Euro. Jnl of Applied Mathematics (1998), *vol.* 9, *pp.* 507–525. Printed in the United Kingdom

© 1998 Cambridge University Press

Convective instability of reaction fronts: Linear stability analysis

V. A. VOLPERT[1] and A. I. VOLPERT[2]

[1] *Laboratoire d'analyse numérique, Université Lyon I, UMR 5585 CNRS, 69622 Villeurbanne Cedex, France*
[2] *Department of Mathematics, Technion, 32000 Haifa, Israel*

(*Received 22 November 1997; revised 20 April 1998*)

The paper is devoted to convective instability of reaction fronts. New approaches are developed to study some eigenvalue problems arising in chemical hydrodynamics. For gaseous combustion in the case of equality of transport coefficients, a linear stability analysis of an upward propagating front is carried out. A minimax representation of the stability boundary is obtained.

1 Introduction

The propagation of reaction fronts can be influenced by gravity and, under some conditions, it can lead to a loss of stability of a stable front and to the appearance of a convective structure. The physical mechanism of convective instability is clear. If we consider for example an upward propagating exothermic chemical reaction, then the temperature of the medium below the reaction zone is greater than the temperature above it. Then, under some critical conditions, natural convection can appear. We can make here an analogy with the Rayleigh–Benard problem of convection in a layer of a liquid heated from below. As is well known, if the Rayleigh number is sufficiently large, the stationary temperature distribution loses its stability and natural convection appears. In the case of propagating reaction fronts the exothermic chemical reaction plays the role of heating from below.

Experimentally, convective instability of reaction fronts was observed first in the experiments on frontal polymerization [1, 2]. Later it was studied experimentally and theoretically in a number of works on frontal polymerization [3–5] for other types of reaction fronts [6–15] and for chemical reactors [16–18]. The theoretical studies of convective instability of reaction fronts usually deal with one-step kinetics and develop approximate approaches based on some special properties of the problems considered.

In this work we develop new approaches to study convective instability of reaction fronts. They allow us to analyse the structure of the spectrum of the problem linearized about a stationary solution and to draw conclusions about its stability. These approaches are applicable for models of gaseous combustion with complex kinetics, for branching chain flames, and for other types of chemical waves.

https://doi.org/10.1017/S095679259800357X Published online by Cambridge University Press

We consider a chemical reaction in the general form

$$\sum_{j=1}^{m} \alpha_{ij} A_j \to \sum_{j=1}^{m} \beta_{ij} A_j, \quad i = 1, \dots, n, \tag{1.1}$$

where A_j, $j = 1, \dots, m$ denote chemical species and their concentrations, and α_{ij} and β_{ij} are stoichometric coefficients. The reaction–diffusion system describing the distribution of the temperature and of the concentrations has the form

$$\frac{\partial T}{\partial t} + v\nabla T = \kappa \Delta T + (q, W) \tag{1.2}$$

$$\frac{\partial A}{\partial t} + v\nabla A = d\Delta A + \Gamma W. \tag{1.3}$$

Here T is the temperature, $A = (A_1, \dots, A_m)$ the vector of the concentrations, κ the coefficient of thermal diffusivity, d the matrix of diffusion coefficients. We suppose that it is diagonal with diagonal elements d_i, and we set

$$(q, W) = \sum_{i=1}^{n} q_i W_i,$$

where q_i is the adiabatic heat release of the ith reaction, W_i is the rate of the ith reaction,

$$W_i = K_i(T) A_1^{\alpha_{i1}} \times \dots \times A_m^{\alpha_{im}};$$

$K_i(T)$ is the temperature coefficient of the reaction rate, often considered in the Arrhenius form,

$$K_i(T) = k_i \mathrm{e}^{-E_i/RT},$$

where E_i is the activation energy of the ith reaction, R is the gas constant, k_i are constant factors; Γ is a $m \times n$ matrix with the elements $\gamma_{ij} = \beta_{ij} - \alpha_{ij}$; finally, $v = (v_x, v_z)$ is the velocity of the medium,

$$\Delta = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial z^2}, \quad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z} \right),$$

$-\infty < x, z < +\infty$.

To describe the motion of the medium we consider the Navier–Stokes equations under the Boussinesq approximation

$$\frac{\partial v}{\partial t} + (v.\nabla)v = -\frac{1}{\rho}\nabla p + \nu \Delta v + \sigma \tau (T - T_0) g, \tag{1.4}$$

$$\operatorname{div} v = 0. \tag{1.5}$$

Here ρ is the density, ν the kinematic viscosity, σ the coefficient of thermal expansion, τ the unit vector in the vertical direction, T_0 is a characteristic value of temperature, g is the acceleration due to gravity, and p is the pressure.

We recall that in the Boussinesq approximation the medium is supposed to be incompressible and the density is constant in all terms except for the last term in (1.4) which appears as a result of linearization of the function $\rho(T)$. This approximation is applicable if the density dependence on temperature and pressure is not very strong. For the gaseous

https://doi.org/10.1017/S095679259800357X Published online by Cambridge University Press

combustion it was derived in Matkowsky & Sivashinsky [19] and used to study the influence of natural convection on cellular flames in Matkowsky & Sivashinsky [20].

We note that all solutions of the reaction–diffusion system

$$\frac{\partial T}{\partial t} = \kappa \Delta T + (q, W) \tag{1.6}$$

$$\frac{\partial A}{\partial t} = d\Delta A + \Gamma W \tag{1.7}$$

are also solutions of (1.2)–(1.5) with $v = 0$. So the question arises whether stationary or propagating solutions that are stable as solutions of (1.6), (1.7) are also stable as solutions of (1.2)–(1.5). If σ is zero, then this is obviously true. If σ is positive, then, as will be shown below, it is not necessarily the case. Stable solutions of the reaction–diffusion system can lose their stability because of the hydrodynamics and, more precisely, because of gravity. We study this question in this paper and find, in particular, the stability conditions. We consider travelling wave solutions of the system (1.2)–(1.5), though this is not necessary and the same methods can be applied to study stability of stationary solutions in bounded domains.

Suppose that the system (1.6), (1.7) has a one-dimensional travelling wave solution, i.e. a solution of the form

$$T(x, z, t) = \tilde{T}(z - ct), \quad A(x, z, t) = \tilde{A}(z - ct).$$

Here we assume that the wave propagates in the vertical direction. The constant c is the wave velocity which is unknown and should be found as a solution of the problem. The functions $\tilde{T}(\zeta)$ and $\tilde{A}(\zeta)$ satisfy the following system of equations:

$$\kappa T'' + cT' + (q, W) = 0, \tag{1.8}$$

$$dA'' + cA' + \Gamma W = 0. \tag{1.9}$$

Here, $'$ denotes differentiation with respect to ζ. The existence of one-dimensional travelling waves for reaction–diffusion systems is well understood (see Volpert *et al.* [21]).

