RECIPROCATING THE REGULAR POLYTOPES

H. S. M. COXETER

In fond memory of John Brian Wilker (1943–1995).

Abstract

For reciprocation with respect to a sphere $\sum x^2 = c$ in Euclidean n-space, there is a unitary analogue: Hermitian reciprocation with respect to an antisphere $\sum \bar{u}u = c$. This is now applied, for the first time, to complex polytopes.

When a regular polytope Π has a palindromic Schläfli symbol, it is self-reciprocal in the sense that its reciprocal Π', with respect to a suitable concentric sphere or antisphere, is congruent to Π. The present article reveals that Π and Π' *usually* have together the same vertices as a third polytope Π^+ and the same facet-hyperplanes as a fourth polytope Π^- (where Π^+ and Π^- are again regular), so as to form a 'compound', $\Pi^+[2\Pi]\Pi^-$. When the geometry is real, Π^+ is the convex hull of Π and Π', while Π^- is their common content or 'core'. For instance, when Π is a regular p-gon $\{p\}$, the compound is

$$\{2p\}[2\{p\}]\{2p\}.$$

The exceptions are of two kinds. In one, Π^+ and Π^- are not regular. The actual cases are when Π is an n-simplex $\{3, 3, \ldots, 3\}$ with $n \geqslant 4$ or the real 4-dimensional 24-cell $\{3, 4, 3\} = 2\{3\}2\{4\}2\{3\}2$ or the complex 4-dimensional Witting polytope $3\{3\}3\{3\}3\{3\}3$. The other kind of exception arises when the vertices of Π are the poles of its own facet-hyperplanes, so that Π, Π', Π^+ and Π^- all coincide. Then Π is said to be *strongly* self-reciprocal.

1. *Introduction*

Kepler gave the name *stella octangula* to the compound of two centrally inverse tetrahedra $\alpha_3 = \{3, 3\}$ inscribed in a cube $\gamma_3 = \{4, 3\}$. The inscribed sphere of the cube reciprocates the two tetrahedra α_3 into each other: each face of either tetrahedron lies in the polar plane of a vertex of the other tetrahedron. In terms of Cartesian coordinates, the sphere

$$x_1^2 + x_2^2 + x_3^2 = 1$$

reciprocates the tetrahedra with vertices

$$(-1, 1, 1), \quad (1, -1, 1), \quad (1, 1, -1), \quad (-1, -1, -1),$$
$$(1, -1, -1), \quad (-1, 1, -1), \quad (-1, -1, 1), \quad (1, 1, 1)$$

into each other: the face plane $x_1 + x_2 + x_3 = 1$ of the former has for its pole the vertex $(1, 1, 1)$ of the latter, and so on. The same sphere reciprocates the whole cube into the octahedron $\beta_3 = \{3, 4\}$, whose vertices

$$(\pm 1, 0, 0), \quad (0, \pm 1, 0), \quad (0, 0, \pm 1)$$

are the poles of the faces $\pm x_1 = 1$, $\pm x_2 = 1$, $\pm x_3 = 1$.

This octahedron is the 'core' or 'set-theoretic intersection' of Kepler's two reciprocal tetrahedra. We may also say that the tetrahedron, or any other regular simplex, is a *self-reciprocal* polytope: its reciprocal (with respect to a suitable concentric sphere) is a congruent figure in the centrally inverted position. The equilateral triangle $\{3\}$, like any other regular polygon $\{p\}$, is self-reciprocal, although, when p is even, the central inversion has to be replaced by a rotation.

Received 2 May 1995; revised 22 August 1995.

1991 *Mathematics Subject Classification* 51M20.

J. London Math. Soc. (2) 55 (1997) 549–557

https://doi.org/10.1112/S0024610797004833 Published online by Cambridge University Press

The star-pentagon or *pentagram* $\{\frac{5}{2}\}$ may be called *strongly* self-reciprocal because, with respect to a suitable circle (see Figure 1), its edges belong to the polar lines of *its own* vertices. More generally, the star polygon $\{p\}$ is strongly self-reciprocal whenever the rational number p has an odd numerator and an even denominator.

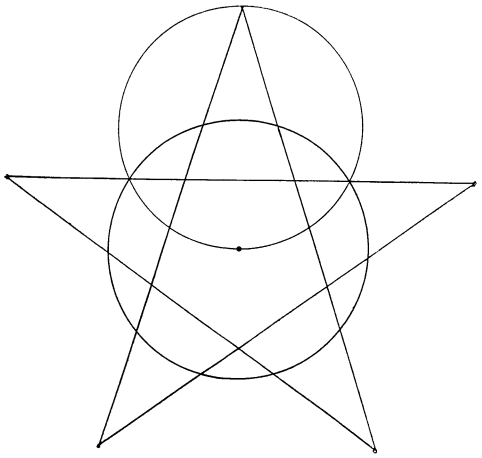


FIG. 1. The pentagram $\{\frac{5}{2}\}$ with a construction for its self-reciprocating circle.

