The isotomic disc and the isogonal disc conjecture

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The isogonal disc conjecture

Let ABC be a triangle with incentre I, circumcentre O, orthocentre H, centroid G and symmedian point K. In standard notation, the triangle ABC has sides a, b, c, semiperimeter s, circumradius R and inradius r. Euler's well-known result that the incentre I is always within the orthocentroidal disc D_{GH}, the disc with diameter GH, is probably the first result about the location of the incentre in some disc formed by triangle centres. Investigating the location of the incentre I in other discs, in [1] we proved that the incentre is interior to the Brocard disc D_{OK}, that is, the disc with diameter OK. The disc is named after the French military meteorologist and geometer Henri Brocard (1845-1922), known in triangle geometry for the Brocard points and the Brocard angle (see [2]).

In [3] we proved that the incentre I lies on the symmedicentroidal disc D_{GK}, the disc with diameter GK (see Figure 1). The symmedian point K, sometimes also called the Lemoine point after the French civil engineer and geometer Emile Lemoine (1840-1912), is the point of concurrency of the three symmedians. A symmedian is a cevian which is a reflection of the median in the corresponding angle bisector. In general, for three concurrent cevians at P, their reflections around the corresponding angle bisectors are also concurrent at a point P^*, called the isogonal conjugate of P (see [2]). Clearly the incentre I is self-isogonal. Since G and are isogonal conjugates and I is interior to the disc D_{GK}, in [3] we made the following conjecture.

Isogonal disc conjecture

For any interior point P of a triangle ABC with isogonal conjugate P^*, the incentre I is within the disc with diameter PP^*.

This conjecture seems to be as far from a purely synthetic geometric proof as ever. One computer proof with the use of *Mathematica* is given in [4].

Isotomic conjugacy and the isotomic disc conjecture

If the point P is defined by the point of concurrency of the cevians AL, BM, CN and if L', M', N' are the reflections of L, M, N in the midpoints of the sides BC, CA, AB, respectively, then the cevians AL', BM', CN' are concurrent at P', the isotomic conjugate of P (see [2]). Clearly the centroid G is self-isotomic.

The Nagel point, N, of a triangle is the point of concurrency of the Nagel cevians, connecting each vertex to the point of contact of the corresponding excircle and the opposite side. It is named after the German geometer Christian Heinrich von Nagel (1803-1882). The three Gergonne cevians connecting each vertex of a triangle to the point of contact of the incircle and the opposite side are concurrent at the Gergonne point Γ (usual

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notation is Ge), named after the French geometer Joseph Diaz Gergonne (1771-1859). Interesting inequalities for the Nagel and Gergonne cevians are given in [5] and [6].

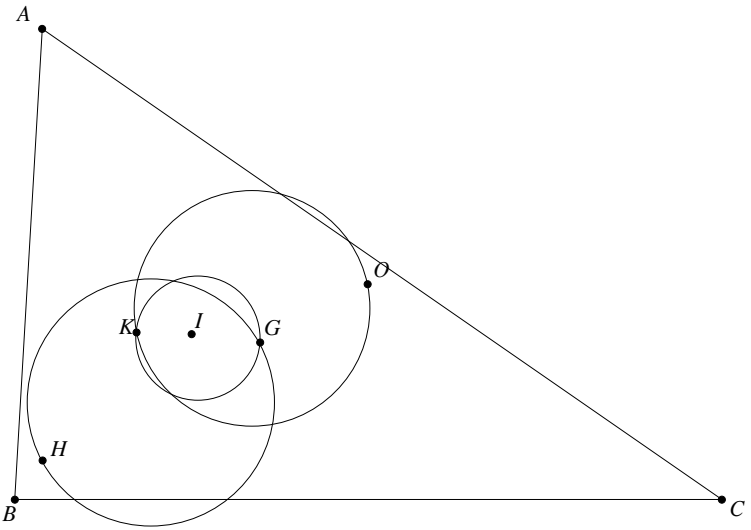


FIGURE 1: The three discs. $I \in D_{GH} \cap D_{OK} \cap D_{GK}$

The Nagel point N and the Gergonne point Γ are isotomic conjugates (see [2]). For the disc $D_{N\Gamma}$ with diameter $N\Gamma$ which we call the Nagel-Gergonne disc, we have the following result.

Theorem 1: The centroid G lies within the Nagel-Gergonne disc $D_{N\Gamma}$.