The vector-function $T = \tilde{T},\ A = \tilde{A},\ v = 0$ is a stationary solution of the system

$$\frac{\partial T}{\partial t} + v\nabla T = \kappa \Delta T + c\frac{\partial T}{\partial z} + (q, W), \tag{1.10}$$

$$\frac{\partial A}{\partial t} + v\nabla A = d\Delta A + c\frac{\partial A}{\partial z} + \Gamma W, \tag{1.11}$$

$$\frac{\partial v}{\partial t} + (v\nabla)v = -\frac{1}{\rho}\nabla p + \nu \Delta v + c\frac{\partial v}{\partial z} + \sigma\tau(T - T_0)g, \tag{1.12}$$

$$\operatorname{div} v = 0. \tag{1.13}$$

We study the spectrum of the system linearized about this stationary solution. The contents of the paper are as follows. In the next section we recall some of our previous results concerning chemical waves with complex kinetics. They allow us to reduce the reaction–diffusion system to a so-called monotone system. This reduction plays an important role because it allows us to consider chemical waves with complex kinetics. In § 3 we linearize the system about the stationary solution and show that it can be transformed to a special form. This transformation is the key point of our approach because the transformed

https://doi.org/10.1017/S095679259800357X Published online by Cambridge University Press

system possesses some special spectral properties of the Krein–Rutman type. In § 4 we discuss these spectral properties. In § 5 we obtain stability conditions for ascending fronts. Conclusions are given in § 6.

2 Reaction–diffusion systems with complex kinetics

In this section we consider the reaction-diffusion system (1.10)–(1.11) assuming that all diffusion coefficients d_i are equal to each other and equal to the coefficient of thermal diffusivity κ. We show that, by a change of variables, it can be reduced to the form

$$\frac{\partial \Theta}{\partial t} + v\nabla\Theta = a\Delta\Theta + c\frac{\partial \Theta}{\partial z} + F(\Theta), \tag{2.1}$$

where a is a diagonal matrix with positive diagonal elements, $F = (F_1, \dots, F_k)$, $\Theta = (\Theta_1, \dots, \Theta_k)$,

$$\frac{\partial F_i}{\partial \Theta_j} \geqslant 0, \quad i \neq j. \tag{2.2}$$

The last condition on the function F plays an important role in providing the stability of chemical waves with complex kinetics in the case of equality of transport coefficients.

We consider the kinetic system of equations

$$\frac{dA}{dt} = \Gamma W, \tag{2.3}$$

where A is the vector of concentrations, W is the vector of reaction rates, Γ the matrix of stoichiometric coefficients,

$$\Gamma = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{n1} \\ . & . & . \\ \gamma_{1m} & \cdots & \gamma_{nm} \end{pmatrix}, \qquad \gamma_{ij} = \beta_{ij} - \alpha_{ij}.$$

Denote by r the rank of the matrix Γ. Without loss of generality we can suppose that the first r columns of this matrix are linearly independent. We introduce the new functions u_i, $i = 1, \dots, n$ defined by

$$A_j = \sum_{i=1}^{r} \gamma_{ij} u_i + A_j^+, \tag{2.4}$$

where A_j^+ are positive constants which determine the balance polyhedron, i.e. the domain of physically relevant concentrations. Since the columns $r+1, \dots, n$ of the matrix Γ are linearly dependent on the first r columns, then we have

$$\gamma_{kj} = \sum_{1}^{r} \lambda_{ki}\gamma_{ij}, \quad k = r+1, \dots, n, \tag{2.5}$$

where λ_{ki} are some numbers. Substituting these expressions into (2.3), we obtain after some transformations

$$\frac{du_i}{dt} = W_i + \sum_{i=1}^{r} \lambda_{ki} W_k, \quad i = 1, \dots, r. \tag{2.6}$$

If the reactions are not isothermic, we need also the expression of the temperature in the

https://doi.org/10.1017/S095679259800357X Published online by Cambridge University Press

new variables:

$$T = \sum_{i=1}^{r} q_i u_i + T^+, \tag{2.7}$$

where T^+ determines the value of the adiabatic temperature and q_i is the heat release of the ith reaction.

After making the indicated substitutions, we denote the right-hand side of (2.6) by $F_i(u)$:

$$F_i(u) = W_i + \sum_{i=1}^{r} \lambda_{ki} W_k, \quad i = 1, \ldots, r. \tag{2.8}$$

We find now the conditions of monotonicity, i.e. the conditions of validity of the inequality (2.2). We assume that all functions $K_i(T)$ are nondecreasing. This condition is satisfied, in particular, for the Arrhenius temperature dependence of the reaction rate.

Direct computations give

$$\frac{\partial F_i}{\partial u_j} = W_i \left[\sum_{k=1}^{m} \frac{\alpha_{ik}\gamma_{jk}}{A_k} + \frac{K_i'(T)}{K_i(T)} q_j \right] + \sum_{l=r+1}^{n} \lambda_{li} W_i \left[\sum_{k=1}^{m} \frac{\alpha_{lk}\gamma_{jk}}{A_k} + \frac{K_i'(T)}{K_i(T)} q_j \right].$$

Sufficient conditions of monotonicity take the form

$$\alpha_{ik}\gamma_{jk} \geqslant 0, \quad \lambda_{li}\alpha_{lk}\gamma_{jk} \geqslant 0, \quad q_j \geqslant 0, \quad \lambda_{li} q_j \geqslant 0$$

for all $i, j = 1, \ldots, r,\ i \neq j,\ l = r+1, \ldots, n,\ k = 1, \ldots, m$, but these conditions are not necessary. For example, we can consider q_j of alternating signs. Moreover these conditions are not convenient for application. They can be given in a different form and simplified [21], but here we restrict ourselves to some more simple particular cases and examples.

An important particular case is that of *linearly independent* reactions. This means that the rank of the matrix Γ is n and its columns are linearly independent. In this case the conditions of monotonicity hold in particular for reactions without parallel stages where each species is consumed only in one reaction:

If each species is consumed in no more than one reaction and all reactions are exothermic, then the condition of monotonicity is satisfied.

Several examples of reaction that satisfy this condition are:

(i) Independent reaction:

$$A_1 + A_2 \to \ldots,\ A_3 + A_4 \to \ldots,\ A_5 + A_6 \to \ldots;$$

(ii) Sequential reactions:

$$A_1 \to A_2 \to A_3 \to \ldots$$

or

$$A_1 + A_2 \to A_3,\ A_3 + A_4 \to A_5,\ A_5 + A_6 \to A_7;$$

(iii) Nonbranching chain reactions, for example

$$A_2 + B \to AB + A, \quad B_2 + A \to AB + B.$$

https://doi.org/10.1017/S095679259800357X Published online by Cambridge University Press

We note, finally, that we can obtain the conditions of monotonicity for reactions with parallel stages, with heat releases of alternating signs, for linearly dependent reactions, for reversible reactions, and for other classes of reactions [21–25]. If the conditions of reducibility are satisfied, then the chemical wave exists, it is unique, globally stable, and its velocity can be found in the form of the minimax representation.

The system (2.1) can be obtained from (1.10), (1.11) under the change of variables (2.4), (2.7) with $\Theta_i = u_i$.

3 Linearization

Let $\Theta = \Theta_s(z)$, $v = 0$ be a stationary solution of the system

$$\frac{\partial \Theta}{\partial t} + v\nabla\Theta = a\Delta\Theta + c\frac{\partial \Theta}{\partial z} + F(\Theta), \tag{3.1}$$

$$\frac{\partial v}{\partial t} + (v\nabla)v = -\frac{1}{\rho}\nabla p + \nu\Delta v + c\frac{\partial v}{\partial z} + g\tau\sum_{i=1}^{k}\sigma_i(\Theta_i - \Theta_{i0}), \tag{3.2}$$

$$\operatorname{div} v = 0. \tag{3.3}$$

We suppose that the function $\Theta_s(z)$ is monotonically decreasing, i.e. each component $\Theta_{si}(z)$ of this vector-valued function is decreasing. Without loss of generality conditions at infinity for the dimensionless concentrations Θ_s can be considered in the form $\Theta_{si}(-\infty) = 1$, $\Theta_{si}(+\infty) = 0$.