The icosahedron $\{3, 5]$ of edge 2, the great dodecahedron $\{5, \frac{5}{2}\}$ of edge 2, the great icosahedron $\{3, \frac{5}{2}\}$ of edge 2τ $(= 1+\sqrt{5})$ and the (small) stellated dodecahedron $\{\frac{5}{2}, 5\}$ of edge 2τ [**2**, pp. 52, 96, 293] all have the same 12 vertices $(\pm\tau, \pm 1, 0)$, cyclically permuted. With respect to the sphere $x_1^2+x_2^2+x_3^2 = \tau^2$, the polar planes of these twelve points are

$$\pm\tau x_\mu \pm x_\nu = \tau^2,$$

where μ and ν are 1 and 2, or 2 and 3, or 3 and 1. These are the face-planes of the respective reciprocals: the pentagonal dodecahedron $\{5, 3\}$ of edge $2\tau^{-1}$, $\{\frac{5}{2}, 5\}$ of edge $2\tau^2$, the great stellated dodecahedron $\{\frac{5}{2}, 3\}$ of edge $2\tau^4$ and $\{5, \frac{5}{2}\}$ of edge 2τ.

These edge lengths can be computed as follows. Table I(i) of *Regular Polytopes* reveals that an icosahedron of edge 2 has circumradius ${}_0R = 5^{\frac{1}{4}}\tau^{\frac{1}{2}}$ while its reciprocal dodecahedron of edge $2l$ has inradius ${}_2R = 5^{-\frac{1}{4}}\tau^{\frac{5}{2}}l$. The product of these two radii, namely $\tau^3 l$, should be equal to τ^2, the squared radius of the reciprocating sphere; therefore $l = \tau^{-1}$. Similarly, in the other cases $l = \tau^2$ or τ^4 or τ.

2. *The four familiar polytopes* $\alpha_4, \beta_4, \gamma_4, t_1\beta_4$

To reciprocate a real polytope, we simply reverse its Schläfli symbol:

$$\{q_1, q_2, q_3\} \longrightarrow \{q_3, q_2, q_1\}.$$

The self-reciprocal cases are palindromes:

$$\{3, 3, 3\}, \quad \{3, 4, 3\}, \quad \{5, \tfrac{5}{2}, 5\}, \quad \{\tfrac{5}{2}, 5, \tfrac{5}{2}\}.$$

Two reciprocal simplexes $\alpha_4 = \{3, 3, 3\}$, of the same size, are related by the central inversion

$$(x_1, x_2, x_3, x_4) \longleftrightarrow (-x_1, -x_2, -x_3, -x_4).$$

https://doi.org/10.1112/S0024610797004833 Published online by Cambridge University Press

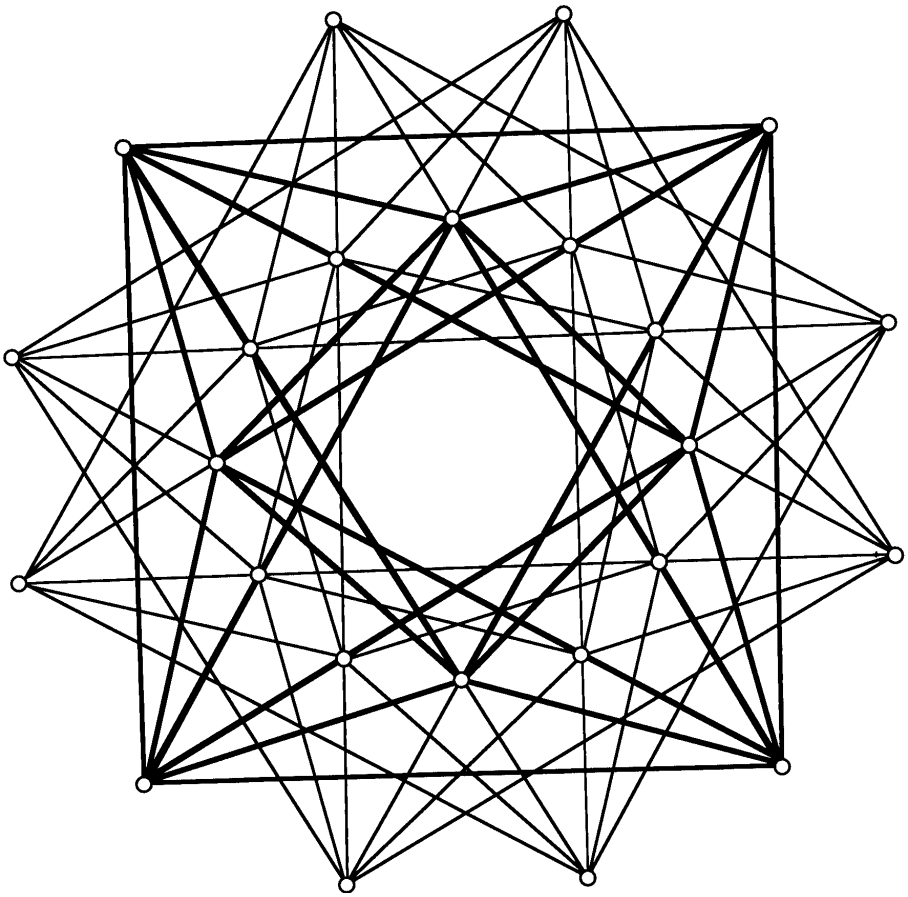


FIG. 2. The compound polytope $\{3,4,3\}[3\beta_4]2\{3,4,3\}$ or the compound polygon $3\{4\}3[3\,3\{3\}3]3\{4\}3$.

