Ergod. Th. & Dynam. Sys. (2004), **24**, 1425–1441
DOI: 10.1017/S0143385703000725
© *2004 Cambridge University Press*
Printed in the United Kingdom

Further rigidity properties of conformal Anosov systems

R. DE LA LLAVE
Department of Mathematics, University of Texas, Austin, TX 78712, USA
(e-mail: llave@math.utexas.edu)

(*Received* 26 *September* 2002 *and accepted in revised form* 13 *November* 2003)

To the memory of M. R. Herman

Abstract. We consider systems that have some hyperbolicity behavior and which preserve conformal structures on the stable and unstable bundles. We show that two such systems that are topologically conjugate are smoothly conjugate. This is somewhat more general than a conjecture of the author in 2002. Related results have also been obtained by Kalinin and Sadovskaia.

1. *Introduction*

The goal of this paper is to establish some rigidity properties of systems with some hyperbolicity behavior which preserve conformal structures along stable and unstable foliations.

The main result is valid only for Anosov diffeomorphisms and flows but many of the other results are valid under less restrictive assumptions on the hyperbolicity of the system. The only obstruction to obtaining the full result for partially hyperbolic systems is a technical regularity result, which we believe is true and within reach. We discuss these generalizations in Remark 3.1.

The main result of this paper is the following theorem.

THEOREM 1.1. *Let f, g be C^r transitive Anosov diffeomorphisms (respectively X_t, Y_t be transitive Anosov flows) of a compact manifold M of dimension d. $r \in \mathbb{N} + [0, 1)$, $r > 1$.*

Assume that:

(i) $\dim(E^s), \dim(E^u) \geq 2$;

(ii) *there exists a homeomorphism h such that*

$$f \circ h = h \circ g \tag{1}$$

(respectively

$$X_t \circ h = h \circ Y_t); \tag{2}$$

(iii) *each of the diffeomorphisms f, g preserve C^1 conformal structures on the stable and unstable bundles.*

https://doi.org/10.1017/S0143385703000725 Published online by Cambridge University Press

Then

$$h \in C^{r'} \tag{3}$$

where

$$r = \begin{cases} r - \epsilon & \text{if } r \in \mathbb{N}, \\ r & \text{if } r \in \mathbb{N} + (0,1). \end{cases}$$

Remark 1.2. We note that assumption (iii) can be weakened in three directions.

(a) In [**dlL02b**], it is shown that if $M = \mathbb{T}^d$—or has some triviality properties of the tangent bundle—the fact that f preserves a smooth conformal structure on the stable and unstable bundles holds when f is sufficiently C^1-close to a linear automorphism and

$$\begin{gathered} f^n(x) = x \implies Df^n(x)|_{E^s_x} = (\mu^s_x)^n \operatorname{Id}_{E^s_x}, \\ Df^n(x)|_{E^u_x} = (\mu^u_x)^n \operatorname{Id}_{E^u_x}. \end{gathered} \tag{4}$$

Note that in (4) we do not assume that the μ_x are independent of x.
One of the consequences of Theorem 1.1 is that if (4) holds and f is C^1-close to a linear automorphism of the torus, then, we have that indeed that μ^s_x are independent of x.

(b) In [**dlLS**], it is shown that the preservation of a continuous conformal structure is implied by the preservation of a structure that belongs to L^p with p high enough and that f is C^1 close to a measure-preserving diffeomorphism. Similarly, in [**dlLS**], one shows that the regularity of the conformal structure can be bootstrapped until it is roughly $r - 1$.

(c) The papers [**Sad02, KS03**] do not require that the systems are transitive. One of the first steps of these papers is to show that Anosov diffeomorphisms that preserve conformal structures are volume preserving.

If we used the arguments from these papers for the first part of the arguments, the assumption that the systems are transitive could be eliminated for diffeomorphisms. We have chosen not to follow this route to have more arguments documented in the literature.

We point out, however that we do not use much transitivity. The main place is in the application of the theory of Sinai–Ruelle–Bowen (SRB) measures, which includes transitivity in its assumptions. It is possible that this use of transitivity could be eliminated by considering the arguments in the basins of attraction of the different ergodic components.

Remark 1.3. Theorem 1.1 remains valid if assumption (i) is replaced by $\dim E^s = 1$, $\dim E^u \geq 2$ and, whenever $g^n(x) = x$ and, hence, $f^n \circ h(x) = h(x)$, we have

$$\operatorname{Spec}(Dg^n(x)|_{E^s,x}) = \operatorname{Spec}((Df^n) \circ h(x)|_{E^s_{h(x)}}).$$

We know that $\dim E^s = 1$ implies that the Anosov system is transitive (see [**New70**]). Therefore, for diffeomorphisms, the assumption of transitivity becomes redundant if one modifies assumption (i) as indicated. For flows, the assumption of transitivity is not implied by the manifolds being one-dimensional [**FW80**].

We develop the proof of Theorem 1.1 under the assumptions indicated in this remark in §3.1.

https://doi.org/10.1017/S0143385703000725 Published online by Cambridge University Press

The main technique of this paper is a sequence of arguments bootstrapping the regularity of the conjugacies. Many of the arguments have a similar flavor. We derive formulas for the increments of the derivatives along the (un)stable directions. These formulas will hold in a rather weak sense. Nevertheless, by examining them, we will discover that the derivatives—or the increments—hold in a somewhat stronger sense. Eventually, we will reach a formula so strong that it can be differentiated term by term. Therefore, we establish that the conjugacy is smooth when restricted to the stable and unstable foliations.

Once we have that the conjugacy is smooth along the stable and unstable foliations, we can appeal to a regularity result [**dlLMM86**, Lemma 2.5] that states that functions smooth along the stable and unstable manifold are actually smooth. As is usual in bootstrap arguments, the more complicated parts happen at the beginning, when we are allowed to assume fewer properties of the function. We have indicated the milestones that we will be accomplishing in the titles of the subsections.

Results very similar to Theorem 1.1 have been obtained by more geometric methods in the remarkable papers [**Sad02, KS03**]. The geometric argument in [**Sad02, KS03**] has several advantages with respect to the argument presented here. In particular, in the Anosov case (using transitivity and the fact that the dimension of the stable and unstable manifolds are at least three), the papers show [**Sad02, KS03**] that the stable and unstable foliations are smooth and that the manifold admits an affine connection. Hence, the papers [**Sad02, KS03**] can use the results of [**Kan93, BFL92, Yue96, BCG95**] to obtain, just from the preservation of the conformal structures along the stable and unstable manifolds (an assumption that looks almost local), global information about the manifold.

Besides, being a different method, the present method has the advantage that it can be applied to flows and, we hope, to partially hyperbolic systems provided that a regularity result which we formally conjecture is true.

2. *Proof of Theorem 1.1*

The proof presented here consists of a sequence of arguments that bootstrap the regularity of the conjugacy h using the fact that it satisfies equation (1) (or (2)). As in [**dlLMM86**], we will bootstrap the regularity along the stable and unstable manifolds separately and, then, we will appeal to regularity results to conclude that a function which is regular restricted to the stable and unstable manifolds is regular in the usual sense.