We linearize the system (3.1)–(3.3) about the stationary solution:

$$\frac{\partial \theta}{\partial t} + u_z\Theta_s' = a\Delta\theta + c\frac{\partial \theta}{\partial z} + F'(\Theta_s)\theta, \tag{3.4}$$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\nabla\pi + \nu\Delta u + c\frac{\partial u}{\partial z} + g\tau\sum_{i=1}^{k}\sigma_i\theta_i, \tag{3.5}$$

$$\operatorname{div} u = 0. \tag{3.6}$$

Here θ is a perturbation of the temperature, $u = (u_x, u_z)$ of the velocity, π of the pressure. We look for the solution in the form

$$\theta(x,z,t) = e^{\lambda t}\bar{\theta}(x,z), \quad u(x,z,t) = e^{\lambda t}\bar{u}(x,z), \quad \pi(x,z,t) = e^{\lambda t}\bar{\pi}(x,z).$$

We obtain

$$\lambda\bar{\theta} + \bar{u}_z\Theta_s' = a\Delta\bar{\theta} + c\frac{\partial \bar{\theta}}{\partial z} + F'(\Theta_s)\bar{\theta}, \tag{3.7}$$

$$\lambda\bar{u} = -\frac{1}{\rho}\nabla\bar{\pi} + \nu\Delta\bar{u} + c\frac{\partial \bar{u}}{\partial z} + g\tau\sum_{i=1}^{k}\sigma_i\bar{\theta}_i, \tag{3.8}$$

$$\operatorname{div}\bar{u} = 0. \tag{3.9}$$

Suppose that the functions $\bar{\theta}$, $\bar{u}$, and $\bar{\pi}$ are square summable with respect to x for each z. Then the Fourier transforms of these functions are defined and we denote them by $\tilde{\theta}$, $\tilde{u}$,

https://doi.org/10.1017/S095679259800357X Published online by Cambridge University Press

and $\tilde{\pi}$, respectively. We have from (3.7)–(3.9)

$$\lambda\tilde{\theta} + \tilde{u}_z\Theta_s' = a(-\xi^2\tilde{\theta} + \tilde{\theta}'') + c\tilde{\theta}' + F'(\Theta_s)\tilde{\theta}, \tag{3.10}$$

$$\lambda\tilde{u}_x = -\frac{1}{\rho}i\xi\tilde{\pi} + v(-\xi^2\tilde{u}_x + \tilde{u}_x'') + c\tilde{u}_x', \tag{3.11}$$

$$\lambda\tilde{u}_z = -\frac{1}{\rho}\tilde{\pi}' + v(-\xi^2\tilde{u}_z + \tilde{u}_z'') + c\tilde{u}_z' + g\sum_{i=1}^{k}\sigma_i\tilde{\theta}_i, \tag{3.12}$$

$$i\xi\tilde{u}_x + \tilde{u}_z' = 0. \tag{3.13}$$

To eliminate $\tilde{\pi}$ we differentiate (3.11), multiply (3.12) by $i\xi$ and subtract. Introducing the function

$$w = -\tilde{u}_z'' + \xi^2\tilde{u}_z,$$

we finally obtain

$$a\tilde{\theta}'' + c\tilde{\theta}' - (\lambda E + a\xi^2 - F'(\Theta_s))\tilde{\theta} - \tilde{u}_z\Theta_s' = 0, \tag{3.14}$$

$$\tilde{u}_z'' - \xi^2\tilde{u}_z + w = 0 \tag{3.15}$$

$$vw'' + cw' - (\lambda + v\xi^2)w + \xi^2 g\sum_{i=1}^{k}\sigma_i\tilde{\theta}_i = 0 \tag{3.16}$$

$$i\xi\tilde{u}_x + \tilde{u}_z' = 0. \tag{3.17}$$

We show that the eigenvalue problems (3.10)–(3.13) and (3.14)–(3.17) are equivalent in the following sense.

Proposition 3.1 *The problem* (3.10)–(3.13) *has a nontrivial solution if and only if the problem* (3.14)–(3.16) *has a nontrivial solution.*

Proof Suppose that (3.10)–(3.13) has nonzero solution. Then $\tilde{\theta} \not\equiv 0$ and $\tilde{\theta}$, $\tilde{u}_z$, w is a solution of (3.14)–(3.16).

Suppose now that (3.14)–(3.16) has a nonzero solution. Then (3.10) is satisfied. From (3.13) we find $\tilde{u}_x$ and from (3.11) $\tilde{\pi}$. It remains to verify that (3.12) is also satisfied. We write

$$\phi(z) = \lambda\tilde{u}_z + \frac{1}{\rho}\tilde{\pi}' - v(-\xi^2\tilde{u}_z + \tilde{u}_z'') - c\tilde{u}_z' - g\sum_{i=1}^{k}\sigma_i\tilde{\theta}_i. \tag{3.18}$$

From (3.11) and (3.18),

$$-\xi^2\phi(z) = -\lambda w + v(-\xi^2 w + w'') + cw' + \xi^2 g\sum_{i=1}^{k}\sigma_i\tilde{\theta}_i.$$

It follows from (3.16) that $\phi(z) \equiv 0$.

Since for a nonzero solution $(\tilde{\theta}, \tilde{u}_z, w)$ of the system (3.14)–(3.16), $\tilde{\theta} \not\equiv 0$, then we obtain a nonzero solution of (3.10)–(3.13). The proposition is proved. □

https://doi.org/10.1017/S095679259800357X Published online by Cambridge University Press

We introduce the dimensionless variables and parameters

$$z_1 = \frac{c}{\kappa}z, \quad \theta_1 = \tilde{\theta}, \quad u_1 = \frac{\tilde{u}_z}{c}, \quad \lambda_1 = \frac{\kappa}{c^2}\lambda, \quad \xi_1 = \frac{\kappa}{c}\xi,$$

$$B(z_1) = \frac{\kappa}{c^2}F'(\Theta_s), \quad b(z_1) = -\Theta_s', \quad P = \frac{\nu}{\kappa}, \quad Le_i = \frac{a_i}{\kappa}, \quad R_i = g\sigma_i\frac{\kappa^2}{\nu c^3}.$$

In the new variables we can rewrite the system (3.14)–(3.16) in the form (the subscript 1 is omitted)

$$Le\,\theta'' + \theta' - (\lambda E + Le\,\xi^2 - B(z))\theta + b(z)u = 0, \tag{3.19}$$

$$u'' - \xi^2 u + w = 0, \tag{3.20}$$

$$Pw'' + w' - (\lambda + P\xi^2)w + P\xi^2\sum_{i=1}^{k} R_i\theta_i = 0. \tag{3.21}$$

Here Le is a diagonal matrix with diagonal elements Le_i. We recall that the numbers R_i are positive, the vector-valued function $b(z)$ is also positive since Θ_s is monotonically decreasing, and the off-diagonal elements of the matrix $B(z)$ are non-negative. This means that in each equation of the system (3.19)–(3.21) 'alien' functions enter with positive coefficients. This is a so-called monotone system which possesses some special spectral properties. We discuss them in the next section.