In striking contrast, two reciprocal 24-cells $t_1\beta_4 = \{3,4,3\}$ [**2**, p. 156] are related by the half-turn about a plane. More precisely, the 3-sphere $x_1^2+x_2^2+x_3^2+x_4^2 = 2\sqrt{2}$ reciprocates the 24-cell

$$(\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2})(\pm1,0,0,0)(0,\pm1,0,0)(0,0,\pm1,0)(0,0,0,\pm1)$$

into the congruent 24-cell $(\pm\sqrt{\tfrac{1}{2}},\pm\sqrt{\tfrac{1}{2}},0,0)$ permuted.

These two reciprocal 24-cells are related by the half-turn about a plane which can be located by asking which points are left invariant by the involution

$$(x_1,x_2,x_3,x_4)\longleftrightarrow\left(\frac{x_1+x_2}{\sqrt{2}},\frac{x_1-x_2}{\sqrt{2}},\frac{x_3+x_4}{\sqrt{2}},\frac{x_3-x_4}{\sqrt{2}}\right).$$

Solving the equations

$$x_1\sqrt{2} = x_1+x_2,\quad x_2\sqrt{2} = x_1-x_2,\quad x_3\sqrt{2} = x_3+x_4,\quad x_4\sqrt{2} = x_3-x_4,$$

we obtain the axial plane of the half-turn as being given by $x_1 = (1+\sqrt{2})x_2$, $x_3 = (1+\sqrt{2})x_4$. By permuting the coordinates and reversing signs, we see that any one of $3\times4\times4 = 48$ such planes will serve equally well.

For any regular polytope, with symmetry group of order g and 'Coxeter number' h [**2**, pp. 130, 223–225], the number of Petrie polygons, or of dihedral subgroups D_h,

https://doi.org/10.1112/S0024610797004833 Published online by Cambridge University Press

is $g/2h$. In the case of the 24-cell $\{3,4,3\}$ we have $g = 2\times 24^2$ and $h = 12$ [**2**, pp. 149, 246], so there are 48 Petrie polygons. Each Petrie polygon is associated with a pair of completely orthogonal planes which it shares with a corresponding Petrie polygon of the reciprocal polytope [**2**, pp. 245–247]. In particular, the half-turn about either plane of such a pair will transform the given 24-cell into its reciprocal. On any one of these 48 planes, the relevant Petrie polygon is projected orthogonally into a regular dodecagon $\{12\}$, the 24-cell itself acquires its 'dodecagonal' picture [**5**, p. 47], and each of the three 16-cells β_4 inscribed in the 24-cell (see Figure 2) acquires its 'square' picture [**2**, p. 149; **5**, p. 30].

3. *The twelve pentagonal polytopes*

The remaining regular polytopes in real 4-space have Schläfli symbols involving the number 5 or $\frac{5}{2}$ (or both). All these polytopes have the same symmetry group $[5,3,3]$.

The 600-cell $\{3,3,5\}$ of edge $2\tau^{-1}$, the icosahedral 120-cell $\{3,5,\frac{5}{2}\}$ of edge $2\tau^{-1}$, the great 120-cell $\{5,\frac{5}{2},5\}$ of edge $2\tau^{-1}$, the grand 120-cell $\{5,3,\frac{5}{2}\}$ of edge $2\tau^{-1}$, the stellated 120-cell $\{\frac{5}{2},5,3\}$ of edge 2, the great 120-cell $\{5,\frac{5}{2},3\}$ of edge 2, the great stellated 120-cell $\{\frac{5}{2},3,5\}$ of edge 2τ, the grand stellated 120-cell $\{\frac{5}{2},5,\frac{5}{2}\}$ of edge 2τ, the great icosahedral 120-cell $\{3,\frac{5}{2},5\}$ of edge 2τ and the grand 600-cell $\{3,3,\frac{5}{2}\}$ of edge 2τ [**2**, pp. 156–157, 293–294; **5**, pp. 42–46] all have the same $16+8+96 = 120$ vertices

$$(\pm 1,\pm 1,\pm 1,\pm 1),\quad (\pm 2,0,0,0),\quad (\pm\tau,\pm 1,\pm\tau^{-1},0),$$

evenly permuted. With respect to the 3-sphere $x_1^2+x_2^2+x_3^2+x_4^2 = 2\tau^2$, the polar hyperplanes of these 120 points are

$$\pm x_1\pm x_2\pm x_3\pm x_4 = 2\tau^2,\quad x_v = \pm\tau^2\quad (\text{for } v = 1,2,3,4),$$
$$\pm\tau x_1\pm x_2\pm\tau^{-1}x_3 = 2\tau^2,\ \text{etc.}$$

These are the facet-hyperplanes of the respective reciprocals: the 120-cell $\{5,3,3\}$ of edge $2\tau^{-2}$, $\{\frac{5}{2},5,3\}$ of edge 2τ, $\{5,\frac{5}{2},5\}$ of edge 2, $\{\frac{5}{2},3,5\}$ of edge $2\tau^3$, $\{3,5,\frac{5}{2}\}$ of edge 2, $\{3,\frac{5}{2},5\}$ of edge $2\tau^4$, $\{5,3,\frac{5}{2}\}$ of edge 2τ, $\{\frac{5}{2},5,\frac{5}{2}\}$ of edge $2\tau^4$, $\{5,\frac{5}{2},3\}$ of edge $2\tau^3$, and the great grand stellated 120-cell $\{\frac{5}{2},3,3\}$ of edge $2\tau^6$.