We emphasize that the fact that we can bootstrap the regularity along the stable and unstable foliations is independent of the fact that they are transversal. Hence, this part of the argument can be carried out for partially hyperbolic systems or for Anosov flows. (We use at certain stages the fact that the system admits a SRB measure.)

In the case of Anosov flows, however, since the conjugacies also have to be smooth along the direction of the flow, we are in a situation where we can apply the regularity results of [**dlLMM86**] and obtain the desired conclusions.

We conjecture that the required regularity result—an analogue of Lemma 2.5 of [**dlLMM86**]—is true for partially hyperbolic systems satisfying the accessibility property. If such a conjecture was true, then Theorem 1.1 will hold for partially hyperbolic systems that have accessibility and SRB measures.

https://doi.org/10.1017/S0143385703000725 Published online by Cambridge University Press

We think that it would be extremely interesting to integrate the arguments presented here with the arguments in [**Sad02, KS03**].

In the following, we will present the proof with the notation of Anosov diffeomorphisms but we call attention to the fact that the bootstrap of the regularity only uses the fact that the manifolds are stable and unstable. Hence, it is applicable for stable manifolds of a flow and a partially hyperbolic system.

We will denote the stable manifold of the point $x \in M$ under the map f as $W^{s,(f)}_x$ but we will suppress (f) if it is understood from the context and will not lead to confusion.

2.1. *h is conformal along the stable and unstable leaves.* The first step of the proof of Theorem 1.1 is Lemma 2.1.

LEMMA 2.1. *$h : W^{s,(g)}_x \to W^{s,(f)}_{h(x)}$ is quasi-conformal when $W^{s,(g)}_x$, $W^{s,(f)}_{h(x)}$ are endowed with the conformal structures preserved by g, f respectively.*

Of course, there is an analogous result for the unstable manifolds, which can be deduced from Lemma 2.1 by considering the inverse mappings (a direct proof only requires trivial modifications).

Lemma 2.1 has already been established in [**dlL02b**, §3.2]. We recall that the main idea of the argument was to show that we can construct a map k such that $k : W^{s,(g)}_x \to W^{s,(f)}_{h(x)}$, $k|_{W^{s(g)}_x}$ has uniformly bounded first derivatives and $d_s(h(x), k(x))$ is bounded, where we denote by d_s the distance measured along $W^{s,(g)}_x$. Such a map—note that we do not require that k is a local homeomorphism—is constructed by convolution with suitable kernels after local coordinates have been taken.

By an argument quite similar to that in [**Shu69**], we have

$$h = \lim_{n\to\infty} f^n \circ k \circ g^{-n} \tag{5}$$

and the limit is reached uniformly.

Using the conformality of f, g and the fact that k is uniformly quasi-conformal in all the leaves, we obtain that the distortion of $f^n \circ k \circ g^{-n}|_{W^{s(g)}_x}$ is uniformly bounded. Since the limit h is a homeomorphism, standard results from the theory of quasi-conformal maps show that h is quasi-conformal. (See [**Väi71**, Theorem 37.2, Corollary 37.3, p. 125 ff].)

We emphasize that the argument to establish Lemma 2.1 does not assume that the eigenvalues at corresponding periodic points are the same. We only assume that there is a conformal structure present. □

2.2. *The stable Jacobian of the conjugacy is bounded.* We will introduce the notation $J^{s,(g)}[h]$ to denote the Jacobian of h restricted to the stable leaves of g. Similar notation will be used for the Jacobians of other maps for restrictions to the unstable foliations. If it is clear which map is being indicated, we will suppress the relevant superscript in parentheses.

So far, we have shown that $J^{s,(g)}[h]$ is well defined but we have not yet established any control on its size.

https://doi.org/10.1017/S0143385703000725 Published online by Cambridge University Press

It is not difficult to show that $J^{s,(g)}[h]$ is uniformly integrable on balls of size r of the stable leaves. It suffices to observe that

$$\int_{B_r \cap W_x^{s,(g)}} J^{s,(g)}[h](x)\, dx = |h(B_r \cap W^{s,(g)})| \geq C(r) > 0, \tag{6}$$

where by $|\cdot|$ we mean the Riemannian volume with respect to the metric in $W^{s,(f)}_{h(x)}$.

The desired result (6) follows because we obtain, from the uniform continuity of h, h^{-1}, the fact that $|h(B_r \cap W^{s,(g)})|$ contains a ball of radius larger than zero.

For subsequent arguments, it will be more useful to obtain control of $\log J^s[h](x)$.

In order to obtain control on $\log J^s[h](x)$, we will use an argument similar to that in [**LS72**]. This argument uses the theory of SRB measures. We refer to [**Sin72, Bow75, Rue76, BR75**] for the classical theory for Anosov or Axiom A systems and to [**PS82**] for results for partially hyperbolic systems.

We point out that, in this section, the fact that the systems are conformal will not play any role. The arguments apply just as well to any Anosov system such that the conjugacy is absolutely continuous along stable manifolds.

We denote by $\mu^{+,(g)}$, $\mu^{-,(g)}$ the future and past SRB measures for the map g. If the map g is understood, we will suppress it from the notation.

One of the characterizations of $\mu^{-,(g)}$ is that it has a density along the stable directions. By the fact that h is absolutely continuous on the stable manifolds and uniformly integrable on balls of size one, we obtain that map $h_*\mu^{-,(g)}$ is a measure which is invariant under f and that it has densities along the stable manifolds of f. Hence, we obtain

$$\mu^{-,(g)} = h_*\mu^{-,(f)}. \tag{7}$$

We also recall that another characterization of SRB measure $\mu^{-,(g)}$ is that it is Gibbs measure with respect to the potential $\log J^s[g]$. This is also true for $\mu^{-,(f)}$. The measure $h_*\mu^{-,(f)}$ is invariant under g and the Gibbs measure with respect to the potential $\log J^s[f] \circ h$. The fact that we have (7) implies, by the theory in [**Sin72**, Theorem 4]—see also [**Bow75**, Proposition 4.5]—that, for a Hölder function Ψ and a constant K, we have

$$\log J^s[f] \circ h - \log J^s[g] = \Psi \circ g - \Psi + K. \tag{8}$$

Moreover, K is the difference between the topological pressures of the functions $\log J^s[f] \circ u$ and $\log J^s[g]$.

It is not hard to show, using the thermodynamic formalism, that $K = 0$, i.e. using the fact that f, g are conjugate and, therefore, have the same entropy. Nevertheless, we will present an elementary proof later. We will show that, if $K = 0$, then $\log J^s[h]$ is bounded. The same argument will reach a contradiction with (6) under the assumption that $K \neq 0$.

We start by showing that if K in (8) is 0, then $\log J^s[h]$ is bounded. The argument will be very similar to the one we applied to see that the function h is quasi-conformal. It was used in [**dlL87**] for one-dimensional manifolds.