4 Spectral properties of linear monotone systems

We consider in this section a problem in a more general formulation than (3.19)–(3.21). Let L_1 and L_2 be elliptic operators:

$$L_1u_1 = a^{(1)}(x)\Delta u_1 + \sum_{j=1}^{m} c_j^{(1)}(x)\frac{\partial u_1}{\partial x_j} + b^{(1)}(x)u_1,$$

$$L_2u_2 = a^{(2)}(x)\Delta u_2 + \sum_{j=1}^{m} c_j^{(2)}(x)\frac{\partial u_2}{\partial x_j} + b^{(2)}(x)u_2,$$

where $x = (x_1, \ldots, x_m) \in \Omega$, Ω is an infinite cylinder with the axis in the x_1-direction and with sufficiently smooth boundary, $u_1 = (u_1^1, \ldots, u_1^k)$, $u_2 = (u_2^1, \ldots, u_2^n)$, $a^{(i)}$, $b^{(i)}$ and $c_j^{(i)}$ are smooth square matrices, $a^{(i)}$ and $c_j^{(i)}$ are diagonal matrices, $a^{(i)}$ with positive diagonal elements, $b^{(i)}$ with non-negative off-diagonal elements. We suppose that all these matrices have limits as $x_1 \to \pm\infty$, and the matrices

$$b_{\pm}^{(i)} = \lim_{x_1\to\pm\infty} b^{(i)}(x)$$

have all eigenvalues in the left half-plane. The last limit is supposed to be uniform in x', the variable in the section of the cylinder.

We consider the following eigenvalue problem:

$$L_1v_1 + d_1v_2 = \lambda v_1, \quad \frac{\partial v_1}{\partial \nu}\,|_{\partial\Omega} = 0, \tag{4.1}$$

$$L_2v_2 + d_2v_1 = 0, \quad \frac{\partial v_2}{\partial \nu}\,|_{\partial\Omega} = 0. \tag{4.2}$$

https://doi.org/10.1017/S095679259800357X Published online by Cambridge University Press

Here $d_1(x)$ and $d_2(x)$ are smooth $k \times n$ and $n \times k$, respectively, matrices with nonnegative elements. The pecularity of this problem is that it consists of two parts and the eigenvalue λ enters only in one of them.

We assume that the block matrix

$$\begin{pmatrix} b^{(1)} & d_1 \\ d_2 & b^{(2)} \end{pmatrix}$$

is functionally irreducible and its principal eigenvalue is negative. The functional irreducibility means that if we replace all elements of a functional matrix by their norms, then we obtain a numerical matrix, which is supposed to be irreducible.

We introduce also the function $v = (v_1, v_2)$ and the operator

$$Lv = (L_1 v_1 + d_1 v_2, L_2 v_2 + d_2 v_1).$$

We begin with some auxiliary results.

Lemma 4.1 *If $u(x) \geqslant 0$, $x \in \bar{\Omega}$ is a nonzero solution of the problem*

$$Lu \leqslant 0, \quad \frac{\partial u}{\partial v} \mid_{\partial\Omega} = 0, \tag{4.3}$$

then $u(x) > 0$ in $\bar{\Omega}$.

Lemma 4.2 *Let $q_\pm = (q_{\pm 1}, q_{\pm 2})$ be positive vectors such that*

$$b_\pm^{(1)} q_{\pm 1} + d_{\pm 1} q_{\pm 2} < 0, \quad b_\pm^{(2)} q_{\pm 2} + d_{\pm 2} q_{\pm 1} < 0.$$

Suppose that a number r is such that

$$b^{(1)}(x) q_{+1} + d_1(x) q_{+2} < 0, \quad b^{(2)}(x) q_{+2} + d_2(x) q_{+1} < 0$$

for $x_1 > r$ and

$$b^{(1)}(x) q_{-1} + d_1(x) q_{-2} < 0, \quad b^{(2)}(x) q_{-2} + d_2(x) q_{-1} < 0$$

for $x_1 < -r$. If the vector-valued function $u(x)$ satisfies the problem

$$L_1 u_1 + d_1 u_2 \leqslant \gamma u_1, \quad L_2 u_2 + d_2 u_1 \leqslant 0, \quad (\gamma \geqslant 0), \quad \frac{\partial u}{\partial v} = 0 \tag{4.4}$$

for $| x_1 | \geqslant r$ and conditions

$$u \mid_{x_1 = \pm r} \geqslant 0, \quad \lim_{N \to \pm\infty} \inf_{x_1 \geqslant N (x_1 \leqslant N)} u(x) \geqslant 0,$$

then

$$u(x) \geqslant 0 \ \text{for} \ \ | x_1 | \geqslant r.$$

Lemma 4.3 *With the conditions of the previous lemma, a bounded solution u of the problem*

$$L_1 u_1 + d_1 u_2 = \gamma u_1, \quad L_2 u_2 + d_2 u_1 = 0, \quad u \mid_{x_1 = r} = 0, \quad \frac{\partial u}{\partial v} \mid_{x \in \partial\Omega} = 0$$

for $x_1 \geqslant r$ is identically zero.

https://doi.org/10.1017/S095679259800357X Published online by Cambridge University Press

The proofs of this lemmas are similar to the proofs of the corresponding results in Volpert & Volpert [26].

Lemma 4.4 *Suppose that the spectrum of the operator L_2 lies in the left half-plane. Then for the problem in the half-cylinder $\Omega \cap \{x_1 \geqslant 0\}$*

$$\frac{\partial u_1}{\partial t} = L_1 u_1 + d_1 u_2, \quad L_2 u_2 + d_2 u_1 = 0, \tag{4.5}$$

$$\frac{\partial u}{\partial \nu} \mid_{\partial\Omega} = 0, \quad u \mid_{x_1=0} = \phi(x) \mid_{x_1=0}, \quad u_1 \mid_{t=0} = \phi_1(x) \tag{4.6}$$

if $\phi(x) \geqslant 0$, then $u(x) > 0$. Here $\phi(x) = (\phi_1(x), \phi_2(x))$ is sufficiently smooth, and $\partial\phi/\partial\nu = 0$.

Proof We show the positivity of the solution. We express u_2 from the second equation in (4.5),

$$u_2 = -M(d_2 u_1),$$

and substitute into the first equation. We have the following problem:

$$\frac{\partial u_1}{\partial t} = L_1 u_1 - d_1 M(d_2 u_1), \quad \frac{\partial u_1}{\partial \nu} \mid_{\partial\Omega} = 0, \quad u_1 \mid_{x_1=0} = \phi_1(x). \tag{4.7}$$

Consider the problems

$$\frac{\partial v_1}{\partial t} = L_1 v_1 + f(x), \quad \frac{\partial v_1}{\partial \nu} \mid_{\partial\Omega} = 0, \quad v_1 \mid_{x_1=0} = \phi_1(x) \tag{4.8}$$

$$f(x) = \max(0, -d_1 M(d_2 v_1)),$$

and

$$\frac{\partial w_1}{\partial t} = L_1 w_1, \quad \frac{\partial w_1}{\partial \nu} \mid_{\partial\Omega} = 0, \quad w_1 \mid_{x_1=0} = \phi_1(x) \tag{4.9}$$

with the same initial condition $\phi_1(x)$. Then

$$v_1(x,t) \geqslant w_1(x,t) > 0, \quad x \in \Omega, t > 0.$$

Then

$$-d_1 M(d_2 v_1) > 0$$

and the solution of the problem (4.8) coincides with the solution of the problem (4.7). The lemma is proved. □

Remark A comparison theorem follows from this lemma.