These edge lengths can be computed as follows. Table I(ii) of *Regular Polytopes* [**2**, p. 295] reveals that a $\{\frac{5}{2},5,3\}$ of edge 2 has circumradius ${}_0R = 2$ while its reciprocal $\{3,5,\frac{5}{2}\}$ of edge $2l$ has inradius ${}_3R = \tau^2 l$. The product of these two radii, namely $2\tau^2 l$, should be equal to $2\tau^2$, the squared radius of the reciprocating 3-sphere; therefore $l = 1$. The hyperplane $x_1 = \tau^2$ contains one facet of this $\{3,5,\frac{5}{2}\}$ of edge 2, namely the icosahedron $(\tau^2;\pm\tau,\pm 1,0)$, where the last three coordinates are cyclically permuted. This is the fifth of the ten cases. The complete list of values of l is

$$\tau^{-2},\quad \tau,\quad 1,\quad \tau^3,\quad 1,\quad \tau^4,\quad \tau,\quad \tau^4,\quad \tau^3,\quad \tau^6.$$

It follows that the two self-reciprocal star-polytopes, $\{5,\frac{5}{2},5\}$ and $\{\frac{5}{2},5,\frac{5}{2}\}$, are *strongly* self-reciprocal: the unit 3-sphere $x_1^2+x_2^2+x_3^2+x_4^2 = 1$ will reciprocate, into themselves, a great 120-cell of edge $2^{\frac{1}{2}}\tau^{-\frac{3}{2}}$ and a grand stellated 120-cell of edge $2^{\frac{1}{2}}\tau^{\frac{3}{2}}$.

4. *Reciprocating the generalized cube* γ_n^p

Real reciprocation has a complex analogue in which the $(n-1)$-sphere $x_1^2+\ldots+x_n^2 = c$ is replaced by the 'antisphere' $\bar{u}_1u_1+\ldots+\bar{u}_nu_n = c$ in unitary n-space. In fact, *Hermitian reciprocation* relates each point $(u_1',\ldots,u_n')$ to its Hermitian polar hyperplane $\bar{u}_1'u_1+\ldots+\bar{u}_n'u_n = c$, where c is a positive constant.

https://doi.org/10.1112/S0024610797004833 Published online by Cambridge University Press

In particular [**5**, p. 119; **9**, p. 96], we can reciprocate the complex cross-polytope $\beta_n^p = 2\{3\}2\{3\}\dots 2\{4\}p$, whose pn vertices, in terms of $\varepsilon = \exp(2\pi i/p)$, are

$$(\varepsilon^{\mu_1}, 0, \dots, 0), (0, \varepsilon^{\mu_2}, 0, \dots, 0), \dots, (0, \dots, 0, \varepsilon^{\mu_n}) \quad (\text{for } \mu_k = 0, 1, \dots, p-1),$$

to obtain the complex n-cube $\gamma_n^p = p\{4\}2\{3\}2\dots\{3\}2$, whose pn facets lie in the hyperplanes $\varepsilon^{-\mu_1}u_1 = 1, \dots, \varepsilon^{-\mu_n}u_n = 1$.

In this case $c = 1$, as it is again when we reverse the procedure by reciprocating γ_n^p, whose p^n vertices are $(\varepsilon^{\mu_1}, \dots, \varepsilon^{\mu_n})$, to obtain the β_n^p whose p^n facets lie in the hyperplanes

$$\varepsilon^{-\mu_1}u_1 + \dots + \varepsilon^{-\mu_n}u_n = 1.$$

5. *The trigonal family of complex polygons*

Along with $2\{4\}p = \beta_2^p$, there is another 'complex octagon': the Möbius–Kantor polygon $3\{3\}3$ [**3**, pp. 281, 290; **9**, p. 93] whose 8 vertices are

$$(a\eta^j, b\eta^{3j}) \quad (\text{for } j = 0, 2, 4, 6),$$
$$(b\eta^j, a\eta^{3j}) \quad (\text{for } j = 1, 3, 5, 7),$$

where $\eta = \exp(\pi i/4) = (1+i)/\sqrt{2}$, while a and b are given by

$$2a^2 = 1 + \sqrt{\tfrac{1}{3}}, \quad 2b^2 = 1 - \sqrt{\tfrac{1}{3}}$$

[**2**, p. 246]. This 'self-inscribed octagon' may be regarded as a *configuration* 8_3: it has 8 vertices and 8 edges, with 3 edges through each vertex and 3 vertices on each edge!

For simplicity, we now write the coordinates as (u, v) instead of (u_1, u_2). One of the 8 edge-lines [**6**, §2] has the equation $(u/b) - (v/a) = \sqrt{2}$; since $ab = \sqrt{\frac{1}{6}}$, this becomes $au - bv = \sqrt{\frac{1}{3}}$. Since this line is the polar of the point $(a, -b)$ with respect to the anti-circle $\bar{u}u + \bar{v}v = \sqrt{\frac{1}{3}}$, we are not surprised to find that the reciprocal $3\{3\}3$ has the 8 vertices

$$(a\eta^j, -b\eta^{3j}) \quad (\text{for } j = 0, 2, 4, 6),$$
$$(b\eta^j, -a\eta^{3j}) \quad (\text{for } j = 1, 3, 5, 7).$$

In other words, the two reciprocal $3\{3\}3$s are related by the transformation $v \to -v$.