We first note that to show that $\log J^s[h]$ is bounded, it suffices to show that, with the same function k as in (5), we have

$$|\log J^{s,(g)}[f^n \circ k \circ g^{-n}](x)| \leq C \tag{9}$$

where C is independent of n, x.

https://doi.org/10.1017/S0143385703000725 Published online by Cambridge University Press

Indeed, given a ball B_r in a stable manifold of g and $\sigma > 0$, by the uniform convergence of $h_n \equiv f^n \circ k \circ g^{-n}$, we can find n such that $B_r \subset h^{-1} \circ h_n(B_{r+\sigma})$ (where $B_{r+\sigma}$ is a ball with the same center and radius $r+\sigma$). Then, $h(B_r) \subset h_n(B_{r+\sigma})$. By the assumption that $J^s[h_n]$ is uniformly bounded, we can bound $|h(B_r)| \leq C|B_{r+\sigma}|$, where, again, $|S|$ denotes the Riemannian volume on $W^{s,(f)}$. Taking $\sigma = 1/10r$, we can bound $|B_{r+\sigma}| \leq C|B_r|$. Hence, under (9), we have shown that $|h(B_r)| \leq C|B_r|$.

To establish (9), we expand the logarithm in the left-hand side (LHS):

$$\log J^{s,(g)}[f^n \circ k \circ g^{-n}](x) = \sum_{i=1}^{n} \log J^s[f] \circ f^i \circ k \circ g^{-n}(x) + \log J^s[k] \circ g^{-n}(x) + \sum_{i=1}^{n} \log J^s[g^{-1}] \circ g^{-i+n} \circ g^{-n}(x). \tag{10}$$

We want to show that the LHS of (10) is bounded uniformly in n, x.

Since $\log J^s[k]$ is bounded and, for every function Γ, we have $\sup \Gamma \circ g^{-n}(x) = \sup \Gamma(x)$, to show that the LHS of (10) is bounded, it suffices to show—relabeling the indexes in the last sum—that

$$\sum_{i=1}^{n} \log J^s[f] \circ f^i \circ k + \sum_{i=1}^{n} \log J^s[g^{-1}] \circ g^i \tag{11}$$

is bounded.

The boundedness of (10) follows from the identity, derived by applying (8) repeatedly,

$$\sum_{i=1}^{n} \log J^s[f] \circ f^i \circ h + \sum_{i=1}^{n} \log J^s[g^{-1}] \circ g^i = \Psi \circ g^i - \Psi \tag{12}$$

which is bounded.

We also observe that $d(f^i \circ h(x), f^i \circ k(x)) \leq C\lambda^i$—where $\lambda < 1$, C are independent of x—and that $J^s[f]$ is Hölder so that we can bound

$$|\log J^s[f] \circ f^i \circ h - \log J^s[f] \circ f^i \circ| \leq C\lambda^{i\alpha}.$$

Hence, the difference between (10) and (12) can be bounded by a geometric series.

To reach a contradiction with the fact that $K \neq 0$, we note that the argument presented earlier remains the same except that (12) is replaced by

$$\sum_{i=1}^{n} \log J^s[f] \circ f^i \circ h + \sum_{i=1}^{n} \log J^s[g^{-1}] \circ g^i = \Psi \circ g^i - \Psi + nK. \tag{13}$$

Therefore, we obtain, even in the case when $K \neq 0$, that $J^s[f^n \circ k \circ g^{-n}](x)e^{-nK}$ remains bounded. If $K > 0$, this is in contradiction with the fact that, given a ball $B \subset W_x^{s,(g)}$, $f^n \circ k \circ g^{-n} B$ converges uniformly to $h(B)$, which is a bounded set. In particular, the measure of $f^n \circ k \circ g^{-n}(B)$ should remain bounded.

We can reach a similar contradiction with $K < 0$ by reversing the roles of f, g and those of h, h^{-1} in the previous argument.

https://doi.org/10.1017/S0143385703000725 Published online by Cambridge University Press

Remark 2.2. Once we have shown that $\log J^s[h]$ is bounded, we note that $\log J^s[h]$ satisfies the cohomology equations (8).

It is well known [**PY99, dlL01**] that the bounded measurable solutions of (8) for systems that have ergodic SRB measures are unique up to an additive constant and, furthermore, are Hölder. This shows that $\log J^s[h] = \Psi + C$. Hence, $\log J^s[h]$ is Hölder.

Of course, in the one-dimensional case, where the Jacobian can be identified with the derivative, this shows that h is $C^{1+\alpha}$.

Later we will do better by exploiting the quasi-conformal structure.

Remark 2.3. Another corollary of the arguments of this section is that if the Jacobians along stable directions match and the system g is Anosov transitive, then the conjugacy is absolutely continuous on stable leaves.

To establish this result, we only need to point out that we can use Livsic's theorem to show that there exists a Ψ satisfying (8) with $K = 0$. From them on, we can use the same argument as the one presented here.

2.3. *The derivative of the conjugacy is bounded.* One reason why the estimates of the Jacobians are so useful for us is the fact that, for conformal mappings, estimates on the Jacobian imply estimates on the derivatives.

Let M, $\tilde{M}$ be two manifolds and E, $\tilde{E}$ be sub-bundles of TM, $T\tilde{M}$ respectively. Let σ, $\tilde{\sigma}$ be Riemannian metrics on E, $\tilde{E}$ continuous over the corresponding manifolds. Given a mapping $F : M \to \tilde{M}$ such that $DF(x) : E_x \to \tilde{E}_{F(x)}$ we define

$$K_{E,\tilde{E},\sigma,\tilde{\sigma}}(x) = \max_{\substack{v \in E_x \\ \|v\|_\sigma = 1}} \|DF(x)v\|_{\tilde{\sigma}} \Big/ \min_{\substack{v \in E_x \\ \|v\|_\sigma = 1}} \|DF(x)v\|_{\tilde{\sigma}}.$$

We say that F is A-quasi-conformal when $K_{E,\tilde{E},\sigma,\tilde{\sigma}}(x) \leq A$.

PROPOSITION 2.4. *Let $F : M \to \tilde{M}$ be E, $\tilde{E}$, σ, $\tilde{\sigma}$, A-quasi-conformal then, for a constant C which depends only on σ, $\tilde{\sigma}$ and the background metric, we have*

$$\|DF(x)\| \leq CA^{(\nu-1)/\nu} J_F^{1/\nu}.$$

Proof. The proof of the proposition is very easy. Using the singular value decomposition of linear algebra, we know that there are matrices U, V such that U is $\sigma'_{F(x)}$ unitary, V is σ_x unitary and $UDF(x)V \equiv D$ is diagonal with pointwise entries $d_1 \leq d_2 \leq \cdots \leq d_\nu$.