Theorem 4.5 *Suppose that the problem*

$$Lw = 0, \quad \frac{\partial w}{\partial \nu} \mid_{\partial\Omega} = 0 \tag{4.10}$$

has a positive solution w. If

$$\lim_{x_1 \to \pm\infty} w(x) = 0, \tag{4.11}$$

uniformly in x', then the following assertions hold:

https://doi.org/10.1017/S095679259800357X Published online by Cambridge University Press

(1) *The problem*

$$L_1u_1 + d_1u_2 = \lambda u_1, \quad L_2u_2 + d_2u_1 = 0, \quad \frac{\partial u}{\partial \nu} \mid_{\partial\Omega} = 0, \tag{4.12}$$

where $u(x) \to 0$ *as* $x_1 \to \pm\infty$ *uniformly in* x', *only has trivial solutions for* $\operatorname{Re}\lambda \geqslant 0$, $\lambda \neq 0$.

(2) *Each solution of the problem*

$$Lu = 0, \quad \frac{\partial u}{\partial \nu} \mid_{\partial\Omega} = 0 \tag{4.13}$$

has the form $u(x) = kw(x)$, *where* k *is a constant.*

(3) *There is no nonzero solution of the problem*

$$L^s u = 0, \quad L^{s-1}u \neq 0, \quad \frac{\partial (L^k u)}{\partial \nu} = 0, \quad k = 0, \ldots, s-1$$

for an integer $s > 1$.

Proof

(1) Consider first the case where in (4.12) $\lambda = \alpha + i\beta$, $\alpha \geqslant 0$, $\beta \neq 0$. Suppose that there exists a nonzero solution $u(x) = u^{(1)}(x) + iu^{(2)}(x)$ of this problem.

We consider also the problem

$$\frac{\partial v_1}{\partial t} = L_1v_1 + d_1v_2 - \alpha v_1, \quad L_2v_2 + d_2v_1 = 0, \quad \frac{\partial v}{\partial \nu} \mid_{\partial\Omega} = 0, \quad v_1(x,0) = u^1(x). \tag{4.14}$$

Its solution is

$$v(x,t) = v^{(1)}(x)\cos(\beta t) - v^{(2)}(x)\sin(\beta t).$$

We write $\hat{v} = (\mid v_1 \mid, \ldots, \mid v_n \mid)$, where $v = (v_1, \ldots, v_n)$. We can choose a number $r > 0$ which satisfies the conditions of Lemma 4.2 (see Volpert *et al.* [21, Chap. 4, Lemma 5.1]). Further, we can choose a positive τ such that

$$\hat{v}(x) \leqslant \tau w(x) \quad \text{for} \quad \mid x_1 \mid \leqslant r \tag{4.15}$$

and there are k and $x^0 \in \bar{\Omega}$, $\mid x_1^0 \mid \leqslant r$ for which

$$\mid v_k(x^0) \mid = \tau w_k(x^0). \tag{4.16}$$

In the domain $\Omega_r = \Omega \cap \{x_1 \geqslant r\}$, consider the problem

$$\frac{\partial y_1}{\partial t} = L_1y_1 + d_1y_2 - \alpha y_1, \quad L_2y_2 + d_2y_1 = 0, \quad y(x,t) \mid_{x_1=r} = \hat{v}(x) \mid_{x_1=r}, \tag{4.17}$$

$$\frac{\partial y}{\partial \nu} \mid_{x\in\partial\Omega} = 0, \tag{4.18}$$

$$y_1(x,0) = \hat{v}_1(x), \tag{4.19}$$

and the corresponding stationary problem

$$L_1\bar{y}_1 + d_1\bar{y}_2 - \alpha\bar{y}_1 = 0, \quad L_2\bar{y}_2 + d_2\bar{y}_1 = 0, \tag{4.20}$$

$$\bar{y}(x) \mid_{x_1=r} = \hat{v}(x) \mid_{x_1=r}, \quad \bar{y}(x) \mid_{x_1=\infty} = 0, \quad \frac{\partial \bar{y}}{\partial \nu} \mid_{x\in\partial\Omega} = 0.$$

https://doi.org/10.1017/S095679259800357X Published online by Cambridge University Press

It follows from Lemma 4.3 that the solution of this problem is unique, and we will show that the solution of the problem (4.17)–(4.19) converges to it. Let $\tilde{y} = y - \bar{y}$. The function $\tilde{y}$ is a solution of the problem

$$\frac{\partial \tilde{y}}{\partial t} = L_1\tilde{y}_1 + d_1\tilde{y}_2 - \alpha\tilde{y}_1, \quad L_2\tilde{y}_2 + d_2\tilde{y}_1 = 0, \quad \tilde{y}(x,t)\,|_{x_1=r} = 0, \tag{4.21}$$

$$\frac{\partial \tilde{y}}{\partial \nu}\,|_{x\in\partial\Omega} = 0, \tag{4.22}$$

$$\tilde{y}_1(x,0) = \hat{v}_1(x) - \hat{y}_1(x). \tag{4.23}$$

We need to show that $\tilde{y}$ converges to zero as t increases. To do this we estimate the solution of the problem (4.21)–(4.23) by the solution $y^*(x,t)$ of the problem

$$\frac{\partial y_1^*}{\partial t} = L_1 y_1^* + d_1 y_2^* - \alpha y_1^*, \quad L_2 y_2^* + d_2 y_1^* = \mathrm{e}^{-t}\tau(L_2 q_{+2} + d_2 q_{+1}), \quad y^*(x,t)\,|_{x_1=r} = \mathrm{e}^{-t}\tau q_{+1}, \tag{4.24}$$

$$\frac{\partial y^*}{\partial \nu}\,|_{x\in\partial\Omega} = 0, \tag{4.25}$$

$$y_1^*(x,0) = \tau q_{+1} \tag{4.26}$$

for positive and sufficiently large τ. By virtue of the inequality

$$L_1 q_{+1} + d_1 q_{+2} < 0, \quad L_2 q_{+2} + d_2 q_{+1} < 0,$$

$y^*(x,t)$ becomes less than $\tau\, q_+$, and consequently, decreases in time for every x and converges to some stationary solution. From Lemma 4.3, it follows that it is identically zero.

We have

$$y_*(x,t) \leqslant \tilde{y}(x,t) \leqslant y^*(x,t),$$

where $y_*(x,t)$ is a solution of the similar problem with a negative and sufficiently large in the absolute value τ. Since $y_*(x,t)$ and $y^*(x,t)$ tend to zero as $t \to \infty$, we finally obtain

$$\lim_{t\to\infty} y(x,t) = \bar{y}(x).$$

Since $v(x,t) \leqslant \hat{u}(x)$ for $x_1 \geqslant r$, then

$$v(x,t) \leqslant y(x,t) \ \textit{for} \ \ x_1 \geqslant r, \ t \geqslant 0.$$

From this,

$$v(x,t) = v\left(x, t + \frac{2\pi n}{\beta}\right) \leqslant y\left(x, t + \frac{2\pi n}{\beta}\right).$$

Passing to the limit as $n \to \infty$, we have

$$v(x,t) \leqslant \bar{y}(x) \ \textit{for} \ \ x_1 \geqslant r, \ t \geqslant 0.$$

From Lemma 4.2 applied to the function $\tau w(x) - \bar{y}(x)$, it follows that $\bar{y}(x) \leqslant \tau w(x)$ for $x_1 \geqslant r$. Hence

$$v(x,t) \leqslant \tau w(x) \tag{4.27}$$

for $x_1 \geqslant r, \ t \geqslant 0$. Similarly, this inequality can be proved for $x_1 \leqslant -r$. From this and (4.15), it follows that (4.27) is valid for all $x \in \Omega$.

https://doi.org/10.1017/S095679259800357X Published online by Cambridge University Press

The function $z = \tau w - v \geqslant 0$ is a solution of the system

$$\frac{\partial z_1}{\partial t} = L_1 z_1 + d_1 z_2 - \alpha z_1 + \alpha \tau w_1, \quad L_2 z_2 + d_2 z_1 = 0.$$

Since $z(x,t) \geqslant 0$, it is not identically zero, and it is periodic in t, then, from the positiveness theorem, it follows that $z(x,t) > 0$ for all $x \in \bar{\Omega}$ and $t > 0$. But this contradicts (4.16). Indeed, we can choose t_0 such that for a kth element z^k of the vector-valued function z

$$z^k(x_0, t) = \tau w^k(x_0) - \mid v^k(x_0) \mid = 0.$$

This contradiction proves the theorem for nonreal λ.

