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RESEARCH ARTICLE

# On local Galois deformation rings: generalised reductive groups

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**Abstract**

We study deformation theory of mod $p$ Galois representations of $p$-adic fields with values in generalised reductive group schemes, such as $L$-groups and $C$-groups. We show that the corresponding deformation rings are complete intersections of expected dimension. We determine their irreducible components in many cases and show that they and their special fibres are normal and complete intersection.

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## 1. Introduction

Let $p$ denote any prime number, let $F$ be a finite extension of $\mathbb{Q}_p$, and let $\Gamma_F$ denote its absolute Galois group. Let $L$ be another finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, uniformiser $\varpi$ and residue field $k = \mathcal{O}/\varpi$.

Let $G$ be a smooth affine group scheme over $\mathcal{O}$, such that its neutral component $G^0$ is reductive and the component group $G/G^0$ is a finite group scheme over $\mathcal{O}$. We call such group schemes *generalised reductive*. We do not make any assumptions on the prime $p$ regarding $G$.

We fix a continuous representation $\overline{\rho} : \Gamma_F \to G(k)$ and denote by $D_{\overline{\rho}}^{\square} : \mathfrak{A}_{\mathcal{O}} \to \mathrm{Set}$ the functor from the category $\mathfrak{A}_{\mathcal{O}}$ of local artinian $\mathcal{O}$-algebras with residue field $k$ to the category of sets, such that for $(A, \mathfrak{m}_A) \in \mathfrak{A}_{\mathcal{O}}$, $D_{\overline{\rho}}^{\square}(A)$ is the set of continuous representations $\rho_A : \Gamma_F \to G(A)$, such that $\rho_A(\gamma) \equiv \overline{\rho}(\gamma) \pmod{\mathfrak{m}_A}$, for all $\gamma \in \Gamma_F$. The functor $D_{\overline{\rho}}^{\square}$ of framed deformations of $\overline{\rho}$ is pro-represented by a complete local noetherian $\mathcal{O}$-algebra $R_{\overline{\rho}}^{\square}$ with residue field $k$.

Böckle, Iyengar and VP have studied in [6] ring theoretic properties of $R_{\overline{\rho}}^{\square}$, when $G = \mathrm{GL}_d$. In this paper we extend the results of [6] to an arbitrary generalised reductive group $G$.

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**Theorem 1.1** (Corollaries 13.27, 15.29). *The ring $R^{\square}_{\overline{\rho}}$ is a local complete intersection, flat over $\mathcal{O}$ and of relative dimension* $\dim G_k([F:\mathbb{Q}_p]+1)$. *In particular, every continuous representation $\overline{\rho}:\Gamma_F\to G(k)$ has a lift to characteristic zero. Moreover, $R^{\square}_{\overline{\rho}}$ is reduced and $R^{\square}_{\overline{\rho}}[1/p]$ is normal.*

Obstruction theory provides a presentation $R^{\square}_{\overline{\rho}}=\mathcal{O}[\![x_1,\dots,x_r]\!]/(f_1,\dots,f_s)$ with $r$ equal to the dimension of the tangent space and $s$ equal to $\dim_k H^2(\Gamma_F,\operatorname{ad}\overline{\rho})$. The Euler–Poincaré characteristic formula from local class field theory gives

$$r-s=\dim G_k([F:\mathbb{Q}_p]+1). \tag{1}$$

Our theorem proves that $\dim R^{\square}_{\overline{\rho}}/\varpi$ is given by this cohomological quantity, the *expected dimension* in the spirit of the *Dimension Conjecture* of Gouvêa from [31, Lecture 4]. Having the expected dimension implies that $\varpi, f_1,\dots,f_s$ is a regular sequence and that $R^{\square}_{\overline{\rho}}$ is a local complete intersection. It also implies (see [29, Lemma 7.5]) that the derived deformation ring of $\overline{\rho}$ as introduced by Galatius and Venkatesh in [29] is homotopy discrete, which means the derived deformation theory of $\overline{\rho}$ does not contain more information than the usual deformation theory of $\overline{\rho}$.

We expect that our results will play an important role in the categorical $p$-adic local Langlands correspondence. Conjecturally [27, Conjecture 6.1.14] on the Galois side one should consider a derived category of coherent sheaves (or some $\infty$-category version of it) on the Emerton–Gee stack. The local rings considered in Theorem 1.1 should arise as versal rings at finite type points of the Emerton–Gee stack, when $G$ is an $L$-group or a $C$-group of a connected reductive group defined over $F$ (see [26, Proposition 3.6.3], when $G=\mathrm{GL}_d$). In other words, they are expected to describe the local properties of the Emerton–Gee stack. An instance of this is a paper of Yu Min [45], where he uses the analogue of Theorem 1.1 proved in [6] for $\mathrm{GL}_d$ to show that if $G=\mathrm{GL}_d$ then the derived version of the Emerton–Gee stack is equivalent to the classical Emerton–Gee stack, obtaining an analogue of the result explained above, that derived deformation rings are homotopy discrete, over the whole of the stack. The Emerton–Gee stack has been recently defined and studied for $\mathrm{GSp}_4$ in [39] and for tame groups in [40]. We infinitesimally guarantee that the results of [45] will extend to this context.

In the paper we introduce and study a scheme $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$; its definition is reviewed in Section 1.4.1 below. Its $\overline{k}$-points correspond to continuous representations $\rho:\Gamma_F\to G(\overline{k})$ with $G$-semisimplification equal to $\overline{\rho}^{\mathrm{ss}}$. It carries an action of $G^0$ and the quotient stack $[X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}/G^0]$ is an analogue of the stacks studied by Wang-Erickson in [62] for $G=\mathrm{GL}_d$. We expect that the disjoint union of the stacks $[X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}/G^0]$ (or rather their formal versions) taken over all possible $\overline{\rho}^{\mathrm{ss}}$ will be the largest substack of the Emerton–Gee stack on which the universal $(\varphi,\Gamma)$-module can be realised as a representation of $\Gamma_F$, that is, the situation described in [26, Remark 6.7.4] for $\mathrm{GL}_d$ will continue to hold in a more general setting.

### *1.1. The conjecture of Böckle–Juschka*

Our second main result proves a generalisation of a conjecture of Böckle–Juschka concerning the irreducible components of $R^{\square}_{\overline{\rho}}$. We assume that $G^0$ is split over $\mathcal{O}$ and $G/G^0$ is a constant group scheme. This can be achieved after replacing $L$ by a finite unramified extension. Let $\Gamma_E$ be the kernel of the homomorphism $\Gamma_F\xrightarrow{\overline{\rho}}G(k)\to(G/G^0)(k)$. We assume that $\mathcal{O}$ contains $\mu_{p^\infty}(E)$. One may further assume that this map is surjective, as replacing $G$ with a subgroup does not change the deformation problem, so that $(G/G^0)(k)=\mathrm{Gal}(E/F)$.

Let $G'$ be the derived group scheme of $G^0$, let $G'_{\mathrm{sc}}\to G'$ be the simply connected central cover of $G'$ and let $\pi_1(G')$ be the kernel of this map. Then $\pi_1(G')$ is a finite diagonalisable subgroup scheme of the centre of $G'_{\mathrm{sc}}$ and is isomorphic to $\prod_{i=1}^r\mu_{n_i}$ for some integers $n_i$. The group scheme $\pi_1(G')$ is étale if and only if $p$ does not divide $n_i$ for all $i$.

Our assumptions on $G$ imply that $G^0/G'$ is a split torus. The action of $G/G'$ on $G^0/G'$ by conjugation induces an action of $\mathrm{Gal}(E/F)$ on its *character lattice* $M:=X^*(G^0/G')$.

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Let $\overline{\psi} : \Gamma_F \to (G/G')(k)$ be the representation $\overline{\psi} = \varphi \circ \overline{\rho}$, where $\varphi : G \to G/G'$ is the quotient map. Let $R^{\square}_{\overline{\psi}}$ be the universal deformation ring of $\overline{\psi}$.

**Theorem 1.2** (Corollaries 15.28, 15.30). *The natural map $R^{\square}_{\overline{\psi}} \to R^{\square}_{\overline{\rho}}$, induced by composing a deformation of $\overline{\rho}$ with $\varphi : G \to G/G'$, is flat and if $\pi_1(G')$ is étale then it induces a bijection between the sets of irreducible components.*

In a companion paper [48] we deal with the case, when $G^0$ is a torus. In particular, we show that the irreducible components of $R^{\square}_{\overline{\psi}}$ can be labelled by characters $\chi : (\mu_{p^\infty}(E) \otimes M)^{\mathrm{Gal}(E/F)} \to \mathcal{O}^\times$.

We assume that $\pi_1(G')$ is étale until the end of this subsection. Let $R^{\square,\chi}_{\overline{\rho}}$ be an irreducible component of $R^{\square}_{\overline{\rho}}$ corresponding to a character $\chi$ under Theorem 1.2.

**Theorem 1.3** (Corollary 15.20). *If $\pi_1(G')$ is étale then the rings $R^{\square,\chi}_{\overline{\rho}}$, $R^{\square,\chi}_{\overline{\rho}}/\varpi$ are complete intersection, regular in codimension $[F : \mathbb{Q}_p]$, normal domains.*

Theorem 1.3 and a theorem of Grothendieck on factoriality of complete intersections imply:

**Corollary 1.4** (Corollary 15.22). *If $\pi_1(G')$ is étale and $[F : \mathbb{Q}_p] \geq 3$ then $R^{\square,\chi}_{\overline{\rho}}$ and $R^{\square,\chi}_{\overline{\rho}}/\varpi$ are factorial.*

If a further hypothesis on $\overline{\rho}$ and $G$ is satisfied (for example, if $G/Z(G^0)$ does not have $\mathrm{PGL}_2$ as a factor) then we show in Corollary 15.20 that $R^{\square,\chi}_{\overline{\rho}}$ and $R^{\square,\chi}_{\overline{\rho}}/\varpi$ are regular in codimension $2[F : \mathbb{Q}_p] - 1$ and hence Corollary 1.4 holds if $[F : \mathbb{Q}_p] = 2$. If $F = \mathbb{Q}_p$ then [6, Remark 4.24, Corollary 4.25] provide examples, where Corollary 1.4 fails.

### *1.2. Deformation problems with fixed partial determinant*

In fact, Theorems 1.1, 1.2, 1.3 and Corollary 1.4 hold in a more general setting, which might be called deformation problems with *'fixed partial determinant'*. Assume that we have a $\mathrm{Gal}(E/F)$-invariant decomposition $X^*(G^0/G') = M_1 \oplus M_2$ and let $H_1$ be the quotient of $G/G'$ such that their component groups coincide and the character lattice of $H_1^0$ is equal to $M_1$. We fix a continuous representation $\psi_1 : \Gamma_F \to H_1(\mathcal{O})$, such that $\psi_1 \equiv \varphi_1 \circ \overline{\rho} (\mathrm{mod}\ \varpi)$, where $\varphi_1 : G \to H_1$ is the composition of $\varphi$ with the quotient map $G/G' \twoheadrightarrow H_1$. Let $D^{\square,\psi_1}_{\overline{\rho}}$ be a subfunctor of $D^{\square}_{\overline{\rho}}$, such that

$$D^{\square,\psi_1}_{\overline{\rho}}(A) = \{\rho_A \in D^{\square}_{\overline{\rho}}(A) : \varphi_1 \circ \rho_A = \psi_1 \otimes_{\mathcal{O}} A\}.$$

The functor $D^{\square,\psi_1}_{\overline{\rho}}$ is pro-represented by a quotient $R^{\square,\psi_1}_{\overline{\rho}}$ of $R^{\square}_{\overline{\rho}}$. Then the above results hold with $R^{\square}_{\overline{\rho}}$ replaced with $R^{\square,\psi_1}_{\overline{\rho}}$, see Corollary 15.14, Theorems 15.18, 15.19 and their Corollaries.

If $\pi_1(G')$ is étale then the irreducible components of $R^{\square,\psi_1}_{\overline{\rho}}$ can be indexed by characters $\chi : (\mu_{p^\infty}(E) \otimes M_2)^{\mathrm{Gal}(E/F)} \to \mathcal{O}^\times$. There are always two interesting cases: if $M_1 = 0$ then we impose an empty condition and $R^{\square,\psi_1}_{\overline{\rho}} = R^{\square}_{\overline{\rho}}$; if $M_1 = X^*(G^0/G')$ then $R^{\square,\psi_1}_{\overline{\rho}}$ parameterises deformations with 'fixed determinant' equal to $\psi_1$ and it follows from above that $R^{\square,\psi_1}_{\overline{\rho}}$ and $R^{\square,\psi_1}_{\overline{\rho}}/\varpi$ are integral domains. If $G = \mathrm{GL}_d$ then $X^*(G^0/G') = \mathbb{Z}$ and these are the only cases that can occur. However, in general there are further interesting cases naturally appearing in the Langlands program, as we discuss in Section 16.

If $H$ is a connected reductive group over $F$ (or more generally over a number field) which splits over a finite Galois extension $E$ of $F$ then it is expected in [13] that the Galois representations attached to $C$-algebraic automorphic forms on $H$ take values in the $C$-group ${}^C H$, which is a generalised reductive group scheme over $\mathcal{O}$ with component group the constant group scheme $\mathrm{Gal}(E/F)$. Moreover, they should satisfy the condition $d \circ \rho = \chi_{\mathrm{cyc}}$, where $\chi_{\mathrm{cyc}}$ is the $p$-adic cyclotomic character and $d$ is the canonical map $d : {}^C H \to \mathbb{G}_m$. We explain in Theorem 16.5 that the functor $D^{\square,\psi_1}_{\overline{\rho}}$ with $G = {}^C H$, $H_1 = \mathrm{Gal}(E/F) \times \mathbb{G}_m$ and $\psi_1 : \Gamma_F \to H_1(\mathcal{O})$, $\gamma \mapsto (\gamma|_E, \chi_{\mathrm{cyc}}(\gamma))$ parameterises deformations $\rho_A$

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of $\overline{\rho} : \Gamma_F \to {}^C H(k)$ satisfying $d \circ \rho_A = \chi_{\mathrm{cyc}} \otimes_{\mathscr{O}} A$. We show that in this case, under the assumption that $\pi_1(\widehat{H}')$ is étale, the irreducible components are canonically in bijection with characters of $p$-power torsion subgroup of $Z(H)^0(F)$, where $Z(H)^0$ is the neutral component of the centre of $H$.

### *1.3. Deformation space of Lafforgue's pseudocharacters*

We refer the reader to Section 7 for the definition of V. Lafforgue's $G$-pseudocharacters. We single out some properties for the purpose of this introduction. To every representation $\rho : \Gamma_F \to G(A)$ one may attach an $A$-valued $G$-pseudocharacter $\Theta_\rho$. Moreover, conjugation of $\rho$ with elements of $G^0(A)$ does not change the associated $G$-pseudocharacter. If $\Theta$ is a $G$-pseudocharacter with values in an algebraically closed field $\kappa$ then there exists a representation $\rho : \Gamma_F \to G(\kappa)$, uniquely determined up to $G^0(\kappa)$-conjugation, such that $\rho$ is $G$-semisimple (Definition 2.22) and $\Theta_\rho = \Theta$. If $A$ and $\kappa$ carry a topology then there is a notion of continuous $G$-pseudocharacters, which interacts well with continuous representations.

Let $\overline{\Theta}$ be the $G$-pseudocharacter attached to $\overline{\rho}$. JQ has shown in [50] as part of his PhD thesis under the direction of Gebhard Böckle that the functor $D^{\mathrm{ps}} : \mathfrak{A}_{\mathscr{O}} \to \mathrm{Set}$, such that $D^{\mathrm{ps}}(A)$ is the set of continuous $A$-valued $G$-pseudocharacters deforming $\overline{\Theta}$ is pro-represented by a complete local noetherian $\mathscr{O}$-algebra $R^{\mathrm{ps}}_G$ with residue field $k$. The proof that $R^{\mathrm{ps}}_G$ is noetherian uses the fact that $\Gamma_F$ is topologically finitely generated.

If $G = \mathrm{GL}_d$ then Emerson–Morel have shown in [25] that there is a natural bijection between $G$-pseudocharacters and Chenevier's $d$-dimensional determinant laws defined in [15]. Thus in this case the ring $R^{\mathrm{ps}}_{\mathrm{GL}_d}$ coincides with the deformation ring studied by Böckle–Juschka in [7]. The results of that paper are an important input in [6]. In our paper we extend the main result [7, Theorem 5.5.1] to $G$-pseudocharacters, also obtaining a new proof of it, when $G = \mathrm{GL}_d$.

**Theorem 1.5** (Corollary 13.31). *The rings $R^{\mathrm{ps}}_G$ and $R^{\mathrm{ps}}_G/\varpi$ are equidimensional of dimension $d+1$ and $d$, respectively, where $d = \dim G_k\,[F : \mathbb{Q}_p] + \dim Z(G)_k$.*

Let $U^{\mathrm{ps}} := (\mathrm{Spec}\, R^{\mathrm{ps}}_G) \setminus \{\text{the closed point}\}$. The non-special absolutely irreducible locus $U^{\text{n-spcl}}$ is an open subscheme of $U^{\mathrm{ps}}$ such that a geometric point $x : \mathrm{Spec}\,\kappa \to U^{\mathrm{ps}}$ lies in $U^{\text{n-spcl}}$ if and only if the representation $\rho_x : \Gamma_F \to G(\kappa)$ associated to the specialisation of the universal $G$-pseudocharacter at $x$ is $G$-irreducible (Definition 2.26) and $H^0(\Gamma_F, (\mathrm{Lie}\, G'_{\mathrm{sc}})^*_x(1)) = 0$, where $\Gamma_F$ acts on $(\mathrm{Lie}\, G'_{\mathrm{sc}})_x$ via $\rho_x$ composed with the adjoint action. If $\pi_1(G')$ is étale or $\mathrm{char}(\kappa) = 0$ then $(\mathrm{Lie}\, G'_{\mathrm{sc}})_x = (\mathrm{Lie}\, G')_x$ and the last condition is equivalent to $H^0(\Gamma_F, (\mathrm{ad}^0\,\rho_x)^*(1)) = 0$.

**Theorem 1.6** (Corollary 13.32). *$U^{\text{n-spcl}}$ is Zariski dense in* $\mathrm{Spec}\, R^{\mathrm{ps}}_G$ *and its special fibre is Zariski dense in* $\mathrm{Spec}\, R^{\mathrm{ps}}_G/\varpi$.

We also show in Corollary 15.25 that the reduced subscheme $(\mathrm{Spec}\, R^{\mathrm{ps}}_G[1/p])^{\mathrm{red}}$ is normal. If $\pi_1(G')$ is étale then we compute the set of irreducible components of $\mathrm{Spec}\, R^{\mathrm{ps}}_G$ in Corollary 15.27, which together with Theorem 1.2 implies that $\mathrm{Spec}\, R^{\square}_{\overline{\rho}} \to \mathrm{Spec}\, R^{\mathrm{ps}}_G$ induces a bijection between the sets of irreducible components.

### *1.4. Complete intersection*

We will now sketch the proof of Theorem 1.1. Let $\mathrm{Rep}^{\Gamma_F}_G : \mathscr{O}\text{-alg} \to \mathrm{Set}$ be the functor, which maps an $\mathscr{O}$-algebra $A$ to the set of group homomorphisms $\Gamma_F \to G(A)$. The scheme $G^0$ acts on $\mathrm{Rep}^{\Gamma_F}_G$ by conjugation. We construct an affine scheme $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} = \mathrm{Spec}\, A^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ over $\mathscr{O}$, which is a $G^0$-invariant subfunctor of $\mathrm{Rep}^{\Gamma_F}_G$ and the following desiderata hold:

(D1) the specialisation of the universal representation $\rho^{\mathrm{univ}} : \Gamma_F \to G(A^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}})$ at $x \in X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}(\overline{k})$ induces a bijection $x \mapsto \rho_x$ between $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}(\overline{k})$ and the set of continuous representations $\rho : \Gamma_F \to G(\overline{k})$ such that the $G$-semisimplification (Definition 2.21) of $\rho$ is equal to $\overline{\rho}^{\mathrm{ss}}$;

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(D2) the completion of a local ring at $x \in X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}(\overline{k})$ is isomorphic to $R^{\square}_{\rho_x}$;

(D3) the GIT quotient $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} /\!\!/ G^0$ is the spectrum of a complete local noetherian $\mathcal{O}$-algebra $R^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$ with residue field $k$.

We then bound the dimension of the special fibre $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ from above

$$\dim \overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} \leq \dim G_k([F:\mathbb{Q}_p]+1). \tag{2}$$

Once we can do this the first part of Theorem 1.1 follows immediately as (D2) implies that $\dim R^{\square}_{\overline{\rho}}/\varpi \leq \dim G_k([F:\mathbb{Q}_p]+1)$ and the result follows from (1) and a little commutative algebra (Lemma 3.9). So the overall strategy is similar to the strategy in [6], however its execution requires substantially new ideas.

**1.4.1. Definition**

The difficulty in defining $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ stems from the fact that we would like to work with continuous representations of $\Gamma_F$, but for arbitrary $A$ there seems to be no sensible topology to put on $G(A)$ which allows us to express the desired continuity condition by simply asking the map $\Gamma_F \to G(A)$ to be continuous. The solution to this problem is inspired by the analogous definition by Fargues–Scholze [28] in the $\ell \neq p$ case, which uses condensed mathematics.

One may define $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ as a functor from $R^{\mathrm{ps}}_G$-alg $\to$ Set such that $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}(A)$ is the set of representations $\rho : \Gamma_F \to G(A)$, such that $\Theta_\rho$ is the specialisation of the universal $G$-pseudocharacter $\Theta^u$ along $R^{\mathrm{ps}}_G \to A$ and $\rho$ extends to a homomorphism of condensed groups

$$\tilde{\rho} : \underline{\Gamma_F} \to G(A_{\mathrm{disc}} \otimes_{(R^{\mathrm{ps}}_G)_{\mathrm{disc}}} \underline{R^{\mathrm{ps}}_G}).$$

We refer the reader to Section 4.2 for the notation. In Lemma 4.8 we show that the condensed condition can be reformulated as follows: for some (equivalently any) closed immersion $\tau : G \hookrightarrow \mathbb{A}^n$ the image $\tau(\rho(\Gamma_F))$ is contained in a finitely generated $R^{\mathrm{ps}}_G$-submodule $M \subseteq A^n = \mathbb{A}^n(A)$ such that the map $\tau \circ \rho : \Gamma_F \to M$ is continuous for the canonical topology on $M$ as an $R^{\mathrm{ps}}_G$-module. We call such representations $R^{\mathrm{ps}}_G$-*condensed*. This definition of $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is aesthetically pleasing and helpful to prove that $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is functorial in $G$. However, it is not immediately clear (at least to the authors) from the definition that the functor is representable by a finite type $R^{\mathrm{ps}}_G$-algebra.

If $G = \mathrm{GL}_d$ then one may linearise representations and work instead with Cayley–Hamilton homomorphisms of $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebras $E^u \to M_d(A)$, where $E^u$ is a universal Cayley–Hamilton quotient of the completed group algebra $R^{\mathrm{ps}}_{\mathrm{GL}_d}[\![\Gamma_F]\!]$. We refer the reader to Section 5.1 for more details. Wang-Erickson has shown in [62] that the algebra $E^u$ is a finitely generated $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-module. This result is the reason why $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-condensed representations do not explicitly appear in [6]. In Lemma 5.2 we show that $X^{\mathrm{gen}}_{\mathrm{GL}_d,\overline{\rho}^{\mathrm{ss}}}$ coincides with the scheme $X^{\mathrm{gen}}$ defined in [6]. In particular, it is represented by a finite type $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebra $A^{\mathrm{gen}}_{\mathrm{GL}_d}$.

In the general case, we fix a closed immersion $\tau : G \hookrightarrow \mathrm{GL}_d$ of $\mathcal{O}$-group schemes and define $X^{\mathrm{gen},\tau}_G$ as a closed subscheme of $X^{\mathrm{gen}}_{\mathrm{GL}_d} := X^{\mathrm{gen}}_{\mathrm{GL}_d,(\tau\circ\overline{\rho})^{\mathrm{ss}}}$, such that $\rho \in X^{\mathrm{gen}}_{\mathrm{GL}_d}(A)$ lies in $X^{\mathrm{gen},\tau}_G(A)$ if and only if $\rho(\Gamma_F)$ is contained in $\tau(G(A))$. The functor $X^{\mathrm{gen},\tau}_G$ is representable by a finite type $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebra $A^{\mathrm{gen},\tau}_G$. We show in Proposition 8.3 that the connected component of $X^{\mathrm{gen},\tau}_G$ containing the point corresponding to $\overline{\rho} : \Gamma_F \to G(k)$ is canonically isomorphic to the functor $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ defined via condensed mathematics as above. In particular, the connected component is independent of the chosen embedding $\tau$. Using the construction via $X^{\mathrm{gen},\tau}_G$ we show that desiderata (D1), (D2), (D3) hold. The condensed description allows to show in Proposition 8.5 that $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is functorial in $G$.

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A key input in the above argument is the following fundamental finiteness theorem proved by Vinberg [61], when $S$ is a spectrum of a field of characteristic zero, by Martin [42], when $S$ is a spectrum of a field of characteristic $p$, and by Cotner [21] in general.

**Theorem 1.7** (Cotner, [21]). *Let $S$ be a locally noetherian scheme and let $f : H \to G$ be a finite morphism of generalised reductive smooth affine $S$-group schemes. Then the induced map on the GIT quotients*

$$H^N /\!\!/ H \to G^N /\!\!/ G$$

*is finite, where the action on both sides is given by conjugation.*

The homomorphism $R^{\mathrm{ps}}_G \to A^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is $G^0$-equivariant for the trivial action on the source. Hence, its image is contained in $G^0$-invariants $R^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$, which represents the GIT quotient $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} /\!\!/ G^0$.

**Proposition 1.8** (Proposition 7.4). *The morphism $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} /\!\!/ G^0 \to \operatorname{Spec} R^{\mathrm{ps}}_G$ is a finite universal homeomorphism.*

We expect that the morphism is adequate in the sense of Alper [1], that is, moreover $R^{\mathrm{ps}}_G[1/p] \cong R^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$, and hope to return to this question in future work[1]. If this were true then one could show that $\operatorname{Spec} R^{\mathrm{ps}}_G[1/p]$ is reduced and hence normal, as explained in Section 1.3.

We define $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ for any profinite group $\Gamma$ satisfying Mazur's $p$-finiteness condition. However, starting with Section 12 we use in an essential way that $\Gamma = \Gamma_F$: besides Euler–Poincaré characteristic formula, it is crucial to our argument that using local Tate duality we may transfer obstructions to lifting from $H^2$ to $H^0$.

### 1.4.2. Bounding the dimension

We will now sketch how we obtain the bound (2). Let $Y_{\overline{\rho}^{\mathrm{ss}}}$ be the preimage of the closed point of $X^{\mathrm{ps}}_G := \operatorname{Spec} R^{\mathrm{ps}}_G$ in $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$. We bound the dimension of $Y_{\overline{\rho}^{\mathrm{ss}}}$ in Corollary 12.18. Its complement $V$ and the special fibre $\overline{V}$ of $V$ are Jacobson schemes and residue fields of closed points $x \in V$ are local fields. Moreover, in Lemma 5.17 we relate the completions of local rings $\mathcal{O}_{V,x}$ to the deformation rings $R^{\square}_{G,\rho_x}$, and completions of local rings $\mathcal{O}_{\overline{V},x}$ to $R^{\square}_{G,\rho_x}/\varpi$. The upshot is that if we control the dimension of $R^{\square}_{G,\rho_x}/\varpi$ then we control the dimension of $\mathcal{O}_{\overline{V},x}$ and if we can do this at every closed point then we can bound the dimension of $\overline{V}$ and the dimension of its closure in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$. In Proposition 3.6 we establish a presentation

$$R^{\square}_{G,\rho_x} \cong \frac{R^{\square}_{G/G',\varphi\circ\rho_x}[\![x_1,\dots,x_r]\!]}{(f_1,\dots,f_s)}, \tag{3}$$

where $s = \dim_{\kappa(x)} H^2(\Gamma_F, \mathrm{ad}^0\,\rho_x)$ and $r - s = \dim G'_k([F:\mathbb{Q}_p]+1)$. We deduce from the results of [48] that if $x \in \overline{V}$ is a closed point then $R^{\square}_{G/G',\varphi\circ\rho_x}/\varpi$ is formally smooth over the group algebra $\kappa(x)[\mu]$ of dimension $\dim(G/G')_k([F:\mathbb{Q}_p]+1)$, where $\mu = (\mu_{p^\infty}(E)\otimes M)^{\mathrm{Gal}(E/F)}$. This is easy if $G = \mathrm{GL}_d$ or more generally if $G$ is connected, but requires a non-trivial argument in the non-connected case, which we carry out in [48]. This allows us to conclude that the closure in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ of the locus in $\overline{V}$, where $H^2(\Gamma_F, \mathrm{ad}^0\,\rho_x)$ vanishes, has dimension $\dim G_k([F:\mathbb{Q}_p]+1)$, if non-empty. This locus will be referred to as the $(\mathrm{Lie}\,G'_k)^*$*-non-special locus* later on.

One is left to understand the locus, where $H^2(\Gamma_F, \mathrm{ad}^0\,\rho_x) \neq 0$. By local Tate duality this is equivalent to $H^0(\Gamma_F, (\mathrm{ad}^0\,\rho_x)^*(1)) \neq 0$. Let us suppose that $\rho_x$ is absolutely $G$-irreducible. If $G = \mathrm{GL}_d$ then Böckle–Juschka show in [7] that non-vanishing of $H^0(\Gamma_F, (\mathrm{ad}^0\,\rho_x)^*(1))$ implies that $\rho_x$ is induced

[1] We have proved this assertion in [47].

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from $\Gamma_{F'}$, where $F'$ is a proper abelian extension of $F$. We know the dimension of $\overline{X}^{\mathrm{ps},\Gamma_{F'}}_{\mathrm{GL}_{d'}}$, where $d' = \frac{d}{[F':F]}$, inductively and using this it is shown in [6] that the closure of this locus has positive codimension. Such arguments might work for classical groups (JQ has studied the case $G = \mathrm{Sp}_{2n}$ under the assumption $p > 2$ in [50, Section 7]), but it seems impossible to get a characterisation like this for an arbitrary generalised reductive group.