Proof: For the centroid G to be in the Nagel-Gergonne disc $D_{N\Gamma}$, it is necessary and sufficient the following inequality to hold:

$$N\Gamma^2 \geqslant GN^2 + GT^2. \tag{1}$$

For the proof we use barycentric coordinates. It is well known that the distance between the centroid G and the incentre I is $9GI^2 = s^2 - 16Rr + 5r^2$ (see [7]). From the collinearity of the incentre I, centroid G and the Nagel point N and the relation $IG : GN = 1 : 2$ [2], we obtain

$$9GN^2 = 4\left(s^2 - r(16R + 5r)\right). \tag{2}$$

To continue with the proof we need the distances for $N\Gamma$ and $G\Gamma$ given in the next two theorems.

Theorem 2: The distance between the centroid G and the Gergonne point Γ is given by

$$9G\Gamma^2 = \frac{4\left[(2R - r)(2R + 5r)s^2 - r(4R + r)^3\right]}{(4R + r)^2}. \tag{3}$$

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Proof: We use barycentric calculus and the important formula for the distance PQ between two points, given the areal coordinates of the one point $P = (x : y : z)$, that is $x + y + z = 1$, and the distances of the other point Q to the vertices of the triangle. Then we have

$$PQ^2 = xQA^2 + yQB^2 + zQC^2 - yza^2 - zxb^2 - xyc^2. \tag{4}$$

The areal coordinates of the Gergonne point Γ are (see [2]):

$$\Gamma = \left(\frac{(s-b)(s-c)}{D} : \frac{(s-c)(s-a)}{D} : \frac{(s-a)(s-b)}{D}\right),$$

where $D = \sum (s-b)(s-c) = r(4R+r)$. The distance AG, where $9AG^2 = 2(b^2+c^2) - a^2$, follows from the well-known length of the median. Using (4), $G\Gamma^2$ is

$$G\Gamma^2 = \sum xAG^2 - \sum yza^2.$$

To find the part $\sum yza^2$, we use the identity $\sum a^2(s-a) = 4rs(R+r)$ which follows from the well-known identities for the sum of the squares of the sides $\sum a^2 = 2(s^2 - r(4R+r))$ and the sum of the cubes of the sides $\sum a^3 = 2s(s^2 - 3r(2R+r))$. Then by Heron's formula for the area $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ and $\Delta = rs$ it follows that

$$\sum yza^2 = \frac{1}{r^2(4R+r)^2} \sum a^2(s-a)^2(s-b)(s-c) = \frac{4rs^2(R+r)}{(4R+r)^2}.$$

Utilising the identity (see [8])

$$\sum a^2(s-b)(s-c) = 4rs^2(R-r),$$

we obtain

$$\begin{aligned} 9G\Gamma^2 &= \frac{1}{r(4R+r)} \sum \left[2(b^2+c^2) - a^2\right](s-b)(s-c) - 9\sum yza^2 \\ &= 4(s^2 - r(4R+r)) - 3\cdot\frac{4rs^2(R-r)}{r(4R+r)} - 9\cdot\frac{4rs^2(R+r)}{(4R+r)^2} \\ &= \frac{4}{(4R+r)^2}\left[(4R+r)^2 - 3(R-r)(4R+r) - 9r(R+r)\right]s^2 - 4r(4R+r) \\ &= \frac{4\left[(2R-r)(2R+5r)s^2 - r(4R+r)^3\right]}{(4R+r)^2}. \end{aligned}$$

The last distance that we need to find is $N\Gamma$.

Theorem 3: The distance between the Nagel point N and the Gergonne point Γ is given by

$$N\Gamma^2 = 16R\left(\frac{s^2(R+r)}{(4R+r)^2} - r\right). \tag{5}$$

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Proof: By (4)

$$N\Gamma^2 = \sum xAN^2 - \sum yza^2$$

where $(x : y : z)$ are the areal coordinates of the Gergonne point and the part $\sum yza^2$ is computed above. The distance from the Nagel point to the vertices can be found similarly as in [9]

$$AN^2 = a^2 - \frac{4a}{s}(s-b)(s-c). \tag{6}$$

The triangle identity used in the proof (see [9] p. 25) is:

$$\sum a(s-b)^2(s-c)^2 = 2r^2s\left(8R^2 + 2Rr - s^2\right). \tag{7}$$

We find by (6) and (7):