The $8+8$ vertices of these two reciprocal Möbius-Kantor polygons, taken together, are the 16 vertices $(a\eta^j, \pm b\eta^{3j})$ (for $j = 0, 2, 4, 6$), and $(b\eta^j, \pm a\eta^{3j})$ (for $j = 1, 3, 5, 7$) of the polygon $2\{6\}3$ [**6**, §3], just as the $4+4$ vertices of two reciprocal tetrahedra $\{3, 3\}$ are the eight vertices of a cube in Kepler's *stella octangula*, $\{4, 3\}[2\{3, 3\}]\{3, 4\}$ [**2**, p. 48]. The '$\{3, 4\}$' on the right indicates that the $4+4$ face planes of the two tetrahedra are the 8 face planes of an octahedron $\{3, 4\}$. Thus, the two reciprocal $3\{3\}3$s form a *compound* polygon $2\{6\}3[2\ 3\{3\}3]3\{6\}2$ in which the $8+8$ edge-lines of the two $3\{3\}3$ are the 16 edge-lines of a $3\{6\}2$.

Instead of using the above procedure in order to reciprocate $2\{6\}3$, it is convenient to use, for the 24 vertices of $3\{6\}2$, the more symmetyrical coordinates $(a\zeta^k, \pm b\eta^{-k})$ (for $k = 1, 3, 5, \dots, 23$), and $(b\zeta^k, \pm a\eta^{-k})$ (for $k = 0, 2, 4, \dots, 22$), where $\zeta = \exp(\pi i/12)$ [**6**, §3] so that $\eta = \zeta^3$. Now 12 of the 16 edge lines are

$$\zeta^{-k}u\sqrt{2} + \eta^k v = a\sqrt{3} \quad (\text{for } k = 1, 3, 5, \dots, 23)$$

containing the 3 vertices $(b\zeta^k, a\eta^{-k})$ and $(a\zeta^{k\pm1}, b\eta^{-(k\pm1)})$, while the remaining 4 are $\eta^k v = a$ (for $k = 0, 2, 4, 6$) containing the 3 vertices $(b\zeta^{k\pm8}, a\eta^{-k})$. Reciprocation now yields the $12+4$ vertices of $2\{6\}3$ in the form $(\zeta^k\sqrt{2}, \eta^{-k})$ (for $k = 1, 3, 5, \dots, 23$), and $(0, \eta^{-k}\sqrt{3})$ (for $k = 0, 2, 4, 6$).

https://doi.org/10.1112/S0024610797004833 Published online by Cambridge University Press

These elegant coordinates for 2{6}3 were originally obtained by G. C. Shephard in his Ph.D. dissertation of 1952. They are illustrated (by projection onto the complex line $v = 0$) by a regular dodecagon with its six diameters, the last four vertices coinciding at the centre. This supersedes the ill-conceived Figure 24 of [**3**, p. 294].

6. *The tetragonal family of complex polygons*

The 24 vertices of 4{3}4 have the 'even-odd' coordinates $(a\zeta^k, b\zeta^{5k})$ (for $k = 0, 2, 4, \dots, 22$), and $(b\zeta^k, a\zeta^{5k})$ (for $k = 1, 3, 5, \dots, 23$) [**6**, §4]. They lie by fours on 24 edge-lines

$$\zeta^{-k}\frac{u}{a} - \zeta^{-5k}\frac{v}{b} = \sqrt{3} \quad (\text{for } k \text{ even}), \quad \zeta^{-k}\frac{u}{b} - \zeta^{-5k}\frac{v}{a} = \sqrt{3} \quad (\text{for } k \text{ odd})$$

or, since $ab = \sqrt{\frac{1}{6}}$, we have $b\zeta^{-k}u - a\zeta^{-5k}v = \sqrt{\frac{1}{2}}$ for k even, and $a\zeta^{-k}u - b\zeta^{-5k}v = \sqrt{\frac{1}{2}}$ for k odd. Thus the reciprocal 4{3}4 has the 24 'odd-even' coordinates $(b\zeta^k, -a\zeta^{5k})$ for k even, and $(a\zeta^k, -b\zeta^{5k})$ for k odd: 4{3}4 is self-reciprocal, but not strongly self-reciprocal.

The 2×24 vertices of two reciprocal 4{3}4s, taken together [**6**, §5], are the 48 vertices of the polygon 2{6}4 and we have a second instance of a compound polygon:

$$2\{6\}4[2\ 4\{3\}4]4\{6\}2.$$

Another complex 24-gon 3{4}3 [**6**, §6] has, for its vertices, the subtly altered 'even-odd' coordinates $(a\zeta^k, b\zeta^{-5k})$ for $k = 0, 2, 4, \dots, 22$, and $(b\zeta^k, a\zeta^{-5k})$ for $k = 1, 3, 5, \dots, 23$. These 24 points lie by threes on 24 edge-lines

$$\zeta^{-k}\frac{u}{a} + \zeta^{5k}\frac{v}{b} = 2 \quad (\text{for } k \text{ even}), \quad \zeta^{-k}\frac{u}{b} + \zeta^{5k}\frac{v}{a} = 2 \quad (\text{for } k \text{ odd})$$