Our key observation is that instead of having such a precise description of $\rho_x$, it is enough to observe that $\rho_x$ has *'small' image*. Namely, let $H$ be the Zariski closure of $\rho_x(\Gamma_F)$ in $G(\kappa)$, where $\kappa$ is the algebraic closure of $\kappa(x)$. Since $\rho_x$ is absolutely $G$-irreducible, its image is not contained in any parabolic subgroup of $G_\kappa$. A theorem of Martin [42] asserts that $H$ is again generalised reductive. In Section 10 we prove two results:

**Proposition 1.9** (Corollary 10.2). *If a generalised reductive subgroup $H$ of $G_\kappa$ does not contain $G'_\kappa$ then* $\dim G_\kappa - \dim H \geq 2$.

Our original proof of the above result used Matsushima's theorem. The current simpler proof was suggested by both Sean Cotner and the anonymous referee.

**Lemma 1.10** (Lemma 10.5). *The $G'_\kappa$-invariants in* $(\mathrm{Lie}\, G'_{\mathrm{sc}})^*_\kappa$ *are zero.*

We note that the above Lemma would be false in small characteristics if we did not take the dual. We first prove it for $\mathrm{SL}_2$ and then use this together with arguments with roots to prove it in the general case.

Let us go back to the situation at hand, where $H$ is the Zariski closure of $\rho_x(\Gamma_F)$ in $G(\kappa)$. If $x$ lies in the special fibre, then the twisting operation by the cyclotomic character becomes trivial after the restriction to $\Gamma_{F(\zeta_p)}$. Using this we show that non-vanishing of $H^0(\Gamma_F, (\mathrm{ad}\,\rho_x)^*(1))$ implies that the neutral component $H^0$ of $H$ has non-zero invariants in $(\mathrm{Lie}\, G')^*_\kappa$. If $\pi_1(G')$ is étale or $\mathrm{char}(\kappa) = 0$ then $(\mathrm{Lie}\, G')^*_\kappa = (\mathrm{Lie}\, G'_{\mathrm{sc}})^*_\kappa$ and Proposition 1.9 and Lemma 1.10 imply that $\dim G_\kappa - \dim H^0 \geq 2$ and hence $\dim G_\kappa - \dim H \geq 2$. The fact that the difference can be bounded below by 2 (and not by 1) plays an important role in verifying Serre's criterion for normality in the proof of Theorem 1.3. As you can see there is an induction argument shaping up. To explain it we need two further ingredients.

Let $W$ be an algebraic representation of $G$ on a finite free $\mathcal{O}$-module, and let $\rho : \Gamma_F \to G(A)$ be a representation, where $A$ is a noetherian $\mathcal{O}$-algebra. We show in Lemma 13.8 that there is a closed subscheme $X_{W,j}$ of $X = \mathrm{Spec}\, A$ such that

$$x \in X_{W,j} \iff \dim_{\kappa(x)} H^0(\Gamma_F, W \otimes_{\mathcal{O}} \kappa(x)(1)) \geq j + 1.$$

We refer to $X_{W,j}$ as the *$W$-special locus of level $j$* and just the *$W$-special locus* when $j = 0$. The complement of the $W$-special locus is called *$W$-non-special*. The proof amounts to reformulating these conditions in terms of some finitely generated $A$-module and a little of commutative algebra. We apply this notion to $W = (\mathrm{Lie}\, G'_{\mathrm{sc}})^*$ and if the image of $\rho$ is contained in $P(A)$, where $P$ is a parabolic subgroup scheme of $G$ with unipotent radical $U$, then we apply it to $W = (\mathrm{Lie}\, U)^*$. When working with the special fibre we consider representations $W$ over $k$ instead of $\mathcal{O}$.

The second ingredient, which uses Theorem 1.7 as an input, is the following finiteness result proved in Proposition 8.6. If $H \to G_k$ is a finite morphism of generalised reductive $k$-group schemes which maps an $H$-pseudocharacter $\overline{\Psi}$ to $\overline{\Theta}$ then the natural map $\overline{X}^{\mathrm{ps}}_{H,\overline{\Psi}} \to \overline{X}^{\mathrm{ps}}_{G}$ is finite. Moreover, there are only finitely many $H$-pseudocharacters $\overline{\Psi}$ mapping to $\overline{\Theta}$. In particular, the dimension of the union $\overline{X}^{\mathrm{ps}}_{HG}$ of the scheme theoretic images can be bounded by the largest $\dim \overline{X}^{\mathrm{ps}}_{H,\overline{\Psi}}$.

We are now in a position to sketch the proof of the bound in (2) when $\overline{\rho}$ is absolutely $G$-irreducible. From (3) and the relation between $R^{\square}_{G,\rho_x}/\varpi$ and the completions of $\mathcal{O}_{\overline{V},x}$ we obtain a lower bound

$$\dim \overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} \geq \dim G_k([F : \mathbb{Q}_p] + 1). \tag{4}$$

We then construct a pair $(G_1, \overline{\rho}_1)$ such that $\dim G_1 \leq \dim G$, $\pi_1(G'_1)$ is étale and if (2) holds for $(G_1, \overline{\rho}_1)$ then it also holds for $(G, \overline{\rho})$. For example, if $G = \mathrm{PGL}_d$ then $G_1 = \mathrm{SL}_d/\mu_m$, where $d = p^e m$ and $p \nmid m$,

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and $\overline{\rho}_1 = \overline{\rho}$ as $G(k) = G_1(k)$. In general, the presence of the component group of $G$ complicates the construction of $G_1$, see Section 2.3. This reduction step allows us to assume that $\pi_1(G')$ is étale, which implies that $\mathrm{Lie}\, G'_{\mathrm{sc}} = \mathrm{Lie}\, G'$. We then show that there are finitely many generalised reductive subgroup schemes $H_i$ of $G_k$, defined over some finite extension of $k$, such that $\dim G_k - \dim H_i \geq 2$ and the $(\mathrm{Lie}\, G'_{\mathrm{sc}})^*_k$-special locus in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is contained in the preimage of the union of $\overline{X}^{\mathrm{ps}}_{H_iG}$. The assumption that $\overline{\rho}$ is absolutely $G$-irreducible implies that every fibre $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{ps}}_G$ has dimension $\dim G_k - \dim Z(G)_k$. Inductively, we may bound the dimension of $\overline{X}^{\mathrm{ps}}_{H_iG}$ by $\dim H_i[F : \mathbb{Q}_p] + \dim Z(H_i)$. Using this we show that the codimension of the $(\mathrm{Lie}\, G'_{\mathrm{sc}})^*_k$-special locus in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is at least $2[F : \mathbb{Q}_p]$. It follows from (4) that the $(\mathrm{Lie}\, G')^*_k$-non-special locus in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is non-empty and as explained above its closure has dimension $\dim G_k([F : \mathbb{Q}_p] + 1)$. This lets us conclude that (4) is an equality.

Let us briefly indicate how we find the subgroup schemes $H_i$ in the above argument. The $(\mathrm{Lie}\, G'_{\mathrm{sc}})^*_k$-special locus of level $j$ is non-empty only for finitely many $j$. The group schemes $H_i$ correspond to the generic points of these loci for $j \geq 0$. Given such a generic point $\eta$ we let $H$ be the Zariski closure of $\rho_\eta(\Gamma_F)$ in $G(\overline{\kappa(\eta)})$. Since $\overline{\rho}^{\mathrm{ss}}$ is assumed to be absolutely $G$-irreducible $H$ is not contained in any parabolic subgroup of $G_{\overline{\kappa(\eta)}}$ and hence is generalised reductive. A conjugate of $H$ by an element of $G(\overline{\kappa(\eta)})$ can be defined over $\overline{k}$ and hence over a finite extension of $k$. Our arguments with generalised reductive groups over algebraically closed fields use results of Martin [42] and Bate–Martin–Röhrle [3] in an essential way.

The general case is an elaboration of this inductive argument. If $\overline{\rho}^{\mathrm{ss}}$ is not absolutely $G$-irreducible then the dimensions of the fibres $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{ps}}_G$ can vary. We bound their dimensions in Section 12. We stratify $\overline{X}^{\mathrm{ps}}_G$ by $\overline{X}^{\mathrm{ps}}_{LG}$, where $L$ runs over the Levi subgroup schemes of $G$ containing a fixed maximal split torus. If $y \in \overline{X}^{\mathrm{ps}}_{LG}$, but does not lie in any $\overline{X}^{\mathrm{ps}}_{L'G}$ for a proper Levi $L' \subsetneq L$ then for typical $y$ the fibre will have dimension $\dim G_k - \dim Z(L)_k + \dim U_k[F : \mathbb{Q}_p]$, where $U$ is the unipotent radical of any parabolic subgroup with Levi $L$. By induction we know that $\dim \overline{X}^{\mathrm{ps}}_{LG}$ is at most $\dim L_k[F : \mathbb{Q}_p] + \dim Z(L)_k$. The sum of these numbers does not exceed $\dim G_k([F : \mathbb{Q}_p] + 1)$. In fact, the difference is equal to $\dim U_k[F : \mathbb{Q}_p]$. We deal with the fibres, which have dimension exceeding $\dim G_k - \dim Z(L)_k + \dim U_k[F : \mathbb{Q}_p]$ by considering the $(\mathrm{Lie}\, U_k)^*$-special locus inside $X^{\mathrm{ps}}_L$. The inductive argument is carried out in Section 13.

If $X^{\mathrm{ps}}_{LG}$ is non-empty for a Levi subgroup $L$ satisfying $\dim G - \dim L = 2$ then the bound $\dim U_k[F : \mathbb{Q}_p]$ is not good enough to apply the Serre's criterion for normality, when $F = \mathbb{Q}_p$, as the assumption on $L$ forces $\dim U_k = 1$. We analyse the $(\mathrm{Lie}\, G'_{\mathrm{sc}})^*$-non-special locus in the preimage of $X^{\mathrm{ps}}_{LG}$ in $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ in this case in detail in Section 11. In Section 14 we show that the $(\mathrm{Lie}\, G'_{\mathrm{sc}})^*$-special locus in $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and also in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ has codimension of at least $1 + [F : \mathbb{Q}_p]$ and of at least $2[F : \mathbb{Q}_p]$ if $X^{\mathrm{ps}}_{LG}$ is empty for all Levi subgroups $L$ of codimension 2.

### *1.5. Irreducible components*

We will sketch the proof of Theorem 1.3. Using the results of [48] we show that

$$A^{\mathrm{gen}}_{G/G',\overline{\psi}^{\mathrm{ss}}} \cong \mathcal{O}[\mu][\![x_1, \ldots, x_m]\!][t_1^{\pm 1}, \ldots, t_l^{\pm 1}], \tag{5}$$

where $l = \dim(G/G')_k - \dim Z(G/G')_k$, $m + l = \dim(G/G')_k([F : \mathbb{Q}_p] + 1)$ and $\mu = (\mu_{p^\infty}(E) \otimes M)^{\mathrm{Gal}(E/F)}$. Let $\mathrm{X}(\mu)$ be the group of characters $\chi : \mu \to \mathcal{O}^\times$. If $A$ is an $A^{\mathrm{gen}}_{G/G',\overline{\psi}^{\mathrm{ss}}}$-algebra then we use (5) to define $A^\chi := A \otimes_{\mathcal{O}[\mu],\chi} \mathcal{O}$ and if $X = \mathrm{Spec}\, A$ we write $X^\chi := \mathrm{Spec}\, A^\chi$. It follows from (5) that the irreducible components of $X^{\mathrm{gen}}_{G/G',\overline{\psi}^{\mathrm{ss}}}$ are given by $X^{\mathrm{gen},\chi}_{G/G',\overline{\psi}^{\mathrm{ss}}}$ for $\chi \in \mathrm{X}(\mu)$ and are regular. In particular, the completions of local rings of $X^{\mathrm{gen},\chi}_{G/G',\overline{\psi}^{\mathrm{ss}}}$ are also regular. Thus the presentation (3) gives us a presentation

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$$R^{\square,\chi}_{G,\rho_x} \cong \frac{R^{\square,\chi}_{G/G',\varphi\circ\rho_x}[\![x_1,\dots,x_r]\!]}{(f_1,\dots,f_s)}, \tag{6}$$

where $R^{\square,\chi}_{G/G',\varphi\circ\rho_x}$ is a regular ring. Further, if $x$ is in the $(\mathrm{Lie}\, G')^*$-non-special locus then $s$ in the above presentation is 0 and we can conclude that $R^{\square,\chi}_{G,\rho_x}$ is a regular ring. We deduce that the $(\mathrm{Lie}\, G')^*$-non-special locus in $X^{\mathrm{gen},\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$ is contained in the regular locus. In general, the $(\mathrm{Lie}\, G')^*$-non-special locus might be empty, for example this happens if $G = \mathrm{PGL}_p$. On the other hand the complement of $(\mathrm{Lie}\, G'_{\mathrm{sc}})^*$-non-special locus has codimension of at least $1+[F:\mathbb{Q}_p]$ in $X^{\mathrm{gen},\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$. If $\pi_1(G')$ is étale then these loci coincide and this implies that $X^{\mathrm{gen},\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$ is regular in codimension $[F:\mathbb{Q}_p]$. Since $X^{\mathrm{gen},\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$ is excellent the same applies to the completions of its local rings, and we may deduce that the deformation rings $R^{\square,\chi}_{G,\rho_x}$ are regular in codimension $[F:\mathbb{Q}_p]$. To show that they are complete intersection it is enough to bound their dimension from above and then use (6). Since $k[\mu]$ is a local artinian algebra the underlying reduced subschemes of the special fibres of $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and $X^{\mathrm{gen},\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$ coincide and hence they have the same dimension. Serre's criterion for normality implies that $R^{\square,\chi}_{G,\rho_x}$ is normal. Since the ring is local, it is a domain.

Since $(\mathrm{Lie}\, G')_L = (\mathrm{Lie}\, G'_{\mathrm{sc}})_L$ the $(\mathrm{Lie}\, G')^*$-non-special and $(\mathrm{Lie}\, G'_{\mathrm{sc}})^*$-non-special loci in the generic fibre coincide. This allows us to prove that $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$ is normal without any assumption on $\pi_1(G')$.

A similar argument works in the 'fixed partial determinant' setting described in Section 1.2. However, there is one new idea not present in [6]. To bound the dimension of $X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}}$ we show that there is a finite morphism $X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{gen}}_{G/Z_1,\overline{\rho}^{\mathrm{ss}}}$, where $Z_1$ is a central subtorus of $G^0$ such that the composition $Z_1 \to G \to H_1$ is an isogeny. We can then bound the dimension of various subschemes of $X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}}$ by dimensions of the corresponding subschemes in $X^{\mathrm{gen}}_{G/Z_1,\overline{\rho}^{\mathrm{ss}}}$. So for example, to prove the results in the fixed determinant case, when $G = \mathrm{GL}_d$, we use the results proved for $G = \mathrm{PGL}_d$. In [6] one twists by characters to unfix the determinant instead. This argument can be made to work, when $G$ is connected, but will not work in general.

### *1.6. Overview by section*

In Section 2 we review some facts about the generalised reductive $\mathcal{O}$-group schemes and their parabolic subgroup schemes, which we call R-parabolic. We also review $G$-semisimplification and $G$-irreducibility. In Section 3 we recall deformation theory of representations of a profinite group satisfying Mazur's $p$-finiteness condition. Proposition 3.6 is a key result of that section. In Section 4 we discuss and compare notions of continuity via algebra and via condensed sets. As explained in Section 1.4.1 this is used to define $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ later on. In Section 5 we construct the scheme $X^{\mathrm{gen},\tau}_G$ and study the relation between the completion of its local rings and deformation rings of Galois representations. In Section 6 we study the GIT quotient $X^{\mathrm{git},\tau}_G := X^{\mathrm{gen},\tau}_G /\!\!/ G^0$ and define $X^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$ and its GIT quotient $X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$. In Section 7 we introduce Lafforgue's $G$-pseudocharacters and their deformation spaces $X^{\mathrm{ps}}_G$ and show that the natural map $X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{ps}}_G$ is a finite universal homeomorphism. In Section 8 we give a condensed description of $X^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$ as explained in Section 1.4.1. This allows us to conclude that $X^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$ is independent of $\tau$, which we then omit from notation, and is functorial in $G$. In Section 9 we show that without loss of generality one may assume that the map $\Gamma_F \xrightarrow{\overline{\rho}} G(k) \to (G/G^0)(\overline{k})$ is surjective. In this case we define the absolutely $G$-irreducible locus in $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and show that if it is non-empty then the difference between its dimension and the dimension of its GIT quotient is $\dim G_k - \dim Z(G)_k$. In Section 10 we bound the codimension of generalised reductive subgroups of generalised reductive groups over algebraically closed fields. The results of this section are used to bound the dimensions of $(\mathrm{Lie}\, G'_{\mathrm{sc}})^*$-special and $(\mathrm{Lie}\, U)^*$-special loci, as explained in Section 1.4.2. In Section 11 we show that if $G$ is

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a generalised reductive group over an algebraically closed field with an R-Levi $L$ of codimension 2 such that $L/L^0 \to G/G^0$ is an isomorphism then $G/Z(G^0) \cong G_1 \times \mathrm{PGL}_2$ and $L/Z(G) \cong G_1 \times \mathbb{G}_m$. Proposition 11.5 is used to bound the dimension of the $(\mathrm{Lie}\, G'_{\mathrm{sc}})^*$-special locus in Section 14. This section can be omitted if one is only interested in the complete intersection property and does not care for normality and irreducible components. In Section 12 we bound the dimensions of fibres of $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{ps}}_G$. In Section 13, which is the core of the paper, we carry out the induction argument described in Section 1.4.2. In Section 14 we bound the dimension of the $(\mathrm{Lie}\, G'_{\mathrm{sc}})^*$-special locus. In Section 15 we study the irreducible components of $X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}}$. In Section 16 we explain that our results apply, when $G$ is an $L$-group or a $C$-group, for the convenience of the reader. The appendix recalls some background on condensed mathematics, which is used in Section 4.

### *1.7. Notation*

Let $F$ be a finite extension of $\mathbb{Q}_p$. We fix an algebraic closure $\overline{F}$ over $F$. Let $\Gamma_F := \mathrm{Gal}(\overline{F}/F)$ be the absolute Galois group of $F$. We will denote by $\zeta_p$ a primitive $p$-th root of unity in $\overline{F}$.

Let $L$ be another finite extension of $\mathbb{Q}_p$, such that $\mathrm{Hom}_{\mathbb{Q}_p\text{-alg}}(F, L)$ has cardinality $[F : \mathbb{Q}_p]$. Let $\mathcal{O}$ be the ring of integers in $L$, $\varpi$ a uniformiser, and $k$ the residue field. All our schemes are defined over $\mathcal{O}$ (unless stated otherwise). We omit $\mathrm{Spec}\,\mathcal{O}$ from the notation, when we take fibre products over it, so that $X \times Y := X \times_{\mathrm{Spec}\,\mathcal{O}} Y$.

If $A$ is a commutative ring and $\rho : \Gamma \to \mathrm{GL}_d(A)$ is a representation, we let

$$\rho^{\mathrm{lin}} : A[\Gamma] \to M_d(A), \quad \sum_{\gamma\in\Gamma} a_\gamma \gamma \mapsto \sum_{\gamma\in\Gamma} a_\gamma \rho(\gamma)$$

be its *linearisation*, where $M_d(A)$ is the set of $d \times d$ matrices with entries in $A$.

We denote by $A$-alg the category of commutative $A$-algebras.

## 2. $\mathcal{O}$-group schemes

The category of $\mathcal{O}$-group schemes embeds fully faithfully in the category of fppf group sheaves on $(\mathcal{O}\text{-alg})^{\mathrm{op}}$. Whenever an exact sequence is considered it is understood as an exact sequence of fppf group sheaves. If $G$ is a flat $\mathcal{O}$-group scheme of finite presentation and $N \leq G$ is a flat closed normal subgroup, then the quotient group sheaf $G/N$ is representable by a scheme [2, Théorème 4.C]. If $G$ and $N$ are generalised reductive then the quotient is affine by [1, Theorem 9.4.1].

**Lemma 2.1.** *Let $G$ be a flat affine group scheme of finite presentation over $\mathcal{O}$. Then the functor $\underline{Z}(G) : \mathcal{O}\text{-alg} \to$ Set given by*

$$\underline{Z}(G)(A) := \{g \in G(A) \mid \forall A \to A', \forall h \in G(A'), g = hgh^{-1}\}$$

*is representable by a closed subgroup scheme of $G$, which we will denote by $Z(G)$ and call the centre of $G$.*

*Proof.* This is a special case of [22, Exposé VIII, Théorème 6.4] as explained in [22, Exposé VIII, Exemples 6.5 (e)]. □

**Lemma 2.2.** *Let $G$ be a flat affine group scheme of finite presentation over $\mathcal{O}$ and let $A \to B$ be a map of $\mathcal{O}$-algebras. Then $\underline{Z}(G_A)$ and $\underline{Z}(G_B)$ as in Lemma 2.1 are representable and we have $Z(G_A)_B = Z(G_B)$.*

*Proof.* By Lemma 2.1 $\underline{Z}(G)$ is representable by an $\mathcal{O}$-scheme $Z(G)$. By definition $\underline{Z}(G_A)$ is the restriction of $\underline{Z}(G)$ to the category of $A$-algebras and is thus represented by $Z(G)_A$. The same argument shows, that $\underline{Z}(G_B)$ is representable by $Z(G)_B$ and the claim follows. □

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**Lemma 2.3.** *Let $0 \to G_1 \to G_2 \to G_3 \to 0$ be a short exact sequence of affine $\mathcal{O}$-group schemes of finite presentation. If $G_1$ and $G_3$ are flat, then $G_2 \to G_3$ is faithfully flat and in particular $G_2$ is flat.*

*Proof.* This is [30, Exposé VIB, Proposition 9.2 (xi)]. □

**Lemma 2.4.** *If $G$ is a flat $\mathcal{O}$-group scheme of finite presentation then $\dim G_k = \dim G_L$.*

*Proof.* See [30, Exposé VIB, Corollaire 4.3]. □

**Definition 2.5.** A *generalised reductive group scheme* over a scheme $S$ is a smooth affine $S$-group scheme $G$, such that the geometric fibres of $G^0$ are reductive and $G/G^0 \to S$ is finite.

**Remark 2.6.** It follows from [19, Proposition 3.1.3], that for a generalised reductive $\mathcal{O}$-group scheme $G$ the quotient $G/G^0$ is étale. Sean Cotner has shown in [21] using the results of Alper that a smooth affine $S$-group scheme $G$ is generalised reductive if and only if it is geometrically reductive in the sense of [1, Definition 9.1.1]. In particular, this holds over $S = \operatorname{Spec} \mathcal{O}$.

**Lemma 2.7.** *Let $G$ be an affine group scheme of finite type over a perfect field $\kappa$. If $G$ is reduced then $G$ is smooth over $\kappa$. Moreover, if we additionally assume that the unipotent radical of $G$ is trivial then $G$ is generalised reductive.*

*Proof.* The first assertion follows from [59, Tag 047N, Tag 047P]. Since $\kappa$ is perfect, if the unipotent radical of $G$ is trivial then also the unipotent radical of $G_K$ is trivial, for any field extension $K$ of $\kappa$ by [20, Proposition 1.1.9]. Thus $G^0$ is reductive and $G/G^0$ is finite by [19, Proposition 3.1.3]. □

**Proposition 2.8.** *Let $G$ be a generalised reductive $\mathcal{O}$-group scheme. Then there exists a finite unramified extension $L'/L$ with ring of integers $\mathcal{O}'$, such that the following hold:*

(1) *$(G/G^0)_{\mathcal{O}'} = G_{\mathcal{O}'}/G^0_{\mathcal{O}'}$ is a constant group scheme;*
(2) *$G(\mathcal{O}') \to (G/G^0)(\mathcal{O}')$ is surjective;*
(3) *$G^0_{\mathcal{O}'}$ is split.*

*Proof.* We observed in Remark 2.6 that $G/G^0$ is finite étale. Using [59, Tag 0BND] and the fact that the étale fundamental group of $\mathcal{O}$ is the Galois group of the maximal unramified extension of $L$ we find a finite unramified extension $L'$, such that $(G/G^0)_{\mathcal{O}'}$ is constant. Since the extension $\mathcal{O} \to \mathcal{O}'$ is flat the sequence of group schemes $0 \to G^0 \to G \to G/G^0 \to 0$ remains exact after base change to $\mathcal{O}'$.

Let $k'$ be the residue field of $\mathcal{O}'$. After further enlarging $k'$ we may ensure that $G(k') \to (G/G^0)(k')$ is surjective. Since $G$ is smooth and $\mathcal{O}'$ is complete the map $G(\mathcal{O}') \to G(k')$ is surjective. Since $G/G^0$ is a constant group scheme $(G/G^0)(k') = (G/G^0)(\mathcal{O}')$ and hence we obtain part (2).

Part (3) follows by the same argument as in part (1) from [19, Lemma 5.1.3]. □

**Proposition 2.9.** *Let $G$ be a generalised reductive $\mathcal{O}$-group scheme. Then $Z(G) \cap G^0$ is flat over $\mathcal{O}$. In particular, $\dim Z(G)_k = \dim Z(G)_L$.*

*Proof.* By Lemma 2.1 we see that $Z(G)$ exists as a closed subgroup scheme of $G$. For flatness and the equality of dimensions, by Proposition 2.8 we may extend $\mathcal{O}$, such that $G/G^0$ is constant and the map $G \to G/G^0$ has a section $s : G/G^0 \to G$ as $\mathcal{O}$-schemes. We write $\underline{\Delta} = G/G^0$ for some finite group $\Delta$. The $s$ is determined by a map $\Delta \to G(\mathcal{O})$ and for every $\mathcal{O}$-algebra $A$ the map $G(A) \to (G/G^0)(A)$ is surjective.

By [19, Theorem 3.3.4] $Z(G^0)$ is diagonalisable; that is, $Z(G^0) = \operatorname{Spec}(\mathcal{O}[M])$ for some finitely generated abelian group $M$. We have an action of $\Delta$ by conjugation on $Z(G^0)$. Let $Z(G^0)^\Delta$ be the maximal closed subgroup scheme of $Z(G^0)$ on which the action of $\Delta$ is trivial. The kernel of $\mathcal{O}[Z(G^0)] \to \mathcal{O}[Z(G^0)^\Delta]$ is generated by $x - \delta x$ for $x \in M$ and $\delta \in \Delta$. Hence, $\mathcal{O}[Z(G^0)^\Delta] = \mathcal{O}[M_\Delta]$.

The group $Z(G^0)^\Delta$ is equal to $Z(G) \cap G^0$, since for every $\mathcal{O}$-algebra $A$, the group $G(A)$ is generated by $G^0(A)$ and $s((G/G^0)(A))$. In particular, $Z(G) \cap G^0$ is flat over $\mathcal{O}$. We have a short exact sequence

$$1 \to Z(G) \cap G^0 \to Z(G) \to H \to 1 \tag{7}$$

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where $H$ is the image of $Z(G)$ in $G/G^0$, in particular $H$ is finite. It follows from (7) that special (resp. generic) fibres of $Z(G) \cap G^0$ and $Z(G)$ will have the same neutral component, and hence the same dimension. Since $(Z(G) \cap G^0)_k = \dim(Z(G) \cap G^0)_L$ by Lemma 2.4, we deduce that $\dim Z(G)_k = \dim Z(G)_L$. □

**Lemma 2.10.** *Let $G$ be a generalised reductive group scheme over $\mathcal{O}$. Then the natural map $\pi : G \to G/G'$ is smooth.*

*Proof.* By [19, Theorem 5.3.1] $G'$ is a semisimple $\mathcal{O}$-group scheme and in particular smooth. Since $G'$ is the fppf sheafification of the commutator subgroup functor $A \mapsto [G^0(A), G^0(A)]$, it is preserved under any automorphism of $G^0$. So $G'$ is a closed normal subgroup scheme of $G$. By Lemma 2.3 the map $\pi$ is faithfully flat and of finite presentation. By [30, Exposé VIB, Proposition 9.2 (vii)] smoothness of $G'$ implies smoothness of $\pi$. □

**Lemma 2.11.** *Let $G$ be a generalised reductive group scheme over $\mathcal{O}$. Let $G'_{\mathrm{sc}} \to G'$ be the simply connected central cover of $G'$. Then there is a natural action of $G$ on $\mathrm{Lie}(G'_{\mathrm{sc}})$ such that the map $\mathrm{Lie}(G'_{\mathrm{sc}}) \to \mathrm{Lie}\, G'$ is $G$-equivariant.*

*Proof.* Existence and uniqueness of the simply connected central cover is ensured by [19, Exercise 6.5.2 (i)]. For any $\mathcal{O}$-algebra $A$, the map $(G'_{\mathrm{sc}})_A \to (G')_A$ is a simply connected central cover of $(G')_A$. By [19, Exercise 6.5.2 (iii)] an automorphism of $(G')_A$ lifts to a unique automorphism of $(G'_{\mathrm{sc}})_A$, so the conjugation action of $G(A)$ on $(G')_A$ by $A$-group scheme automorphisms extends to an action of $G(A)$ on $(G'_{\mathrm{sc}})_A$. This action is functorial in $A$ and the map $(G'_{\mathrm{sc}})_A \to (G')_A$ is $G(A)$-equivariant. So $G(A)$ acts functorially on $\mathrm{Lie}(G'_{\mathrm{sc}}) \otimes A$, which gives $\mathrm{Lie}(G'_{\mathrm{sc}})$ the structure of a rational $G$-module. This action is clearly compatible with the action of $G$ on $\mathrm{Lie}\, G'$. □

### *2.1. R-parabolics*

We recall some facts about parabolic subgroups of a generalised reductive $\mathcal{O}$-group scheme $G$. The main issue is that we do not assume that $G$ is connected. The letter R stands for Richardson. Let $A$ be an $\mathcal{O}$-algebra and let $\lambda : \mathbb{G}_{m,A} \to G_A$ be a cocharacter. We then define $P_\lambda$ (resp. $U_\lambda$) to be the closed $A$-group scheme of $G_A$ representing the functor of points $g$ of $G$ such that the limit $\lim_{t\to 0} \lambda(t) g \lambda(t)^{-1}$ exists (resp. exists and is equal to the identity), see [20, Lemmas 2.1.4, 2.1.5]. Let $L_\lambda = P_\lambda \cap P_{-\lambda}$, where $-\lambda$ is the cocharacter defined by $-\lambda(t) := \lambda(1/t) = \lambda(t)^{-1}$. We call subgroup schemes of $G$ of the form $P_\lambda$ *R-parabolic* subgroup schemes and $L_\lambda$ *R-Levi* subgroup schemes of $P_\lambda$. We will only consider R-parabolic and R-Levi subgroup schemes over a connected base.