Then we have

$$\begin{aligned} \|DF(x)\|_{\sigma,\sigma'} &= d_\nu \\ K(DF(x)) &= \frac{d_\nu}{d_1} \\ J_{\sigma,\sigma'}(DF(x)) &= d_1 \cdots d_\nu. \end{aligned}$$

Because of the continuity of the metric, the norm and the Jacobian with respect to the background metric are equivalent to those with respect to σ, $\tilde{\sigma}$.

https://doi.org/10.1017/S0143385703000725 Published online by Cambridge University Press

Therefore, we have

$$A^{-(\nu-1)}\|DF(x)\|^{\nu}_{\sigma,\tilde{\sigma}} \leq J_{\sigma,\tilde{\sigma}}\big(DF(x)\big) \leq \|DF(x)\|^{\nu}_{\sigma,\tilde{\sigma}}. \qquad \square$$

As an immediate consequence of Proposition 2.4, we obtain

$$\|D_{s,(g)}h\| \in L^{\infty}, \tag{14}$$

where by $D_{s,(g)}$ we mean, following [**dlLMM86**], the derivative along the leaves of the stable foliation for g. When there is no possibility of confusion, we will omit the map g from the notation.

Of course, a similar argument shows that $D_{u,(g)}h \in L^{\infty}$.

Result (14) implies that h is Lipschitz restricted to the stable and unstable manifolds.

We also note that a similar argument with f, g exchanged shows that $D_s(h^{-1})$ is uniformly bounded. Hence, $\|(D_s h)^{-1}\|$ is also uniformly bounded.

Remark 2.5. For Anosov systems, because of the local product structure, we have shown that h is Lipschitz. When the system is partially hyperbolic, we would have established that some Hölder exponents depend only on the accessibility properties. In particular, they would be independent of h. This will not be a difficulty for the subsequent argument since we will continue bootstrapping the regularity along stable and unstable manifolds separately.

2.4. *Further regularity along stable and unstable directions.* The next part of the argument will be based on the observation that, taking derivatives along the stable directions of g in (1), we have

$$D_s h(x) = Df^{-1} \circ h \circ g(x) D_s h \circ g(x) D_s g(x). \tag{15}$$

(Note that we have omitted the indexes that indicate which maps give rise to the stable foliations that originate the operators D_s. All of them in this case are with respect to foliations invariant under g.)

Equation (15) can be considered as stating that $D_s h$ is a coboundary between $D_s f^{-1} \circ h \circ g(x)$ and $D_s g(x)$.

From the results in the previous section, we already have that $D_s h$ is an L^{∞} coboundary. There are already several papers in the literature which show that coboundaries which are measurable are automatically smooth (see, e.g., [**NT98**] for the case of continuous functions). In this paper we will adapt the method in [**dlL02a**].

We denote by $\mathcal{L}(X, Y)$ the space of linear operators from X to Y endowed with the supremum norm.

We note that $D_s h(x) \in \mathcal{L}(E^{s,(g)}_x, E^{s,(f)}_{h(x)})$. Similarly,

$$Df^{-1} \in \mathcal{L}(E^{s,(f)}_x, E^{s,(f)}_{f^{-1}(x)})$$
$$Dg(x) \in \mathcal{L}(E^{s,(f)}_x, E^{s,f}_{f^{-1}(x)}).$$

We introduce the operator $\eta(x)$:

$$\eta(x) : \mathcal{L}(E^{s,(g)}_{g(x)}, E^{s,(f)}_{h\circ g(x)}) \to \mathcal{L}(E^{s,(g)}_x, E^{s,f}_{h(x)})$$

https://doi.org/10.1017/S0143385703000725 Published online by Cambridge University Press

by

$$\eta(x)\Gamma_{g(x)} = D_s f \circ h \circ g(x)\Gamma_{g(x)} D_s g(x).$$

We also introduce the notation

$$\Psi(x) = D_s h(x).$$

Hence, equation (15) can be written more compactly as

$$\Psi(x) = \eta(x)\Psi \circ g(x). \tag{16}$$

Iterating (15), we have

$$\Psi(x) = \eta(x) \cdots \eta \circ g^n(x)\Psi \circ g^{n+1}(x). \tag{17}$$

The subsequent analysis will heavily use such expressions as (17) to bootstrap the regularity of Ψ and it will be somewhat similar to that in [**dlL02a**]. For the moment, we will just estimate $\eta(x) \cdots \eta \circ g^n(x)$.

The following bounds will be crucial for subsequent analysis.

LEMMA 2.6. *With the previous notation, we have*

$$\|\eta(x) \cdots \eta \circ g^n(x)\| \leq C$$

where C depends only on the properties of the metrics. In particular, it is independent of n.

Proof. Clearly we have

$$\|\eta(x) \cdots \eta \circ g^n(x)\| \leq \|Df^{-n} \circ h \circ g(x)\| \|Dg^n(x)\|. \tag{18}$$

By (18) and Proposition 2.4, to prove Lemma 2.6, it suffices to bound uniformly in n, x.

$$E(n, x) = (J^s[f^{-n}]) \circ h \circ g^n(x) J^s[g^n](x).$$

We note that, because of (1), we have

$$E(n, x) = J^s[h](x) \big/ J^s[h] \circ g^n(x) \tag{19}$$

and the RHS is bounded uniformly in n, x because of the fact that the stable Jacobian is Hölder, and bounded above and below away from zero. □

Remark 2.7. Even if we do not use it here, we note that the fact that the maps are conformal shows that the rates of growth along all directions in the (un)stable bundle of an orbit are the same. The proof of stable foliations in [**HK90**] allows us to conclude that the stable and unstable foliations are C^1.

In fact, following [**HP70**], once we know that the foliations are C^1, we can use the fact that the derivatives of the jet satisfy an equation that can be analyzed with the fiber contraction theorem, to show that the foliations are $C^{1+\gamma}$.

Even if these considerations do not play any role in our discussions, the regularity of the foliations is a very important part of the argument in [**Sad02, KS03**]. These papers contain a different argument which uses more geometric considerations to bootstrap the regularity of the manifolds.

https://doi.org/10.1017/S0143385703000725 Published online by Cambridge University Press

2.5. *$D_s h$ is Hölder.* The following will be the main result of this section.

LEMMA 2.8. *Assume that $D_s g$, $D_s f \circ h$ are $C^{\alpha,(s)}$, $\alpha > 0$ (respectively $C^{\alpha,(u)}$, $\alpha > 0$). Then*

$$D_s h \in C^{\alpha,(s)} \quad (\textit{respectively } C^{\alpha,(u)}).$$

Note that we have assumed $f \in C^r$ with $r > 1$. We have also shown that h is Lipschitz restricted to the unstable and stable foliations. Hence, as the map is Anosov, we can take

$$\alpha = \min(r - 1, \mathrm{Lip}).$$

(When the map is partially hyperbolic, we will need to rearrange the argument slightly.)

In the case that $\alpha = \mathrm{Lip}$—which is the relevant case for higher regularity—we obtain that $D_s h$ is differentiable almost everywhere along the stable and unstable leaves.

We also call attention to the fact that in the course of the proof of Lemma 2.8, we will introduce a remarkable formula (25) giving the 'increments' of $D_s h$ along the stable manifolds. This formula will be the basis for subsequent bootstraps.

We will need to introduce some notation which is very similar to that in [**dlL02a, dlLS**].

Given a vector field u in M, we denote by $\delta_u(x)$ the $\exp_x(u(x))$ where exp denotes the Riemannian geometry exponential.

In subsequent arguments, we will need the vector fields u to be such that δ_u preserves the (un)stable foliation of g and that they be absolutely continuous.