or, more relevantly (since $ab = \sqrt{\frac{1}{6}}$), we have $b\zeta^{-k}u + a\zeta^{5k}v = \sqrt{\frac{2}{3}}$ for k even, and $a\zeta^{-k}u + b\zeta^{5k}v = \sqrt{\frac{2}{3}}$ for k odd. Thus the reciprocal 3{4}3 has the 24 'odd-even' coordinates $(b\zeta^k, a\zeta^{-5k})$ for k even, and $(a\zeta^k, b\zeta^{-5k})$ for k odd, and is derived by a simple rotation. Again we have a self-reciprocal polygon which is not strongly self-reciprocal. The vertices of these two reciprocal 3{4}3s, taken together [**6**, §7], are the 48 vertices of 2{8}3 and we have a third compound polygon

$$2\{8\}3[2\ 3\{4\}3]3\{8\}2.$$

There is actually a fourth (see Figure 2): 3{4}3[3 3{3}3]3{4}3. And we may combine this with the third, to obtain a fifth: 2{8}3[6 3{3}3]3{8}2. Remarkably [**6**, end of §6], there is even a sixth: 3{4}3[3 3{3}3]3{4}3.

7. *The pentagonal family of complex polygons*

To deal with the most complicated complex polygons, we must change the meaning of a and b, using instead four new numbers a, b, c, d [**2**, pp. 247–248] which are given by

$$2a^2 = 1 + 3^{-\frac{1}{2}}5^{-\frac{1}{4}}\tau^{\frac{3}{2}}, \quad 2b^2 = 1 + 3^{-\frac{1}{2}}5^{-\frac{1}{4}}\tau^{-\frac{3}{2}},$$
$$2c^2 = 1 - 3^{-\frac{1}{2}}5^{-\frac{1}{4}}\tau^{-\frac{3}{2}}, \quad 2d^2 = 1 - 3^{-\frac{1}{2}}5^{-\frac{1}{4}}\tau^{\frac{3}{2}}.$$

We use also $\xi = \exp(\pi i/30)$.

https://doi.org/10.1112/S0024610797004833 Published online by Cambridge University Press

The 120 vertices of the self-reciprocal polygon 5{3}5 [**6**, §9] have the 'even-odd' coordinates

$$(a\xi^k, d\xi^{11k}), \quad (d\xi^k, -a\xi^{11k}) \quad (\text{for } k = 0, 2, \dots, 58),$$
$$(b\xi^k, c\xi^{11k}), \quad (c\xi^k, -b\xi^{11k}) \quad (\text{for } k = 1, 3, \dots, 59),$$

and lie by fives on 120 edge-lines such as

$$\frac{u}{b} + \frac{v}{c} = 3^{\frac{1}{2}}$$

or, since $bc = 3^{-\frac{1}{2}}5^{-\frac{1}{4}}\tau^{\frac{1}{2}}$ and $ad = 3^{-\frac{1}{2}}5^{-\frac{1}{4}}\tau^{-\frac{1}{2}}$, we have $cu + bv = 5^{-\frac{1}{4}}\tau^{\frac{1}{2}}$. The complete list of 120 edge-lines is

$$cu\xi^{-k} + bv\xi^{-11k} = 5^{-\frac{1}{4}}\tau^{\frac{1}{2}} \quad (\text{for } k \text{ even}),$$
$$du\xi^{-k} + av\xi^{-11k} = 5^{-\frac{1}{4}}\tau^{\frac{1}{2}} \quad (\text{for } k \text{ odd}),$$
$$au\xi^{-k} - dv\xi^{-11k} = 5^{-\frac{1}{4}}\tau^{\frac{1}{2}} \quad (\text{for } k \text{ odd}),$$
$$bu\xi^{-k} - cv\xi^{-11k} = 5^{-\frac{1}{4}}\tau^{\frac{1}{2}} \quad (\text{for } k \text{ even}).$$

Thus the reciprocal 5{3}5 has the 120 vertices

$$(c\xi^k, b\xi^{11k}) \quad (\text{for } k \text{ even}), \qquad (a\xi^k, -d\xi^{11k}) \quad (\text{for } k \text{ odd}),$$
$$(d\xi^k, a\xi^{11k}) \quad (\text{for } k \text{ odd}), \qquad (b\xi^k, -c\xi^{11k}) \quad (\text{for } k \text{ even}),$$

which are derived by reversing the parity conditions on k and multiplying the second coordinate by -1. It follows that the self-reciprocal 5{3}5 is not strongly self-reciprocal.

The 2×120 vertices of two reciprocal 5{3}5s, taken together [**6**, §10], are the 240 vertices of the polygon 2{6}5 and we have a seventh instance of a compound polygon:

$$2\{6\}5[2\ 5\{3\}5]5\{6\}2.$$

Another complex 120-gon, 3{5}3 [**6**, §12] has the vertices

$$(a\xi^k, d\xi^{-11k}), \quad (d\xi^k, -a\xi^{-11k}) \quad (\text{for } k = 0, 2, \dots, 58),$$
$$(b\xi^k, c\xi^{-11k}), \quad (c\xi^k, -b\xi^{-11k}) \quad (\text{for } k = 1, 3, \dots, 59),$$

lying by threes on 120 edge-lines such as

$$\frac{u}{c} + \frac{v}{b} = 5^{\frac{1}{4}}\tau^{\frac{1}{2}}$$

or, using the known value for bc, we have $bu + cv = 3^{-\frac{1}{2}}\tau$. The complete list of 120 edge-lines is

$$bu\xi^{-k} + cv\xi^{11k} = 3^{-\frac{1}{2}}\tau \quad (\text{for } k \text{ even}), \qquad du\xi^{-k} - av\xi^{11k} = 3^{-\frac{1}{2}}\tau \quad (\text{for } k \text{ odd}),$$
$$au\xi^{-k} + dv\xi^{11k} = 3^{-\frac{1}{2}}\tau \quad (\text{for } k \text{ odd}), \qquad cu\xi^{-k} - bv\xi^{11k} = 3^{-\frac{1}{2}}\tau \quad (\text{for } k \text{ even}).$$