**Lemma 2.12.** *The following hold*

(1) *$L_\lambda$ is the scheme theoretic centraliser of $\lambda$;*
(2) *$U_\lambda$ is a closed normal subgroup scheme of $P_\lambda$;*
(3) *the multiplication map $L_\lambda \ltimes U_\lambda \to P_\lambda$ is an isomorphism of R-groups;*
(4) *$L_\lambda$, $U_\lambda$, $P_\lambda$ are smooth over $\mathcal{O}$.*

*Proof.* The assertions follow from Lemma 2.1.5 and Proposition 2.1.8(2) of [20]. The last assertion is given by Theorem 4.1.7 (4) in [19]. □

**Lemma 2.13.** *Let $P$ be an R-parabolic subgroup of $G$ with R-Levi subgroup $L$, both defined over $\mathcal{O}$. Then $P \cap G^0$ (resp. $L \cap G^0$) is the neutral component of $P$ (resp. $L$) and is a parabolic (resp. Levi) subgroup of $G^0$. In particular, $L$ is a generalised reductive $\mathcal{O}$-group scheme. Moreover, there are only finitely many R-parabolic subgroups $Q$ of $G$ such that $Q^0 = P^0$.*

*Proof.* Let $\lambda : \mathbb{G}_m \to G$ be a cocharacter such that $P = P_{\lambda,G}$ and $L = L_{\lambda,G}$. Since $\mathbb{G}_m$ is connected $\lambda$ factors through $\lambda : \mathbb{G}_m \to G^0$. It follows from [20, Lemma 2.1.5] that $P \cap G^0 = P_{\lambda,G^0}$ and $L \cap G^0 = L_{\lambda,G^0}$. Let $\bar{x}$ be a geometric point of $\mathrm{Spec}\, \mathcal{O}$. Then $(P_{\lambda,G^0})_{\bar{x}} = P_{\lambda_{\bar{x}},G^0_{\bar{x}}}$ and $(L_{\lambda,G^0})_{\bar{x}} = L_{\lambda_{\bar{x}},G^0_{\bar{x}}}$.

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It follows from [20, Proposition 2.2.9] that $P_{\lambda_{\bar{x}},G^0_{\bar{x}}}$ is a parabolic subgroup of $G^0_{\bar{x}}$ with Levi $L_{\lambda_{\bar{x}},G^0_{\bar{x}}}$. Moreover, both groups are connected and $L_{\lambda_{\bar{x}},G^0_{\bar{x}}}$ is reductive by the classical theory of reductive groups over algebraically closed fields. Thus $P \cap G^0$ is the neutral component of $P$ and $L \cap G^0$ is the neutral component of $L$. Moreover, $P^0$ is a parabolic subgroup of $G^0$ in the sense of [19, Definition 5.2.1] and it follows from Lemma 2.12 (3) that $L^0$ is a Levi of $P^0$ in the sense of [19, Definition 5.4.2]. Further, $L^0$ is reductive in the sense of [19, Definition 3.1.1]. The map $L \to G/G^0$ identifies $L/L^0$ with a closed subgroup of $G/G^0$. Hence $L/L^0$ is finite over $\mathcal{O}$. We deduce that $L$ is generalised reductive and $L/L^0$ is finite étale over $\mathcal{O}$ by Remark 2.6.

Since $U$ is contained in $P^0$ the map $L \to P/P^0$ induces an isomorphism $L/L^0 \cong P/P^0$. Thus $P/P^0$ is a closed subgroup of $G/G^0$, which is finite étale over $\mathcal{O}$. Since $G/G^0$ is finite étale over $\mathcal{O}$ by Remark 2.6, there are only finitely many closed subgroup schemes of $G/G^0$, which are finite étale over $\mathcal{O}$. This implies that there are only finitely many R-parabolic subgroups $Q$ such that $Q^0 = P^0$. □

**Example 2.14.** Let $G = (\mathbb{G}_m \times \mathbb{G}_m) \rtimes \mathbb{Z}/2\mathbb{Z}$ with $\mathbb{Z}/2\mathbb{Z}$ acting on $\mathbb{G}_m \times \mathbb{G}_m$ by permuting the coordinates. Let $\lambda, \mu : \mathbb{G}_m \to G$ be the cocharacters $\lambda(t) = ((t,1),0)$ and $\mu(t) = ((1,1),0)$. Then $P_\lambda = G^0$ and $P_\mu = G$, but their neutral components are the same.

**Remark 2.15.** If $G \to S$ is a connected reductive group scheme and $S$ is affine then the notions of R-parabolic and parabolic subgroup schemes coincide; see the discussion at the end of [19, Section 5.4].

**Proposition 2.16.** *Let $G$ be a generalised reductive group over a field, let $L$ be an R-Levi factor of an R-parabolic subgroup $P$, let $U$ be the unipotent radical of $P$ and assume that the natural map $L/L^0 \to G/G^0$ is an isomorphism. Then*

$$\dim L + \dim Z(L) \leq \dim G + \dim Z(G) - \dim U \tag{8}$$

*Proof.* We may assume throughout the proof that the base field is algebraically closed. Let $T \subseteq L^0$ be a maximal torus of $G^0$. Since $G^0$ is generated by $L^0$, the root subgroups of $U$ and their opposites, we have $Z(G^0) = Z(L^0) \cap \bigcap_a \ker(a)$, where $a : T \to \mathbb{G}_m$ varies over all roots, which generate root subgroups $U_a \subseteq U$. The groups $U_{\pm a}$ commute with $\ker(a)$ and $\dim \ker(a) = \dim T - 1$, so

$$\dim Z(L^0) \leq \dim Z(G^0) + \dim U. \tag{9}$$

Since $\Delta := G/G^0$ is constant, we have a scheme theoretic splitting $s : G/G^0 \to G$ and $\Delta$ acts on $Z(G^0)$ by conjugation. Since we are assuming that $L/L^0 = G/G^0$, we have a compatible action of $\Delta$ on $Z(L^0)$. We have a diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z(G^0) & \longrightarrow & Z(L^0) & \longrightarrow & Z(L^0)/Z(G^0) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & Z(G^0)^\Delta & \longrightarrow & Z(L^0)^\Delta & \longrightarrow & Z(L^0)^\Delta/Z(G^0)^\Delta & \longrightarrow & 1 \end{array}$$

and the right vertical map is injective, since $Z(G^0)^\Delta = Z(L^0)^\Delta \cap Z(G^0)$. As in the proof of Proposition 2.9, we have $Z(G^0)^\Delta = Z(G) \cap G^0$ and $Z(L^0)^\Delta = Z(L) \cap L^0$, so $\dim Z(G^0)^\Delta = \dim Z(G)$ and $\dim Z(L^0)^\Delta = \dim Z(L)$. Injectivity of the right vertical map gives

$$\dim Z(L) - \dim Z(G) \leq \dim Z(L^0) - \dim Z(G^0) \tag{10}$$

We conclude from (10) and (9) that

$$\begin{aligned} \dim L + \dim Z(L) &\leq \dim L^0 + \dim Z(L^0) - \dim Z(G^0) + \dim Z(G) \\ &\leq \dim L^0 + \dim U + \dim Z(G) = \dim G + \dim Z(G) - \dim U, \end{aligned}$$

where we use $\dim G^0 = \dim L^0 + 2 \dim U$ in the last step. □

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For the rest of the subsection we assume that $G$ is split and $G/G^0$ is constant. We fix a maximal split torus $T$ of $G$ defined over $\mathcal{O}$. Let $s$ be the closed point of $\operatorname{Spec}\mathcal{O}$ and let $\bar{s}$ be a geometric point above $s$.

**Lemma 2.17.** *Base change induces a bijection between the following sets:*

(1) *R-parabolics of $G$ containing $T$;*
(2) *R-parabolics of $G_s$ containing $T_s$;*
(3) *R-parabolics of $G_{\bar{s}}$ containing $T_{\bar{s}}$.*

*Proof.* We will first establish that $P \mapsto P_{\bar{s}}$ induces the bijection between sets in parts (1) and (3). If $\lambda : \mathbb{G}_m \to G$ is a cocharacter defined over $\mathcal{O}$ then $(P_\lambda)_{\bar{s}} = P_{\lambda_{\bar{s}}}$ by [20, Lemma 2.1.4]. Moreover, if $P$ contains $T$ then $P_{\bar{s}}$ will contain $T_{\bar{s}}$. Hence, the map is well defined.

If $P$ and $Q$ are R-parabolics of $G$ such that $P_{\bar{s}} = Q_{\bar{s}}$ then $P^0_{\bar{s}} = Q^0_{\bar{s}}$. It follows from [19, Corollary 5.2.7 (2)] that there exists a Zariski open subset $U$ of $\operatorname{Spec}\mathcal{O}$ containing $s$ such that $P^0_U = Q^0_U$. Since $\mathcal{O}$ is a DVR we have $U = \operatorname{Spec}\mathcal{O}$ and hence $P^0 = Q^0$. It follows from Lemma 2.13 that $P \cap G^0 = P^0 = Q^0 = Q \cap G^0$. Thus both $P$ and $Q$ are contained in the $G$-normaliser $N_G(P^0)$ of $P^0$ and it is enough to show that the images of $P$ and $Q$ in $N_G(P^0)/P^0$ coincide. Since $N_G(P^0) \cap G^0 = P^0$ it is enough to show that the images of $P$ and $Q$ in $G/G^0$ coincide. Since $G/G^0$ is a constant group scheme, we may check this after base change to $\bar{s}$, which holds as $P_{\bar{s}} = Q_{\bar{s}}$.

Let $\bar{\lambda} : \mathbb{G}_m \to G_{\bar{s}}$ be a cocharacter. Its image is contained in a maximal split torus $T'$ of $G_{\bar{s}}$. Moreover, $T'$ is contained in $P_{\bar{\lambda}}$. If $P_{\bar{\lambda}}$ contains $T_{\bar{s}}$ then there is $g \in P_{\bar{\lambda}}(\kappa(\bar{s}))$ such that $T_{\bar{s}} = gT'g^{-1}$ by [9, Proposition 11.19]. We thus may assume that the image of $\bar{\lambda}$ is contained in $T_{\bar{s}}$. Since both $\mathbb{G}_m$ and $T$ are split tori defined over $\mathcal{O}$ there is a cocharacter $\lambda : \mathbb{G}_m \to T$, such that $\lambda_{\bar{s}} = \bar{\lambda}$. Then $P_\lambda$ contains $T$ and satisfies $(P_\lambda)_{\bar{s}} = P_{\bar{\lambda}}$.

The proof that base change induces a bijection between the sets in parts (2) and (3) is the same. Since $P_{\bar{s}} = (P_s)_{\bar{s}}$ this also implies that base change induces a bijection between the sets in parts (1) and (2). □

**Lemma 2.18.** *There are only finitely many R-parabolic subgroups of $G$ containing $T$. Each of them has a unique R-Levi subgroup containing $T$.*

*Proof.* Lemma 2.17 implies that it is enough to show that there are only finitely many R-parabolics of $G_{\bar{s}}$ containing $T_{\bar{s}}$. Lemma 2.13 implies that we may assume that $G_{\bar{s}}$ is connected. Proposition 11.19 (b) in [9] says that there are only finitely many Borel subgroups of $G^0_{\bar{s}}$ containing $T_{\bar{s}}$. Proposition 14.18 in [9] implies that there are only finitely many parabolic subgroups of $G^0_{\bar{s}}$ containing a given Borel subgroup. This proves the first assertion.

Let $M$ be the unique Levi of $P^0$ containing $T$, [19, Proposition 5.4.5]. Let $\lambda : \mathbb{G}_m \to G$ be a cocharacter such that $P = P_\lambda$. Then $L^0_\lambda$ is also a Levi of $P^0$ by Lemma 2.13. The scheme of Levi subgroups of $P^0$ is a $U_\lambda$-torsor, [19, Corollary 5.4.6]. Thus there is $g \in U_\lambda(\mathcal{O})$ such that $M = gL^0_\lambda g^{-1}$. Then $gL_\lambda g^{-1}$ is an R-Levi of $P$ containing $T$.

If $L$ and $M$ are R-Levi subgroups of $P$ containing $T$ then $M_{\bar{x}} = L_{\bar{x}}$ for every geometric point $\bar{x}$ of $\operatorname{Spec}\mathcal{O}$ by [3, Corollary 6.5]. This implies that $L = M$. □

**Lemma 2.19.** *Let $\bar{x}$ be a geometric point of $\operatorname{Spec}\mathcal{O}$ and let $Q$ be an R-parabolic subgroup of $G_{\bar{x}}$ with R-Levi $M$. Then there is an R-parabolic subgroup $P$ of $G$ defined over $\mathcal{O}$ with R-Levi $L$ containing $T$ and $g \in G^0(\kappa(\bar{x}))$, such that $gQg^{-1} = P_{\bar{x}}$ and $gMg^{-1} = L_{\bar{x}}$.*

*Proof.* Let $\lambda : \mathbb{G}_{m,\bar{x}} \to G_{\bar{x}}$ be a cocharacter such that $Q = P_\lambda$ and $M = L_\lambda$. There is a maximal torus $T'$ of $G_{\bar{x}}$, such that the image of $\lambda$ is contained in $T'$. Since tori are connected, both $T'$ and $T_{\bar{x}}$ are contained in $G^0_{\bar{x}}$. Since $\kappa(\bar{x})$ is algebraically closed any two maximal tori in $G^0_{\bar{x}}$ are conjugate, and hence after conjugating $Q$ we may assume that $\lambda : \mathbb{G}_{m,\bar{x}} \to T_{\bar{x}}$. Since $T$ is split over $\mathcal{O}$, there is a cocharacter $\mu : \mathbb{G}_m \to T$ defined over $\mathcal{O}$ such that $\lambda = \mu_{\bar{x}}$. We have $P_\mu \times_{\operatorname{Spec}\mathcal{O}} \bar{x} = P_{\mu_{\bar{x}}}$ by [20, Lemma 2.1.4] and $L_\mu \times_{\operatorname{Spec}\mathcal{O}} \bar{x} = L_{\mu_{\bar{x}}}$ by [20, Lemma 2.1.5]. □

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### 2.2. G-semisimplification

In this section let $G$ be a generalised reductive group over an algebraically closed field $\kappa$. Let $\Gamma$ be a group and let $\rho : \Gamma \to G(\kappa)$ be a representation. Let $P$ be an R-parabolic of $G$ minimal with respect to the property that $P(\kappa)$ contains $\rho(\Gamma)$. Let $L$ be an R-Levi of $P$ and let $U$ be the unipotent radical of $P$. The composition $L \to P \to P/U$ is an isomorphism, and we define $c_{P,L} : P(\kappa) \to G(\kappa)$ as the composition $P(\kappa) \to (P/U)(\kappa) \xrightarrow{\cong} L(\kappa) \hookrightarrow G(\kappa)$.

**Proposition 2.20.** *The $G^0(\kappa)$-conjugacy class of $c_{P,L} \circ \rho$ is independent of the choice of $P$ and $L$.*

*Proof.* If $G$ is connected then this is proved in [56, Proposition 3.3 (b)]. The same argument is carried out in our more general setting in [50, Proposition 2.15]. □

**Definition 2.21.** The *G-semisimplification* of $\rho$ is the $G^0(\kappa)$-conjugacy class of the representation $c_{P,L} \circ \rho$. It will be denoted by $\rho^{\mathrm{ss}}$.

In [56, Section 3.2.1] Serre introduced the notion of $G$-complete reducibility in the connected case. This definition extends to generalised reductive group schemes verbatim.

**Definition 2.22.** A subgroup $H$ of $G(\kappa)$ is *G-completely reducible* if for every R-parabolic $P \subseteq G$ with $H \subseteq P(\kappa)$, there exists an R-Levi subgroup $L \subseteq P$ with $H \subseteq L(\kappa)$. We say, that a homomorphism $\rho : \Gamma \to G(\kappa)$ is *G-completely reducible*, if its image is $G$-completely reducible. We will also use the term *G-semisimple* synonymously for $G$-completely reducible.

**Proposition 2.23.** *Let $\rho : \Gamma \to G(\kappa)$ be a homomorphism. Then $\rho$ is G-completely reducible if and only if $\rho^{\mathrm{ss}}$ is the $G^0(\kappa)$-conjugacy class of $\rho$.*

*Proof.* Let $P$ be a minimal R-parabolic, such that $\rho(\Gamma) \subseteq P(\kappa)$. If $\rho$ is $G$-completely reducible, there exists some R-Levi $L$ of $P$, such that $\rho(\Gamma) \subseteq L(\kappa)$. In particular $\rho = c_{P,L} \circ \rho$. We can apply Proposition 2.20 to conclude, $\rho^{\mathrm{ss}}$ is in the $G^0(\kappa)$-conjugacy class of $\rho$. Conversely, suppose that $\rho \in \rho^{\mathrm{ss}}$, let $L$ be any R-Levi of $P$. The assumption implies that $\rho$ is conjugate to $c_{P,L} \circ \rho$ by an element of $g \in G^0(\kappa)$. Minimality of $P$ and [3, Corollary 6.4] imply that the image of $c_{P,L} \circ \rho$ is $L$-irreducible, which means that it is not contained in any proper R-parabolic of $L$. Hence, $\rho(\Gamma)$ is $gLg^{-1}$-irreducible. It follows from the equivalence of parts (iv) and (v) of [3, Corollary 3.5], which as explained in [3, Section 6.3] also holds for non-connected groups, that $\rho(\Gamma)$ is $G$-completely reducible. □

**Proposition 2.24.** *Let $\rho : \Gamma \to G(\kappa)$ be a homomorphism. Then the determinant laws attached to $(\tau \circ \rho)^{\mathrm{lin}}$ and $(\tau \circ \rho^{\mathrm{ss}})^{\mathrm{lin}}$ agree.*

*Proof.* Let $P$ be an R-parabolic of $G$, minimal with respect to the property that $P(\kappa)$ contains $\rho(\Gamma)$. Let $U$ be the unipotent radical of $P$ and let let $\lambda$ be a cocharacter, such that $P = P_\lambda$. Then $\lim_{t\to 0} \lambda(t)U\lambda(t)^{-1} = \{1\}$ and hence $c_{P,L_\lambda} \circ \rho = \lim_{t\to 0} \lambda(t)\rho\lambda(t)^{-1}$. Let $D_{\tau\circ\rho} : \kappa[\Gamma] \to \kappa$ be the determinant law attached to $(\tau \circ \rho)^{\mathrm{lin}}$ and let $D_{\tau\circ\rho^{\mathrm{ss}}}$ be the determinant law attached to $(\tau \circ c_{P,L_\lambda} \circ \rho)^{\mathrm{lin}}$. We note that replacing $c_{P,L_\lambda} \circ \rho$ by a conjugate with an element of $G^0(\kappa)$ does not change the determinant law, as they are invariant under conjugation.

We also have a family of determinant laws $D : \kappa[\Gamma] \to \kappa[t, t^{-1}]$ over $\mathbb{G}_m$ given by

$$D_A : A[\Gamma] \to A[t, t^{-1}], \quad r \mapsto \det(((\tau(\lambda(t))(\tau \circ \rho)\tau(\lambda(t))^{-1}) \otimes \mathrm{id}_A)(r)),$$

which is actually constant in $t$ and equal to $D_{\tau\circ\rho}$. So this family extends uniquely to a family over $\mathbb{A}^1$. Since the limit of $\lambda(t)\rho\lambda(t)^{-1}$ as $t \to 0$ exists and formation of the determinant is algebraic, we obtain $D^{t=0} = D_{\tau\circ\rho^{\mathrm{ss}}}$ and hence $D_{\tau\circ\rho} = D_{\tau\circ\rho^{\mathrm{ss}}}$. □

**Remark 2.25.** In general $(\tau \circ \rho)^{\mathrm{ss}}$ is not isomorphic to $\tau \circ \rho^{\mathrm{ss}}$. But it follows from Proposition 2.24, that $D_{\tau\circ\rho^{\mathrm{ss}}} = D_{\tau\circ\rho}$ and in particular from [15, Theorem 2.12] that $(\tau \circ \rho^{\mathrm{ss}})^{\mathrm{ss}}$ coincides with $(\tau \circ \rho)^{\mathrm{ss}}$.

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**Definition 2.26.** A subgroup $H$ of $G(\kappa)$ is *G-irreducible* if $H$ is not contained in any proper R-parabolic subgroup of $G$. We say, that a homomorphism $\rho : \Gamma \to G(\kappa)$ is *G-irreducible*, if its image is $G$-irreducible.

**Proposition 2.27.** *Let $L$ and $M$ be R-Levi subgroups of $G$ and let $H$ be a closed subgroup of $L \cap M$ such that $H$ is both $L$-irreducible and $M$-irreducible. Then there is $g \in Z_G(H)^0(\kappa)$ such that $L = gMg^{-1}$.*

*Proof.* Let $\lambda, \mu : \mathbb{G}_m \to G$ be cocharacters such that $L = L_\lambda$ and $M = L_\mu$. Since $H$ is contained in $L$ and in $M$, Lemma 2.12 (1) implies that the both $\lambda$ and $\mu$ factor through the inclusion $Z_G(H) \subseteq G$. Let $S$ (resp. $T$) be a maximal torus of $Z_G(H)$ containing the image of $\lambda$ (resp. $\mu$). It follows from the proof of [9, Proposition 8.18] that $Z_G(S)$ is an R-Levi of $G$ containing $H$ and contained in $L$. Since $H$ is $L$-irreducible by assumption we get that $L = Z_G(S)$. Similarly, we obtain that $M = Z_G(T)$. There is $g \in Z_G(H)^0(\kappa)$ such that $S = gTg^{-1}$ by [9, Proposition 11.19]. Thus $L = Z_G(gTg^{-1}) = gZ_G(T)g^{-1} = gMg^{-1}$. □

### *2.3. Extensions*

Let $N$ and $\Delta$ be abstract groups. A *rigidified extension of $\Delta$ by $N$* is a group law $*$ on the set $N \times \Delta$, such that the maps $\iota : N \to N \times \Delta$, $g \mapsto (g, 1)$ and $\pi : N \times \Delta \to \Delta$, $(g, \delta) \mapsto \delta$ are group homomorphisms.

**Definition 2.28.** A *generalised 2-cocycle* of $\Delta$ with coefficients in $N$ is a pair $(\omega, c)$, where $\omega : \Delta \to \mathrm{Aut}(N)$ and $c : \Delta \times \Delta \to N$ are maps (of sets) satisfying the following:

(1) $\omega(\delta_1) \circ \omega(\delta_2) = \mathrm{Int}_N(c(\delta_1, \delta_2)) \circ \omega(\delta_1\delta_2)$ in $\mathrm{Aut}(N)$;
(2) $\omega(1) = \mathrm{id}_N$;
(3) $c(\delta_1, \delta_2)c(\delta_1\delta_2, \delta_3) = \omega(\delta_1)(c(\delta_2, \delta_3))c(\delta_1, \delta_2\delta_3)$ in $N$;
(4) $c(\delta_1, 1) = c(1, \delta_2) = 1$;

where for $g \in N$, $\mathrm{Int}_N(g)$ is the inner automorphism $x \mapsto gxg^{-1}$ of $N$.

We emphasise that, since $N$ is not assumed to be abelian, $\omega$ is not going to be a group homomorphism in general. Let $(N \times \Delta, *)$ be a rigidified extension of $\Delta$ by $N$. It is a pleasant exercise in algebra to verify that if we let

$$\omega^*(\delta)(x) := s(\delta)xs(\delta)^{-1}, \quad c^*(\delta_1, \delta_2) := s(\delta_1)s(\delta_2)s(\delta_1\delta_2)^{-1},$$

for all $x \in N$ and all $\delta, \delta_1, \delta_2 \in \Delta$, where $s(\delta) = (1, \delta)$, then the group axioms for $*$ imply that $(\omega^*, c^*)$ is a generalised 2-cocycle.

**Lemma 2.29.** *Mapping $(N \times \Delta, *)$ to $(\omega^*, c^*)$ induces a bijection from the set of rigidified extensions of $\Delta$ by $N$ to the set of generalised 2-cocycles of $\Delta$ with values in $N$.*

*Proof.* If $N$ is abelian then $\mathrm{Int}_N = \mathrm{id}_N$ and the result is well known. In general, the assertion follows from [54, Satz I], a nice exposition is given in [65, Section 2.2] and the lemma follows from [65, Lemma 2.11]. One could also deduce the assertion from [24]: property (3) implies that the element denoted by $f_3(x, y, z)$ in [24, Section 7] is $1_N$. Theorem 8.1 in [24] then implies that every generalised 2-cocycle comes from a rigidified extension. □

**Proposition 2.30.** *Let $G$ be a generalised reductive group over a perfect field $\kappa$ of characteristic $p$, such that $G^0$ is split semisimple, $G/G^0 = \underline{\Delta}$ is constant and the map $G(\kappa) \to (G/G^0)(\kappa)$ is surjective. Then there exists a surjection of generalised reductive groups $H \to G$ over $\kappa$ with kernel $\mu$, such that the following hold:*

(1) *$H \to G$ induces an isomorphism $H/H^0 \cong G/G^0$;*
(2) *$0 \to \mu \to H^0 \to G^0 \to 0$ is a central extension with $\mu(\overline{\kappa}) = 1$;*
(3) *$\pi_1(H^0)$ is étale.*

*Moreover, $H(\kappa) \cong G(\kappa)$.*

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*Proof.* We may choose a set-theoretic section $s : \Delta \to G(\kappa)$ such that $s(1) = 1$. It induces an isomorphism of $\kappa$-schemes

$$G^0 \times \underline{\Delta} \xrightarrow{\sim} G, \quad (g, \delta) \mapsto gs(\delta). \tag{11}$$

We define a map $\omega : \Delta \to \mathrm{Aut}(G^0)$ by $\omega(\delta)(g) := s(\delta)gs(\delta)^{-1}$. We further define a map $c : \Delta \times \Delta \to G^0(\kappa)$ by $c(\delta_1, \delta_2) := s(\delta_1)s(\delta_2)s(\delta_1\delta_2)^{-1}$.

For a $\kappa$-algebra $A$, we get a map $\omega_A : \Delta \to \mathrm{Aut}(G^0(A))$ by composing with the homomorphism $\mathrm{Aut}(G^0) \to \mathrm{Aut}((G^0)_A) \to \mathrm{Aut}(G^0(A))$ with $\omega$ and a map $c_A : \Delta \times \Delta \to G^0(A)$ by composing $c$ with the map $G^0(\kappa) \to G^0(A)$.

The natural map $\Delta \to \underline{\Delta}(A)$ is a group homomorphism. The isomorphism (11) induces a bijection $G^0(A) \times \underline{\Delta}(A) \xrightarrow{\sim} G(A)$. We use the bijection to transport the group structure on $G^0(A) \times \underline{\Delta}(A)$. When $A \neq 0$, Lemma 2.29 implies that $(\omega_A, c_A)$ is a generalised 2-cocycle of $\Delta$ with values in $G^0(A)$ corresponding to the restriction of this group law to the subset $G^0(A) \times \Delta$. Thus, for all $\delta_1, \delta_2, \delta_3 \in \Delta$ we obtain the following identities:

(C1) $\omega(\delta_1) \circ \omega(\delta_2) = \mathrm{Int}_{G^0}(c(\delta_1, \delta_2)) \circ \omega(\delta_1\delta_2)$ in $\mathrm{Aut}(G^0)$,
(C2) $\omega(1) = \mathrm{id}_{G^0}$ in $\mathrm{Aut}(G^0)$,
(C3) $c(\delta_1, \delta_2)c(\delta_1\delta_2, \delta_3) = \omega_\kappa(\delta_1)(c(\delta_2, \delta_3))c(\delta_1, \delta_2\delta_3)$ in $G^0(\kappa)$, and
(C4) $c(1, \delta_1) = c(\delta_1, 1) = 1$ in $G^0(\kappa)$,

where for $g \in G^0(\kappa)$, $\mathrm{Int}_{G^0}(g)$ is the inner automorphism $x \mapsto gxg^{-1}$ of $G^0$. Here (C1) and (C2) follow by considering all $A$. Identities (C3) and (C4) follow by considering the case $A = \kappa$.

The simply connected central cover $\pi_{\mathrm{sc}} : (G^0)_{\mathrm{sc}} \to G^0$ has finite multiplicative kernel $K$. Let $K_0$ be the prime to $p$ part of $K$ and define $H_0 := (G^0)_{\mathrm{sc}}/K_0$. The kernel $\mu$ of $\pi : H_0 \to G^0$ is isomorphic to $K/K_0$ and is thus a connected finite multiplicative $\kappa$-group scheme. Since $G^0$ is split, $\mu$ is a product of group schemes of the form $\mu_{p^r}$. We have an exact sequence of non-abelian fppf cohomology groups

$$0 \to \mu(\kappa) \to H_0(\kappa) \to G^0(\kappa) \to H^1_{\mathrm{fppf}}(\mathrm{Spec}(\kappa), \mu).$$

Since $H^1_{\mathrm{fppf}}(\mathrm{Spec}(\kappa), \mu_{p^n}) = \kappa^\times/(\kappa^\times)^{p^n}$ by [59, Tag 040Q] and $\kappa$ is a perfect field of characteristic $p$, we have $\mu(\kappa) = 1$ and $H^1_{\mathrm{fppf}}(\mathrm{Spec}(\kappa), \mu) = 0$. So we can see $c$ as a map to $H_0(\kappa)$, which we will denote by $\tilde{c}$. If $A$ is a $\kappa$-algebra we let $\tilde{c}_A$ be the composition of $c$ with the map $H_0(\kappa) \to H_0(A)$.

By [19, Exercise 6.5.2 (iii)] any automorphism $\varphi : G^0 \to G^0$ lifts uniquely to an automorphism $\varphi_{\mathrm{sc}} : (G^0)_{\mathrm{sc}} \to (G^0)_{\mathrm{sc}}$ such that $\pi_{\mathrm{sc}} \circ \varphi_{\mathrm{sc}} = \varphi \circ \pi_{\mathrm{sc}}$ and $\varphi_{\mathrm{sc}}$ preserves $K_0$, so $\varphi$ lifts uniquely to an automorphism $L(\varphi) : H_0 \to H_0$ such that $\pi \circ L(\varphi) = \varphi \circ \pi$ and we obtain an injective group homomorphism $L : \mathrm{Aut}(G^0) \hookrightarrow \mathrm{Aut}(H_0)$.