The reason to require δ to be absolutely continuous is that we are going to consider $\Psi \circ \delta$ where $\Psi \in L^\infty$ and evaluate identities that hold for general points x at $\delta(x)$.

We note that maps satisfying these requirements are plentiful because of the absolute continuity of the foliation. This comprises the following proposition.

PROPOSITION 2.9. *If $d(x, y) \le \varepsilon_0$, $y \in W_x^{s,(g)}$, then there exists δ as before, such that*

$$\delta(x) = y, \quad \|\delta - \mathrm{Id}\|_{L^\infty} \le C d(x, y), \quad \|J[\delta]\|_{L^\infty} \le C.$$

Proof. The proof of the proposition is obvious if we consider a coordinate patch such as those introduced in Theorem 9 in [**Ano69**]. There, it is shown how it is possible to make changes in the variables that reduce the stable foliations to a product foliation in $\mathbb{R}^d$. These changes in variables are, moreover, absolutely continuous. Hence, the problem considered in Proposition 2.9 can be seen just as a trivial foliation.

We will also need to compare tangent spaces at nearby points. For this, we just recall the device of '*connectors*' introduced in [**HPPS70**].

Let x, y be points which are close enough that we can find a unique shortest geodesic.

Define by $S_x^y : T_x M \to T_y M$ to be the map obtained by transporting the vectors from $T_x M$ to $T_y M$ along the shortest geodesic.

Note that since we are assuming that the background metric is analytic, we obtain that S_x^y depends analytically on x, y provided that $d(x, y) \le \varepsilon_0$ for some $\varepsilon_0 > 0$.

Note that, if both sides can be defined, we have

$$S_x^z = S_y^z \circ S_x^y.$$

In particular,

$$\mathrm{Id} = S_y^x \circ S_x^y. \tag{20}$$

https://doi.org/10.1017/S0143385703000725 Published online by Cambridge University Press

Using the notation of $\Psi = D_s h$ and η introduced in §2.3, we will show that

$$\|\Psi(x) - S^x_{\delta_v(x)}\Psi \circ \delta_v(x)\| \leq C\|v\|^\alpha_{C^0}$$

provided that δ_v preserves the stable foliation.

Later we will provide a different argument to show a similar result when δ_v preserves the unstable foliation.

Assuming that δ preserves the stable foliation of g and that $\delta(x) - x$ is small enough so that we can define the transport, then we see that $g^n \circ \delta(x)$, $g^n(x)$ are also close and we can define $S^{g^n(x)}_{g^n\circ\delta(x)}$.

Evaluating (17) at $\delta(x)$ and inserting factors $S^{g^i\circ\delta(x)}_{g^i(x)} S^{g^i(x)}_{g^i\circ\delta(x)}$, which are just the identity, we obtain

$$\begin{aligned} S^x_{\delta(x)}\Psi \circ \delta(x) = {} & S^x_{\delta(x)}\eta \circ \delta(x) S^{g(x)}_{g\circ\delta(x)} S^{g\circ\delta(x)}_{g(x)} \eta \circ g \circ \delta(x) S^{g^2(x)}_{g^2\circ\delta(x)} S^{g^2\circ\delta(x)}_{g^2(x)} \\ & \cdots \\ & \cdot S^{g^n(x)}_{g^n\circ\delta(x)} S^{g^n\circ\delta(x)}_{g^n(x)} \eta \circ g^n \circ \delta(x) \\ & \cdot S^{g^{n+1}(x)}_{g^{n+1}\circ\delta(x)} S^{g^{n+1}\circ\delta(x)}_{g^{n+1}(x)} \Psi \circ g^{n+1} \circ \delta(x). \end{aligned} \tag{21}$$

Introducing this notation,

$$\begin{aligned} \tilde{\eta} \circ \delta(x) &= S^x_{\delta(x)}\eta \circ \delta(x) S^{g\circ\delta(x)}_{g(x)} \\ \tilde{\Psi} \circ \delta(x) &= S^x_{\delta(x)}\Psi \circ \delta(x). \end{aligned} \tag{22}$$

We see that (21) can be written as follows

$$\tilde{\Psi} \circ \delta(x) = \tilde{\eta} \circ \delta(x)\tilde{\eta} \circ g \circ \delta(x) \cdots \tilde{\eta} \circ g^n \circ \delta(x)\tilde{\Psi} \circ g^{n+1} \circ \delta(x). \tag{23}$$

Remark 2.10. There is a nice geometric intuition behind the calculation leading to (21); i.e. we are trying to analyze the objects along the orbit $g^i \circ \delta(x)$ by referring them to objects in the orbit $g^i(x)$.

This is accomplished, of course, by conjugating with the identifications S. This will allow us to compare operators along two neighboring orbits by referring them to common points.

Adding and subtracting in (17) and (21), we obtain

$$\begin{aligned} &\Psi(x) - \tilde{\Psi}\circ\delta(x) \\ &\quad = [\eta(x) - \tilde{\eta}\circ\delta(x)]\tilde{\eta}\circ g\circ\delta(x)\cdots\tilde{\eta}\circ g^n\circ\delta(x)\tilde{\Psi}\circ g^{n+1}\circ\delta(x) + \cdots \\ &\qquad + \eta(x)\cdots\eta\circ g^{i-1}(x)[\eta\circ g^i(x) - \tilde{\eta}\circ g^i\circ\delta(x)]\tilde{\eta}\circ g^n\circ\delta(x)\tilde{\Psi}\circ g^{n+1}\delta(x) + \cdots \\ &\qquad + \eta(x)\cdots\eta\circ g^{n-1}(x)[\eta\circ g^n(x) - \tilde{\eta}\circ g^n\circ\delta(x)]\tilde{\Psi}\circ g^{n+1}\delta(x) \\ &\qquad + \eta(x)\cdots\eta\circ g^n(x)[\Psi\circ g^{n+1}(x) - \tilde{\Psi}\circ g^{n+1}\circ\delta(x)]. \end{aligned} \tag{24}$$

We denote by $T_i(x)$ the general term in (24), namely

$$T_i(x) = \eta(x)\cdots\eta\circ g^{i-1}(x)[\eta\circ g^i(x) - \tilde{\eta}\circ g^i\circ\delta(x)]\Psi\circ g^{i+1}\circ\delta(x).$$

https://doi.org/10.1017/S0143385703000725 Published online by Cambridge University Press

We want to show that

$$\Psi(x) - \tilde{\Psi} \circ \delta(x) = \sum_{i=1}^{\infty} T_i(x) \tag{25}$$

almost everywhere x.

First of all we want to show that $\sum_{i=1}^{\infty} T_i(x)$ converges almost everywhere as well as in the L^∞ sense. This follows from

$$\|T_i\|_{L^\infty} \leq \lambda_c^{i\alpha} C \|\delta - \mathrm{Id}\|_{C^0}^{\alpha}. \tag{26}$$

To prove (26), we use that, by Lemma 2.6, the factors $\eta(x) \cdots \eta \circ g^{i-1}(x)$ are uniformly bounded independently of i, x. $\tilde{\Psi} \circ g^{n+1} \circ \delta(x)$ is also uniformly bounded in L^∞.