Thus the reciprocal 3{5}3 has the 120 'odd-even' vertices $(b\xi^k, c\xi^{-11k}), (c\xi^k, -b\xi^{-11k})$ (for k even), and $(a\xi^k, d\xi^{-11k}), (d\xi^k, -a\xi^{-11k})$ (for k odd), and is derived by a rotation (interchanging even and odd values of k). Again we have a polygon which is not strongly self-reciprocal. The 2×120 vertices of the two reciprocal 3{5}3s, taken together, are the 240 vertices of 2{10}3 [**6**, §13] and we have our final compound polygon

$$2\{10\}3[2\ 3\{5\}3]3\{10\}2.$$

https://doi.org/10.1112/S0024610797004833 Published online by Cambridge University Press

While considering complex polygons, it might seem natural for us to include the two self-reciprocal *star* polygons, $3\{\frac{5}{2}\}3$ and $5\{\frac{5}{2}\}5$, which have the same vertices as $3\{5\}3$ and $5\{3\}5$, respectively, but different edges [**6**, pp. 50, 51, 114, 179]. The necessary computations (which are omitted) are analogous to those described above, and lead to the same conclusion: these polygons again are not strongly self-reciprocal. Just as the octagram $\{\frac{8}{3}\}$ shares its 8 vertices with one octagon $\{8\}$ and its 8 edge-lines with another, so $3\{\frac{5}{2}\}3$ shares its 120 vertices with one $3\{5\}3$ and its 120 edge-lines with another. Likewise $5\{\frac{5}{2}\}5$ shares its 120 vertices with one $5\{3\}5$ and its 120 edge-lines with another.

8. *The Hessian polyhedron*

We now leave the unitary plane in favour of unitary 3-space, in which the most interesting polytope is the *Hessian polyhedron*

$$3\{3\}3\{3\}3$$

[**1**, p. 460; **5**, pp. 119–124; **7**, pp. 240–242; **8**, pp. 19–23; **9**, p. 94] having the 27 vertices

$$(0, \omega^\mu, -\omega^\nu), \quad (-\omega^\nu, 0, \omega^\mu), \quad (\omega^\mu, -\omega^\nu, 0),$$

where $\omega = \exp(2\pi/3) = \frac{1}{2}(-1+i\sqrt{3})$, so that $\omega^3 = 1$ and $\omega - \omega^2 = i\sqrt{3}$. Its 27 faces are Möbius–Kantor polygons $3\{3\}3$, lying in the planes

$$-\omega^{-\mu}v + \omega^{-\nu}w = 1, \quad \omega^{-\nu}u - \omega^{-\mu}\omega = 1, \quad -\omega^{-\mu}u + \omega^{-\nu}v = 1.$$

For instance, the polygon with vertices $(0, \omega, -\omega^2), (0, \omega^2, -\omega), (-\omega^\lambda, 0, 1)$, $(\omega^\lambda, -1, 0)$ (for $\lambda = 0, 1, 2$) lies in the plane $-v + w = 1$ [**3**, p. 292]. With respect to the anti-sphere $\bar{u}u + \bar{v}v + \bar{w}w = 1$, the poles of these 27 planes are the 27 vertices $(0, -\omega^\mu, \omega^\nu)$, $(\omega^\nu, 0, -\omega^\mu)$, $(-\omega^\mu, \omega^\nu, 0)$ of the reciprocal Hessian polyhedron, which is derived by the central inversion (reversal of signs). Thus the Hessian polyhedron again is not strongly self-reciprocal. The 2×27 vertices of the two reciprocal $3\{3\}3\{3\}3$s, taken together, are the 54 vertices $(0, \pm\omega^\mu, \mp\omega^\nu)$, $(\mp\omega^\nu, 0, \pm\omega^\mu)$, $(\pm\omega^\mu, \mp\omega^\nu, 0)$ of the 2-symmetric (that is, centrally symmetric) polyhedron $2\{4\}3\{3\}3$ [**5**, pp. 125, 127] and we have a compound polyhedron:

$$2\{4\}3\{3\}3[2\ 3\{3\}3\{3\}3]3\{3\}3\{4\}2.$$

9. *The Witting polytope*

Unitary 4-space admits the spectacular *Witting polytope*

$$3\{3\}3\{3\}3\{3\}3$$

[**1**, p. 480; **5**, pp. 132–134; **7**, p. 98; **8**, pp. 24–26; **9**, p. 95], which has the 240 vertices $(0, \pm\omega^\mu, \mp\omega^\nu, \pm\omega^\lambda)$, $(\mp\omega^\nu, 0, \pm\omega^\mu, \pm\omega^\lambda)$, $(\pm\omega^\mu, \mp\omega^\nu, 0, \pm\omega^\lambda)$, $(\pm\omega^\lambda, \pm\omega^\mu, \pm\omega^\nu, 0)$, $(\pm i\omega^\lambda\sqrt{3}, 0, 0, 0)$, $(0, \pm i\omega^\lambda\sqrt{3}, 0, 0)$, $(0, 0, \pm i\omega^\lambda\sqrt{3}, 0)$, $(0, 0, 0, \pm i\omega^\lambda\sqrt{3})$, where λ, μ, ν take independently the values $0, 1, 2$. Its 240 facets $3\{3\}3\{3\}3$ lie in the hyperplanes