Let $\tilde{\omega} := L \circ \omega$, that is, for all $\delta \in \Delta$, we have $\omega(\delta) \circ \pi = \pi \circ \tilde{\omega}(\delta)$. We claim that the pair $(\tilde{\omega}, \tilde{c})$ satisfies the identities (C1), (C2), (C3) and (C4) above. Clearly the map $\tilde{c}$ satisfies (C4). Via the isomorphism $H_0(\kappa) \cong G^0(\kappa)$ the maps $\omega_\kappa$ and $\tilde{\omega}_\kappa$ can be identified, so (C3) holds for $(\tilde{\omega}, \tilde{c})$. Property (C2) holds, as $L$ is a homomorphism. By applying $L$ to the identity (C1) for $G^0$, we get

$$\tilde{\omega}(\delta_1) \circ \tilde{\omega}(\delta_2) = L(\mathrm{Int}_{G^0}(c(\delta_1, \delta_2))) \circ \tilde{\omega}(\delta_1\delta_2) \quad \text{in } \mathrm{Aut}(H_0). \tag{12}$$

Since under the isomorphism $H_0(\kappa) \xrightarrow{\sim} G^0(\kappa)$ the element $\tilde{c}(\delta_1, \delta_2)$ is mapped to $c(\delta_1, \delta_2)$, we have $\pi \circ \mathrm{Int}_{H_0}(\tilde{c}(\delta_1, \delta_2)) = \mathrm{Int}_{G^0}(c(\delta_1, \delta_2)) \circ \pi$. Thus $L(\mathrm{Int}_{G^0}(c(\delta_1, \delta_2))) = \mathrm{Int}_{H_0}(\tilde{c}(\delta_1, \delta_2))$ by definition of $L$ and we deduce from (12) that (C1) holds for $H_0$, which finishes the proof of the claim. We deduce that for every $\kappa$-algebra $A$ the pair $(\tilde{\omega}_A, \tilde{c}_A)$ is a generalised 2-cocycle of $\Delta$ with values in $H_0(A)$, which by Lemma 2.29 defines a group law $*$ on $H_0(A) \times \Delta$, natural in $A$.

The functor $A \mapsto (H_0(A) \times \Delta, *)$ defines a presheaf of groups, which we denote by $H^{\mathrm{pre}}$. The map $H^{\mathrm{pre}}(A) \to G(A)$, $(h, \delta) \mapsto \pi(h)s(\delta)$ is a group homomorphism natural in $A$, as $\pi_A \circ \tilde{c}_A = c_A$ and for any $\delta \in \Delta$, we have $\omega(\delta) \circ \pi = \pi \circ \tilde{\omega}(\delta)$. We let $H$ be the Zariski sheafification of $H^{\mathrm{pre}}$. Since $\underline{\Delta}$

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is the sheafification of the constant presheaf $\Delta$, $H = H_0 \times \underline{\Delta}$ and $H^{\mathrm{pre}} \to G$ induces a map of $\kappa$-group schemes $H \to G$. Properties (1) and (2) claimed in the Lemma hold by construction as $H^0 = H_0$ and (3) holds as $\pi_1(H^0)$ is isomorphic to $K_0$. □

## 3. Deformation rings

Let $\Gamma$ be a profinite group such that $H^1(\Gamma', \mathbb{F}_p)$ is finite for all open subgroups $\Gamma'$ of $\Gamma$. This $p$-finiteness condition introduced by Mazur holds, when $\Gamma = \Gamma_F$.

Let $\rho : \Gamma \to G(\kappa)$ be a continuous representation, where $G$ is a smooth affine $\mathcal{O}$-group scheme and $\kappa$ is either a finite extension of $k$, a finite extension of $L$ or a local field of characteristic $p$ containing $k$ equipped with natural topology. We will refer to such fields $\kappa$ as *a finite or a local $\mathcal{O}$-field*. In this section we will study the deformation theory of $\rho$. We first define a ring of coefficients $\Lambda$ over which the deformation problem is defined following [6, Section 3.5].

(1) If $\kappa$ is a finite field then pick an unramified extension $L'$ of $L$ with residue field $\kappa$ and let $\Lambda := \mathcal{O}_{L'}$ denote the ring of integers in $L'$.
(2) If $\kappa$ is a finite extension of $L$ then let $\Lambda := \kappa$, let $\Lambda^0$ be the ring of integers in $\Lambda$ and let $t = \varpi$.
(3) If $\kappa$ is a local field of characteristic $p$ then let $\mathcal{O}_\kappa$ be the ring of integers in $\kappa$ and let $k'$ be its residue field. Since $\mathrm{char}(\kappa) = p$ by choosing a uniformiser we obtain an isomorphism $\mathcal{O}_\kappa \cong k'[\![t]\!]$. Let $L'$ be an unramified extension of $L$ with residue field $k'$, let $\Lambda^0 := \mathcal{O}_{L'}[\![t]\!]$ and let $\Lambda$ be the $p$-adic completion of $\Lambda^0[1/t]$. Then $\Lambda$ is a complete DVR with uniformiser $\varpi$ and residue field $\kappa$. We equip $\Lambda^0$ with its $(\varpi, t)$-adic topology, this induces a topology on $\Lambda^0[1/t]$ and $\Lambda^0[1/t]/p^n\Lambda^0[1/t]$ for all $n \geq 1$. We equip $\Lambda = \varprojlim_n \Lambda^0[1/t]/p^n\Lambda^0[1/t]$ with the projective limit topology.

Let $\mathfrak{A}_\Lambda$ be the category of local artinian $\Lambda$-algebras with residue field $\kappa$. If $(A, \mathfrak{m}_A) \in \mathfrak{A}_\Lambda$ then we define the topology on $A$ in the cases (1), (2) and (3) above as follows:

(1) $A$ is a finite $\mathcal{O}/\varpi^n$-module for some $n \geq 1$ with the discrete topology on $A$;
(2) $A$ is a finite dimensional $L$-vector space with the $p$-adic topology on $A$;
(3) $A$ is a $\Lambda^0[1/t]/\varpi^n\Lambda^0[1/t]$-module of finite length for some $n \geq 1$ and we put the induced topology on $A$.

Since $G$ is smooth the map $G(A) \to G(\kappa)$ is surjective for all $A \in \mathfrak{A}_\Lambda$. Let $D^{\square}_{\rho,G} : \mathfrak{A}_\Lambda \to \mathrm{Set}$ be the functor such that $D^{\square}_{\rho,G}(A)$ is the set of continuous group homomorphisms $\rho_A : \Gamma \to G(A)$ satisfying $\rho_A (\mathrm{mod}\ \mathfrak{m}_A) = \rho$.

**Lemma 3.1.** *Let $V$ be a continuous representation of $\Gamma$ on a finite dimensional $\kappa$-vector space $V$. Then*

$$\dim_\kappa Z^1(\Gamma, V) = h^1(\Gamma, V) + \dim_\kappa V - h^0(\Gamma, V), \tag{13}$$

*where $h^i(\Gamma, V) := \dim_\kappa H^i(\Gamma, V)$. Moreover, if $\Gamma = \Gamma_F$ then*

$$\begin{aligned} \dim_\kappa Z^1(\Gamma_F, V) &= \dim_\kappa V([F : \mathbb{Q}_p] + 1) + h^0(\Gamma_F, V^*(1)) \\ &= \dim_\kappa V([F : \mathbb{Q}_p] + 1) + h^2(\Gamma_F, V). \end{aligned} \tag{14}$$

*Proof.* Mapping $v \in V$ to a coboundary $\gamma \mapsto (\gamma - 1)v$ induces an exact sequence:

$$0 \to V^\Gamma \to V \to Z^1(\Gamma, V) \to H^1(\Gamma, V) \to 0,$$

which implies (13). The second part follows from local Euler–Poincaré characteristic formula and local Tate duality, see for example [7, Theorem 3.4.1]. □

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Let $\kappa[\varepsilon]$ be the ring of dual numbers over $\kappa$. The Lie algebra $\operatorname{Lie} G_\kappa$ of $G_\kappa$ is a finite dimensional $\kappa$-vector space, which sits in the exact sequence of groups

$$0 \to \operatorname{Lie} G_\kappa \to G(\kappa[\varepsilon]) \to G(\kappa) \to 0.$$

The action of $G(\kappa)$ on $G(\kappa[\varepsilon])$ by conjugation induces a $\kappa$-linear action of $G(\kappa)$ on $\operatorname{Lie} G_\kappa$. We let $\operatorname{ad}\rho$ be the representation of $\Gamma$ on $\operatorname{Lie} G_\kappa$ defined by

$$\operatorname{ad}(\rho)(\gamma)(X) := \rho(\gamma) X \rho(\gamma)^{-1}, \quad \forall \gamma \in \Gamma, \quad \forall X \in \operatorname{Lie} G_\kappa.$$

Then $\operatorname{ad}\rho$ is a continuous representation of $\Gamma$ for the topology on $\operatorname{Lie} G_\kappa$ induced by the topology on $\kappa$.

**Lemma 3.2.** *Let $\rho : \Gamma \to G(\kappa)$ be a continuous representation. Then $D^{\square}_{\rho,G}$ is pro-representable by a complete local noetherian $\Lambda$-algebra $R^{\square}_{\rho,G}$ with residue field $\kappa$.*

*Proof.* If $A, B, C \in \mathfrak{A}_\Lambda$, then a homomorphism $\Gamma \to G(A \times_C B)$ is continuous if and only if its projections to $G(A)$ and $G(B)$ are continuous. It follows that the functor $\mathfrak{A}_\Lambda \to \mathrm{Set}$, $A \mapsto \operatorname{Hom}^{\mathrm{cont}}_{\mathrm{Group}}(\Gamma, G(A))$ preserves finite limits. The deformation functor $D^{\square}_{\rho,G}$ is defined as a pullback of the latter, so $D^{\square}_{\rho,G}$ also preserves finite limits. By Grothendieck's criterion, $D^{\square}_{\rho,G}$ is pro-representable by a projective limit of objects in $\mathfrak{A}_\Lambda$.

The map $Z^1(\Gamma, \operatorname{ad}\rho) \to D^{\square}_{\rho,G}(\kappa[\varepsilon])$, $\Phi \mapsto [\gamma \mapsto \rho(\gamma)(1 + \varepsilon\Phi(\gamma))]$ is an isomorphism of $\kappa$-vector spaces. Since $\Gamma$ satisfies Mazur's $p$-finiteness condition [7, Theorem 3.4.1] implies that $h^1(\Gamma, \operatorname{ad}\rho)$ is finite. Lemma 3.1 implies that $Z^1(\Gamma, \operatorname{ad}\rho)$ is finite dimensional, thus $R^{\square}_{\rho,G}$ is noetherian. □

**Lemma 3.3.** *Let $f : G \to H$ be a continuous surjective open homomorphism between locally profinite groups. Then there is a continuous map $s : H \to G$ with $f \circ s = \mathrm{id}_H$.*

*Proof.* If $G$ and $H$ are profinite, this is [55, Section I.1, Proposition 1]. In general, let $N$ be an open profinite subgroup of $G$. Then $N' := f(N)$ is an open profinite subgroup of $H$ and we find a continuous map $s : N' \to N$ with $f|_N \circ s = \mathrm{id}_{N'}$. Let $R \subseteq H$ be a set of $N'$-coset representatives so that $H = \bigcup_{r \in R} rN'$ and choose $g_r \in G$ with $f(g_r) = r$ for each $r \in R$. We can extend $s$ to a continuous map $\tilde{s} : H \to G$ by defining $\tilde{s}(rn') := g_r s(n')$ for all $n' \in N'$ and see, that $f \circ \tilde{s} = \mathrm{id}_H$, as desired. □

**Lemma 3.4.** *Let $\kappa$ be a local $\mathcal{O}$-field, let $\Lambda$ be a coefficient ring for $\kappa$ and let $(A, \mathfrak{m}_A) \in \mathfrak{A}_\Lambda$. Then there is a module-finite $\Lambda^0$-subalgebra $R$ of $A$ with $R \otimes_{\Lambda^0} \Lambda = A$. In particular, $R$ is a profinite open subring of $A$.*

*Proof.* Let $\pi_A : A \to \kappa$ be the projection to the residue field. Pick $\Lambda$-module generators $x_1, \dots, x_n$ of $A$ and we may assume by rescaling with $t$, that $\pi_A(x_i) \in \mathcal{O}_\kappa$ for all $i$. Since $\Lambda^0$ surjects onto $\mathcal{O}_\kappa$, we can pick $y_i \in \Lambda^0$ with $\pi_A(y_i) = \pi_A(x_i)$ for all $i$. Since $\mathfrak{m}_A$ is nilpotent, the finitely many nonzero products of the elements $x_i - y_i \in \mathfrak{m}_A$ generate a multiplicatively closed finitely generated $\Lambda^0$-submodule $I$ of $\mathfrak{m}_A$. In particular $R := \Lambda^0 + I$ is a module-finite $\Lambda^0$-subalgebra of $A$ with $R[1/t] = A$. Since $R$ is a finitely generated $\Lambda^0$-module, $\Lambda^0$ is open in $\Lambda$ and $\Lambda$ is Hausdorff, $R$ is profinite and open. □

**Lemma 3.5.** *Let $\kappa$ be either a finite or a local $\mathcal{O}$-field and let $\Lambda$ be a coefficient ring for $\kappa$. Let $G$ be a smooth affine $\mathcal{O}$-group.*

(1) *For every $A \in \mathfrak{A}_\Lambda$, the topological group $G(A)$ is locally profinite.*
(2) *Let $f : A \to B$ be a surjective morphism in $\mathfrak{A}_\Lambda$. Then the map $\varphi : G(A) \to G(B)$ induced by $f$ is surjective and there is a continuous map $s : G(B) \to G(A)$ with $\varphi \circ s = \mathrm{id}_{G(B)}$.*

*Proof.* When $\kappa$ is finite, both assertions are clear, so we assume that $\kappa$ is local. Let $R$ be a profinite open subalgebra of $A$ with $R \otimes_{\Lambda^0} \Lambda = A$ as constructed in Lemma 3.4.

Part (1). We claim, that $G(R) \subseteq G(A)$ is a profinite open subgroup. Choose a closed immersion $G \hookrightarrow \mathbb{A}^n$ over $\mathcal{O}$. By definition $G(A)$ carries the subspace topology of $\mathbb{A}^n(A)$. Since $\mathbb{A}^n(R)$ is open in $\mathbb{A}^n(A)$, $G(R)$ is open in $G(A)$. Since $R$ is Hausdorff, $G(R)$ is closed in $\mathbb{A}^n(R)$ and since $\mathbb{A}^n(R)$ is profinite, $G(R)$ is profinite.

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Part (2). Surjectivity of $\varphi$ follows from smoothness of $G$. Then $R' := f(R)$ is a compact subalgebra of $B$ with $R' \otimes_{\Lambda^0} \Lambda = B$. Since the topology of $B$ is by definition induced by an arbitrary open $\Lambda^0$-lattice, we see that $R'$ is open in $B$. Since $G$ is smooth, we have a surjection $G(R) \to G(R')$ of open profinite subgroups of $G(A)$ and $G(B)$ respectively. Since any surjective continuous homomorphism between profinite groups is open[2], we see that $\varphi$ is open. The assertion follows from Lemma 3.3. □

**Proposition 3.6.** *Let $\varphi : G \to H$ be a morphism of smooth affine $\mathcal{O}$-group schemes such that the induced morphism on neutral components $G^0 \to H^0$ is smooth and surjective. Let $\rho : \Gamma \to G(\kappa)$ be a continuous representation and let $\mathrm{ad}^{0,\varphi} = \ker(\mathrm{ad}\,\rho \to \mathrm{ad}\,\varphi \circ \rho)$. Then there is a natural isomorphism of $R^{\square}_{\varphi\circ\rho,H}$-algebras*

$$R^{\square}_{\varphi\circ\rho,H}[\![x_1,\dots,x_r]\!]/(f_1,\dots,f_t) \cong R^{\square}_{\rho,G} \tag{15}$$

*where $r = \dim_\kappa Z^1(\Gamma, \mathrm{ad}^{0,\varphi}\rho)$, $t = h^2(\Gamma, \mathrm{ad}^{0,\varphi}\rho)$. Moreover, if $\Gamma = \Gamma_F$ then*

$$r - t = (\dim G_\kappa - \dim H_\kappa)\cdot([F:\mathbb{Q}_p]+1). \tag{16}$$

*Proof.* If $\kappa$ is a finite field then the argument of [6, Proposition 4.3], where $G = \mathrm{GL}_d$, $H = \mathrm{GL}_1$, $\varphi = \det$, goes through verbatim. If $\kappa$ is a local field then one needs to additionally verify that obstruction to lifting can be realised by a continuous 2-cocycle, as noted in [6, Remark 4.4]. The solution to this problem is explained in detail in [17, Lecture 6], when $\kappa$ is a local field of characteristic zero and $H$ is trivial. We sketch how these arguments fit together.

The exact sequence $0 \to \mathrm{ad}^{0,\varphi}\rho \to \mathrm{ad}\,\rho \to \mathrm{ad}\,\varphi\circ\rho \to 0$ of Galois representations induces an exact sequence of abelian groups:

$$0 \to Z^1(\Gamma, \mathrm{ad}^{0,\varphi}\rho) \to Z^1(\Gamma, \mathrm{ad}\,\rho) \to Z^1(\Gamma, \mathrm{ad}\,\varphi\circ\rho)$$

and hence $r = \dim_\kappa \ker(Z^1(\mathrm{ad}\,\varphi))$. The map

$$Z^1(\mathrm{ad}\,\varphi) : Z^1(\Gamma, \mathrm{ad}\,\rho) \to Z^1(\Gamma, \mathrm{ad}\,\varphi\circ\rho)$$

is the induced map on Zariski tangent spaces of the map of deformation rings $R^{\square}_{\varphi\circ\rho} \to R^{\square}_{\rho}$, and thus lifts to a surjection

$$\tilde{\phi} : \tilde{R} := R^{\square}_{\varphi\circ\rho}[\![x_1,\dots,x_r]\!] \twoheadrightarrow R^{\square}_{\rho}.$$

We set $J := \ker\tilde{\phi}$. By Nakayama's lemma, we need to show, that $\dim_\kappa J/\tilde{\mathfrak{m}}J \le t$. The module $J/\tilde{\mathfrak{m}}J$ appears as the kernel of the sequence

$$0 \to J/\tilde{\mathfrak{m}}J \to \tilde{R}/\tilde{\mathfrak{m}}J \to \tilde{R}/J \cong R^{\square}_{\rho,G} \to 0 \tag{17}$$

In view of the above sequence, it is enough construct a homomorphism

$$\alpha : \mathrm{Hom}_\kappa(J/\tilde{\mathfrak{m}}J, \kappa) \to \ker(H^2(\mathrm{ad}\,\varphi) : H^2(\Gamma, \mathrm{ad}\,\rho) \to H^2(\Gamma, \mathrm{ad}\,\varphi\circ\rho)) \tag{18}$$

and show, that $\ker(\alpha)$ injects into $\mathrm{coker}(H^1(\mathrm{ad}\,\varphi))$. This will imply the existence of the presentation in the statement of the Proposition, since then

$$\dim_\kappa J/\tilde{\mathfrak{m}}J \le \dim_\kappa \ker(H^2(\mathrm{ad}\,\varphi)) + \dim_\kappa \mathrm{coker}(H^1(\mathrm{ad}\,\varphi)) = h^2(\Gamma, \mathrm{ad}^{0,\varphi}\rho) \tag{19}$$

where the last equality comes from the long exact cohomology sequence that arises from $0 \to \mathrm{ad}^{0,\varphi}\rho \to \mathrm{ad}\,\rho \to \mathrm{ad}\,\varphi\circ\rho \to 0$.

[2]Indeed, any continuous surjective homomorphism from a compact group onto a Hausdorff group is open by Remark 1 after Proposition 24 in [11, Ch. III, §2, no. 8].

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For all $i > 0$, we define $\tilde{R}_i := \tilde{R}/\tilde{\mathfrak{m}}^i$, $\tilde{\mathfrak{m}}_i := \tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}^i$, $J_i := (J + \tilde{\mathfrak{m}}^i)/\tilde{\mathfrak{m}}^i$ and $R^{\square}_{\rho,G,i} := R^{\square}_{\rho,G}/\tilde{\mathfrak{m}}^i R^{\square}_{\rho,G}$. The Artin–Rees lemma implies that $J \cap (\tilde{\mathfrak{m}}J + \tilde{\mathfrak{m}}^i) = \tilde{\mathfrak{m}}J$ for sufficiently large $i \gg 0$, thus the map $J/\tilde{\mathfrak{m}}J \to J_i/\tilde{\mathfrak{m}}_i J_i$ is an isomorphism. We will fix such $i$ for the rest of the proof. It is therefore enough to work with objects of $\mathfrak{A}_\Lambda$ whose maximal ideal has vanishing $i$-th power, we will denote this category by $\mathfrak{A}_{\Lambda,i}$. The inclusion functor $\mathfrak{A}_{\Lambda,i} \to \mathfrak{A}_\Lambda$ has a left adjoint given by modding out the $i$-th power of the maximal ideal. The ring $R^{\square}_{\rho,G,i}$ represents the restriction of $D^{\square}_{\rho,G}$ to $\mathfrak{A}_{\Lambda,i}$ and we will denote the universal lift by $\rho^{\square}_i : \Gamma \to G(R^{\square}_{\rho,G,i})$. We obtain an exact sequence

$$0 \to J_i/\tilde{\mathfrak{m}}_i J_i \to \tilde{R}_i/\tilde{\mathfrak{m}}_i J_i \to \tilde{R}_i/J_i \cong R^{\square}_{\rho,G,i} \to 0 \tag{20}$$

where $\tilde{R}_i/\tilde{\mathfrak{m}}_i J_i$ and $R^{\square}_{\rho,G,i}$ are objects of $\mathfrak{A}_{\Lambda,i}$. If $u \in \mathrm{Hom}_\kappa(J_i/\tilde{\mathfrak{m}}_i J_i, \kappa)$ then by modding out $\ker(u)$ in (20) we obtain an exact sequence

$$0 \to I_u \to R_u \xrightarrow{\phi_u} R^{\square}_{\rho,G,i} \to 0, \tag{21}$$

The surjection of locally profinite groups $G(R_u) \twoheadrightarrow G(R^{\square}_{\rho,G,i})$ has a continuous section by Lemma 3.5. Once we have this the proof is identical to the proof of [6, Proposition 4.3]. Moreover, if $\Gamma = \Gamma_F$ then (16) follows from Lemma 3.1. □

**Remark 3.7.** We will apply the above proposition in the setting of Lemma 2.10.

**Corollary 3.8.** *There is an isomorphism of local $\Lambda$-algebras*

$$R^{\square}_{\rho,G} \cong \Lambda[\![x_1, \dots, x_r]\!]/(f_1, \dots, f_s) \tag{22}$$

*with $r = \dim_\kappa Z^1(\Gamma, \mathrm{ad}\,\rho)$ and $s = \dim_\kappa H^2(\Gamma, \mathrm{ad}\,\rho)$. Further, if $\Gamma = \Gamma_F$ then*

$$r - s = \dim G_\kappa([F : \mathbb{Q}_p] + 1).$$

*Proof.* The assertion follows from Lemma 3.2 and Proposition 3.6 applied with $H$ equal to the trivial group. □

Proposition 3.6 will be often used together with the following lemma.

**Lemma 3.9.** *Let $A$ be a complete local noetherian $\mathcal{O}$-algebra with residue field $k$ and let $B = A[\![x_1, \dots, x_r]\!]/(f_1, \dots, f_t)$. If $A$ is complete intersection and $\dim B \leq \dim A + r - t$ then $B$ is complete intersection and $\dim B = \dim A + r - t$. Moreover, if $\dim k \otimes_A B \leq r - t$ then $B$ is a flat $A$-algebra and $\dim k \otimes_A B = r - t$.*

*Proof.* Since $A$ is complete intersection we may write $A = S/(g_1, \dots, g_s)$, where $S$ is a complete local regular ring and $g_1, \dots, g_s$ is a regular sequence in $S$. We choose $\tilde{f}_1, \dots, \tilde{f}_t \in S[\![x_1, \dots, x_r]\!]$ that map to $f_1, \dots, f_t$. Since when we quotient out by a relation the dimension either goes down by one or stays then same, the assumption on the dimension of $B$ implies that $\dim B = \dim A + r - t$ and $g_1, \dots, g_s, \tilde{f}_1, \dots, \tilde{f}_t$ can be extended to a system of parameters of $S[\![x_1, \dots, x_r]\!]$, and hence is a regular sequence in the regular ring $S[\![x_1, \dots, x_r]\!]$. Thus $B$ is complete intersection and also $C := S[\![x_1, \dots, x_r]\!]/(\tilde{f}_1, \dots, \tilde{f}_t)$ is complete intersection. Since $k \otimes_S B = k \otimes_A B$, it follows from miracle flatness [43, Theorem 23.1] that $C$ is flat over $S$, and hence $B = C/(g_1, \dots, g_s)$ is flat over $A = S/(g_1, \dots, g_s)$. □

**Lemma 3.10.** *Let $\rho : \Gamma \to G(\kappa)$ be a continuous representation, where $\kappa$ is either local or finite $\mathcal{O}$-field. Let $\kappa'$ be a finite extension of $\kappa$ and let $\rho' : \Gamma \to G(\kappa')$ be the representation obtained by composing $\rho$ with the inclusion $G(\kappa) \subseteq G(\kappa')$. Then there is a natural isomorphism of local $\Lambda'$-algebras:*

$$\Lambda' \otimes_\Lambda R^{\square}_{G,\rho} \cong R^{\square}_{G,\rho'},$$

*where $\Lambda$ and $\Lambda'$ are coefficient rings for $\kappa$ and $\kappa'$, respectively.*

*Proof.* The argument explained on page 457 of [64] also applies in our setting. □

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**Lemma 3.11.** *Let $\kappa$ be a local $\mathcal{O}$-field with ring of integers $\mathcal{O}_\kappa$ and residue field $k$. Let $\rho : \Gamma \to G(\kappa)$ be a continuous representation, such that its image is contained in $G(\mathcal{O}_\kappa)$, and let $\overline{\rho} : \Gamma \to G(k)$ be the composition of $\rho$ with the reduction map $G(\mathcal{O}_\kappa) \to G(k)$. Let $\varphi : R^{\square}_{G,\overline{\rho}} \to \mathcal{O}_\kappa \hookrightarrow \kappa$ be the map induced by considering $\rho$ as a deformation of $\overline{\rho}$ to $\mathcal{O}_\kappa$. Let $\mathfrak{q}$ be the kernel of the natural map*

$$\Lambda \otimes_{\mathcal{O}} R^{\square}_{G,\overline{\rho}} \to \kappa, \quad \lambda \otimes a \mapsto \bar{\lambda}\varphi(a),$$

*where $\Lambda$ is the coefficient ring for $\kappa$. Then there is a natural isomorphism of local $\Lambda$-algebras between $R^{\square}_{G,\rho}$ and the completion $\Lambda \otimes_{\mathcal{O}} R^{\square}_{G,\overline{\rho}}$ with respect to $\mathfrak{q}$.*

*Proof.* The proof is the same as the proof of [7, Theorem 3.3.1] for $\mathrm{GL}_d$, which in turn is based on [36, Proposition 9.5], where the case $G = \mathrm{GL}_d$ and $\mathrm{char}(\kappa) = 0$ is treated. □

### *3.1. Central extensions*

In this section, we assume that $G$ is generalised reductive and let $H = G/Z$, where $Z$ is a flat closed subgroup scheme of $Z(G^0)$, which is normal in $G$. Part (i) of [19, Exercise 5.5.9] implies that $H$ is also generalised reductive. We let $\varphi : G \to H$ be the quotient map. We fix a continuous representation $\rho : \Gamma \to G(\kappa)$ and drop it from the notation. We will write $R^{\square}_G$ for $R^{\square}_{\rho,G}$, $R^{\square}_H$ for $R^{\square}_{\varphi\circ\rho,H}$, etc.

**Corollary 3.12.** *If $Z$ is finite étale then $R^{\square}_H = R^{\square}_G$.*

*Proof.* Since $Z$ is finite étale its Lie algebra is zero. We thus get $\mathrm{ad}^{0,\varphi}\,\rho = 0$ and the assertion follows from Proposition 3.6. □

**Proposition 3.13.** *If $Z$ is a torus such that $Z \cap G'$ is étale then there is a natural isomorphism of local $\Lambda$-algebras $R^{\square}_{G/G'} \widehat{\otimes}_{R^{\square}_{H/H'}} R^{\square}_H \xrightarrow{\sim} R^{\square}_G$.*

*Proof.* Let $D : \mathfrak{A}_\Lambda \to \mathrm{Set}$ be the functor such that $D(A)$ is the set of pairs $(\psi, \xi) \in D^{\square}_{G/G'}(A) \times D^{\square}_H(A)$ such that $\pi \circ \psi = \theta \circ \xi$, where $\theta : H \to H/H'$ and $\pi : G/G' \to H/H'$ are the quotient maps. The Proposition amounts to the claim that the map

$$D^{\square}_G(A) \to D(A), \quad \rho_A \mapsto (\varphi \circ \rho_A, q \circ \rho_A) \tag{23}$$

is bijective. Since $Z \cap G'$ is étale by assumption, using Corollary 3.12 we may assume that $Z \cap G'$ is trivial. The map $G^0 \to H^0$ is surjective with kernel $Z$ and the map $G' \to H'$ is surjective with kernel $Z \cap G'$. The map of group schemes

$$G \to G/G' \times_{H/H'} H \tag{24}$$

has kernel $Z \cap G'$, so it is injective. If $([g], h) \in G(A)/G'(A) \times_{H(A)/H'(A)} H(A)$ then, since $G \to H$ is surjective, there exists a finite flat extension $A \hookrightarrow B$ and $\tilde{g} \in G(B)$, which maps to $h \in H(A) \subseteq H(B)$. Then $\tilde{g}g^{-1} \in H'(B) = G'(B)$, so $[\tilde{g}] = [g]$ in $G(B)/G'(B)$. By sheafification this implies, that (24) is bijective.