Let now λ be real and positive, and $u(x)$ be a nonzero solution of (4.12). Without loss of generality we can assume that at least one of the components of the vector-valued function $u(x)$ has negative values. Otherwise, we can change its sign. Consider the function $v = u + \tau w$ and choose a positive τ such that $v(x) \geqslant 0$ for $\mid x_1 \mid \leqslant r$, but such that it is not strictly positive, and $v^k(x^0) = 0$ for some k and x^0. We have

$$L_1 v_1 + d_1 v_2 = \lambda v_1 - \lambda \tau w_1, \quad L_2 v_2 + d_2 v_1 = 0, \quad \frac{\partial v}{\partial v} \mid_{\partial\Omega} = 0. \tag{4.28}$$

By virtue of Lemma 4.2, $v(x) \geqslant 0$ in the whole Ω. If $\lambda > 0$ and x^0 is an interior point, then the kth equation in (4.28) gives a contradiction. So $v^k(x) > 0$ for $x \in \Omega$ and, consequently, $x_0 \in \partial\Omega$. From the inequality

$$a_k \Delta v^k + \sum_{j=1}^{m} c_{kj} \frac{\partial v^k}{\partial x_j} + b_{kk} v^k - \lambda v^k \leqslant 0,$$

it follows that at $x = x_0$, $\partial v^k / \partial v < 0$. This contradicts the boundary condition in (4.28).

(2) If $\lambda = 0$, we obtain as above that $v(x)$ is non-negative and not strictly positive solution of the equation $Lv = 0$. Then $v(x) \equiv 0$.

(3) Let now $s > 1$. Since the only solution of the equation $Lu = 0$ up to a constant factor is $w(x)$, then

$$L^{s-1} u = k w(x).$$

We put $k = -1$ and denote $v = L^{s-2} u$. Then $Lv = -w(x)$. We can choose τ such that $z = v + \tau w \geqslant 0$ but not strictly positive, and $z(x_0) = 0$. The τ here is not necessarily positive.

We have $w(x) > 0$ in $\bar{\Omega}$ and $Lz < 0$. We obtain a contradiction in signs for $x_0 \in \Omega$ and a contradiction with the boundary condition for $x_0 \in \partial\Omega$ and the theorem is proved. □

Thus, we have shown that if the problem

$$L_1 v_1 + d_1 v_2 = \lambda v_1, \quad L_2 v_2 + d_2 v_1 = 0, \quad \frac{\partial v}{\partial v} \mid_{\partial\Omega} = 0, \quad v \mid_{x_1 = \pm\infty} = 0 \tag{4.29}$$

https://doi.org/10.1017/S095679259800357X Published online by Cambridge University Press

has a real eigenvalue λ_0 with a positive eigenfunction, then all other eigenvalues lie in the half-plane $\mathrm{Re}\,\lambda < \lambda_0$. Thus λ_0 is the principal eigenvalue.

We consider now the eigenvalue problem

$$L_1 v = \lambda v, \quad \frac{\partial v}{\partial \nu} \mid_{\partial\Omega} = 0$$

with the same operator L_1 as above. We recall that the principal eigenvalue λ_0 of this operator is real and that the corresponding eigenfunction $w(x)$ is positive in $\bar{\Omega}$ [26].

Everywhere below the limit as $x_1 \to \infty$ is supposed to be uniform in x'.

Theorem 4.6 *Let λ_0 be the principal eigenvalue of the problem*

$$L_1 v = \lambda v, \quad \frac{\partial v}{\partial \nu} \mid_{\partial\Omega} = 0, \quad v \mid_{x_1 = \pm\infty} = 0$$

and all other points of spectrum lie in the half-plane $\mathrm{Re}\,\lambda < \lambda_0 - \epsilon$ for some positive ϵ. Then the following inequality holds:

$$\inf_{x\in\Omega, i} \frac{(L_1 u)_i}{u_i} \leqslant \lambda_0 \leqslant \sup_{x\in\Omega, i} \frac{(L_1 u)_i}{u_i}. \tag{4.30}$$

Here u is an arbitrary sufficiently smooth function positive in $\bar{\Omega}$ that satisfies the conditions

$$\frac{\partial u}{\partial \nu} \mid_{\partial\Omega} = 0, \quad u \mid_{x_1 = \pm\infty} = 0.$$

Proof Suppose that for some function $u(x)$, the right inequality in (4.30) is not satisfied. Then $Lu < \lambda_0 u$. We can choose τ, not necessarily positive, such that the function $v(x) = u(x) + \tau w(x)$ is non-negative for $|\, x_1 \,| \leqslant r$, but not strictly positive. Here r is from Lemma 4.2. Since $L_1 v < \lambda_0 v$, then, by virtue of this lemma, $v(x) \geqslant 0$ in Ω. We obtain a contradiction with the maximum principle.

Suppose now that the left inequality in (4.30) is not satisfied for some function $u(x)$. Then there exists $\lambda_1 > \lambda_0$ such that

$$L_1 u > \lambda_1 u. \tag{4.31}$$

Consider the problem

$$\frac{\partial v}{\partial t} = L_1 v - \lambda_1 v, \quad \frac{\partial v}{\partial \nu} \mid_{\partial\Omega} = 0, \quad v(x,0) = u(x).$$

From (4.31), it follows that $v(x,t)$ increases in t for every x. Then

$$v(x,t) \geqslant u(x), \quad x \in \bar{\Omega}, \quad t \geqslant 0.$$

On the other hand, all spectrum of the operator $L_1 - \lambda_1 I$ is in the left half-plane. Hence $v(x,t) \to 0$ as $t \to \infty$. This contradiction proves the theorem. □

Corollary *The following minimax representation of the principal eigenvalue holds:*

$$\lambda_0 = \inf_{u\in K} \sup_{x\in\Omega, i} \frac{(L_1 u)_i}{u_i} = \sup_{u\in K} \inf_{x\in\Omega, i} \frac{(L_1 u)_i}{u_i}. \tag{4.32}$$

Here K is a class of sufficiently smooth functions satisfying the boundary conditions and positive in $\bar{\Omega}$.

https://doi.org/10.1017/S095679259800357X Published online by Cambridge University Press

For the proof it is sufficient to note that $w(x) \in K$.

The next theorem determines the behaviour of the principal eigenvalue as the matrix $b^{(1)}(x)$ increases.