$$\pm\omega^\mu v \mp \omega^\nu w \pm \omega^\lambda x = i\sqrt{3}, \quad \mp\omega^\nu u \pm \omega^\mu w \pm \omega^\lambda x = i\sqrt{3},$$
$$\pm\omega^\mu u \mp \omega^\nu v \pm \omega^\lambda x = i\sqrt{3}, \quad \pm\omega^\lambda u \pm \omega^\mu v \pm \omega^\nu w = i\sqrt{3},$$
$$u = 1, \quad v = 1, \quad w = 1, \quad x = 1.$$

https://doi.org/10.1112/S0024610797004833 Published online by Cambridge University Press

For instance, the last hyperplane contains the Hessian polyhedron

$$(0, \omega^{\mu}, -\omega^{\nu}, 1)(-\omega^{\nu}, 0, \omega^{\mu}, 1)(\omega^{\mu}, -\omega^{\nu}, 0, 1),$$

while the hyperplane $u+v+w = \mathrm{i}\sqrt{3}$ (given by $\lambda = \mu = \nu = 0$) contains the 27 vertices

$$(0, \omega, -\omega^2, \omega^{\lambda}), \quad (0, -\omega^2, \omega, -\omega^{\lambda}), \quad (-\omega^2, 0, \omega, \omega^{\lambda}), \quad (\omega, 0, -\omega^2, -\omega^{\lambda}),$$
$$(\omega, -\omega^2, 0, \omega^{\lambda}), \quad (-\omega^2, \omega, 0, -\omega^{\lambda}), \quad (1, \omega, \omega, 0), \quad (\omega, 1, \omega, 0), \quad (\omega, \omega, 1, 0),$$
$$(-1, -\omega^2, -\omega^2, 0), \quad (-\omega^2, -1, -\omega^2, 0), \quad (-\omega^2, -\omega^2, -1, 0),$$
$$(\mathrm{i}\sqrt{3}, 0, 0, 0), \quad (0, \mathrm{i}\sqrt{3}, 0, 0), \quad (0, 0, \mathrm{i}\sqrt{3}, 0)$$

of another such facet of the Witting polytope.

With respect to the anti-sphere $\bar{u}u+\bar{v}v+\bar{w}w+\bar{x}x = \sqrt{3}$, the poles of those 240 hyperplanes are the same 240 vertices with all their coordinates multiplied by i: the reciprocal $3\{3\}3\{3\}3\{3\}3$ is derived by an imaginary dilatation. In the triacontagonal projection [**5**, frontispiece], this transformation appears as a quarter-turn or, since the whole drawing is invariant for rotation through $\pi/30$, it appears as a rotation through $\pi/60$, reciprocating the peripheral $\{30\}$. Thus two reciprocal Witting polytopes are related by this imaginary dilatation. Again we have a self-reciprocal polytope which, though '6-symmetric' (invariant for multiplication by $-\omega$), is not strongly self-reciprocal.

We conclude that, among all the regular polytopes, real and unreal, the only cases of strong self-reciprocity are real: the polygons $\{\substack{\text{odd}\\\text{even}}\}$ and two of the sixteen regular polytopes in 4 dimensions, the great 120-cell $\{5, \frac{5}{2}, 5\}$ and the grand stellated 120-cell $\{\frac{5}{2}, 5, \frac{5}{2}\}$.

Acknowledgement. The author thanks Professor G. C. Shephard for drawing Figure 2.

References

1. H. S. M. Coxeter, 'The polytope 2_{21}, whose twenty-seven vertices correspond to the lines on the general cubic surface', *Amer. J. Math.* 62 (1940) 457–486.
2. H. S. M. Coxeter, *Regular polytopes*, 3rd edition (Dover, New York, 1973).
3. H. S. M. Coxeter, 'The Pappus configuration and the self-inscribed octagon', *Nederl. Akad. Wetensch. Proc. Ser.* A 80 (1977) 256–300.
4. H. S. M. Coxeter, 'Surprising relationships among unitary reflection groups', *Proc. Edinburgh Math. Soc.* 27 (1984) 185–194.
5. H. S. M. Coxeter, *Regular complex polytopes*, 2nd edition (Cambridge University Press, 1991).
6. H. S. M. Coxeter, J. Chris Fisher and J. B. Wilker, 'Coordinates for the regular complex polygons', *J. London Math. Soc.* (2) 55 (1997) 527–548.
7. H. S. M. Coxeter and G. C. Shephard, 'Portraits of a family of complex polytopes', *Leonardo* 25 (1992) 239–244; 26 (1993) 98, 191.
8. Michele Emmer, *The visual mind* (MIT Press, Cambridge, Mass., 1993).
9. G. C. Shephard, 'Regular complex polytopes', *Proc. London Math. Soc.* (3) 2 (1952) 82–97.

Department of Mathematics
University of Toronto
Toronto
Ontario M5S 1A1
Canada

https://doi.org/10.1112/S0024610797004833 Published online by Cambridge University Press