This implies that (23) is bijective, since then given $(\psi, \xi) \in D(A)$ for every $\gamma \in \Gamma$ there is a unique $\rho'(\gamma) \in G(A)$ such that $\varphi(\rho'(\gamma)) = \psi(\gamma)$ and $\pi(\rho'(\gamma)) = \xi(\gamma)$. Uniqueness implies that $\rho' \in D^{\square}_G(A)$, which maps to $(\psi, \xi)$. □

**Corollary 3.14.** *If $\Gamma = \Gamma_F$, $\kappa$ is finite and we are in the setting of Proposition 3.13 then* $\dim R^{\square}_G/\varpi = \dim R^{\square}_H/\varpi + ([F : \mathbb{Q}_p] + 1)(\dim G_k - \dim H_k)$.

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*Proof.* Let $A' = (R^{\square}_{H/H'}/\varpi)^{\mathrm{red}}$, $B' = (R^{\square}_{G/G'}/\varpi)^{\mathrm{red}}$, $A = (R^{\square}_{H}/\varpi)^{\mathrm{red}}$, where the superscript red indicates reduced rings. We have shown in [48, Corollary 9.9] that $A' \to B'$ is flat and the dimension $d$ of the fibre $\kappa \otimes_{A'} B'$ is equal to

$$(\dim(G/G')_k - \dim(H/H')_k)([F:\mathbb{Q}_p]+1) = (\dim G_k - \dim H_k)([F:\mathbb{Q}_p]+1).$$

Hence, $A \to B' \widehat{\otimes}_{A'} A$ is flat and the fibre has dimension $d$. Thus $\dim A + d = \dim B' \widehat{\otimes}_{A'} A$ by [59, Tag 00ON]. Proposition 3.13 implies that $(B' \widehat{\otimes}_{A'} A)^{\mathrm{red}} \cong (R^{\square}_{G}/\varpi)^{\mathrm{red}}$. Hence, $\dim R^{\square}_{G}/\varpi = \dim A + d = \dim R^{\square}_{H}/\varpi + d$. □

For the rest of the section we let $Z$ be a finite connected diagonalisable subgroup scheme of $Z(G^0)$. If $\mathrm{char}(\kappa) = 0$ then $Z$ is trivial so we assume that $\mathrm{char}(\kappa) = p$. In this case there is a finite abelian $p$-group $\mathrm{m}$ such that $Z(A) = \mathrm{Hom}(\mathrm{m}, A^{\times})$ for $A \in \mathcal{O}$-alg.

**Lemma 3.15.** *$R^{\square}_{H} \to R^{\square}_{G}$ is finite.*

*Proof.* Let $A := \kappa \otimes_{R^{\square}_{H}} R^{\square}_{G}$ and let $\mathfrak{p}$ be a prime ideal of $A$. Since $A/\mathfrak{p}$ is a domain of characteristic $p$, we have $Z(A/\mathfrak{p}) = 0$ and thus $G(A/\mathfrak{p}) \subseteq H(A/\mathfrak{p})$. Let $\rho_{\mathfrak{p}}$ be the specialisation of the universal deformation $\Gamma \to G(R^{\square}_{G})$ at $\mathfrak{p}$. Since $\varphi \circ \rho_{\mathfrak{p}} = \varphi \circ \rho$ by definition of $A$, we get that $\rho_{\mathfrak{p}}(\Gamma)$ is contained in $G(\kappa)$. Hence, $\mathfrak{p}$ is the maximal ideal of $A$. We conclude that $A$ is zero dimensional and, since $R^{\square}_{G}$ is noetherian, $A$ is a finite dimensional $\kappa$-vector space. The claim follows from Nakayama's Lemma. □

**Proposition 3.16.** $\dim R^{\square}_{H}/\varpi = \dim R^{\square}_{G}/\varpi$.

*Proof.* For any local $\kappa$-algebra $A$ we have an exact sequence of non-abelian fppf cohomology groups

$$0 \to Z(A) \to G(A) \to H(A) \to H^1_{\mathrm{fppf}}(\mathrm{Spec}(A), Z) \tag{25}$$

Let $e \geq 0$ be an integer such that $p^e \cdot \mathrm{m} = 0$. Let $A = R^{\square}_{H}/\varpi$ and let $\mathfrak{m}$ be the maximal ideal of $A$. We choose elements $a_1, \dots, a_r \in \mathfrak{m}$ such that their images form a basis of $\mathfrak{m}/\mathfrak{m}^2$ as a $\kappa$-vector space. Let $B = A[x_1, \dots, x_n]/I$, where $I$ is the ideal generated by $x_i^{p^e} - a_i$ for $1 \leq i \leq r$. Then $B$ is finite flat over $A$ and every $x \in 1 + \mathfrak{m}$ is a $p^e$-th power in $B^{\times}$. Let $A^{\mathrm{red}}$ and $B^{\mathrm{red}}$ be the maximal reduced quotients of $A$ and $B$, respectively. Then $A^{\mathrm{red}}$ is a subring of $B^{\mathrm{red}}$ and every $x \in 1 + \mathfrak{m}_{A^{\mathrm{red}}}$ is a $p^e$-th power in $B^{\mathrm{red}}$. Let $\rho' : \Gamma \to H(A^{\mathrm{red}})$ be the specialisation of the universal deformation $\Gamma \to H(R^{\square}_{H})$ along $R^{\square}_{H} \to A^{\mathrm{red}}$.

Since $Z$ is a finite product of $\mu_{p^n}$ with $n \leq e$ and $B^{\mathrm{red}}$ is reduced and of characteristic $p$, we have $Z(B^{\mathrm{red}}) = 1$. Since $H^1_{\mathrm{fppf}}(\mathrm{Spec}(A), \mu_{p^n}) = A^{\times}/(A^{\times})^{p^n}$ by [59, Tag 040Q], it follows from (25) that there exists a continuous representation $\tilde{\rho} : \Gamma \to G(B^{\mathrm{red}})$ deforming $\rho$ such that $\varphi \circ \tilde{\rho} = \rho' \otimes_{A^{\mathrm{red}}} B^{\mathrm{red}}$. This induces a map of $A$-algebras $R^{\square}_{G}/\varpi \to B^{\mathrm{red}}$. This map is finite as $B$ is a finite $A$-algebra. Hence $\dim(R^{\square}_{G}/\varpi) \geq \dim B^{\mathrm{red}} = \dim B = \dim A$, where the last equality follows as $B$ is finite and flat over $A$. Lemma 3.15 implies that $\dim(R^{\square}_{G}/\varpi) \leq \dim A$. □

## 4. Continuity

In this section let $\mathcal{O}$ be any ring, let $R \to A$ be a map of $\mathcal{O}$-algebras, with $R$ noetherian and profinite. Let $\rho : \Gamma \to G(A)$ be a group homomorphism, where $\Gamma$ is a profinite group and $G$ is an affine $\mathcal{O}$-group scheme of finite type.

In Section 4.1 we define an algebraic notion of continuity for such representations, such that whenever $A$ is a topological $R$-algebra the algebraic notion of continuity implies that the representation $\rho : \Gamma \to G(A)$ is continuous for the topology on the target induced by the topology of $A$. Our construction is based on the fact that if $M$ is a finitely generated $R$-module then since $R$ is noetherian and profinite there is a unique topology on $M$ making $M$ into a Hausdorff topological $R$-module; this topology is profinite. For example, if $G = \mathbb{G}_a$ then our algebraic notion of continuity is the requirement that $\rho(\Gamma)$ is contained in a finitely generated $R$-submodule $M \subseteq A = \mathbb{G}_a(A)$ and the map $\rho : \Gamma \to M$ is continuous.

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We extend this example following Conrad's exposition in [18] on how to topologise $X(A)$, when $A$ is a topological ring.

In Section 4.2 we reformulate this notion in terms of condensed sets. In fact, one could reprove the results of Section 4.1 entirely within the framework of condensed sets. However, we have chosen not to do so as the arguments in Section 4.1 are elementary and suffice for the paper.

### *4.1. Continuity via algebra*

The following is the key lemma.

**Lemma 4.1.** *Let $f \in A[y_1, \dots, y_n]$ be a polynomial, which we view as a function on $A^n$ and let $V(f) := \{\underline{a} \in A^n : f(\underline{a}) = 0\}$. Let $M$ be a finitely generated $R$-submodule of $A^n$. Then $f|_M : M \to A$ is a continuous map with values in a finitely generated $R$-submodule of $A$. In particular, $V(f) \cap M$ is a closed subset of $M$ for the natural topology on $M$ as an $R$-module.*

*Proof.* We first focus on the case, when $f$ is a monomial $y_1^{d_1} \cdots y_n^{d_n}$ of degree $d_1 + \cdots + d_n = d \geq 1$. There exists an $A$-multilinear form $\beta : A^{\otimes_A d} \to A$, such that $f(a_1, \dots, a_n) = \beta(a_1 \otimes \cdots \otimes a_1 \otimes \cdots \otimes a_n \otimes \cdots \otimes a_n)$ for all $a_1, \dots, a_n \in A$, where in the argument of $\beta$, $a_i$ is repeated $d_i$ times. Write $\iota : M \to A$ for the inclusion map. We have a natural $R$-linear map $\iota^{\otimes} : M^{\otimes_R d} \to A^{\otimes_A d}$ and we denote by $N$ the image of $M^{\otimes_R d}$ under $\tilde{\beta} := \beta \circ \iota^{\otimes}$ in $A$. In particular $f|_M$ takes values in $N$ and arises as the composition

$$M \xrightarrow{\Delta} M^d \xrightarrow{\text{can}} M^{\otimes_R d} \xrightarrow{\tilde{\beta}} N.$$

The diagonal $\Delta$ and $\tilde{\beta}$ are $R$-linear, hence continuous. We are left to show, that the universal $d$-multilinear form $M^d \to M^{\otimes_R d}$ is continuous. Choose an $R$-linear surjection $R^s \twoheadrightarrow M$. We have a commutative square

$$\begin{array}{ccc} (R^s)^d & \longrightarrow & (R^s)^{\otimes_R d} \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ M^d & \longrightarrow & M^{\otimes_R d} \end{array} \tag{26}$$

where the top and the bottom maps are the universal $d$-multilinear forms. The vertical maps are quotient maps. The top map is a polynomial map over $R$, hence continuous and thus the bottom map is continuous, which finishes the proof in this case.

Now assume, that $f$ is a general polynomial and let us write $f = \sum_i a_i f_i$ for monomials $f_i$ and $a_i \in A$. As in the previous step, we obtain multilinear maps $\beta_i : A^{\otimes_A d} \to A$ and we define $\tilde{\beta}_i := \beta_i \circ \iota^{\otimes}$, $N_i := \mathrm{Im}(\tilde{\beta}_i)$ and $N := \sum_i a_i N_i$. The multiplication maps $N_i \to N$, $x \mapsto a_i x$ are $R$-linear, thus $N$ is a finitely generated $R$-module. So $f|_M = \sum_i a_i f_i|_M$ is a sum of continuous maps and hence continuous, which finishes the proof.

Since $R$ is Hausdorff, so is $N$ and we get that $V(f) \cap M$ is closed. □

Let $X = \mathrm{Spec}\, B$ be an affine scheme of finite type over $\mathcal{O}$. A point $x \in X(A)$ corresponds to an $\mathcal{O}$-algebra homomorphism $x : B \to A$. We thus have a canonical map

$$X(A) \times \Gamma(X, \mathcal{O}_X) \to A, \quad (x, b) \mapsto \mathrm{ev}_b(x) := x(b).$$

This induces a canonical injection

$$\iota_X := \prod_{b \in B} \mathrm{ev}_b : X(A) \hookrightarrow A^B := \prod_{b \in B} A.$$

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We will denote the projection map onto the $b$-th component by $p_b : A^B \to A$. As explained above any finitely generated $R$-module $M$ has a unique topology, which makes it into a Hausdorff topological $R$-module. Moreover, $M$ is profinite with respect to this topology.

**Lemma 4.2.** *Let $S$ be a profinite set and let $f : S \to X(A)$ be a map of sets. Then the following are equivalent:*

(1) *for each $b \in B$ there exists a finitely generated $R$-submodule $M_b$ of $A$, such that $\iota_X(f(S))$ is contained in $\prod_{b\in B} M_b$ and the map $\iota_X \circ f : S \to \prod_{b\in B} M_b$ is continuous for the product topology on the target.*
(2) *for all $n \geq 1$ and all closed immersions $\tau : X \hookrightarrow \mathbb{A}^n$ of $\mathcal{O}$-schemes there exists a finitely generated $R$-submodule $M \subseteq \mathbb{A}^n(A) = A^n$ such that $\tau(f(S)) \subseteq M$ and $\tau \circ f : S \to M$ is continuous for the canonical $R$-module topology on the target.*
(3) *there exists a closed immersion $\tau : X \hookrightarrow \mathbb{A}^n$ of $\mathcal{O}$-schemes and a finitely generated $R$-submodule $M \subseteq \mathbb{A}^n(A) = A^n$ such that $\tau(f(S)) \subseteq M$ and $\tau \circ f : S \to M$ is continuous for the canonical $R$-module topology on the target.*

*Proof.* A closed immersion $\tau : X \hookrightarrow \mathbb{A}^n$ corresponds to a surjective homomorphism of $\mathcal{O}$-algebras $\mathcal{O}[x_1, \dots, x_n] \twoheadrightarrow B$. Let $b_1, \dots, b_n$ be the images of $x_1, \dots, x_n$ in $B$. Then the map $X(A) \xrightarrow{\tau} \mathbb{A}^n(A) = A^n$ factors as $X(A) \to A^B \xrightarrow{\prod_{i=1}^n p_{b_i}} A^n$. The image of $\prod_b M_b$ under this map is $\prod_{i=1}^n M_{b_i}$, which is a finitely generated $R$-module. Moreover, this projection is continuous. Hence, part (1) implies part (2).

Part (2) trivially implies part (3).

Let us assume that (3) holds. Let $M_i \subseteq A$ be the image of $M \subseteq \prod_{i=1}^n A$ under the projection onto the $i$-th component. Then $M \subseteq \prod_{i=1}^n M_i$ and $\prod_{i=1}^n M_i$ is a finitely generated $R$-module. Hence, we may replace $M$ with $\prod_{i=1}^n M_i$. Let $b_i \in B$ be the images of $x_i$ as before. If $b \in B$ then there is $g \in \mathcal{O}[x_1, \dots, x_n]$ such that $b = g(b_1, \dots, b_n)$. The image of $\prod_{i=1}^n M_i$ under the map $\varphi_g : A^n \to A$, $(a_1, \dots, a_n) \mapsto g(a_1, \dots, a_n)$ is contained in a finitely generated $R$-module $M_b$ by Lemma 4.1. Since $\tau(f(S)) \subseteq \prod_{i=1}^n M_i$ we have $\varphi_g(\tau(f(S))) \subseteq M_b$. Moreover, the induced map $S \to M_b$ is continuous since polynomial maps are continuous. The composition $\varphi_g \circ \tau : X(A) \to A$ is the map given by evaluation at $b$. Hence, (3) implies (1). □

**Lemma 4.3.** *Let $g : X \to Y$ be a morphism of affine schemes of finite type over $\mathcal{O}$. Let $S$ be a profinite set and let $f : S \to X(A)$ be a map of sets. If the equivalent conditions of Lemma 4.2 hold for $f$ then they also hold for $g \circ f$.*

*Proof.* Let us assume that $\iota_X(f(S))$ is contained in $\prod_{b\in\Gamma(X,\mathcal{O}_X)} M_b$. We claim that $\iota_Y(g(f(S)))$ is contained in $\prod_{c\in\Gamma(Y,\mathcal{O}_Y)} M_{g^\sharp(c)}$. Indeed, if $c \in \Gamma(Y, \mathcal{O}_Y)$ and $x \in X(A)$ then $\mathrm{ev}_c(g(x)) = \mathrm{ev}_{g^\sharp(c)}(x)$. □

**Lemma 4.4.** *Let $S$ be a profinite set and let $f_1 : S \to X_1(A)$ and $f_2 : S \to X_2(A)$ be maps of sets, where $X_1$ and $X_2$ are affine schemes of finite type over $\mathcal{O}$. If the equivalent conditions of Lemma 4.2 hold for $f_1$ and $f_2$ then they also hold for the product $f_1 \times f_2 : S \to X_1(A) \times X_2(A) = (X_1 \times X_2)(A)$.*

*Proof.* We pick closed embeddings $X_1 \hookrightarrow \mathbb{A}^{n_1}$, $X_2 \hookrightarrow \mathbb{A}^{n_2}$. This induces a closed embedding $X_1 \times X_2 \hookrightarrow \mathbb{A}^{n_1} \times \mathbb{A}^{n_2} \cong \mathbb{A}^{n_1+n_2}$. If $f_1(S) \subseteq M_1 \subseteq A^{n_1}$ and $f_2(S) \subseteq M_2 \subseteq A^{n_2}$ then $(f_1 \times f_2)(S) \subseteq M_1 \times M_2 \subseteq A^{n_1} \times A^{n_2}$. Moreover, if $f_1 : S \to M_1$ and $f_2 : S \to M_2$ are continuous then $f_1 \times f_2 : S \to M_1 \times M_2$ is continuous. □

**Lemma 4.5.** *Let $R'$ be a noetherian profinite $R$-algebra together with a map of $R$-algebras $R' \to A$. Let $S$ be a profinite set and let $f : S \to X(A)$ be a map of sets. If the equivalent conditions of Lemma 4.2 hold for $f$ with respect to $R$ then they also hold with respect to $R'$.*

*Proof.* It is immediate that part (1) of Lemma 4.2 holds with $M'_b$ equal to the $R'$-submodule of $A$ generated by $M_b$. □

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### *4.2. Continuity via condensed sets*

We will now express the equivalent conditions in Lemma 4.2 in terms of condensed sets. Recall that these are certain functors from the category of profinite sets to the category of sets, see [53] and Appendix A. We may view profinite sets as condensed sets via the Yoneda embedding $S \mapsto \underline{S}$, where $\underline{S}(T) := \mathscr{C}(T, S)$. The underlying set of a condensed set is by definition that functor evaluated at a point.

Recall further, that every profinite group (ring) is naturally a condensed group (ring), as the functor from topological spaces to condensed sets commutes with finite limits [53, Proposition 1.7]. We can also equip a profinite ring $R$ with the discrete topology and write $R_{\mathrm{disc}}$ to denote the associated discrete condensed ring. We have a natural map of condensed rings $R_{\mathrm{disc}} \to \underline{R}$. An abstract $R$-module $M$ can be seen as a discrete condensed $R_{\mathrm{disc}}$-module $M_{\mathrm{disc}}$, so that $M_{\mathrm{disc}}(T) = \mathscr{C}(T, M)$, where the target $M$ is given the discrete topology. The tensor product of condensed $R_{\mathrm{disc}}$-modules $M_{\mathrm{disc}} \otimes_{R_{\mathrm{disc}}} \underline{R}$ is a condensed $R$-module.[3]

**Lemma 4.6.** *Let $N$ be an abstract $R$-module. Then $N_{\mathrm{disc}} \otimes_{R_{\mathrm{disc}}} \underline{R}$ is an inductive limit of profinite submodules along quasi-compact injections. In particular, $N_{\mathrm{disc}} \otimes_{R_{\mathrm{disc}}} \underline{R}$ is quasi-separated.*

*Proof.* We write $N = \varinjlim_i M_i$ where $M_i$ are finitely generated $R$-submodules of $N$. As the tensor product of condensed $R_{\mathrm{disc}}$-modules is left adjoint to the internal Hom functor of condensed $R_{\mathrm{disc}}$-modules, it commutes with arbitrary colimits. Thus

$$N_{\mathrm{disc}} \otimes_{R_{\mathrm{disc}}} \underline{R} = (\varinjlim_i (M_i)_{\mathrm{disc}}) \otimes_{R_{\mathrm{disc}}} \underline{R} = \varinjlim_i ((M_i)_{\mathrm{disc}} \otimes_{R_{\mathrm{disc}}} \underline{R}).$$

Since $M_i$ are finitely generated they are profinite and we have a natural isomorphism $(M_i)_{\mathrm{disc}} \otimes_{R_{\mathrm{disc}}} \underline{R} \cong \underline{M_i}$. Moreover, the transition maps are quasi-compact injections, so $N_{\mathrm{disc}} \otimes_{R_{\mathrm{disc}}} \underline{R}$ is quasi-separated by Lemma A.5. □

Let $X$ be an affine scheme of finite type over $\mathcal{O}$ as before and let $S$ be a profinite set. If $C$ is a commutative condensed $\mathcal{O}$-algebra then by evaluating it at a profinite set $T$ we obtain a commutative $\mathcal{O}$-algebra $C(T)$ and then we may evaluate $X$ at $C(T)$ to obtain a set $X(C(T))$. One may show that $X(C) := X \circ C$ is a condensed set. If $C$ is quasi-separated, then $X(C)$ is quasi-separated (see Lemma A.6).

**Definition 4.7.** We will say that a map of sets $f : S \to X(A)$ is *$R$-condensed* if there is a morphism of condensed sets $\tilde{f} : \underline{S} \to X(A_{\mathrm{disc}} \otimes_{R_{\mathrm{disc}}} \underline{R})$, whose map on underlying sets is $f$.

**Lemma 4.8.** *Let $f : S \to X(A)$ be a map of sets. Then $f$ is $R$-condensed if and only if the equivalent conditions of Lemma 4.2 hold.*

*Proof.* Suppose $f$ is $R$-condensed. To verify (1) of Lemma 4.2, let $b \in B$, where $X = \mathrm{Spec}(B)$. This determines a map of schemes $\varphi_b : X \to \mathbb{A}^1$. We obtain a map of condensed sets

$$\underline{S} \xrightarrow{\tilde{f}} X(A_{\mathrm{disc}} \otimes_{R_{\mathrm{disc}}} \underline{R}) \xrightarrow{\varphi_b} \mathbb{A}^1(A_{\mathrm{disc}} \otimes_{R_{\mathrm{disc}}} \underline{R}) = A_{\mathrm{disc}} \otimes_{R_{\mathrm{disc}}} \underline{R}.$$

By Lemma 4.6 $A_{\mathrm{disc}} \otimes_{R_{\mathrm{disc}}} \underline{R}$ is an inductive limit of condensed $R$-submodules of the form $\underline{M}$ for some finitely generated $R$-submodule $M \subseteq A$. By Lemma A.1 $\underline{S}$ is a compact object in the category of condensed sets. In particular there exists a finitely generated $R$-submodule $M_b \subseteq A$, such that $\varphi_b \circ \tilde{f}$ factors over a map $\underline{S} \to \underline{M_b}$. It follows, that $\varphi_b(f(S))$ is contained in $M_b$. We deduce condition (1) by applying the universal property of product spaces.

Assume (2) of Lemma 4.2, i.e. there is a closed immersion $\tau : X \hookrightarrow \mathbb{A}^n$ of $\mathcal{O}$-schemes, a finitely generated $R$-submodule $M \subseteq \mathbb{A}^n(A)$, such that $\tau(f(S)) \subseteq M$ and $\tau \circ f : S \to M$ is continuous. We obtain a map of condensed sets

$$g : \underline{S} \to \underline{M} = M_{\mathrm{disc}} \otimes_{R_{\mathrm{disc}}} \underline{R} \to \mathbb{A}^n(A_{\mathrm{disc}} \otimes_{R_{\mathrm{disc}}} \underline{R}).$$

[3] Fargues–Scholze in [28, Section VIII] call such modules *relatively discrete* $R$-modules.

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By Lemma A.8 the map $X(A_{\mathrm{disc}} \otimes_{R_{\mathrm{disc}}} \underline{R}) \to \mathbb{A}^n(A_{\mathrm{disc}} \otimes_{R_{\mathrm{disc}}} \underline{R})$ is a quasi-compact injection. It follows from [52, Proposition 4.13], that

$$X(A_{\mathrm{disc}} \otimes_{R_{\mathrm{disc}}} \underline{R})(S) \cong \mathbb{A}^n(A_{\mathrm{disc}} \otimes_{R_{\mathrm{disc}}} \underline{R})(S) \times_{\mathrm{Map}(S, \mathbb{A}^n(A))} \mathrm{Map}(S, X(A)).$$

The pair $(g, f)$ is an element in the right hand side. So by Yoneda there exists a map of condensed sets $\tilde{f} : \underline{S} \to X(A_{\mathrm{disc}} \otimes_{R_{\mathrm{disc}}} \underline{R})$ with underlying map $f$. □

**Lemma 4.9.** *Let $A$ be a topological $R$-algebra and let $f : S \to X(A)$ be an $R$-condensed map. Then $f$ is continuous.*

*Proof.* Let $\tau : X \hookrightarrow \mathbb{A}^n$ be a closed immersion. Since $X(A)$ has the subspace topology of $A^n$, it is enough to show that $\tau \circ \rho : \Gamma \to A^n$ is continuous for the product topology on $A^n$. Since $\rho$ is $R$-condensed, there is a finitely generated $R$-submodule $N \subseteq A^n$, such that $\tau(\rho(\Gamma)) \subseteq N$ and such that $\tau \circ \rho : \Gamma \to N$ is continuous for the unique Hausdorff topology on $N$. Let us write $N^s$ for $N$ equipped with the subspace topology of $A^n$. We claim, that the identity $\mathrm{id}_N : N \to N^s$ is continuous for the Hausdorff topology on the source. It is enough to show that the projection $N \to N^s/\overline{\{0\}}$ is continuous, as every closed subset of $N^s$ is a union of translates of $\overline{\{0\}}$, where the closure is taken in $N^s$. But the quotient topology on $N^s/\overline{\{0\}}$ is Hausdorff, so $N \to N^s/\overline{\{0\}}$ is continuous. □

### *4.3. R-condensed representations*

Let $\Gamma$ be a profinite group, and let $G$ be an affine group scheme of finite type over $\mathcal{O}$.

**Definition 4.10.** A representation $\rho : \Gamma \to G(A)$ is *$R$-condensed* if $\rho$ is $R$-condensed as a map of sets.

**Remark 4.11.** Lemma 4.8 implies that $\rho$ is $R$-condensed if and only if there is a closed immersion $\tau : G \hookrightarrow \mathbb{A}^n$ of $\mathcal{O}$-schemes such that $\tau(\rho(\Gamma))$ is contained in a finitely generated $R$-submodule $M$ of $A^n = \mathbb{A}^n(A)$ and the map $\tau \circ \rho : \Gamma \to M$ is continuous for the canonical topology on $M$. For the purposes of the paper we could take this last statement as the definition of $R$-condensed representations, and proceed without actually making use of condensed mathematics. The main point of Lemma 4.2 is that this definition is independent of the chosen embedding $\tau$.

**Lemma 4.12.** *Let $\rho : \Gamma \to G(A)$ be an $R$-condensed representation. Then there is a unique homomorphism of condensed groups $\tilde{\rho} : \underline{\Gamma} \to G(A_{\mathrm{disc}} \otimes_{R_{\mathrm{disc}}} \underline{R})$ whose map on underlying sets is $\rho$.*

*Proof.* Since $\rho$ is $R$-condensed, there is a map of condensed sets

$$\tilde{\rho} : \underline{\Gamma} \to G(A_{\mathrm{disc}} \otimes_{R_{\mathrm{disc}}} \underline{R})$$

whose map on underlying sets is $\rho$. Moreover, this map is unique by Lemma A.7. We need to show, that the diagram of condensed sets

$$\begin{array}{ccc} \underline{\Gamma} \times \underline{\Gamma} & \xrightarrow{\tilde{\rho} \times \tilde{\rho}} & G(A_{\mathrm{disc}} \otimes_{R_{\mathrm{disc}}} \underline{R}) \times G(A_{\mathrm{disc}} \otimes_{R_{\mathrm{disc}}} \underline{R}) \\ \downarrow & & \downarrow \\ \underline{\Gamma} & \xrightarrow{\tilde{\rho}} & G(A_{\mathrm{disc}} \otimes_{R_{\mathrm{disc}}} \underline{R}) \end{array} \tag{27}$$

commutes, where the vertical maps are the multiplications. The diagram commutes on underlying sets, so the claim follows from uniqueness Lemma A.7. □

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**Lemma 4.13.** *Let $\rho : \Gamma \to G(A)$ be a representation and let $\Gamma'$ be an open subgroup of $\Gamma$. Then the following are equivalent:*

(1) *$\rho$ is $R$-condensed;*
(2) *the restriction of $\rho$ to $\Gamma'$ is $R$-condensed.*

*Proof.* Since $\rho(\Gamma') \subseteq \rho(\Gamma)$ part (1) trivially implies part (2).

Since $\Gamma'$ is an open subgroup of $\Gamma$ we may write $\Gamma = \bigcup_i c_i\Gamma'$ for $c_1, \dots, c_r \in \Gamma$ a finite set of coset representatives. Let $\tau : G \hookrightarrow \mathrm{GL}_d$ be a homomorphism of $\mathcal{O}$-group schemes, which is a closed immersion and let $\delta : \mathrm{GL}_d \to M_d \times M_d$ be the closed immersion $g \mapsto (g, g^{-1})$. Then $\delta \circ \tau : G \hookrightarrow M_d \times M_d$ is a closed immersion. If (2) holds then $\delta(\tau(\rho(\Gamma'))$ is contained in a finitely generated $R$-submodule of $M_d(A) \times M_d(A)$. Since the map

$$M_d(A) \times M_d(A) \to M_d(A) \times M_d(A), \quad (X, Y) \mapsto (\tau(\rho(c_i))X, Y\tau(\rho(c_i))^{-1})$$

is $R$-linear, we conclude that $\rho(\Gamma)$ maps into a finitely generated $R$-submodule of $M_d(A) \times M_d(A)$, which implies that $\rho$ is $R$-condensed. □

**Lemma 4.14.** *Let $A$ be a Hausdorff topological $R$-algebra, such that $A$ is a filtered union of closed $R$-subalgebras $B \subseteq A$, where each $B$ contains an open $R$-subalgebra $B^0 \subseteq B$, which is a finitely generated $R$-module. Assume, that $\Gamma$ is a topologically finitely generated profinite group and let $\rho : \Gamma \to G(A)$ be a homomorphism. Then the following are equivalent:*

(1) *$\rho$ is continuous;*
(2) *$\rho$ is $R$-condensed.*

*Proof.* (1) implies (2). Let $\gamma_1, \dots, \gamma_N \in \Gamma$ be generators of a dense subgroup of $\Gamma$. By assumption there is an $R$-subalgebra $B \subseteq A$, which contains an $R$-finite open $R$-subalgebra $B^0$, such that $\rho(\gamma_1), \dots, \rho(\gamma_N) \in G(B)$. Since $G(B)$ is closed in $G(A)$, we have $\rho(\Gamma) \subseteq G(B)$. Since $G$ is of finite type over $\mathcal{O}$, $G(B^0)$ is an open subgroup of $G(B)$. Thus the preimage $\Gamma' := \rho^{-1}(G(B^0))$ is an open subgroup of $\Gamma$. If $\tau : G \to \mathbb{A}^n$ is a closed immersion then $\tau(\rho(\Gamma'))$ is contained in $\mathbb{A}^n(B^0)$, which a finitely generated $R$-module. Thus the restriction of $\rho$ to $\Gamma'$ is $R$-condensed and Lemma 4.13 implies that $\rho$ is $R$-condensed.