Because η is C^α and δ is adapted to the stable foliation, we have

$$\|\eta \circ g^i - \tilde{\eta} \circ g^i \circ \delta\|_{L^\infty} \leq \lambda^{i\alpha} C \|\delta - \mathrm{Id}\|_{L^\infty}^{\alpha}.$$

This proves (26).

To estimate the last term in (24), we follow an argument in [**NP99**]. We note that, by Lusin's theorem, Ψ is continuous in a set of measures at least $1 - \varepsilon$. Because of the ergodicity of g, $\Psi \circ g^{n+1}(x) - \tilde{\Psi} \circ g^{n+1} \circ \delta(x)$ converges to zero along a subsequence.

Hence, we obtain that (25) holds almost everywhere. This shows that $D_s h \in C^{\alpha,s}$.

We note that if Ψ satisfies (15) then we also have

$$\Psi(x) = \eta^{-1} \circ g^{-1}(x) \Psi \circ g^{-1}(x).$$

Hence, applying the result we have for the regularity along the stable directions, we obtain the claim for the unstable directions. □

One consequence that we will use later is that formula (25) holds not only almost everywhere but everywhere. Hence, we do not need to use the device of studying increments by comparing absolutely continuous vector fields.

When $y \in W_x^{s(g)}$, we have

$$\begin{aligned} \Delta(y) &\equiv \Psi(x) - S_y^x \Psi(y) \\ &= \sum_{i-1}^{\infty} \eta(x) \cdots \eta \circ g^i(x) [\eta \circ g^i(x) - \tilde{\eta} \circ g^i(y)] \Psi \circ g^{i+1}(y). \end{aligned} \tag{27}$$

2.6. *$D_s h$ is differentiable.*

LEMMA 2.11. *If $D_s f^{-1} \circ h$, $Dg \in C^{1+\alpha,(s)}$, $\alpha > 0$ (respectively $\in C^{1+\alpha,(u)}$, $\alpha > 0$), then $D_s h \in C^{1+\alpha,(s)}$ (respectively $C^{1+\alpha,(u)}$).*

Proof. The proof of Lemma 2.11 is very simple. Given (27) it is natural to guess the formula for the derivative (see (28)). Then we will prove that this formula is indeed a derivative by estimating the remainder. We call attention to the fact that formula (28) will be the basis for subsequent bootstraps.

https://doi.org/10.1017/S0143385703000725 Published online by Cambridge University Press

We guess

$$G(x) = \sum_{i=1}^{\infty} \eta(x) \cdots \eta \circ g^i(x)(D_s\tilde{\eta} \circ g^i)(x)\Psi \circ g^{i+1}(x)$$
$$= \sum_{i=1}^{\infty} \Psi(x)\Psi^{-1} \circ g^{n-1}(x)(D_s\tilde{\eta} \circ g^i)(x)\Psi \circ g^{i+1}(x). \tag{28}$$

First of all, we observe that the formula in (28) converges uniformly.

The reason is that, because of the chain rule, we have, for a vector field X tangent to the stable direction,

$$D_s(\tilde{\eta} \circ g^i)(x)X(\tilde{\eta} \circ g^i)(x) = (D_s\eta) \circ g^i Dg^i(x)X.$$

Because the vector field X is tangent to the stable direction, we see that $\|SDg^i(x)X\| \leq \lambda^i\|X\|$. Hence, $\|D_s\tilde{\eta} \circ g^i(x)\| \leq C\lambda^i$ and, therefore, the general term of (28) can be bounded by a decreasing exponential.

Once we know that the formula (28) makes sense, to establish that it is a true derivative, we just need to estimate the error in the linear approximations. This will be easy because the formula for the increments is given in (25). When we subtract the linear approximation given by (28) from the increment formula (25), we obtain

$$\begin{aligned}\Psi(x) - S_y^x\Psi(y) - G(x)[y - x] &= \sum_i \eta(x) \cdots \eta \circ g^i(x)[\eta g^i(x) - \tilde{\eta} \circ g^i(y) - D_y\tilde{\eta} \circ g^i(y)]\Psi \circ g^{i+1}(x) \\ &\quad + \sum_i \eta(x) \cdots \eta \circ g^i(x)[\eta \circ g^i(x) - \tilde{\eta} \circ g^i(y)][\Psi \circ g^{i+1}(x) - \tilde{\Psi} \circ g^{i+1}(y)].\end{aligned} \tag{29}$$

The desired result will be established when we estimate (29) and show that it can be bounded by something that is $o(d_s(x, y))$. We will start by bounding the general term of each of the sums in (29) and then we will add up all the estimates obtained for each of the terms.

We again note that the factor $\eta(x) \cdots \eta \circ g^i(x)$ is bounded uniformly. So is $\Psi \circ g^{i+1}(x)$.

Because $\eta \in C^{1+\alpha,(s)}$,

$$|[\eta \circ g^i(x) - \tilde{\eta} \circ g^i(y) - [D_s\tilde{\eta}g^i](x)[x - y]]| \leq \lambda^{(1+\alpha)i}d_s(x - y)^{1+\alpha}.$$

This shows that the first sum in (29) can be estimated by $Cd(x, y)^{1+\alpha}$.

Now we estimate the second sum in (29).

Because Ψ is $C^{\mathrm{Lip},(s)}$, we have

$$|\Psi \circ g^i(x) - \tilde{\Psi} \circ g^i(y)| \leq C\lambda^i d_s(x, y).$$

Since η is Lipschitz, we can bound

$$\|\eta \circ g^i(x) - \tilde{\eta} \circ g^i(y)\| \leq C\lambda^i d_s(x, y).$$

Hence, the second sum can be bounded by $Cd_s(x, y)^2$.

This finishes the proof that (28) is indeed the derivative along the stable direction and the proof of Lemma 2.11. □

https://doi.org/10.1017/S0143385703000725 Published online by Cambridge University Press

2.7. *Higher derivatives along the stable directions.* Note that we have also established in (28) a formula for the derivative along the stable direction.

Establishing the existence of higher-order derivatives is now very easy because we can take derivatives of (28) term by term. The proof is, from now on, quite similar to the proofs in [**dlLMM86**] and [**BdlLW96**].

We just need to show that the expression for the derivative given in (28) can be differentiated term by term. These formal derivatives will converge uniformly. Hence, it is standard to conclude that these sums are indeed the true derivatives. The estimates needed are a particular case of Lemma 4 in [**BdlLW96**]—which is very similar to those in [**dlLMM86**, p. 587].

Following [**dlLMM86**], for $k \in \mathbb{N}$, we say that a function φ is in $C_s^{k,(g)}$ when the restriction to the stable leafs of g are uniformly C^k and the k-jets of these restrictions are continuous over the manifold.

We say that $\varphi \in C_s^{k+\alpha,(g)}$ when $\varphi \in C_s^{k,(g)}$ and the k-jets are uniformly Hölder when restricted to the stable leaves. The space $C_s^{k,(g)}$ is a Banach space when endowed with the norm of the sup of the k-jets. Similarly, $C_s^{k+\alpha,(g)}$ is a Banach space when endowed with the norm obtained taking the maximum of the $C^{k,(g)}$ norm and the semi-norm $H^{\alpha}_{s,(g)}$ for the k-jet.

