. Jech, editor), American Mathematical Society, Providence, 1974, pp. 33–101.

[21] ———, *Models and types of Peano's arithmetic*. ***Annals of Mathematical Logic***, vol. 9 (1976), pp. 223–306.

[22] G. Goldberg and J. Steel, *The structure of $C(aa)$*. ***Monatshefte für Mathematik***, (2025). https://doi.org/10.1007/s00605-025-02084-z.

[23] J. D. Hamkins, *Does ZFC prove the universe is linearly orderable*? MathOverflow, 2012. https://mathoverflow.net/q/110823.

[24] T. Jech, ***Set Theory***, Springer Monographs in Mathematics, Springer, Berlin, 2003.

[25] A. Kanamori, ***The Higher Infinite***, Springer, Amsterdam, 1994.

[26] M. Kaufmann, *Blunt and topless end extensions of models of set theory*. ***Journal of Symbolic Logic***, vol. 48 (1983), pp. 1053–1073.

[27] R. Kaye and T. Wong, *On interpretations of arithmetic and set theory*. ***Notre Dame Journal of Formal Logic***, vol. 48 (2007), pp. 497–510.

[28] H. J. Keisler, *Logic with the quantifier "there exist uncountably many"*. ***Annals of Mathematical Logic***, vol. 1 (1970), pp. 1–93.

[29] ———, *Models with tree structures*, ***Proceedings of the Tarski Symposium*** (L. Henkin and others, editors), American Mathematical Society, Providence, 1974, pp. 331–348.

[30] H. J. Keisler and M. Morley, *Elementary extensions of models of set theory*. ***Israel Journal of Mathematics***, vol. 5 (1968), pp. 49–65.

[31] H. J. Keisler and J. Silver, *End extensions of models of set theory*, ***Axiomatic Set Theory (Part I)*** (D. Scott, editor), Proceedings of Symposia in Pure Mathematics, 13, American Mathematical Society, Providence, 1970, pp. 177–187.

[32] K. Kunen, ***Set Theory***, first ed., North-Holland, Amsterdam, 1983.

[33] R. Mac Dowell and E. Specker, *Modelle der arithmetik*, ***Infinitistic Methods, Proceedings of Symposia in Pure Mathematics (Warsaw, 1959)***, Pergamon, Oxford-London-New York-Paris, 1961, pp. 257–263.

[34] S. Mac Lane, ***Mathematics, Form and Function***, Springer-Verlag, Providence, 1986.

[35] A. R. D. Mathias, *The strength of mac lane set theory*. ***Annals of Pure and Applied Logic*** (J. Bagaria and S. Todorcevic, editors), vol. 110 (2001), pp. 107–234.

[36] ———, *Weak systems of Gandy, Jensen and Devlin*, ***Set Theory***, Trends in Mathematics, Birkhäuser, Basel, 2006, pp. 149–224.

[37] Z. McKenzie, *On the relative strengths of fragments of collection*. ***Mathematical Logic Quarterly***, vol. 65 (2019), pp. 80–94.

[38] J. Myhill and D. Scott, *Ordinal definability*, ***Axiomatic Set Theory, Part I*** (D. Scott, editor), Proceedings of Symposia in Pure Mathematics, 13, vol. I, American Mathematical Society, Providence, 1970, pp. 271–278.

https://doi.org/10.1017/jsl.2026.10199 Published online by Cambridge University Press

[39] R. G. Phillips, *Omitting types in arithmetic and conservative extensions*, ***Victoria Symposium in Nonstandard Analysis***, vol. 369 (A. Hurd and P. Loeb, editors), Springer-Verlag, Berlin, 1974, pp. 195–202.

[40] M. Rubin and S. Shelah, *On the elementary equivalence of automorphisms of Boolean algebras, downward Lowenheim-Skolem theorems, and compactness of related quantifiers*. ***Journal of Symbolic Logic***, vol. 45 (1980), pp. 265–283.

[41] J. Schmerl, *End extensions of models of arithmetic*. ***Notre Dame Journal of Formal Logic***, vol. 33 (1992), pp. 216–219.

[42] ———, *Elementary extensions of models of set theory*. ***Archive for Mathematical Logic***, vol. 39 (2000), pp. 509–514.

[43] N. Schweber, *Are there ill-founded "maximally wide" models of ZFC*? MathOverflow, 2021, https://mathoverflow.net/q/397896.

[44] S. Shelah, *Models with second order properties II: Trees with no undefined branches*. ***Annals of Mathematical Logic***, vol. 14 (1978), pp. 73–87.

[45] C. B. Switzer, *Destructibility and axiomatizability of Kaufmann models*. ***Archive for Mathematical Logic***, vol. 61 (2022), pp. 1091–1111.

[46] M. Takahashi, *$\widetilde{\Delta}_n$-definability in set theory*, ***Conference in Mathematical Logic-London '70*** (W. Hodges, editor), Lecture Notes in Mathematics, 255, Springer-Verlag, Berlin, 1972, pp. 281–304.

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