(2) implies (1) by Lemma 4.9. □

**Lemma 4.15.** *Let $H$ be a closed subgroup scheme of $G$, let $\{\gamma_i\}_{i\in I}$ be a set of topological generators of $\Gamma$ and let $\rho : \Gamma \to G(A)$ be an $R$-condensed representation. Then the following are equivalent:*

(1) *$\rho$ factors through $H(A) \subseteq G(A)$;*
(2) *$\rho(\gamma_i) \in H(A)$ for all $i \in I$.*

*Proof.* Clearly (1) implies (2). For the converse, let $\Delta$ be the abstract subgroup of $\Gamma$ generated by $\{\gamma_i\}_{i\in I}$. Let $\tau : G \hookrightarrow \mathbb{A}^n$ be a closed immersion. Since $\rho$ is $R$-condensed $\rho(\Gamma)$ is contained in a finitely generated $R$-submodule $M \subseteq \mathbb{A}^n(A)$ and the map $\rho : \Gamma \to M$ is continuous. The assumption in (2) implies that $\rho(\Delta) \subseteq H(A)$. Since $H$ is closed in $G$ the restriction of $\tau$ to $H$ is a closed immersion. It follows from Lemma 4.1 that $M \cap H(A)$ is a closed subset of $M$. Its preimage in $\Gamma$ will be a closed subset of $\Gamma$ containing $\Delta$. Since $\Delta$ is dense in $\Gamma$ we conclude that $\rho(\Gamma) \subseteq M \cap H(A)$, which implies part (1). □

**Lemma 4.16.** *Let $\varphi : G \to H$ be a morphism of $\mathcal{O}$-group schemes of finite type, and let $\rho : \Gamma \to G(A)$ be a representation. Then the following hold:*

(1) *if $\rho$ is $R$-condensed then $\varphi \circ \rho : \Gamma \to H(A)$ is $R$-condensed;*
(2) *if $\varphi \circ \rho$ is $R$-condensed and $\varphi$ is a closed immersion then $\rho$ is $R$-condensed.*

*Proof.* Lemma 4.3 implies part (1); part (2) holds trivially. □

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## 5. The space of generic matrices

Let $G$ be a smooth affine $\mathcal{O}$-group scheme and let $\overline{\rho}:\Gamma\to G(k)$ be a continuous representation, where $\Gamma$ is a profinite group satisfying Mazur's $p$-finiteness condition. We will introduce a space of generic matrices[4] for $G$, which will depend on $\overline{\rho}$ and a faithful representation $\tau: G\to \mathrm{GL}_d$.

### 5.1. Definition of $X^{\mathrm{gen}}_{\mathrm{GL}_d}$ and $X^{\mathrm{gen},\tau}_{G}$

We fix[5] a closed immersion $\tau: G\to \mathrm{GL}_d$ of $\mathcal{O}$-group schemes. By composing it with $\overline{\rho}$ we obtain a continuous representation $\tau\circ\overline{\rho}:\Gamma\to \mathrm{GL}_d(k)$. The linearisation $(\tau\circ\overline{\rho})^{\mathrm{lin}}: k[\Gamma]\to M_d(k)$ extends to the completed group ring $(\tau\circ\overline{\rho})^{\mathrm{lin}}: k[\![\Gamma]\!]\to M_d(k)$. Let $\overline{D}: k[\![\Gamma]\!]\to k$ be the continuous determinant law associated to $(\tau\circ\overline{\rho})^{\mathrm{lin}}$. (See [15] and [62, Section 3].)

Let $R^{\mathrm{ps}}_{\mathrm{GL}_d}$ be the universal deformation ring of $\overline{D}$ introduced in [15, Section 3.1] and let $D^u: R^{\mathrm{ps}}_{\mathrm{GL}_d}[\![\Gamma]\!]\to R^{\mathrm{ps}}_{\mathrm{GL}_d}$ be the extension of the universal $d$-dimensional determinant law lifting $\overline{D}$. In particular, $R^{\mathrm{ps}}_{\mathrm{GL}_d}$ is a complete local noetherian $\mathcal{O}$-algebra with residue field $k$, and we equip it with its $\mathfrak{m}$-adic topology. We will write $D^u_{|A}$ for the specialisation of $D^u$ along $R^{\mathrm{ps}}_{\mathrm{GL}_d}\to A$. We let $X^{\mathrm{ps}}_{\mathrm{GL}_d}:=\mathrm{Spec}(R^{\mathrm{ps}}_{\mathrm{GL}_d})$. Note, that $R^{\mathrm{ps}}_{\mathrm{GL}_d}$ corresponds to $R^{\mathrm{ps}}$ of [6] when we take the residual representation to be $\tau\circ\overline{\rho}$.

Let $\mathrm{CH}(D^u)$ be the two-sided ideal of $R^{\mathrm{ps}}_{\mathrm{GL}_d}[\![\Gamma]\!]$ defined in [15, Section 1.17], so that $E^u:=R^{\mathrm{ps}}_{\mathrm{GL}_d}[\![\Gamma]\!]/\mathrm{CH}(D^u)$ is the largest quotient of $R^{\mathrm{ps}}_{\mathrm{GL}_d}[\![\Gamma]\!]$ on which $D^u$ descends to a Cayley–Hamilton determinant law. Since $\Gamma$ satisfies Mazur's $p$-finiteness condition, $E^u$ is a finitely generated $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-module and $\mathrm{CH}(D^u)$ is closed by [62, Proposition 3.6]. We write $D^u_{E^u}: E^u\to R^{\mathrm{ps}}_{\mathrm{GL}_d}$ for the determinant law induced on $E^u$ by $D^u$ and $\pi_{E^u}: R^{\mathrm{ps}}_{\mathrm{GL}_d}[\![\Gamma]\!]\to E^u$ for the projection.

If $f: E^u\to M_d(A)$ is a homomorphism of $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebras for a commutative $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebra $A$ then we say $f$ is a *homomorphism of Cayley–Hamilton algebras* if $\det\circ(f\otimes A): E^u\otimes_{R^{\mathrm{ps}}_{\mathrm{GL}_d}}A\to A$ is equal to $D^u_{E^u}\otimes A$.

**Lemma 5.1.** *The composition $R^{\mathrm{ps}}_{\mathrm{GL}_d}[\Gamma]\to R^{\mathrm{ps}}_{\mathrm{GL}_d}[\![\Gamma]\!]\twoheadrightarrow E^u$ is surjective.*

*Proof.* Let $S$ be the image of $R^{\mathrm{ps}}_{\mathrm{GL}_d}[\Gamma]$ in $E^u$. Since $R^{\mathrm{ps}}_{\mathrm{GL}_d}[\Gamma]$ is a dense subring of $R^{\mathrm{ps}}_{\mathrm{GL}_d}[\![\Gamma]\!]$, $S$ is also dense in $E^u$. It is enough to show that $S$ is closed in $E^u$.

It follows from [62, Proposition 3.6] that $E^u$ is a finitely generated $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-module. This implies that the quotient topology on $E^u$ coincides with the topology inherited from $R^{\mathrm{ps}}_{\mathrm{GL}_d}$ as a finitely generated $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-module, see the proof of [6, Lemma 3.2]. Every $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-submodule of $E^u$ is closed with respect to this topology. Hence, $S$ is closed in $E^u$, and thus $S=E^u$. □

**Lemma 5.2.** *Let $R\to A$ be a map of $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebras such that $R$ is a noetherian profinite topological $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebra. Let $\rho:\Gamma\to \mathrm{GL}_d(A)$ be a representation satisfying $\det\circ\rho^{\mathrm{lin}}=D^u_{|A}$. Then the following are equivalent:*

(1) *$\rho^{\mathrm{lin}}: R^{\mathrm{ps}}_{\mathrm{GL}_d}[\Gamma]\to M_d(A)$ extends to a homomorphism of $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebras $\widehat{\rho}: R^{\mathrm{ps}}_{\mathrm{GL}_d}[\![\Gamma]\!]\to M_d(A)$;*
(2) *$\rho^{\mathrm{lin}}: R^{\mathrm{ps}}_{\mathrm{GL}_d}[\Gamma]\to M_d(A)$ factors through a homomorphism of $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebras $E^u\to M_d(A)$;*
(3) *$\rho$ is $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-condensed;*
(4) *$\rho$ is $R$-condensed.*

*Moreover, if the equivalent conditions hold then $E^u\to M_d(A)$ in (2) is a homomorphism of Cayley–Hamilton algebras.*

[4] The terminology is taken from [6] and is inspired by Procesi's paper [49].
[5] Its existence follows from [30, Exp. $VII_B$, Rem. 11.11.1].

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*Proof.* (1) implies (2). The homomorphism $\widehat{\rho} \otimes A : R^{\mathrm{ps}}_{\mathrm{GL}_d}[\![\Gamma]\!] \otimes_{R^{\mathrm{ps}}_{\mathrm{GL}_d}} A \to M_d(A)$ induces a $d$-dimensional $A$-valued Cayley–Hamilton determinant law

$$\det \circ (\widehat{\rho} \otimes A) : R^{\mathrm{ps}}_{\mathrm{GL}_d}[\![\Gamma]\!] \otimes_{R^{\mathrm{ps}}_{\mathrm{GL}_d}} A \to A$$

which satisfies $\det \circ (\widehat{\rho} \otimes A) = D^u_{|A}$ by assumption.

It follows, that $D^u_{|A}$ is Cayley–Hamilton, hence $\widehat{\rho} \otimes A$ factors through a homomorphism $E^u \otimes_{R^{\mathrm{ps}}_{\mathrm{GL}_d}} A \to M_d(A)$ and so $\rho^{\mathrm{lin}}|_{R^{\mathrm{ps}}_{\mathrm{GL}_d}[\Gamma]}$ factors through a homomorphism of Cayley–Hamilton algebras $E^u \to M_d(A)$.

(2) implies (3). Since $E^u$ is finitely generated as $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-module by considering the closed embedding $\delta : \mathrm{GL}_d \hookrightarrow M_d \times M_d$, $g \mapsto (g, g^{-1})$ we obtain that $\rho$ is $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-condensed.

(3) implies (4) by Lemma 4.5.

(4) implies (1). Since $\rho$ is $R$-condensed, $\delta(\rho(\Gamma))$ is contained in a finitely generated $R$-submodule of $M_d(A) \times M_d(A)$. By projecting onto the first factor we deduce that the the image of $\rho^{\mathrm{lin}} : R^{\mathrm{ps}}_{\mathrm{GL}_d}[\Gamma] \to M_d(A)$ is contained in a finitely generated $R$-module $M$. Since $M$ is profinite the map extends to a continuous map of profinite $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-modules $\widehat{\rho} : R^{\mathrm{ps}}_{\mathrm{GL}_d}[\![\Gamma]\!] \to M$. Since $R^{\mathrm{ps}}_{\mathrm{GL}_d}[\Gamma]$ is a dense subalgebra of $R^{\mathrm{ps}}_{\mathrm{GL}_d}[\![\Gamma]\!]$ we deduce that $\rho^{\mathrm{lin}}$ extends to a map of $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebras $R^{\mathrm{ps}}_{\mathrm{GL}_d}[\![\Gamma]\!] \to M_d(A)$. □

**Definition 5.3.** Let $X^{\mathrm{gen}}_{\mathrm{GL}_d} : R^{\mathrm{ps}}_{\mathrm{GL}_d}\text{-alg} \to \mathrm{Set}$ be the functor which sends $A$ to the set of representations $\rho : \Gamma \to \mathrm{GL}_d(A)$ such that $\det \circ \rho^{\mathrm{lin}} = D^u \otimes_{R^{\mathrm{ps}}_{\mathrm{GL}_d}} A$ and $\rho$ satisfies the equivalent conditions of Lemma 5.2.

We give an alternative description of the points of $X^{\mathrm{gen}}_{\mathrm{GL}_d}$. By the following Lemma, we see that $X^{\mathrm{gen}}_{\mathrm{GL}_d}$ identifies canonically with the scheme $X^{\mathrm{gen}}$ of [6].

**Lemma 5.4.** *Let $A$ be a commutative $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebra.*

(1) *$X^{\mathrm{gen}}_{\mathrm{GL}_d}(A)$ is in natural bijection with the set of homomorphisms of Cayley–Hamilton $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebras $f : E^u \to M_d(A)$. In particular, $X^{\mathrm{gen}}_{\mathrm{GL}_d}$ is representable by a finitely generated commutative $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebra $A^{\mathrm{gen}}_{\mathrm{GL}_d}$.*

(2) *If $A$ is a topological $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebra then every representation $\rho \in X^{\mathrm{gen}}_{\mathrm{GL}_d}(A)$ is continuous.*

*Proof.* Part (1). Recall, that in [6, Lemma 3.1] the $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebra $A^{\mathrm{gen}}$ is proved to represent the functor, that maps an $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebra $A$ to the set of homomorphisms of Cayley–Hamilton algebras $f : E^u \to M_d(A)$. Once we prove this functor to be naturally isomorphic to $X^{\mathrm{gen}}_{\mathrm{GL}_d}$ as in Definition 5.3, the claim about representability follows and $X^{\mathrm{gen}}_{\mathrm{GL}_d}$ is naturally isomorphic to the $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-scheme $X^{\mathrm{gen}} = \mathrm{Spec}\, A^{\mathrm{gen}}$ from [6].

If $\rho \in X^{\mathrm{gen}}_{\mathrm{GL}_d}(A)$ then by Lemma 5.2 (2) we obtain a natural homomorphism of Cayley–Hamilton algebras $f_\rho : E^u \to M_d(A)$. Conversely, a map of Cayley–Hamilton algebras $f : E^u \to M_d(A)$ gives rise to a representation by means of the composition $\rho_f : \Gamma \to (E^u)^\times \xrightarrow{f} \mathrm{GL}_d(A)$. By restriction along $R^{\mathrm{ps}}_{\mathrm{GL}_d}[\![\Gamma]\!] \otimes_{R^{\mathrm{ps}}_{\mathrm{GL}_d}} A \to E^u \otimes_{R^{\mathrm{ps}}_{\mathrm{GL}_d}} A$, we see that

$$\det \circ \rho^{\mathrm{lin}}_f = \det \circ ((f \circ \pi_{E^u}) \otimes A) = D^u_{|A}.$$

Lemma 5.2 (1) implies that $\rho_f \in X^{\mathrm{gen}}_{\mathrm{GL}_d}(A)$. The maps $\rho \mapsto f_\rho$ and $f \mapsto \rho_f$ are mutually inverse.

Part (2) follows from [6, Lemma 3.2] or Lemma 4.9. □

**Lemma 5.5.** *There is a topologically finitely generated quotient $\Gamma \twoheadrightarrow Q$ such that $R^{\mathrm{ps},\Gamma}_{\mathrm{GL}_d} = R^{\mathrm{ps},Q}_{\mathrm{GL}_d}$ and every $\rho \in X^{\mathrm{gen}}_{\mathrm{GL}_d}(A)$, where $A \in R^{\mathrm{ps}}_G$-alg, factors as $\rho : \Gamma \to Q \to \mathrm{GL}_d(A)$.*

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*Proof.* Let $Q = \Gamma/K$, where $K \subseteq \ker(\overline{\rho})$ is the smallest closed normal subgroup of $\Gamma$ such that $\ker(\overline{\rho})/K$ is pro-$p$. Since $\ker(\overline{\rho})$ is open in $\Gamma$ and $\Gamma$ satisfies Mazur's $p$-finiteness condition, $Q$ is topologically finitely generated. Lemma 3.8 in [15] implies that $R^{\mathrm{ps},\Gamma}_{\mathrm{GL}_d} = R^{\mathrm{ps},Q}_{\mathrm{GL}_d}$ and the Cayley–Hamilton algebra $E^u$ does not change if we replace $\Gamma$ by $Q$. The assertion follows from Lemma 5.4 (1). □

As already explained above, [62, Proposition 3.6] implies that $E^u$ is a finitely generated $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-module, and hence $k \otimes_{R^{\mathrm{ps}}_{\mathrm{GL}_d}} E^u$ is a finite dimensional $k$-vector space.

**Lemma 5.6.** *Let $\{\gamma_1, \dots, \gamma_N\}$ be a subset of $\Gamma$, such that its image via the map in Lemma 5.1 spans $k \otimes_{R^{\mathrm{ps}}_{\mathrm{GL}_d}} E^u$ as a $k$-vector space. Then the map*

$$X^{\mathrm{gen}}_{\mathrm{GL}_d} \to \mathrm{GL}_d^N \times X^{\mathrm{ps}}_{\mathrm{GL}_d}, \quad \rho \mapsto (\rho(\gamma_1), \dots, \rho(\gamma_N))$$

*is a closed immersion.*

*Proof.* It follows from Part (1) of Lemma 5.4 that $X^{\mathrm{gen}}_{\mathrm{GL}_d}$ coincides with the scheme $X^{\mathrm{gen}}$, that was contructed in [6, Lemma 3.1] as a closed subscheme of $M_d^n \times X^{\mathrm{ps}}_{\mathrm{GL}_d}$ by choosing generators $\delta_1, \dots, \delta_n$ of $E^u$ as an $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-module and mapping $\rho$ to the $n$-tuple $(\rho(\delta_1), \dots, \rho(\delta_n))$. Since $\rho(\gamma_i)$ are invertible matrices, it is enough to show that the images of $\gamma_1, \dots, \gamma_N$ generate $E^u$ as an $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-module. This follows from Nakayama's lemma. □

Let $Q$ be a quotient of $\Gamma$ as in Lemma 5.5. We fix an $N$-tuple $(\gamma_1, \dots, \gamma_N) \in \Gamma^N$, such that the entries generate a dense subgroup of $Q$ and its image spans $E^u \otimes_{R^{\mathrm{ps}}_{\mathrm{GL}_d}} k$ as a $k$-vector space.

**Definition 5.7.** We define the $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-scheme $X^{\mathrm{gen},\tau}_G$ as the fibre product

$$X^{\mathrm{gen},\tau}_G := X^{\mathrm{gen}}_{\mathrm{GL}_d} \times_{\mathrm{GL}_d^N} G^N, \tag{28}$$

where $G^N \to \mathrm{GL}_d^N$ is the map $(g_1, \dots, g_N) \mapsto (\tau(g_1), \dots, \tau(g_N))$.

Thus $X^{\mathrm{gen},\tau}_G$ is a closed $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-subscheme of $X^{\mathrm{gen}}_{\mathrm{GL}_d}$, and hence a closed subfunctor. We note that $X^{\mathrm{gen},\tau}_G$ is non-empty, as $X^{\mathrm{gen},\tau}_G(k)$ contains the representation $\overline{\rho}$ that we have started with.

**Remark 5.8.** By construction $X^{\mathrm{gen}}_{\mathrm{GL}_d}$ is a closed subscheme of $\mathrm{GL}_d^N \times X^{\mathrm{ps}}_{\mathrm{GL}_d}$. Thus $X^{\mathrm{gen},\tau}_G$ is a closed subscheme of $G^N \times X^{\mathrm{ps}}_{\mathrm{GL}_d}$, which can interpreted as follows.

Let $\mathcal{F}_N$ be a free group generated by elements $x_1, \dots, x_N$. Let $\mathrm{Rep}^{\mathcal{F}_N,\square}_G$ be the affine $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-scheme, which represents the functor, that maps a commutative $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebra $A$ to the set of (abstract) group homomorphisms $\rho : \mathcal{F}_N \to G(A)$. Sending $\rho$ to the $N$-tuple $(\rho(x_1), \dots, \rho(x_N))$ induces an isomorphism of $X^{\mathrm{ps}}_{\mathrm{GL}_d}$-schemes

$$\mathrm{Rep}^{\mathcal{F}_N,\square}_G \cong G^N \times X^{\mathrm{ps}}_{\mathrm{GL}_d}.$$

Mapping $x_i$ to $\gamma_i$ induces a group homomorphism $\varphi : \mathcal{F}_N \to Q$ and it follows from above, that mapping $\rho$ to $\rho \circ \varphi$ induces a closed immersion $X^{\mathrm{gen},\tau}_G \to \mathrm{Rep}^{\mathcal{F}_N,\square}_G$.

**Remark 5.9.** If $L'$ is a finite extension of $L$ with ring of integers $\mathcal{O}'$ then it follows from the definition that $(X^{\mathrm{gen},\tau}_G)_{\mathcal{O}'} \cong X^{\mathrm{gen},\tau}_{G_{\mathcal{O}'}}$. The main technical result of this paper is the computation of the dimension of $X^{\mathrm{gen},\tau}_G$. Since $\mathcal{O}'$ is finite and flat over $\mathcal{O}$ the dimensions of $(X^{\mathrm{gen},\tau}_G)_{\mathcal{O}'}$ and $X^{\mathrm{gen},\tau}_G$ are the same and using Proposition 2.8 we may assume that $G$ is split, $G/G^0$ is constant and $G(\mathcal{O}) \to (G/G^0)(\mathcal{O})$ is surjective.

We are now ready to give a description of the points of $X^{\mathrm{gen},\tau}_G$.

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**Lemma 5.10.** *Let $A$ be a commutative $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebra.*

(1) *$X_G^{\mathrm{gen},\tau}(A)$ is the set of representations $\rho : \Gamma \to G(A)$, such that $\tau \circ \rho \in X^{\mathrm{gen}}_{\mathrm{GL}_d}(A)$.*
(2) *If $A$ is a topological $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebra then every representation $\rho \in X_G^{\mathrm{gen},\tau}(A)$ is continuous.*

*Proof.* Part (1) follows from Lemmas 4.15 and 5.5.

For part (2), we observe that part (1) and Lemma 5.4 (2) imply that that the composition $\tau \circ \rho : \Gamma \to G(A) \to \mathrm{GL}_d(A)$ is continuous. Since $\tau$ is a closed immersion the topology on $G(A)$ is the subspace topology induced by $\mathrm{GL}_d(A)$ and thus $\rho$ is continuous. □

**Remark 5.11.** It follows from Lemma 5.10 that the definition of $X_G^{\mathrm{gen},\tau}$ does not depend on the choice of the quotient $Q$ and $\gamma_1, \dots, \gamma_N$. However, it will depend a priori on the choice of the faithful representation $\tau$. It will follow from Corollaries 6.12 and 8.4 that the connected component of $X_G^{\mathrm{gen},\tau}$ containing the point corresponding to the representation $\overline{\rho} : \Gamma \to G(k)$ is independent of $\tau$.

If $\mathfrak{p}$ is the maximal ideal of $R^{\mathrm{ps}}_{\mathrm{GL}_d}$ then $\kappa(\mathfrak{p}) = k$ and we equip the algebraic closure $\overline{\kappa(\mathfrak{p})}$ with the discrete topology. If $\mathfrak{p}$ is a prime of $R^{\mathrm{ps}}_{\mathrm{GL}_d}$ with $\dim R^{\mathrm{ps}}_{\mathrm{GL}_d}/\mathfrak{p} = 1$ then $\kappa(\mathfrak{p})$ is a local field and the valuation topology on $\kappa(\mathfrak{p})$ makes it into a topological $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebra by [6, Lemma 3.17]. We equip the algebraic closure $\overline{\kappa(\mathfrak{p})}$ with the topology induced by a valuation, which extends the valuation on $\kappa(\mathfrak{p})$.

**Lemma 5.12.** *Let $\mathfrak{p}$ be a prime of $R^{\mathrm{ps}}_{\mathrm{GL}_d}$ such that $\dim R^{\mathrm{ps}}_{\mathrm{GL}_d}/\mathfrak{p} \leq 1$. The set $X_G^{\mathrm{gen},\tau}(\overline{\kappa(\mathfrak{p})})$ is naturally in bijection with the set of continuous representations $\rho : \Gamma \to G(\overline{\kappa(\mathfrak{p})})$, with the topology on $\overline{\kappa(\mathfrak{p})}$ as above, such that the $d$-dimensional determinant law $\det \circ (\tau \circ \rho)^{\mathrm{lin}}$ is equal to the specialisation of $D^u$ along $R^{\mathrm{ps}}_{\mathrm{GL}_d} \to \overline{\kappa(\mathfrak{p})}$.*

*Proof.* Lemma 4.14 implies that a representation $\rho : \Gamma \to G(\overline{\kappa(\mathfrak{p})})$ is continuous if and only if it is $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-condensed. The assertion then follows from Lemma 5.10. □

Let $A^{\mathrm{gen}}_{\mathrm{GL}_d} \twoheadrightarrow A_G^{\mathrm{gen},\tau}$ be the quotient representing $X_G^{\mathrm{gen},\tau}$. We will denote by $\overline{X}_G^{\mathrm{gen},\tau}$ the special fibre of $X_G^{\mathrm{gen},\tau}$ over $\mathcal{O}$.

**Lemma 5.13.** *Let $y : R^{\mathrm{ps}}_{\mathrm{GL}_d} \to \Omega$ be a ring homomorphism into an algebraically closed field $\Omega$ and let $x \in X_G^{\mathrm{gen},\tau}(\Omega)$ be a point above $y$. Let $H$ be the smallest Zariski closed subgroup of $G_\Omega$ containing $\rho_x(\gamma_1), \dots, \rho_x(\gamma_N)$, where $\rho_x$ is the specialisation at $x$ of the universal representation $\rho : \Gamma \to \mathrm{GL}_d(A^{\mathrm{gen}}_{\mathrm{GL}_d})$. Then $H$ is equal to the Zariski closure of $\rho_x(\Gamma)$ in $G_\Omega$.*

*Proof.* Lemmas 4.15 and 5.5 imply that $\rho_x(\Gamma) = \rho_x(Q) \subseteq H(\Omega)$. Hence, its Zariski closure is contained in $H$. The other inclusion follows from the definition of $H$. □

### *5.2. Completion at maximal ideals and deformation problems*

Let $Y \subset X_G^{\mathrm{gen},\tau}$ be the preimage of the closed point of $X^{\mathrm{ps}}_{\mathrm{GL}_d}$ and let $x$ be either a closed point of $Y$ or a closed point of $X_G^{\mathrm{gen},\tau} \setminus Y$ and let $y$ be its image in $\operatorname{Spec} R^{\mathrm{ps}}_{\mathrm{GL}_d}$. It follows from [6, Lemmas 3.17, 3.18] that $\kappa(x)$ is a finite extension of $\kappa(y)$ and there are the following possibilities:

(1) if $x \in Y$ then $\kappa(x)$ is a finite extension of $k$;
(2) if $x \in X_G^{\mathrm{gen},\tau}[1/p]$ then $\kappa(x)$ is a finite extension of $L$;
(3) if $x \in \overline{X}_G^{\mathrm{gen},\tau} \setminus Y$ then $\kappa(x)$ is a local field of characteristic $p$.

The universal property of $A_G^{\mathrm{gen},\tau}$ gives us a continuous Galois representation

$$\rho_x : \Gamma \to G(\kappa(x))$$

for the natural topology on the target. In this section we want to relate the completion of the local ring $\mathcal{O}_{X_G^{\mathrm{gen},\tau},x}$ to a deformation problem for $\rho_x$.

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**Lemma 5.14.** *Let $\kappa$ be either a finite or a local $\mathcal{O}$-field and let $\rho : \Gamma \to G(\kappa)$ be a continuous representation. Let $\Lambda$ be a coefficient ring of $\kappa$ and let $(A, \mathfrak{m}_A) \in \mathfrak{A}_\Lambda$. If $\gamma_1, \dots, \gamma_N$ topologically generate $\Gamma$ then the map*

$$\rho_A \mapsto (\tau \circ \rho_A, (\rho(\gamma_1), \dots, \rho(\gamma_N)))$$

*induces a bijection of sets $D^\square_{\rho,G}(A) \longrightarrow D^\square_{\tau\circ\rho,\mathrm{GL}_d}(A) \times_{\mathrm{GL}_d^N(A)} G^N(A)$.*

*Proof.* Since $\tau : G(A) \to \mathrm{GL}_d(A)$ is injective, the map is an injection. A point in $D^\square_{\tau\circ\rho,\mathrm{GL}_d}(A) \times_{\mathrm{GL}_d^N(A)} G^N(A)$ corresponds to continuous representation $\rho_A : \Gamma \to \mathrm{GL}_d(A)$ such that $\rho_A \equiv \tau \circ \rho (\mathrm{mod}\ \mathfrak{m}_A)$ and $(g_1, \dots, g_N) \in G^N(A)$ such that $\rho(\gamma_i) = \tau(g_i)$ for $1 \le i \le N$. Since $\tau(G(A))$ is a closed subgroup of $\mathrm{GL}_d(A)$, its preimage $\rho_A^{-1}(\tau(G(A)))$ is a closed subgroup of $\Gamma$. Since it contains the topological generators $\gamma_1, \dots, \gamma_N$ we deduce that the image of $\rho_A$ is contained in $\tau(G(A))$. This implies that the map is surjective. □

**Lemma 5.15.** *Let $(A, \mathfrak{m})$ be a local (possibly non-noetherian) ring and let $I$ be a finitely generated ideal of $A$. If the $\mathfrak{m}$-adic completion $\hat{A}$ of $A$ is noetherian then $\widehat{(A/I)} = \hat{A}/I\hat{A}$.*

*Proof.* It follows from [59, Tag 0BNG] that $A/\mathfrak{m}^n$ is a quotient of $\hat{A}$ for all $n \ge 1$. Since $\hat{A}$ is noetherian, $A/\mathfrak{m}^n$ is noetherian, and hence artinian for all $n \ge 1$. By choosing generators of $I = (a_1, \dots, a_s)$ we obtain a surjection $A^s \twoheadrightarrow I$. Let $K_n = \mathrm{Ker}((A/\mathfrak{m}^n)^s \twoheadrightarrow (I + \mathfrak{m}^n)/\mathfrak{m}^n)$. The modules $K_n$ are artinian, since they are finitely generated over the artinian ring $A/\mathfrak{m}^n$, and thus form a Mittag-Leffler system, which implies that the map $\hat{A}^s = \varprojlim_n (A/\mathfrak{m}^n)^s \to \varprojlim_n (I + \mathfrak{m}^n)/\mathfrak{m}^n$ is surjective.