Theorem 4.7 *Let the principal eigenvalues λ and $\tilde{\lambda}$ of the operators L_1 and $\tilde{L} = L_1 + \mathrm{d}(x)$, where $\mathrm{d}(x)$ is a non-zero matrix with $\mathrm{d}(x) \geqslant 0$, be real and the corresponding eigenfunctions be positive. If the matrix $b^{(1)}(x)$ is functionally irreducible, then $\tilde{\lambda} > \lambda$.*

Proof From the minimax representation it follows that $\tilde{\lambda} \geqslant \lambda$. We show that the inequality is strict.

Denote by $w(x)$ and $w_1(x)$, respectively, the eigenfunctions corresponding to the principal eigenvalues of the operators L_1 and $\tilde{L}$. Suppose that $\tilde{\lambda} = \lambda$. Then $L_1 w_1 \leqslant \lambda w_1$. Applying the usual construction, we find a value of τ such that the function $z = w_1 - \tau w$ is nonnegative but not strictly positive in $\bar{\Omega}$. It satisfies equation

$$L_1 z - \lambda z + \mathrm{d}(x) w_1 = 0.$$

From the maximum principle we obtain that $z \equiv 0$ and $\mathrm{d}(x) \equiv 0$. This contradiction proves the theorem. □

5 Ascending fronts

We apply now results of the previous section to study the eigenvalue problem (3.19)–(3.21) in the case $Le_i = 1$. Consider the matrix of zero-order terms of this system

$$A(z) = \begin{pmatrix} B(z) - \xi^2 E_n & 0_n^T & b(z) \\ P\xi^2 R & -P\xi^2 & 0 \\ 0_n & 1 & -\xi^2 \end{pmatrix}.$$

Here E_n is the identity matrix, 0_n is a n-vector consisting of zeros, 0_n^T is the transposed vector, $R = (R_1, \ldots, R_n)$. It is a functionally irreducible matrix with nonnegative off-diagonal elements. The limiting matrix at infinity

$$A_{\pm} = \lim_{z \to \pm\infty} A(z) = \begin{pmatrix} B_{\pm} - \xi^2 E_n & 0_n^T & 0 \\ P\xi^2 R & -P\xi^2 & 0 \\ 0_n & 1 & -\xi^2 \end{pmatrix}$$

has the eigenvalues

$$\mu_i^{\pm} - \xi^2, \ i = 1, \ldots, k, \ -P\xi^2, \ -\xi^2,$$

where $\mu_i^{\pm}$ are the eigenvalues of the matrix $B_{\pm}$. We suppose that they all have negative real parts. Then conditions of the previous section are satisfied and we can apply its results.

Denote by L_λ the operator which corresponds to the left-hand side of (3.19)–(3.21). If λ is real, then the principal eigenvalue $\eta_0(\lambda)$ of this operator can be found as

$$\eta_0(\lambda) = \inf_{\rho \in K} \sup_{z,i} \frac{(L_\lambda \rho)_i}{\rho_i} = \sup_{\rho \in K} \inf_{z,i} \frac{(L_\lambda \rho)_i}{\rho_i}.$$

We define the convective instability boundary as the values of parameters for which the

https://doi.org/10.1017/S095679259800357X Published online by Cambridge University Press

eigenvalue problem (3.19)–(3.21) has a simple eigenvalue $\lambda = 0$ and all other eigenvalues lie in the left half-plane.

Theorem 5.1 *The convective instability boundary is given by*

$$\eta_0(0) = 0. \tag{5.1}$$

Proof Let (5.1) hold. Then $\lambda = 0$ is the eigenvalue of the problem (3.19)–(3.21) and the corresponding eigenfunction is positive. From Theorem 4.5, it follows that $\lambda = 0$ is the principal eigenvalue.

We show now that the inverse is also correct: all points on the convective instability boundary satisfy (5.1). Suppose that $\lambda = 0$ is an eigenvalue of the problem (3.19)–(3.21) and there are no eigenvalues with $\operatorname{Re}\lambda > 0$. Then $\eta = 0$ is an eigenvalue of the operator L_0. We need to prove that it is the principal eigenvalue. Suppose it is not so. Then $\eta_0(0) > 0$. We show that there exists $\lambda > 0$ such that $\eta_0(\lambda) = 0$.

Indeed, $\eta_0(\lambda)$ is a continuous function of λ. Thus it is sufficient to prove that $\eta_0(\lambda) \leqslant 0$ for λ large enough. To show this we use the minimax representation of the principal eigenvalue and put

$$\tilde{\theta}_i = \tilde{u}_z = \frac{1}{1+z^2}, \quad w = -\mu\tilde{u}_z.$$

For this test function

$$\sup_{z,i} (L_0\rho)_i \leqslant 0$$

if λ and μ are sufficiently large, and the principal eigenvalue is nonpositive.

Thus we have shown that $\eta_0(\lambda) = 0$ for some positive λ. This λ is an eigenvalue of (3.19)–(3.21). The theorem is proved. □

We illustrate an application of this theorem with the following example. We consider the system (3.19)–(3.21) with $Le = 1$ and $\lambda = 0$, in the case of a single reaction. In this case we have a system of three equations. Instead of the corresponding eigenvalue problem, we consider the following eigenvalue problem with constant coefficients:

$$\theta'' + \theta' + (B_m - \xi^2)\theta + b_m u = \eta\theta, \tag{5.2}$$

$$u'' - \xi^2 u + w = \eta u, \tag{5.3}$$

$$Pw'' + w' - P\xi^2 w + PR\xi^2\theta = \eta w, \tag{5.4}$$

where

$$B_m = \max B(z), \quad b_m = \max b(z).$$

It is easy to verify that if the spectrum of this problem lies in the left half-plane, then it is also the case for the original problem with variable coefficients. In its turn the problem (5.2)–(5.4) has the spectrum in the left half-plane if the principal eigenvalue of the matrix

$$\begin{pmatrix} B_m - \xi^2 & 0 & b_m \\ P\xi^2 R & -P\xi^2 & 0 \\ 0 & 1 & -\xi^2 \end{pmatrix}$$

https://doi.org/10.1017/S095679259800357X Published online by Cambridge University Press

is negative. Thus, we obtain a sufficient condition of absence of convective instability. In this case it has the form

$$R < \frac{\xi^2(\xi^2 - B_m)}{b_m}, \quad (\xi^2 > B_m).$$

6 Conclusions

In this work we have studied the influence of natural convection on the stability of an upward propagating reaction front in case of equality of transport coefficients. Without convection the front is stable. We have shown that convection can lead to instability of the front, and we obtained a minimax representation for the stability boundary.

One of the conventional methods to study stability of combustion waves is based on approximation of an infinitely narrow reaction zone. However if the reaction zone is not narrow, as for example for branching chain flames, or if the chemical kinetics are complex, this method is not easily applicable.

We have developed another approach to the problem. We have shown that the system linearized about a stationary solution can be reduced to so-called monotone system, for which the principal eigenvalue is real and can be found in the form of a minimax representation. This allows us to obtain an exact representation for the stability boundary. This representation can be used to study qualitative properties of the problem, and in some cases its quantitative characteristics. We note in particular that $\eta_0(\lambda)$ is a decreasing function of λ and an increasing function of R. Then we can conclude that there is a critical value of the Rayleigh number, R_c such that the spectrum lies in the left half-plane for $R < R_c$ and there are points of the spectrum in the right half-plane for all $R > R_c$.

Another example is given in the end of the previous section. It shows that perturbations with sufficiently large wavenumbers decay. Both conclusions are in good agreement with the results obtained by the infinitely narrow reaction zone method for reaction fronts in liquids in the case of one-step kinetics [27].