Clearly, we have

$$\varphi \in C_s^{k,(g)} \iff D_{s,(g)}\varphi \in C_s^{k-1,(g)}$$

and the same with fractional exponents.

LEMMA 2.12. *Assume that* Ψ, $D_s\eta \in C_s^{t,(g)}$ $t \in \mathbb{N} + [0, \mathrm{Lip}]$ *and that* $g \in C^r$, $r \geq t$, $t > 0$. *Then, the sum in (28) converges uniformly in* $C_s^{t,(g)}$.

Proof. This lemma is very similar to the estimates in [**dlLMM86**, p. 587] and [**BdlLW96**, Lemma 4]. The estimates in these papers give

$$\begin{aligned} \|D_s^j \Psi^{-1} \circ g^{n-1}\|_{L^\infty} &\leq C_j \|\Psi^{-1}\|_{C_s^j} \lambda^{j(n+1)} j^{n+1}, \\ \|D_s \tilde{\eta} \circ g^i\|_{L^\infty} &\leq C_j \|D_s \tilde{\eta}\|_{C_s^j} \lambda^{ji} j^i, \\ \|\Psi \circ g^{i+1}\|_{L^\infty} &\leq C_j \|\Psi\|_{C_s^j} \lambda^{j(i+1)} j^{i+1}, \end{aligned} \tag{30}$$

where C_j are numbers that depend on j and on $\|g\|_{C^j}$ but which are independent of i, n.

Using Leibniz' rules for the derivatives, we obtain that, as claimed, the $C_s^{r,(g)}$ norm of the general term in (28) is bounded by a decreasing exponential.

When we need to consider fractional derivatives, we denote by $d_s(x, y)$ the distance between two points in the same leaf of the stable foliation measured along the leaves. We also denote

$$H_s^{\alpha}(\eta) = \sup_{0 \leq d_s(x,y) \leq \gamma} (\eta(x) - \eta(y))/d_s(x, y)^{\alpha}.$$

We will assume, without loss of generality, that the metric is adapted. That is, $d_s(f(x), f(y)) \leq \lambda d_s(x, y)$ for some $\lambda < 1$. This immediately leads to

$$H_s^{\alpha}(\eta \circ f) \leq H_s^{\alpha}(\eta)\lambda^{\alpha}. \tag{31}$$

The bounds for the H_s^{α} of the high derivatives follow by estimating the same formulas that we used for the integer derivatives. □

https://doi.org/10.1017/S0143385703000725 Published online by Cambridge University Press

By applying Lemma 2.12 repeatedly, we obtain the desired result. Note that, if we assume that $\Psi, \eta \in C_s^{j,(g)}$, then Lemma 2.12 shows that the sum in (28) converges in $C_s^{j,(g)}$; therefore, $G = D_s h$ is in $C_s^{j,(g)}$ and, therefore, $h \in C_s^{j+1,(g)}$ and, therefore, $\Psi, \eta \in C_s^{j+1,(g)}$. The induction stops when we reach the regularity assumed in g, f.

Remark 2.13. Note that the argument presented to obtain the higher derivatives is very reminiscent of the argument presented for the bootstrap of the regularity of conjugacies in [**dlL92**]. In Theorem 6.1 in [**dlL92**], it is established that once a conjugacy is smoother than a critical value, determined from the bunching of the expansion and contraction spectrum, then the conjugacy is as smooth as the map. (See also the last parts of [**NT98**].)

For maps preserving a conformal metric, the expansion of all vectors along the stable manifold is roughly the same. Hence, the critical value for Theorem 6.1 in [**dlL92**]—expressed in terms of the bunching properties of the expansion properties—is $1+\epsilon$. Hence, once we have that $h \in C^{1+\alpha}$, it is also possible just to adapt the arguments in [**dlL92**]. □

3. *Regularity in the usual classes*

We have so far shown that the conjugacy $h \in C_s^{r,(g)}$. A similar argument shows that $h \in C_u^{r,(g)}$.

For the case of diffeomorphisms, this is done by invoking a regularity lemma that states that functions with these properties are $C^{r'}$. In [**dlLMM86**] one can find a proof of this result using elliptic regularity theory. Other proofs are available in [**Jou88**]. (Other versions which are not so good technically appear in [**HK90, dlL92, dlL97**].)

For flows, we point out that exactly the same argument applies for the stable and unstable manifolds of the flow. It is also trivial to observe that if (2) holds, then the conjugacy is smooth along the direction of the flow. Then, we can apply the regularity results to obtain that the conjugacy is $C_{cs}^{r',X}$—where the subscript cs refers to the center-stable direction. Then, a further application of the regularity result now considering the cs foliation and the unstable one finishes the proof of Theorem 1.1.

Remark 3.1. It is interesting to note that, except for §3, all the results in previous sections work for partially hyperbolic maps that have SRB measures with ergodic properties.

It is reasonable to conjecture that there should be an analogue of the regularity result for foliations of Anosov systems for foliations of partially hyperbolic systems that have a certain accessibility property. Following the scheme of the proof in [**dlLMM86**], it seems that it would be enough to use some version of the theory of hypo-elliptic equations rather than the elliptic regularity theory used in [**dlLMM86**].

One would expect that the regularity recovered would be a fraction of the regularity of the derivatives obtained.

Once such result is at hand, then a version of Theorem 1.1 for partially hyperbolic systems will automatically follow. We hope to come to this problem in the future.

3.1. *Results for systems with one-dimensional foliations.*

We will present two arguments that are very similar to the arguments in [**dlL87, dlLM88**] and in [**Pol88, dlL92**].

https://doi.org/10.1017/S0143385703000725 Published online by Cambridge University Press

In the first argument, we note that, because of the condition on the eigenvalues, the fact that the system is transitive and using the result in [**Liv72**], we can find a function Ψ satisfying (8) (with $K = 0$). Then, we can repeat the argument as in §2.2. We obtain that the Jacobian of the derivative is bounded. Of course, in the one-dimensional case, the Jacobian of the derivative is the same as the derivative. Hence, showing that the derivative is bounded is obvious.

An alternative argument is based on the fact that the SRB measure is also characterized as the weak limit of measures based on periodic orbits with weights which are proportional to the stable Jacobians. Given the equality of the stable derivatives at corresponding periodic points, we obtain that the SRB measure is transported by the conjugacy.

From then, one can use that the densities are transported and use the fact that they are smooth to show that the derivatives are bounded. The argument presented before also applies to the one-dimensional case.

Using the commutativity of the derivatives, one can use simpler arguments. Several different arguments to bootstrap the regularity taking advantage of the fact that the stable manifold is one-dimensional can be found in [**dlL87, dlL92, dlL97**]. We refer to [**dlL97**] for a comparison between these one-dimensional arguments.