The map $\hat{A} \to \widehat{(A/I)}$ is surjective by [59, Tag 0315] and the kernel is equal to $\varprojlim_n (I + \mathfrak{m}^n)/\mathfrak{m}^n$, [59, Tag 02N1]. It follows from above that this projective limit is equal to $I\hat{A}$, which implies the assertion. □

**Proposition 5.16.** *Let $x \in X_G^{\mathrm{gen},\tau}(\kappa)$, where $\kappa$ is either a finite or a local $\mathcal{O}$-field. Let $\mathfrak{q}$ be the kernel of the map*

$$\Lambda \otimes_{\mathcal{O}} A_G^{\mathrm{gen},\tau} \to \kappa, \quad \lambda \otimes a \mapsto \bar{\lambda}\bar{a},$$

*where $\bar{\lambda}$ and $\bar{a}$ denote the images of $\lambda$ and $a$ in $\kappa$ and $\Lambda$ is a coefficient ring for $\kappa$. The completion of $\Lambda \otimes_{\mathcal{O}} A_G^{\mathrm{gen},\tau}$ with respect to $\mathfrak{q}$ is naturally isomorphic to $R^\square_{\rho_x,G}$.*

*Proof.* If $\rho_A \in D^\square_{\rho,G}(A)$ then [6, Proposition 3.33] implies that $A$ is naturally an $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebra and $\tau \circ \rho \in X^{\mathrm{gen}}_{\mathrm{GL}_d}(A)$. Lemma 5.5 implies that there is a topologically finitely generated quotient $Q$ of $\Gamma$ such that any deformation of $\rho$ factor through $Q$. Hence, we may assume without loss of generality that $\gamma_1, \dots, \gamma_N$ are topological generators of $\Gamma$.

Let $I$ be the kernel of the surjection $\mathcal{O}(\mathrm{GL}_d^N) \twoheadrightarrow \mathcal{O}(G^N)$. We note that $I$ is a finitely generated ideal. Let $\mathfrak{m}_G$ and $\mathfrak{m}_{\mathrm{GL}_d}$ be the maximal ideals of $R^\square_{\rho_x,G}$ and $R^\square_{\tau\circ\rho_x,\mathrm{GL}_d}$, respectively. The rings $R^\square_{\rho_x,G}/\mathfrak{m}_G^n$ and $R^\square_{\tau\circ\rho_x,\mathrm{GL}_d}/\mathfrak{m}^n_{\mathrm{GL}_d}$ represent the restrictions of functors $D^\square_{\rho_x,G}$ and $D^\square_{\tau\circ\rho_x,\mathrm{GL}_d}$ to $\mathfrak{A}_{\Lambda,n}$, respectively. It follows from Lemma 5.14 that

$$R^\square_{\rho_x,G}/\mathfrak{m}_G^n \cong (R^\square_{\tau\circ\rho_x,\mathrm{GL}_d}/\mathfrak{m}^n_{\mathrm{GL}_d}) \otimes_{\mathcal{O}(\mathrm{GL}_d^N)} \mathcal{O}(G^N).$$

By passing to the limit and using Lemma 5.15 with $A = R^\square_{\tau\circ\rho_x,\mathrm{GL}_d}$ we obtain

$$R^\square_{\rho_x,G} \cong R^\square_{\tau\circ\rho_x,\mathrm{GL}_d} \otimes_{\mathcal{O}(\mathrm{GL}_d^N)} \mathcal{O}(G^N).$$

If $G = \mathrm{GL}_d$ and $\tau = \mathrm{id}$ then the proposition is [6, Proposition 3.33], so that $R^\square_{\tau\circ\rho_x,\mathrm{GL}_d}$ is naturally isomorphic to the completion of the local ring of $\Lambda \otimes_{\mathcal{O}} A^{\mathrm{gen}}_{\mathrm{GL}_d}$ at $\mathfrak{q}$. Since $A_G^{\mathrm{gen},\tau} = A^{\mathrm{gen}}_{\mathrm{GL}_d} \otimes_{\mathcal{O}(\mathrm{GL}_d^N)} \mathcal{O}(G^N) =$

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$A^{\mathrm{gen}}_{\mathrm{GL}_d}/IA^{\mathrm{gen}}_{\mathrm{GL}_d}$ by definition, and $R^{\square}_{\tau\circ\rho_x,\mathrm{GL}_d}$ is noetherian by Proposition 3.8, the assertion follows from Lemma 5.15 applied to $A = (\Lambda \otimes_{\mathcal{O}} A^{\mathrm{gen}}_{\mathrm{GL}_d})_{\mathfrak{q}}$. □

**Lemma 5.17.** *Let $X := X_G^{\mathrm{gen},\tau}$ and let $x$ be either a closed point of $Y$ or a closed point of $X \setminus Y$. Let $\mathcal{O}_{X,x}$ be the local ring at $x$ and let $\hat{\mathcal{O}}_{X,x}$ be its completion with respect to the maximal ideal. Then we have the following isomorphisms of local rings:*

(1) *if $x \in X[1/p]$ or $x \in Y$ then $\hat{\mathcal{O}}_{X,x} \cong R^{\square}_{G,\rho_x}$;*
(2) *if $x \in \overline{X} \setminus Y$ then $\hat{\mathcal{O}}_{X,x}[\![T]\!] \cong R^{\square}_{G,\rho_x}$.*

*In particular, $\mathcal{O}_{X,x}$ is complete intersection if and only if $R^{\square}_{G,\rho_x}$ is.*

*Proof.* The assertion follows from Proposition 5.16 and [6, Lemmas 3.36, 3.37]. □

## 6. GIT quotients

Let $G$ be a generalised reductive $\mathcal{O}$-group scheme. If $A$ is an $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebra then the group $G(A)$ acts on the set $X_G^{\mathrm{gen},\tau}(A)$ by conjugation. This defines an action of $G$ on $X_G^{\mathrm{gen},\tau}$. We let

$$X_G^{\mathrm{git},\tau} := X_G^{\mathrm{gen},\tau} /\!\!/ G^0, \quad \overline{X}_G^{\mathrm{git},\tau} := \overline{X}_G^{\mathrm{gen},\tau} /\!\!/ G^0. \tag{29}$$

**Remark 6.1.** The closed immersion $\overline{X}_G^{\mathrm{gen},\tau} \hookrightarrow X_G^{\mathrm{gen},\tau}$ induces a morphism $\overline{X}_G^{\mathrm{git},\tau} \to (X_G^{\mathrm{git},\tau})_k$, which is finite by [57, Theorem 2] and induces a bijection on geometric points by [57, Theorem 3]. We deduce that this map is a homeomorphism.

We will show that $X_G^{\mathrm{git},\tau}$ is the spectrum of a complete semi-local $\mathcal{O}$-algebra. We use its connected components to decompose $X_G^{\mathrm{gen},\tau}$ and show in Lemma 6.12 that the resulting decomposition is the decomposition of $X_G^{\mathrm{gen},\tau}$ into connected components. The component containing the point corresponding to a representation $\overline{\rho} : \Gamma_F \to G(k)$ that we have started with in Section 5.1 will be denoted $X^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$. We then evaluate both functors at finite and local fields and study the corresponding Galois representations.

In Corollary 6.5 we prove an important finiteness result, which is used in an essential way in the induction argument in Section 13. This in turn rests on the following fundamental finiteness theorem proved by Vinberg [61], when $S$ is a spectrum of a field of characteristic zero, by Martin [42], when $S$ is a spectrum of a field of characteristic $p$, and by Cotner [21] in general.

**Theorem 6.2** [21]. *Let $S$ be a locally noetherian scheme and let $f : H \to G$ be a finite morphism of generalised reductive smooth affine $S$-group schemes. Then the induced map*

$$H^N /\!\!/ H \to G^N /\!\!/ G$$

*is finite.*

**Corollary 6.3.** *Let us assume the setting of Theorem 6.2. Then the induced map*

$$H^N /\!\!/ H^0 \to G^N /\!\!/ G^0$$

*is also finite.*

*Proof.* Since $G^N /\!\!/ G^0 \to S$ and $H^N /\!\!/ H^0 \to S$ are affine and $S$ is locally noetherian, we may assume that $S$ is affine and noetherian. Then $G$, $H$, $G^N$, $H^N$ as well as their respective GIT quotients are all affine schemes. Since $H$ is generalised reductive $H/H^0$ is a finite étale $S$-group scheme. Since

$$H^N /\!\!/ H = (H^N /\!\!/ H^0) /\!\!/ (H/H^0),$$

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[30, Exp. V, Thm. 4.1 (ii)] applied with $X_0 = H^N /\!\!/ H^0$ and $X_1 = (H/H^0) \times_S X_0$, implies that the morphism $H^N /\!\!/ H^0 \to H^N /\!\!/ H$ is integral. It follows from [1, Theorem 6.3.3] applied to the quotient stack $[H^N/H^0]$ that $H^N /\!\!/ H^0$ is of finite type over $S$. (If $S = \operatorname{Spec} R$, where $R$ is of finite type over a universally Japanese integral domain then the assertion also follows from [57, Theorem 2 (i)]. This setting is sufficient for our paper as we only need it over $\mathcal{O}$.) Hence, $H^N /\!\!/ H^0 \to H^N /\!\!/ H$ is of finite type, and [59, Tag 01WJ] implies that $H^N /\!\!/ H^0 \to H^N /\!\!/ H$ is finite.

Since the composition of finite morphisms is finite, using Theorem 6.2 we obtain that $H^N /\!\!/ H^0 \to G^N /\!\!/ G$ is finite. Since this map factors as $H^N /\!\!/ H^0 \to G^N /\!\!/ G^0 \to G^N /\!\!/ G$ and all the schemes are affine we obtain the claim. □

**Proposition 6.4.** *The maps $X_G^{\mathrm{git},\tau} \to X_{\mathrm{GL}_d}^{\mathrm{ps}}$ and $\overline{X}_G^{\mathrm{git},\tau} \to \overline{X}_{\mathrm{GL}_d}^{\mathrm{ps}}$ are finite.*

*Proof.* We consider the following diagram:

$$\begin{array}{ccc} X_G^{\mathrm{git},\tau} & \longrightarrow & (X_{\mathrm{GL}_d}^{\mathrm{ps}} \times G^N) /\!\!/ G^0 \\ \downarrow & & \downarrow \\ X_{\mathrm{GL}_d}^{\mathrm{ps}} & \longrightarrow & (X_{\mathrm{GL}_d}^{\mathrm{ps}} \times \mathrm{GL}_d^N) /\!\!/ \mathrm{GL}_d \end{array}$$

By Corollary 6.3 the right map is finite. Since the map $X_G^{\mathrm{gen},\tau} \to X_{\mathrm{GL}_d}^{\mathrm{ps}} \times G^N$ defined in Definition 5.7 is a closed immersion, it follows from [57, Theorem 2 (ii)], that the induced map of GIT quotients, which is the top map in the diagram, is finite. It follows that the left map in the diagram is finite. The same argument over $k$ proves the last assertion. □

**Corollary 6.5.** *Let $H$ be a closed generalised reductive subgroup scheme of $G$ (resp. $G_k$). Then the map $X_H^{\mathrm{git},\tau} \to X_G^{\mathrm{git},\tau}$ (resp. $\overline{X}_H^{\mathrm{git},\tau} \to \overline{X}_G^{\mathrm{git},\tau}$) is finite.*

*Proof.* It follows from Proposition 6.4 that the map $X_H^{\mathrm{git},\tau} \to X_{\mathrm{GL}_d}^{\mathrm{ps}}$ is finite. Moreover, it factors as $X_H^{\mathrm{git},\tau} \to X_G^{\mathrm{git},\tau} \to X_{\mathrm{GL}_d}^{\mathrm{ps}}$. Since these are affine schemes we deduce that the map $X_H^{\mathrm{git},\tau} \to X_G^{\mathrm{git},\tau}$ is finite. The same argument over $k$ gives finiteness of $\overline{X}_H^{\mathrm{git},\tau} \to \overline{X}_G^{\mathrm{git},\tau}$. □

**Proposition 6.6** [21, Lemma 4.3]. *Let $\kappa$ be a local $\mathcal{O}$-field and let $\rho : \Gamma_F \to G(\kappa)$ be a continuous representation for the topology on the target induced by the valuation on $\kappa$. Then there exists a finite extension $\kappa'$ of $\kappa$ and $g \in G^0(\kappa')$ such that $g\rho(\gamma)g^{-1} \in G(\mathcal{O}_{\kappa'})$ for all $\gamma \in \Gamma_F$.*

### 6.1. Points in $X_G^{\mathrm{gen},\tau}$

We recall some terminology following [42]. Let $\kappa$ be an algebraically closed field, which is an $\mathcal{O}$-algebra. Let $H$ be a closed reduced subgroup of $G_\kappa$. We say that $H$ is *$G$-irreducible* in $G_\kappa$ if it is not contained in any proper R-parabolic subgroup of $G_\kappa$. Let $S$ be a maximal torus in the centraliser $Z_G(H)$. We say that $H$ is *strongly reductive* in $G$ if $H$ is $Z_G(S)$-irreducible in $Z_G(S)$.

**Proposition 6.7.** *Let $x$ be a geometric point of $X_G^{\mathrm{gen},\tau}$, and let $H$ be the Zariski closure of $\rho_x(\Gamma_F)$ in $G(\kappa(x))$. Then the following are equivalent*

(1) *the orbit $G^0 \cdot x$ is closed in its fibre over $X_G^{\mathrm{git},\tau}$;*
(2) *the orbit $G \cdot x$ is closed in its fibre over $X_G^{\mathrm{git},\tau}$;*
(3) *$H$ is strongly reductive in $G_{\kappa(x)}$;*
(4) *$\rho_x : \Gamma_F \to G(\kappa(x))$ is $G$-completely reducible.*

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*Proof.* Since $G^0$ is of finite index in $G$ we may pick $g_1, \dots, g_n \in G(\kappa(x))$ such that

$$G \cdot x = \bigcup_{i=1}^{n} G^0 \cdot (g_i x) = \bigcup_{i=1}^{n} g_i \cdot (G^0 \cdot x). \tag{30}$$

It follows from (30) that (1) implies (2). If $G \cdot x$ is closed, then as it is $G^0$-invariant it will contain a closed $G^0$-orbit. We deduce from (30) that $G^0 \cdot x$ is closed, so that (2) implies (1). We have established in Lemma 5.13 that $H$ is equal to the Zariski closure in $G(\kappa(x))$ of the subgroup generated by the $N$-tuple $(\rho_x(\gamma_1), \dots, \rho_x(\gamma_N))$. The equivalence of parts (2), (3) has been established in [51, Theorem 16.4]. The equivalence of (3) and (4) follows from [3, Theorem 3.1], see the begining of [3, Section 6.3] for the non-connected case. □

**Lemma 6.8.** *Let $y$ be a geometric point in $X_G^{\mathrm{git},\tau}$. Then the fibre $(X_G^{\mathrm{gen},\tau})_y$ contains a unique closed $G^0$-orbit.*

*Proof.* This follows from [57, Theorem 3]. □

**Proposition 6.9.** *Let $x$ be a geometric point of $X_G^{\mathrm{gen},\tau}$. Then there is $z \in \overline{G^0 \cdot x}$ such that the $G^0$-orbit of $\rho_z$ is equal to the $G$-semisimplification of $\rho_x$. Moreover, $G^0 \cdot z$ is the unique closed $G^0$-orbit contained in $\overline{G^0 \cdot x}$. Here the closures are taken in the fibre of $x$ over $X_G^{\mathrm{git},\tau}$.*

*Proof.* Let $z$ be a point in the closure of the $G^0$-orbit of $x$ in $G^N(\kappa(x))$, such that the $G^0$-orbit of $z$ is closed. Since $X_G^{\mathrm{gen},\tau}$ is closed in $X_{\mathrm{GL}_d}^{\mathrm{ps}} \times G^N$ by Remark 5.8, we deduce that $z \in X_G^{\mathrm{gen},\tau}(\kappa(x))$. Lemma 5.13 implies that the $G^0$-orbit of $\rho_z$ is the $G$-semisimplification of $\rho_x$. Thus $\rho_z$ is $G$-completely reducible by Proposition 2.23. Thus the orbit $G \cdot z$ is closed by Proposition 6.7. Let $y$ be the image of $x$ in $X_G^{\mathrm{git},\tau}$. The fibre $(X_G^{\mathrm{gen},\tau})_y$ is a closed $G^0$-invariant subscheme $X_G^{\mathrm{gen},\tau}$ and thus it will contain $\overline{G^0 \cdot x}$. The last assertion follows from Lemma 6.8. □

**Corollary 6.10.** *Let $y$ be a geometric point of $X_{\mathrm{GL}_d}^{\mathrm{ps}}$ and let $x, x' \in X_G^{\mathrm{gen},\tau}(\kappa(y))$. Then $x$ and $x'$ map to the same point in $X_G^{\mathrm{git},\tau}$ if and only if $\rho_x$ and $\rho_{x'}$ have the same $G$-semisimplification.*

*Proof.* Let $x$ and $x'$ map to the same point in $X_G^{\mathrm{git},\tau}$. By Lemma 6.8 there is a unique closed $G^0$-orbit in the fibre of $y$, let $z \in X_G^{\mathrm{git},\tau}$ be a point in this orbit. It follows from Proposition 6.9 that the $G$-semisimplification of $\rho_x$ and of $\rho_{x'}$ are equal to the $G^0$-orbit of $\rho_z$. The other implication follows from directly from Proposition 6.9. □

**Corollary 6.11.** *Let $y$ be a geometric point of $X_G^{\mathrm{git},\tau}$. Assume that $G$ is split and let $T$ be a maximal split torus of $G$ defined over $\mathcal{O}$. Then there is an R-Levi $L$ of $G$ defined over $\mathcal{O}$, which contains $T$, and $x \in X_G^{\mathrm{gen},\tau}(\kappa(y))$ such that the following hold:*

(1) *$x$ maps to $y$;*
(2) *$\rho_x(\Gamma_F) \subseteq L(\kappa(y))$;*
(3) *$\rho_x$ is $L_{\kappa(y)}$-irreducible.*

*Moreover, if $x' \in X_G^{\mathrm{gen},\tau}(\kappa(y))$ maps to $y$ and $Q$ is a minimal R-parabolic of $G_{\kappa(y)}$ containing $\rho_{x'}(\Gamma_F)$ with an R-Levi $M$ then there is $g \in G^0(\kappa(y))$ such that $M$ and $L_{\kappa(y)} = gMg^{-1}$ and $\rho_x(\gamma) = g c_{Q,M}(\rho_{x'}(\gamma)) g^{-1}$ for all $\gamma \in \Gamma_F$.*

*Proof.* Lemma 6.8 implies that there exists $x \in X_G^{\mathrm{gen},\tau}(\kappa(y))$ such that $G^0 \cdot x$ is a closed. The representation $\rho_x$ is $G$-completely reducible by Proposition 6.7. Let $P$ be a minimal R-parabolic of $G_{\kappa(x)}$ containing $\rho_x(\Gamma_F)$. Since $\rho_x$ is $G$-completely reducible the image of $\rho_x(\Gamma_F)$ is contained in an R-Levi $L$ of $P$. Lemma 2.19 implies that after replacing $L$ and $P$ by a conjugate by $g \in G^0(\kappa(y))$ we may assume that $L$ and $P$ are defined over $\mathcal{O}$ and $L$ contains $T$.

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Let $Q$ be a minimal R-parabolic containing $\rho_x$ and let $M$ be an R-Levi of $Q$. Since $x$ and $x'$ both map to $y$, Corollary 6.10 implies that $\rho_x$ and $\rho_{x'}$ have the same $G$-semisimplification. It follows from Proposition 2.23 that $\rho_x$ is conjugate to $\rho' := c_{Q,M} \circ \rho_{x'}$ by $g \in G^0(\kappa(y))$. After replacing $(Q, M)$ by a conjugate, we may assume that $\rho_x = \rho'$, so that $\rho_x(\Gamma_F) \subseteq M$, and $Q$ is minimal with respect to the property containing $\rho_x(\Gamma_F)$. Let $H$ be the Zariski closure of $\rho_x(\Gamma_F)$ in $G(\kappa(y))$. Minimality of $P$ and $Q$ implies that $H$ is both $L$-irreducible and $M$-irreducible. Proposition 2.27 implies that $L$ and $M$ are conjugate by an element $h \in Z_G(H)^0(\kappa(y)) \subseteq G^0(\kappa(y))$. Since the image of $\rho'$ is contained in $H$, we have $h\rho'(\gamma)h^{-1} = \rho'(\gamma) = \rho(\gamma)$ for all $\gamma \in \Gamma_F$. □

### 6.2. *Definition of* $X^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$

Proposition 6.4 implies that the map $R^{\mathrm{ps}}_{\mathrm{GL}_d} \to (A^{\mathrm{gen},\tau}_G)^{G^0}$ is finite. Since $R^{\mathrm{ps}}_{\mathrm{GL}_d}$ is a complete local ring with residue field $k$, part (1) of [59, Tag 04GH] implies that

$$R^{\mathrm{git},\tau}_G := (A^{\mathrm{gen},\tau}_G)^{G^0} \cong R_1 \times \ldots \times R_n, \tag{31}$$

where each $R_j$ is a complete local noetherian $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebra with residue field a finite extension of $k$. The product can be indexed by the set of $\mathrm{Gal}(\overline{k}/k)$-orbits in $X^{\mathrm{git},\tau}_G(\overline{k})$. From (31) we obtain a $G^0$-equivariant decomposition

$$X^{\mathrm{gen},\tau}_G \cong \coprod_j X^{\mathrm{gen},\tau}_G \times_{X^{\mathrm{git},\tau}_G} \mathrm{Spec}\, R_j. \tag{32}$$

Moreover, we have

$$(X^{\mathrm{gen},\tau}_G \times_{X^{\mathrm{git},\tau}_G} \mathrm{Spec}\, R_j) /\!\!/ G^0 \cong \mathrm{Spec}\, R_j. \tag{33}$$

**Lemma 6.12.** *Each $X^{\mathrm{gen},\tau}_G \times_{X^{\mathrm{git},\tau}_G} \mathrm{Spec}\, R_j$ is connected. In particular, (32) is the decomposition of $X^{\mathrm{gen},\tau}_G$ into connected components.*

*Proof.* Since a connected component is a union of irreducible components, [6, Lemma 2.1] implies that the connected components of $X^{\mathrm{gen},\tau}_G \times_{X^{\mathrm{git},\tau}_G} \mathrm{Spec}\, R_j$ are $G^0$-invariant. It follows from [57, Theorem 3 (iii)] that if there were more than one then their images in $\mathrm{Spec}\, R_j$ under (33) would disconnect it. This is not possible as $R_j$ is a local ring. □

The representation $\overline{\rho} : \Gamma_F \to G(k)$ that we have fixed in Section 5 gives us a point $x_0 \in X^{\mathrm{gen},\tau}_G(k)$. We denote the unique connected component of $X^{\mathrm{gen},\tau}_G$ which contains $x_0$ by $X^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$, where $\overline{\rho}^{\mathrm{ss}}$ is the $G$-semisimplification of the representation $\Gamma_F \xrightarrow{\overline{\rho}} G(k) \hookrightarrow G(\overline{k})$. The justification for this notation is given in the lemma below. Let

$$R^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}} := R_{j_0}, \quad X^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}} := \mathrm{Spec}\, R^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}},$$

where $j_0$ is the index corresponding to the image of $x_0$ in $X^{\mathrm{git},\tau}_G(k) \subseteq X^{\mathrm{git},\tau}_G(\overline{k})$. Since $x_0$ is $k$-rational, the residue field of $R^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$ is equal to $k$.

**Lemma 6.13.** *Let $x$ be a geometric point of $X^{\mathrm{gen},\tau}_G$ above the closed point of $X^{\mathrm{ps}}_{\mathrm{GL}_d}$. Then $x \in X^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$ if and only if the $G$-semisimplification of $\rho_x$ is equal to $\overline{\rho}^{\mathrm{ss}}$.*

*Proof.* Let $z \in X^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}(\kappa(x))$ be a point above $x_0$. It follows from Lemma 6.12 that $x$ and $z$ lie on the same connected component of $X^{\mathrm{gen},\tau}_G$ if and only if their images in $X^{\mathrm{git},\tau}_G$ lie on the same connected

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component of $X_G^{\mathrm{git},\tau}$. Since $z \in X_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{gen},\tau}$ and the residue field of $R_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{git},\tau}$ is $k$, $X_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{git},\tau}(\kappa(x))$ is a point, and thus this is equivalent to $z$ and $x$ mapping to the same point in $X_G^{\mathrm{git},\tau}$. Corollary 6.10 implies that this is equivalent to $\rho_x$ and $\rho_z$ having the same $G$-semisimplification. □

**Remark 6.14.** After replacing $L$ by an unramified extension we may assume that all the factors in (31) have residue field $k$. It follows from Lemma 6.13 that in this case the connected components of $X_G^{\mathrm{gen},\tau}$ are indexed by $G^0(\overline{k})$-conjugacy classes of continuous $G$-completely reducible representations $\overline{\rho}^{\mathrm{ss}} : \Gamma_F \to G(\overline{k})$ such that $\det \circ (\tau \circ \overline{\rho}^{\mathrm{ss}})^{\mathrm{lin}} = \overline{D}$. Lemma 6.12 implies that after replacing $L$ by a finite unramified extension we may assume that every connected component of $X_G^{\mathrm{gen},\tau}$ has a $k$-rational point.

## 7. Lafforgue's $G$-pseudocharacters

In his work on the local Langlands correspondence for function fields [37] Vincent Lafforgue introduced a notion of *$G$-pseudocharacter* for a reductive group $G$. If $G = \mathrm{GL}_d$ then Emerson and Morel [25] have shown that a $\mathrm{GL}_d$-pseudocharacter after Lafforgue is equivalent to a determinant law after Chenevier [15]. If $d!$ is invertible in the base ring then the trace of representation valued in $\mathrm{GL}_d$ determines its $\mathrm{GL}_d$-pseudocharacter. In the arithmetic setting this was first used by Wiles [63] and Taylor [60].

### *7.1. Definition of G-pseudocharacter*

The definition of $G$-pseudocharacter we use is a slight modification of Lafforgue's original definition [37, Section 11], in that we work over the base ring $\mathcal{O}$ and allow arbitrary (disconnected) generalised reductive group schemes for $G$.

**Definition 7.1.** Let $\Gamma$ be an abstract group and let $A$ be a commutative $\mathcal{O}$-algebra. A *$G$-pseudocharacter* $\Theta$ of $\Gamma$ over $A$ is a sequence $(\Theta_n)_{n\geq 1}$ of $\mathcal{O}$-algebra maps

$$\Theta_n : \mathcal{O}[G^n]^{G^0} \to \mathrm{Map}(\Gamma^n, A)$$

for $n \geq 1$, satisfying the following conditions[6]:

(1) For each $n, m \geq 1$, each map $\zeta : \{1, \dots, m\} \to \{1, \dots, n\}$, $f \in \mathcal{O}[G^m]^{G^0}$ and $\gamma_1, \dots, \gamma_n \in \Gamma$, we have

$$\Theta_n(f^\zeta)(\gamma_1, \dots, \gamma_n) = \Theta_m(f)(\gamma_{\zeta(1)}, \dots, \gamma_{\zeta(m)})$$

where $f^\zeta(g_1, \dots, g_n) = f(g_{\zeta(1)}, \dots, g_{\zeta(m)})$.

(2) For each $n \geq 1$, for each $\gamma_1, \dots, \gamma_{n+1} \in \Gamma$ and each $f \in \mathcal{O}[G^n]^{G^0}$, we have

$$\Theta_{n+1}(\hat{f})(\gamma_1, \dots, \gamma_{n+1}) = \Theta_n(f)(\gamma_1, \dots, \gamma_n\gamma_{n+1})$$

where $\hat{f}(g_1, \dots, g_{n+1}) = f(g_1, \dots, g_n g_{n+1})$.

We denote the set of $G$-pseudocharacters of $\Gamma$ over $A$ by $\mathrm{PC}_G^\Gamma(A)$. If $f : A \to B$ is a homomorphism of $\mathcal{O}$-algebras, then there is an induced map $f_* : \mathrm{PC}_G^\Gamma(A) \to \mathrm{PC}_G^\Gamma(B)$. For $\Theta \in \mathrm{PC}_G^\Gamma(A)$, the image $f_*(\Theta)$ is called the *specialisation* of $\Theta$ along $f$ and is denoted by $\Theta \otimes_A B$. When $\varphi : G \to H$ is a homomorphism of generalised reductive $\mathcal{O}$-group schemes, the induced maps $\varphi_n^* : \mathcal{O}[H^n]^{H^0} \to \mathcal{O}[G^n]^{G^0}$ give rise to an $H$-pseudocharacter $(\Theta_n \circ \varphi_n^*)_{n\geq 1}$. By analogy with the notation for representations we denote this $H$-pseudocharacter by $\varphi \circ \Theta$. So we also have an induced map $\mathrm{PC}_G^\Gamma(A) \to \mathrm{PC}_H^\Gamma(A)$. It is easy to verify that specialisation along $f : A \to B$ commutes with composition with $\varphi$, that is, $(\varphi \circ \Theta) \otimes_A B = \varphi \circ (\Theta \otimes_A B)$.

[6] Here $G$ acts on $G^n$ by $g \cdot (g_1, \dots, g_n) = (gg_1g^{-1}, \dots, gg_ng^{-1})$. This induces a rational action of $G$ on the affine coordinate ring $\mathcal{O}[G^n]$ of $G^n$. The submodule $\mathcal{O}[G^n]^{G^0} \subseteq \mathcal{O}[G^n]$ is defined as the rational invariant module of the $G^0$-representation $\mathcal{O}[G^n]$. It is an $\mathcal{O}$-subalgebra, since $G$ acts by $\mathcal{O}$-linear automorphisms.

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When $\Gamma$ is a topological group, $A$ is a topological ring and for all $n \geq 1$, the map $\Theta_n$ has image in the set $\mathscr{C}(\Gamma^n, A)$ of continuous maps $\Gamma^n \to A$, we say that $\Theta$ is *continuous*. We denote the set of continuous $G$-pseudocharacters of $\Gamma$ with values in $A$ by $\mathrm{cPC}^{\Gamma}_{G}(A)$.