$$N\Gamma^2 = \sum xAN^2 - \sum yza^2$$

$$= \frac{1}{r(4R+r)}\sum a^2(s-b)(s-c) - \frac{4}{rs(4R+r)}\sum a(s-b)^2(s-c)^2 - \sum yza^2$$

$$= \frac{4s^2(R-r) - 4\cdot 2r\left(8R^2+2Rr-s^2\right)}{4R+r} - \frac{4rs^2(R+r)}{(4R+r)^2}$$

$$= \frac{4s^2}{(4R+r)^2}\left[(R-r)(4R+r) + 2r(4R+r) - (R+r)\right] - 16Rr$$

$$= 16R\left(\frac{s^2(R+r)}{(4R+r)^2} - r\right).$$

We are now ready to prove the inequality (1), or equivalently $9N\Gamma^2 \geqslant 9GN^2 + 9N\Gamma^2$, which multiplied by $(4R+r)^2$ and by (2), (3) and (5) is

$$144R\left((R+r)s^2 - r(4R+r)^2\right)$$
$$\geqslant 4\left(s^2 - r(16R+5r)\right) + 4\left((2R-r)(2R+5)s^2 - r(4R+r)^3\right).$$

The last inequality can be rewritten as

$$\left(4R^2 + 5Rr + r^2\right)s^2 \geqslant r(4R+r)^3$$

and this is $s^2 \geqslant \dfrac{r(4R+r)^2}{R+r}$. We use the fundamental Gerretsen's inequality $s^2 \geqslant r(16R-5r)$ (see [7]). Then

$$s^2 \geqslant r(16R-5r) \geqslant \frac{r(4R+r)^2}{R+r}.$$

The last inequality is true since

$$(16R-5R)(R+r) - (4R+r)^2 = 3r(R-2x) \geqslant 0$$

by Euler's inequality $R \geqslant 2r$ [10]. The proof of Theorem 1 is complete.

The isotomic disc conjecture

The fact that the centroid G lies in the Nagel-Gergonne disc $D_{N\Gamma}$ and the two centres are isotomic conjugates makes the following conjecture plausible.

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Isotomic disc conjecture: For any interior point P of a triangle ABC with isotomic conjugate Q, the centroid G is within the disc with diameter PQ.

Although analogous to the isogonal disc conjecture, this is not true. A counter-example is provided by the nine-point centre X_5 and its isotomic conjugate X_{95}, see Figure 2.

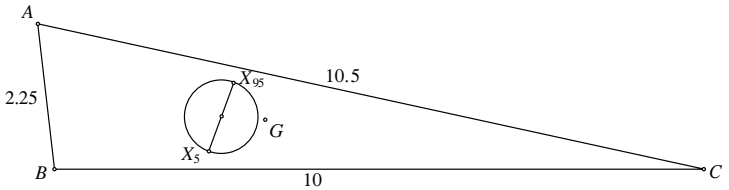


FIGURE 2: Counterexample to the isotomic disc conjecture

References

1. M. Lukarevski, J. A. Scott, On the Brocard disc, *Math. Gaz*. **105** (July 2021) pp. 327-329.
2. G. Leversha, *The geometry of the triangle*, UKMT (2013).
3. M. Lukarevski, J. A. Scott, Three discs for the incentre, *Math. Gaz*. **106** (July 2022) pp. 322-325.
4. S. Rabinowitz, Proof of the isogonal disc conjecture, *International Journal of Computer Discovered Mathematics* (IJCDM) **8** (2023) pp. 75-77.
5. M. Lukarevski, D. S. Marinescu, An inequality for cevians and applications, *Elem. Math*. **75**(4) (2020), pp. 166-171.
6. M. Lukarevski, D. S. Marinescu, Isogonal cevians and inequalities for quotients of cevians, *Elem. Math*. **78**(2) (2023), pp. 62-67.
7. M. Lukarevski, An alternate proof of Gerretsen's inequalities, *Elem. Math*. **72**(1) (2017) pp. 2-8.
8. M. Lukarevski, Proximity of the incentre to the inarc centres, *Math. Gaz*. **105** (March 2021) pp. 142-147.
9. M. Lukarevski, Three discs for the Mittenpunkt, *Math. Gaz*. **108** (July 2024) pp. 209-218.
10. M. Lukarevski, Exradii of the triangle and Euler's inequality, *Math. Gaz*. **101** (March 2017) p. 123.

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