Acknowledgements. The work of the author has been supported by NSF grants. I am also grateful to V. Sadoskaia and B. Kalinin for making very valuable comments on [**dlL02b**]. In particular, spotting incomplete arguments in support of Conjecture 1.3 there, which is a particular case of Theorem 1.1 of this paper. I also thank them for very valuable discussions and for encouragement on the subject of this paper. I thank A. González for a careful reading of this paper.

References

[Ano69] D. V. Anosov. *Geodesic Flows on Closed Riemann Manifolds with Negative Curvature.* American Mathematical Society, Providence, RI, 1969.

[BCG95] G. Besson, G. Courtois and S. Gallot. Entropies et rigidités des espaces localement symétriques de courbure strictement négative. *Geom. Funct. Anal.* **5**(5) (1995), 731–799.

[BdlLW96] A. Banyaga, R. de la Llave and C. E. Wayne. Cohomology equations near hyperbolic points and geometric versions of Sternberg linearization theorem. *J. Geom. Anal.* **6**(4) (1997), 613–649.

[BFL92] Y. Benoist, P. Foulon and F. Labourie. Flots d'Anosov à distributions stable et instable différentiables. *J. Amer. Math. Soc.* **5**(1) (1992), 33–74.

[Bow75] R. Bowen. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms (Lecture Notes in Mathematics, 470).* Springer, Berlin, 1975.

[BR75] R. Bowen and D. Ruelle. The ergodic theory of Axiom A flows. *Invent. Math.* **29**(3) (1975), 181–202.

[dlL87] R. de la Llave. Invariants for smooth conjugacy of hyperbolic dynamical systems. II. *Comm. Math. Phys.* **109**(3) (1987), 369–378.

[dlL92] R. de la Llave. Smooth conjugacy and S-R-B measures for uniformly and non-uniformly hyperbolic systems. *Comm. Math. Phys.* **150**(2) (1992), 289–320.

[dlL97] R. de la Llave. Analytic regularity of solutions of Livsic's cohomology equation and some applications to analytic conjugacy of hyperbolic dynamical systems. *Ergod. Th. & Dynam. Sys.* **17**(3) (1997), 649–662.

https://doi.org/10.1017/S0143385703000725 Published online by Cambridge University Press

[dlL01] R. de la Llave. Remarks on Sobolev regularity in Anosov systems. *Ergod. Th. & Dynam. Sys.* **21**(4) (2001), 1139–1180.

[dlL02a] R de la Llave. Bootstrap of regularity for integrable solutions of cohomology equations. *Preprint*, 2002. MP_ARC # 02-378.

[dlL02b] R. de la Llave. Rigidity of higher-dimensional conformal Anosov systems. *Ergod. Th. & Dynam. Sys.* **22**(6) (2002), 1845–1870.

[dlLM88] R. de la Llave and R. Moriyón. Invariants for smooth conjugacy of hyperbolic dynamical systems. IV. *Comm. Math. Phys.* **116**(2) (1988), 185–192.

[dlLMM86] R. de la Llave, J. M. Marco and R. Moriyón. Canonical perturbation theory of Anosov systems and regularity results for the Livšic cohomology equation. *Ann. Math. (2)* **123**(3) (1986), 537–611.

[dlLS] R. de la Llave and V. Sadovskaia. On regularity of integrable conformal structures invariant under Anosov systems. *Preprint*, 2003. MP_ARC # 03-463.

[FW80] J. Franks and B. Williams. Anomalous Anosov flows. *Global Theory of Dynamical Systems (Proc. Int. Conf., Northwestern University, Evanston, IL., 1979) (Lecture Notes in Mathematics, 819)*. Springer, Berlin, 1980, pp. 158–174.

[HK90] S. Hurder and A. Katok. Differentiability, rigidity and Godbillon–Vey classes for Anosov flows. *Inst. Hautes Études Sci. Publ. Math.* **72** (1990), 5–61.

[HP70] M. W. Hirsch and C. C. Pugh. Stable manifolds and hyperbolic sets. *Global Analysis (Proc. Symp. Pure Math., Vol. XIV, Berkeley, CA, 1968)*. American Mathematical Society, Providence, RI, 1970, pp. 133–163.

[HPPS70] M. Hirsch, J. Palis, C. Pugh and M. Shub. Neighborhoods of hyperbolic sets. *Invent. Math.* **9** (1969/1970), 121–134.

[Jou88] J.-L. Journé. A regularity lemma for functions of several variables. *Rev. Mat. Iberoamericana* **4**(2) (1988), 187–193.

[Kan93] M. Kanai. Differential-geometric studies on dynamics of geodesic and frame flows. *Japan. J. Math. (N.S.)* **19**(1) (1993), 1–30.

[KS03] B. Kalinin and V. Sadovskaia. On local and global rigidity of quasiconformal Anosov diffeomorphisms. *J. Inst. Mat. Jussieu* **2**(4) (2003), 567–582.

[Liv72] A. N. Livšic. The homology of dynamical systems. *Usp. Mat. Nauk* **27**(3(165)) (1972), 203–204.

[LS72] A. N. Livšic and Ja. G. Sinaĭ. Invariant measures that are compatible with smoothness for transitive C-systems. *Dokl. Akad. Nauk SSSR* **207** (1972), 1039–1041.

[New70] S. E. Newhouse. On codimension one Anosov diffeomorphisms. *Amer. J. Math.* **92** (1970), 761–770.

[NP99] M. Nicol and M. Pollicott. Measurable cocycle rigidity for some non-compact groups. *Bull. London Math. Soc.* **31**(5) (1999), 592–600.

[NT98] V. Niţică and A. Török. Regularity of the transfer map for cohomologous cocycles. *Ergod. Th. & Dynam. Sys.* **18**(5) (1998), 1187–1209.

[Pol88] M. Pollicott. C^r-rigidity theorems for hyperbolic flows. *Israel J. Math.* **61**(1) (1988), 14–28.

[PS82] Ya. B. Pesin and Ya. G. Sinaĭ. Gibbs measures for partially hyperbolic attractors. *Ergod. Th. & Dynam. Sys.* **2**(3–4) (1982), 417–438.

[PY99] M. Pollicott and M. Yuri. Regularity of solutions to the measurable Livsic equation. *Trans. Amer. Math. Soc.* **351**(2) (1999), 559–568.

[Rue76] D. Ruelle. A measure associated with axiom-A attractors. *Amer. J. Math.* **98**(3) (1976), 619–654.

[Sad02] V. Sadovskaia. On uniformly quasiconformal Anosov systems. *Preprint*, 2002.

[Shu69] M. Shub. Endomorphisms of compact differentiable manifolds. *Amer. J. Math.* **91** (1969), 175–199.

[Sin72] Ja. G. Sinaĭ. Gibbs measures in ergodic theory. *Usp. Mat. Nauk* **27**(4(166)) (1972), 21–64.

[Väi71] J. Väisälä. *Lectures on n-dimensional Quasiconformal Mappings (Lecture Notes in Mathematics, 229)*. Springer, Berlin, 1971.

[Yue96] C. Yue. Quasiconformality in the geodesic flow of negatively curved manifolds. *Geom. Funct. Anal.* **6**(4) (1996), 740–750.

https://doi.org/10.1017/S0143385703000725 Published online by Cambridge University Press