If we consider (5.1) as a curve in a plane of parameters, then the real principal eigenvalue intersects the imaginary axis at zero when the parameters intersect this curve. We can put the question whether the loss of stability always happens in this way, or it is possible that there exists another stability boundary where a pair of complex conjugate eigenvalues intersect the imaginary axis at pure imaginary values. As is well known, these two cases are essentially different. The first case can lead to a bifurcation of a stationary convective structure, while the second to appearance of periodic in time modes (Hopf bifurcation). Theorem 5.1 does not give an answer to this question.

We can expect, however, that the principal eigenvalue of the problem (4.1), (4.2) is real. This result can be probably obtained as generalization of the corresponding results in Volpert & Volpert [26]. From this it would follow that the loss of stability occurs because the real eigenvalue intersects the imaginary axis at zero. The mathematical justification of this conjecture is a subject for further analysis.

Our last remark concerns location of the continuous spectrum. In the beginning of the previous section we have assumed that all eigenvalues of the matrices A_+ and A_- are in the left half-plane. Then the continuous spectrum of the operator L_λ lies in the left half-plane, and we can use results of § 4. If all eigenvalues of the matrices B_+ and B_- are

https://doi.org/10.1017/S095679259800357X Published online by Cambridge University Press

in the left half-plane then this condition is satisfied. However it can be also satisfied under weaker conditions on $B_{\pm}$. Obviously the principal eigenvalue of the matrix $A_{\pm}$ is negative if the principal eigenvalue of the matrix $B_{\pm}$ is less than ξ^2. If the principal eigenvalue of the matrix $B_{\pm}$ is greater than ξ^2, then we can introduce a weighted norm which shifts the continuous spectrum to the left.

References

[1] Chechilo, N. M., Khvilivitsky, R. Ya. & Enikolopyan, N. S. (1972) The phenomenon of propagation of the polymerization reaction. *Dokl. Phys. Chemistry* **204**(4–6), 512–513.

[2] Chechilo, N. M. & Enikolopyan, N. S. (1974) Structure of the polymerization wave front and propagation mechanism of the polymerization reaction. *Dokl. Phys. Chemistry* **214**(4–6), 174–176.

[3] Bowden, G., Garbey, M., Ilyashenko, V., Pojman, J., Solovyov, S., Taik, A. & Volpert, V. (1997) The effect of convection on a propagating front with a solid product: comparison of theory and experiments. *J. Chem. Physics B* **101**, 678–686.

[4] Garbey, M., Taik, A. & Volpert, V. (1996) Linear stability analysis of reaction fronts in liquids. *Quart. Appl. Math.* **54**(2), 225–247.

[5] Pojman, J. A., Craven, R., Khan, A. & West, W. (1992) Convective instabilities in traveling fronts of addition polymerization. *J. Phys. Chem.* **96**, 7466–7472.

[6] Edwards, B. F., Wilder, J. W. & Showalter, K. (1991) Onset of convection for autocatalytic reaction fronts: lateraly unbounded system. *Phys. Rev. A* **43**, 749.

[7] Masere, J., Vasquez, D. A., Edwards, B. F., Wilder, J. W. & Showalter, K. (1994) Nonaxisymmetric and axisymmetric convection in propagating reaction–diffusion fronts. *J. Phys. Chem.* **98**, 6505–6508.

[8] Pojman, J. A. & Epstein, I. R. (1990) Convective effects on chemical waves. 1. Mechanisms and stability criteria. *J. Phys. Chem.* **94**(12), 4966–4972.

[9] Pojman, J. A., Epstein, I. R., McManus, T. J. & Scowalter, K. (1991) Convective effects on chemical waves. 2. Simple convection in the iodate-arsenous acid system. *J. Phys. Chem.* **95**(3), 1299–1306.

[10] Pojman, J. A., Nagy, I. P. & Epstein, I. R. (1991) Convective effects in chemical waves. 3. Multicomponent convection in the iron(II) – nitric acid system. *J. Phys. Chem.* **95**(3), 1306–1311.

[11] Vasquez, D. A., Wilder, J. W. & Edwards, B. F. (1993) Hydrodynamic instability of chemical waves. *J. Chem. Phys.* **98**(3), 2138–2143.

[12] Vasquez, D. A., Edwards, B. F. & Wilder, J. W. (1991) Onset of convection for autocatalytic reaction fronts: Laterally bounded systems. *Physical Review A* **43**, 6694–6699.

[13] Vasquez, D. A., Littley, J. M., Wilder, J. W. & Edwards, B. F. (1994) Chemical Wave Velocity Near the Onset of Convection. *Phys. Rev. E* **50**, 280–284.

[14] Vasquez, D. A., Littley, J. M., Wilder, J. W. & Edwards, B. F. (1994) Convection in Chemical Waves. *Phys. Rev. E* **50**, 280–284.

[15] Wilder, J. W., Edwards, B. F., Vasquez, D. A. & Sivashinsky, G. I. (1994) Derivation of a nonlinear front evolution equation for chemical waves involving convection. *Physica D* **73**, 217–226.

[16] Gutkowicz-Krusin, D. & Ross, J. (1980) Rayleigh–Benard instability in reactive binary fluids. *J. Chem. Phys.* **72**, 3577–3587.

[17] Gutkowicz-Krusin, D. & Ross, J. (1980) Rayleigh–Benard instability in n-component reactive fluids. *J. Chem. Phys.* **72**, 3588–3595.

[18] Viljoen, H. J., Gatica, J. E. & Hlavacek, V. (1990) Bifurcation analysis of chemically driven convection. *Chem. Eng. Sci.* **45**(2), 503–517.

[19] Matkowsky, B. J. & Sivashinsky, G. I. (1979) An asymptotic derivation of two models in

https://doi.org/10.1017/S095679259800357X Published online by Cambridge University Press

flame theroy associated with the constant density approximation. *SIAM J. Appl. Math.* **37**, 686–699.

[20] Matkowsky, B. J. & Sivashinsky, G. I. (1979) Acceleration effects on the stability of flame propagation. *SIAM J. Appl. Math.* **37**, 669–685.

[21] Volpert, A. I., Volpert, V. A. & Volpert, V. A. (1994) *Travelling wave solutions of parabolic systems.* American Math. Society, Translation of Mathematical Monographs, Vol. 140, Providence, RI.

[22] Volpert, V. A. & Volpert, A. I. (1989) Existence and stability of traveling waves in chemical kinetics. In: *Dynamics of Chemical and Biological Systems*, pp. 56–131. Nauka, Novosibirsk (in Russian).

[23] Volpert, V. A. & Volpert, A. I. (1989) Waves of chemical transformation having complex kinetics . *Dokl. Phys. Chemistry* **309**(1–3), 877–879.

[24] Volpert, V. A. & Volpert, A. I. (1990) Existence and stability of waves in chemical kinetics. *Khimicheskaya Fizika* **9**(2), 238–245 (in Russian).

[25] Volpert, V. A. & Volpert, A. I. (1990) Some mathematical problems of wave propagation in chemical active media. *Khimicheskaya Fizika* **9**(8), 1118–1127 (in Russian).

[26] Volpert, V. A. & Volpert, A. I. (1996) Location of spectrum and stability of solutions for monotone parabolic systems. Preprint No. 222, University Lyon 1, France.

[27] Garbey, M., Taik, A. & Volpert, V. (1998) Influence of natural convection on stability of reaction fronts in liquids. *Quart. Appl. Math.* **56**(1), 1–35.

https://doi.org/10.1017/S095679259800357X Published online by Cambridge University Press