When $\mathcal{O}'$ is a commutative $\mathcal{O}$-algebra and $A$ is a commutative $\mathcal{O}'$-algebra the natural maps $\mathcal{O}[G^n]^{G^0} \to \mathcal{O}'[G^n]^{G^0}$ induce a map $\mathrm{PC}^{\Gamma}_{G_{\mathcal{O}'}}(A) \to \mathrm{PC}^{\Gamma}_{G}(A)$. This map is always a bijection, as we will now explain. When $\mathcal{O}'$ is flat over $\mathcal{O}$ the map $\mathcal{O}[G^n]^{G^0} \otimes_{\mathcal{O}} \mathcal{O}' \to \mathcal{O}'[G^n]^{G^0}$ is an isomorphism by the universal coefficient theorem in rational cohomology, see [35, Proposition I.4.18 (a)]. The case of a general $\mathcal{O}$-algebra is treated in [50, Proposition 3.21]. We will use this fact whenever we need to extend the base ring $\mathcal{O}$.

### 7.2. Deformations of G-pseudocharacters

We adopt the deformation theoretic setup of subsection 5.2. Let $\kappa$ be either a finite or a local field, which is an $\mathcal{O}$-algebra, equipped with its natural topology. Let $\Lambda$ be the coefficient ring for $\kappa$ and let $\mathfrak{A}_\Lambda$ be the category of local artinian $\Lambda$-algebras with residue field $\kappa$, equipped with the natural topology.

In this section $G$ is a generalised reductive group over $\mathcal{O}$. We consider the following deformation problem for $G$-pseudocharacters.

**Definition 7.2.** Let $\overline{\Theta} \in \mathrm{cPC}^{\Gamma}_{G}(\kappa)$ be a continuous $G$-pseudocharacter of $\Gamma$ with values in $\kappa$. We define the *deformation functor*

$$D_{\overline{\Theta}} : \mathfrak{A}_\Lambda \to \mathrm{Set}, \quad A \mapsto \{\Theta \in \mathrm{cPC}^{\Gamma}_{G}(A) \mid \Theta \otimes_A \kappa = \overline{\Theta}\}$$

that sends an object $A \in \mathfrak{A}_\Lambda$ to the set of continuous $G$-pseudocharacters $\Theta$ of $\Gamma$ over $A$ with $\Theta \otimes_A \kappa = \overline{\Theta}$.

The deformation problem for $G$-pseudocharacters admits a universal deformation ring, which we will denote by $R^{\mathrm{ps}}_{G}$ when $\overline{\Theta} = \Theta_{\overline{\rho}}$. We will denote the universal $G$-pseudocharacter by $\Theta^{u}_{G}$ and will drop the index $G$, when it is clear which group scheme we are working with.

### 7.3. Comparison of $X^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$ with $X^{\mathrm{ps}}_{G}$

In this section we assume that $\Gamma$ is a profinite topologically finitely generated group. It has been shown by the second author in [50, Theorem A] that $R^{\mathrm{ps}}_{G}$ is a complete noetherian local $\Lambda$-algebra.

Let $A$ be a commutative $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebra. We have a natural transformation

$$\vartheta_A : X^{\mathrm{gen},\tau}_{G}(A) \to \mathrm{PC}^{\Gamma}_{G}(A), \quad \rho \mapsto \Theta_\rho,$$

that attaches to a representation $\rho$ its $G$-pseudocharacter $\Theta_\rho$. By [50, Theorem 3.19] $\mathrm{PC}^{\Gamma}_{G}$ is representable by an affine scheme, so $\vartheta : X^{\mathrm{gen},\tau}_{G} \to \mathrm{PC}^{\Gamma}_{G}$ factors over $\theta : X^{\mathrm{git},\tau}_{G} \to \mathrm{PC}^{\Gamma}_{G}$.

**Lemma 7.3.** *Let $\rho : \Gamma \to G(A^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}})$ be the universal representation. Then $\Theta_\rho \in \mathrm{cPC}^{\Gamma}_{G}(R^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}})$ for the $\mathfrak{m}$-adic topology on $R^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$.*

*Proof.* Lemma 5.10 implies that $\rho$ is continuous for the $\mathfrak{m}_{R^{\mathrm{ps}}_{\mathrm{GL}_d}}$-adic topology on $A^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$. The $G$-pseudocharacter $\Theta_\rho$ takes values in $R^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$ and is thus continuous for the subspace topology on $R^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$. Since

$$\mathfrak{m}^n_{R^{\mathrm{ps}}_{\mathrm{GL}_d}} R^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}} \subseteq \mathfrak{m}^n_{R^{\mathrm{ps}}_{\mathrm{GL}_d}} A^{\mathrm{gen},\tau}_{G} \cap R^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}, \quad \forall n \geq 1,$$

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$\Theta_\rho$ is continuous for the $\mathfrak{m}_{R^{\mathrm{ps}}_{\mathrm{GL}_d}}$-adic topology on $R^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$. Since by Proposition 6.4, $R^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$ is a finite $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebra, this topology coincides with the topology defined by the maximal ideal of $R^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$. □

Let $\Theta_\rho$ be the $G$-pseudocharacter of the universal representation over $A^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$ as in Lemma 7.3. Recall that $R^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$ has residue field $k$ and $\Theta_\rho \otimes_{R^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}} k = \Theta_{\overline{\rho}}$. It follows from Lemma 7.3, that $\Theta_\rho$ is indeed a continuous deformation of $\Theta_{\overline{\rho}^{\mathrm{ss}}}$, so we have a local homomorphism $R^{\mathrm{ps}}_G \to R^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$ which induces a map

$$\nu : X^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{ps}}_G \tag{34}$$

of $\mathcal{O}$-schemes.

**Proposition 7.4.** *$\nu : X^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{ps}}_G$ is a finite universal homeomorphism.*

*Proof.* By [59, Tag 04DF], it suffices to show, that $\nu$ is finite, universally injective and surjective. By Proposition 6.4 $R^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$ is a finite $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-module. From the embedding $\tau : G \to \mathrm{GL}_d$ we obtain a factorisation $R^{\mathrm{ps}}_{\mathrm{GL}_d} \to R^{\mathrm{ps}}_G \to R^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$. In particular, $R^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$ is a finite $R^{\mathrm{ps}}_G$-module.

The closed point of $X^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$ is mapped to the closed point of $X^{\mathrm{ps}}_G$, so we only need to show surjectivity of $\dot{\nu} : U^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}} := X^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}} \setminus \{*\} \to U^{\mathrm{ps}}_G := X^{\mathrm{ps}}_G \setminus \{*\}$. By [59, Tag 02J6] $U^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$ and $U^{\mathrm{ps}}_G$ are Jacobson. By Chevalley's theorem [59, Tag 054K] the image of $\dot{\nu}$ is locally constructible and thus also its complement. Therefore for surjectivity of $\nu$ it is sufficient to show, that the closed points of $U^{\mathrm{ps}}_G$ are all contained in the image of $\dot{\nu}$.

Let $y \in U^{\mathrm{ps}}_G$ be a closed point. By [6, Lemma 3.17] $\kappa(y)$ is a local field. By [50, Theorem 5.7], $R^{\mathrm{ps}}_G$ is noetherian, and thus $\kappa(y)$ with its natural topology is a topological $R^{\mathrm{ps}}_G$-algebra and the image of $y : R^{\mathrm{ps}}_G \to \kappa(y)$ is contained in the ring of integers $\mathcal{O}_{\kappa(y)}$ by [6, Lemma 3.17]. We obtain a continuous $G$-pseudocharacter $\Theta_y \in \mathrm{cPC}^{\Gamma}_G(\mathcal{O}_{\kappa(y)})$, which is a deformation of $\Theta_{\overline{\rho}}$ to $\mathcal{O}_{\kappa(y)}$. By the continuous reconstruction theorem [50, Theorem 3.8], there is a continuous representation $\rho' : \Gamma \to G(\overline{\kappa})$ with $\Theta_{\rho'} = \Theta_y$, where $\overline{\kappa}$ is an algebraic closure of $\kappa(y)$. Lemma 5.12 implies that $\rho' \in X^{\mathrm{gen},\tau}_G(\overline{\kappa})$. The $G(\overline{\kappa})$-conjugacy class of $\rho'$ defines a point $y' \in X^{\mathrm{git},\tau}_G(\overline{\kappa})$ which maps to $y$. We have to show that $y'$ lies on the same connected component of $X^{\mathrm{git},\tau}_G$ as $\overline{\rho}^{\mathrm{ss}}$. Since $X^{\mathrm{gen},\tau}_G$ is of finite type over $X^{\mathrm{ps}}_{\mathrm{GL}_d}$, $\rho'$ takes values in $G(\kappa')$ for some finite extension of $\kappa(y)$ in $\overline{\kappa}$. Proposition 6.6 implies that after conjugating $\rho'$ with an element of $G^0(\overline{\kappa})$ we may assume that $\rho'$ takes values in $G(\mathcal{O}_{\overline{\kappa}})$. Let $\overline{\rho}'$ be the composition of $\rho'$ with the reduction map $G(\mathcal{O}_{\overline{\kappa}}) \to G(\overline{k})$. Then $\Theta_{\overline{\rho}'} = \Theta_y \otimes_{\mathcal{O}_{\kappa(y)}} \overline{k} = \Theta_{\overline{\rho}^{\mathrm{ss}}}$. The uniqueness part of the reconstruction theorem [50, Theorem 3.7] implies that $G$-semisimplification of $\overline{\rho}'$ is equal to $\overline{\rho}^{\mathrm{ss}}$, and Lemma 6.12 implies that $y' \in X^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$.

For universal injectivity it suffices by [59, Tag 01S4] to show that the diagonal morphism $X^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}} \times_{X^{\mathrm{ps}}_G} X^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$ is surjective. Let $x \in X^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}} \times_{X^{\mathrm{ps}}_G} X^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$. We obtain by Proposition 6.7 and Lemma 6.8 two $G$-semisimple representations $\rho_x, \rho'_x \in X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}(\overline{\kappa(x)})$, such that their associated $G$-pseudocharacters coincide. This means by the uniqueness part of the reconstruction theorem and Corollary 6.10, their image in $X^{\mathrm{git}}_G(\overline{\kappa(x)})$ coincides and therefore $x$ is in the image of the diagonal. □

**Lemma 7.5.** *$\nu \otimes_{\mathbb{Z}_p} \mathbb{Q}_p : X^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p] \to X^{\mathrm{ps}}_G[1/p]$ is a closed immersion.*

*Proof.* Let $\mathcal{F}_N$ be a free group generated by elements $x_1, \ldots, x_N$. Let $\mathrm{Rep}^{\mathcal{F}_N,\square}_G$ be the affine $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-scheme, which represents the functor, that maps a commutative $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebra $A$ to the set of (abstract) homomorphisms $\rho : \mathcal{F}_N \to G(A)$. Mapping $x_i$ to $\gamma_i$ induces a group homomorphism $\mathcal{F}_N \to \Gamma$ and it follows from Remark 5.8, that we have a closed immersion $X^{\mathrm{gen},\tau}_G \to \mathrm{Rep}^{\mathcal{F}_N,\square}_G$ and hence a closed immersion $X^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}} \to \mathrm{Rep}^{\mathcal{F}_N,\square}_G$.

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By [25, Proposition 2.11 (i)] the vertical map on the left hand side of the following diagram is an isomorphism.

$$\begin{array}{ccc} R^{\mathrm{ps}}_{\mathrm{GL}_d}[1/p][\mathrm{PC}_G^{\mathcal{F}_N}] & \longrightarrow & R^{\mathrm{ps}}_G[1/p] \\ \downarrow\cong & & \downarrow \nu^* \\ R^{\mathrm{ps}}_{\mathrm{GL}_d}[1/p][\mathrm{Rep}_G^{\mathcal{F}_N,\square}]^{G^0} & \twoheadrightarrow & R^{\mathrm{git},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p] \end{array} \tag{35}$$

The bottom map is surjective, since $R^{\mathrm{ps}}_{\mathrm{GL}_d}[1/p][\mathrm{Rep}_G^{\mathcal{F}_N,\square}] \to A^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$ is surjective as explained above and taking $G^0$-invariants is exact in characteristic 0. Since the diagram commutes the right vertical arrow is surjective, and we obtain the claim. □

**Corollary 7.6.** *The map $\nu \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ induces an isomorphism of underlying reduced schemes. In particular, if $X^{\mathrm{ps}}_G[1/p]$ is reduced, then $\nu \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an isomorphism.*

*Proof.* Proposition 7.4 implies that $\nu \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a universal homeomorphism and Lemma 7.5 that it is a closed immersion, which implies the first claim. The second claim is then immediate. □

**Remark 7.7.** In [47], building on the results of this paper, we show that $\nu \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an isomorphism.

## 8. Functoriality

The main result of this section is Proposition 8.3, which gives an intrinsic description of $X^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$ as a functor. As a consequence in Section 8.1 we show that $X^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$ is independent of the choice of an embedding $\tau : G \hookrightarrow \mathrm{GL}_d$ and is functorial in $G$. In Section 8.2 we investigate functoriality in $G$ under finite maps. In Section 8.3 we study the case when $G$ is a product of two generalised reductive $\mathcal{O}$-groups. In Section 8.4 we study the morphisms induced by restriction of pseudocharacters and representations to an open subgroup of $\Gamma$.

We assume in this section that $\Gamma$ is topologically finitely generated. This assumption ensures that $R^{\mathrm{ps}}_G$ is noetherian. If $\Gamma$ is only assumed to satisfy Mazur's $p$-finiteness condition[7] then results of this section can be reformulated by replacing $\Gamma$ by a topologically finitely generated quotient as in Lemma 5.5. We note that $\Gamma_F$ is topologically finitely generated by [34, Satz 3.6].

### 8.1. Independence of $\tau$ and functoriality in $G$

**Lemma 8.1.** *Let $A$ be an $R^{\mathrm{ps}}_G$-algebra and let $\rho \in X^{\mathrm{gen},\tau}_G(A)$. Then $\rho \in X^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}(A)$ if and only if the $G$-pseudocharacter $\Theta_\rho$ is equal to the specialisation of $\Theta^u$ at $A$.*

*Proof.* Let $\rho \in X^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}(A)$. Then $\rho = \rho^u \otimes_{A^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}} A$ where $\rho^u \in X^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}(A^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}})$ is the universal representation. Taking $G$-pseudocharacters, we obtain $\Theta_\rho = \Theta_{\rho^u} \otimes_{A^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}} A$. Using the map $X^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{ps}}_G$ given by composing the map in (34) with the GIT quotient map, we see that $\Theta_{\rho^u} = \Theta^u \otimes_{R^{\mathrm{ps}}_G} A^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$, so $\Theta_\rho = \Theta^u \otimes_{R^{\mathrm{ps}}_G} A$.

Conversely, let $\rho \in X^{\mathrm{gen},\tau}_G(A)$ satisfy $\Theta_\rho = \Theta^u \otimes_{R^{\mathrm{ps}}_G} A$. By factoring $\rho$ as $\Gamma \to G(A^{\mathrm{gen},\tau}_G) \to G(A)$ over the universal representation on $A^{\mathrm{gen},\tau}_G$ and using that $A^{\mathrm{gen},\tau}_G$ is of finite type over $R^{\mathrm{ps}}_{\mathrm{GL}_d}$, we see that $\rho$ actually takes values in a finitely generated $R^{\mathrm{ps}}_G$-subalgebra $A' \subseteq A$. We want to show, that the image of $\mathrm{Spec}(A') \to X^{\mathrm{gen},\tau}_G$ is entirely contained in the component $X^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$, which amounts to showing that an idempotent $e \in A^{\mathrm{gen},\tau}_G$ associated to a component $X^{\mathrm{gen},\tau}_{G,\overline{\rho}_2^{\mathrm{ss}}}$ associated with a residual

[7] In [46], using the results of this paper, we show that if $\Gamma_F$ satisfies Mazur's $p$-finiteness condition then $R^{\mathrm{ps}}_G$ is noetherian and the results of this section still hold in this more general setting.

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representation $\overline{\rho}_2^{\mathrm{ss}}$ which is non-conjugate to $\overline{\rho}^{\mathrm{ss}}$ acts as zero on $A'$. By contradiction, we assume that $eA'$ is nonzero. As $A'$ is of finite type over $R_G^{\mathrm{ps}}$, there is a dimension $\leq 1$ prime ideal $\mathfrak{p} \in \mathrm{Spec}(eA')$ with residue field $\kappa(\mathfrak{p})$ finite or local by [6, Lemma 3.17]. By Lemma 5.12, we know that the representation $\rho_{\mathfrak{p}} : \Gamma \to G(\overline{\kappa(\mathfrak{p})})$ is continuous and that $\Theta_{\rho_{\mathfrak{p}}} = \Theta^u \otimes_{R_G^{\mathrm{ps}}} \overline{\kappa(\mathfrak{p})}$. If $\kappa(\mathfrak{p})$ is a finite field then we deduce that the $G$-semisimplification of $\rho_{\mathfrak{p}}$ is $\overline{\rho}^{\mathrm{ss}}$, which yields a contradiction to the choice of $e$ via Lemma 6.13. If $\kappa(\mathfrak{p})$ is a local field, by Proposition 6.6 the image of $\rho_{\mathfrak{p}}$ lies up to conjugation in the ring of integers of a finite extension of $\kappa(\mathfrak{p})$. The $G$-semisimplification of the reduction modulo the maximal ideal is $\overline{\rho}^{\mathrm{ss}}$, since $\Theta_{\rho_{\mathfrak{p}}}$ specialises $\Theta^u$, which yields a contradiction to the choice of $e$ via Lemma 6.13. We conclude, that $\rho \in X_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{gen},\tau}(A') \subseteq X_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{gen},\tau}(A)$. □

**Lemma 8.2.** *Let $R \to A$ be a map of $R_G^{\mathrm{ps}}$-algebras such that $R$ is a noetherian profinite topological $R_G^{\mathrm{ps}}$-algebra. Let $\rho : \Gamma \to G(A)$ be a representation satisfying $\Theta_\rho = \Theta^u_{|A}$. Then the following are equivalent:*

(1) $(\tau \circ \rho)^{\mathrm{lin}}|_{R_G^{\mathrm{ps}}[\Gamma]}$ *factors through* $R_G^{\mathrm{ps}} \otimes_{R_{\mathrm{GL}_d}^{\mathrm{ps}}} E^u$;
(2) $\rho$ *is* $R_G^{\mathrm{ps}}$*-condensed;*
(3) $\rho$ *is* $R$*-condensed.*

*Proof.* It follows from Lemma 4.16 that $\rho$ is $R$-condensed if and only if $\tau \circ \rho$ is $R$-condensed. As explained in section 7.3 the assumption $\Theta_\rho = \Theta^u_{|A}$ implies that $\det \circ (\tau \circ \rho)^{\mathrm{lin}} = D^u_{|A}$. The equivalence now follows from Lemma 5.2. □

**Proposition 8.3.** *Let $A$ be an $R_G^{\mathrm{ps}}$-algebra. There is canonical bijection between the following sets:*

(1) $X_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{gen},\tau}(A)$;
(2) *the set of representations $\rho : \Gamma \to G(A)$, such that the $G$-pseudocharacter $\Theta_\rho$ is equal to the specialisation of $\Theta^u$ at $A$ and the equivalent conditions of Lemma 8.2 hold.*

*Proof.* If $\rho \in X_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{gen},\tau}(A)$ then $\Theta_\rho = \Theta^u_{|A}$ by Lemma 8.1. Moreover, $\tau \circ \rho$ is $R_{\mathrm{GL}_d}^{\mathrm{ps}}$-condensed by Lemma 5.10 (1). Lemmas 4.16 and 4.5 imply that $\rho$ is $R_G^{\mathrm{ps}}$-condensed.

If $\rho : \Gamma \to G(A)$ satisfies the conditions of part (2) then $\det \circ (\tau \circ \rho)^{\mathrm{lin}} = D^u_{|A}$ and $\tau \circ \rho$ is $R_G^{\mathrm{ps}}$-condensed. Thus $\rho$ satisfies part (4) of Lemma 5.2 with $R = R_G^{\mathrm{ps}}$ and thus $\rho \in X_G^{\mathrm{gen},\tau}(A)$ by Lemma 5.10 (1). Since $\Theta_\rho = \Theta^u_{|A}$, Lemma 8.1 implies that $\rho \in X_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{gen},\tau}(A)$. □

**Corollary 8.4.** *The $R_G^{\mathrm{ps}}$-algebra $A_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{gen},\tau}$ does not depend on the chosen embedding $\tau : G \hookrightarrow \mathrm{GL}_d$.*

*Proof.* The assertion follows from Proposition 8.3, as the condition in part (2) of Lemma 8.2 is independent of $\tau$. □

As a consequence of the Corollary we will drop $\tau$ from the notation and write $A_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{gen}}$, $R_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{git}}$, $X_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{gen}}$, $X_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{git}}$ for $A_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{gen},\tau}$, $R_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{git},\tau}$, $X_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{gen},\tau}$ and $X_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{git},\tau}$, respectively.

We now turn to functoriality.

**Proposition 8.5.** *Let $\varphi : G \to H$ be a homomorphism of generalised reductive $\mathcal{O}$-group schemes. Then there is a natural map*[8] *of $R_H^{\mathrm{ps}}$-schemes $X_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{gen}} \to X_{H,\varphi\circ\overline{\rho}^{\mathrm{ss}}}^{\mathrm{gen}}$, such that for every $R_G^{\mathrm{ps}}$-algebra $A$, the map $X_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{gen}}(A) \to X_{H,\varphi\circ\overline{\rho}^{\mathrm{ss}}}^{\mathrm{gen}}(A)$ is given by $\rho \mapsto \varphi \circ \rho$.*

*Proof.* Let $\Theta_G^u$ be the universal $G$-pseudocharacter deforming $\Theta_{\overline{\rho}^{\mathrm{ss}}}$ and let $\Theta_H^u$ be the universal $H$-pseudocharacter deforming $\Theta_{\varphi\circ\overline{\rho}^{\mathrm{ss}}}$. Since $\varphi(\Theta_G^u)$ is a deformation of $\Theta_{\varphi\circ\overline{\rho}^{\mathrm{ss}}}$ to $R_G^{\mathrm{ps}}$, we obtain a natural homomorphism of local $\mathcal{O}$-algebras $R_H^{\mathrm{ps}} \to R_G^{\mathrm{ps}}$ such that $\varphi(\Theta_G^u) = \Theta_H^u \otimes_{R_H^{\mathrm{ps}}} R_G^{\mathrm{ps}}$.

[8] As pointed out by referee the target of the map should be denoted by $X_{H,(\varphi\circ\overline{\rho}^{\mathrm{ss}})^{\mathrm{ss}}}^{\mathrm{gen}}$, as $\varphi \circ \overline{\rho}^{\mathrm{ss}}$ need not be $H$-semisimple in general. For ease of reading, we will continue to use this abuse of notation.

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If $\rho \in X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}(A)$ then $\Theta_\rho = \Theta^u_G \otimes_{R^{\mathrm{ps}}_G} A$ and by applying $\varphi$, we get

$$\Theta_{\varphi\circ\rho} = \varphi(\Theta_\rho) = \varphi(\Theta^u_G) \otimes_{R^{\mathrm{ps}}_G} A = \Theta^u_H \otimes_{R^{\mathrm{ps}}_H} A. \tag{36}$$

Lemma 4.16 implies that $\varphi \circ \rho$ is $R^{\mathrm{ps}}_G$-condensed. Since $\varphi \circ \rho$ satisfies (36), Lemma 8.2 (3) applied with $R = R^{\mathrm{ps}}_G$ and Proposition 8.3 imply that $\varphi \circ \rho \in X^{\mathrm{gen}}_{H,\varphi\circ\overline{\rho}^{\mathrm{ss}}}(A)$. □

### 8.2. *Finite maps*

Let $\varphi : G \to H$ be a morphism of generalised reductive $\mathcal{O}$-group schemes. By functoriality of $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ proved in Propositions 8.5 we obtain a morphism $f : X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{gen}}_{H,\varphi\circ\overline{\rho}^{\mathrm{ss}}}$.

**Proposition 8.6.** *If $\varphi$ is finite then the map $X^{\mathrm{ps}}_G \to X^{\mathrm{ps}}_H$ is also finite.*

*Proof.* By topological Nakayama's lemma it is enough to show, that the fibre ring $C := k \otimes_{R^{\mathrm{ps}}_H} R^{\mathrm{ps}}_G$ is a finite-dimensional $k$-vector space. Since $\Gamma$ is topologically finitely generated, we know by [50, Theorem 5.7], that $R^{\mathrm{ps}}_G$ is noetherian, hence $C$ is a complete local noetherian ring and we are left to show, that $C$ has Krull dimension 0.

If $C$ is not 0-dimensional then there is a point $x \in \mathrm{Spec}(C)$ of dimension 1. The residue field $\kappa(x)$ is a local field of characteristic $p$ by [6, Lemma 3.17] and the corresponding homomorphism $C \to \kappa(x)$ gives a $G$-pseudocharacter $\Theta \in \mathrm{PC}^\Gamma_G(\kappa(x))$, such that $\varphi \circ \Theta \in \mathrm{PC}^\Gamma_H(\kappa(x))$ arises by scalar extension from the residual pseudocharacter attached to $\varphi \circ \overline{\rho}$.

For all $m \geq 0$, the map $(\varphi \circ \Theta)_m = \Theta_m \circ \varphi^*$ is the composition

$$\mathcal{O}[H^m]^{H^0} \xrightarrow{\varphi^*} \mathcal{O}[G^m]^{G^0} \xrightarrow{\Theta_m} \mathrm{Map}(\Gamma^m, \kappa(x))$$

and has image in $\mathrm{Map}(\Gamma^m, k)$. By Corollary 6.3 $\varphi^*$ is finite, so $\Theta_m$ has image in $\mathrm{Map}(\Gamma^m, \overline{k})$. This implies that $\kappa(x)$ is algebraic over $k$ and contradicts the assumption that $x$ is a dimension 1 point. □

**Proposition 8.7.** *If $\varphi : G \to H$ is finite then $f : X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{gen}}_{H,\varphi\circ\overline{\rho}^{\mathrm{ss}}}$ is also finite.*

*Proof.* The induced map $X^{\mathrm{ps}}_G \to X^{\mathrm{ps}}_H$ is finite by Proposition 8.6. Since $G \to H$ is finite by assumption the map $G^N_{X^{\mathrm{ps}}_G} \to H^N_{X^{\mathrm{ps}}_H}$ is also finite. We have a commutative diagram

$$\begin{array}{ccc} X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} & \longrightarrow & G^N_{X^{\mathrm{ps}}_G} \\ \downarrow f & & \downarrow \\ X^{\mathrm{gen}}_{H,\varphi\circ\overline{\rho}^{\mathrm{ss}}} & \longrightarrow & H^N_{X^{\mathrm{ps}}_H}. \end{array} \tag{37}$$

with horizontal arrows closed immersions. Since closed immersions are finite maps and the composition of finite maps are again finite, the diagram implies the assertion. □

### 8.3. *Products*

In this subsection suppose that $G = G_1 \times G_2$, where the fibre product is taken over $\mathrm{Spec}\,\mathcal{O}$ and $G_1$ and $G_2$ are generalised reductive group schemes over $\mathcal{O}$. Let $p_i : G \to G_i$ denote the projection map onto the $i$-th component. Then $\overline{\rho}^{\mathrm{ss}}$ is of the form $\overline{\rho}^{\mathrm{ss}}_1 \times \overline{\rho}^{\mathrm{ss}}_2$, where $\overline{\rho}^{\mathrm{ss}}_i : \Gamma \to G_i(k)$ is $G_i$-semisimple.

**Lemma 8.8.** *The map $\Theta \mapsto (p_1 \circ \Theta, p_2 \circ \Theta)$ induces a bijection of functors $\mathrm{PC}^\Gamma_G \xrightarrow{\sim} \mathrm{PC}^\Gamma_{G_1} \times \mathrm{PC}^\Gamma_{G_2}$ and $\mathrm{cPC}^\Gamma_G \xrightarrow{\sim} \mathrm{cPC}^\Gamma_{G_1} \times \mathrm{cPC}^\Gamma_{G_2}$.*

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*Proof.* For an integer $m \geq 0$, we have $\mathcal{O}[G^m]^{G^0} = \mathcal{O}[G_1^m]^{G_1^0} \otimes_{\mathcal{O}} \mathcal{O}[G_2^m]^{G_2^0}$, since the $\mathcal{O}[G_i^m]$ are flat $\mathcal{O}$-modules.

Recall that a $G$-valued pseudocharacter is a family of $\mathcal{O}$-algebra homomorphisms $\Theta_m : \mathcal{O}[G^m]^{G^0} \to \mathrm{Map}(\Gamma^m, A)$ satisfying certain compatibility conditions. Restriction along the map $\mathcal{O}[G_i^m]^{G_i^0} \to \mathcal{O}[G^m]^{G^0}$ defines $G_i$-pseudocharacters $\Theta^{(i)}$ satisfying the same compatibility conditions; this can be directly seen from the description of pseudocharacters in terms of functors on the category of finitely generated free groups [50, Proposition 3.18]. Conversely given $\Theta^{(i)} \in \mathrm{PC}^{\Gamma}_{G_i}(A)$, we can define a $G$-pseudocharacter $\Theta^{(1)} \otimes \Theta^{(2)}$ by defining $(\Theta^{(1)} \otimes \Theta^{(2)})_m : \mathcal{O}[G^m]^{G^0} \to \mathrm{Map}(\Gamma^m, A)$ as the tensor product $\Theta^{(1)}_m \otimes \Theta^{(2)}_m : \mathcal{O}[G_1^m]^{G_1^0} \otimes_{\mathcal{O}} \mathcal{O}[G_2^m]^{G_2^0} \to \mathrm{Map}(\Gamma^m, A)$. This implies that the first map is bijective.

If $A$ is a topological $\mathcal{O}$-algebra then a map $f : \Gamma^m \to G(A)$ is continuous if and only if $p_1 \circ f$ and $p_2 \circ f$ are continuous. This implies the second assertion. □

**Lemma 8.9.** $R^{\mathrm{ps}}_G \cong R^{\mathrm{ps}}_{G_1} \widehat{\otimes}_{\mathcal{O}} R^{\mathrm{ps}}_{G_2}$.

*Proof.* Let $\overline{\Theta}_i$ be the $G_i$-pseudocharacter associated to $\overline{\rho}_i^{\mathrm{ss}}$; $i = 1, 2$. The ring $R^{\mathrm{ps}}_{G_1} \widehat{\otimes}_{\mathcal{O}} R^{\mathrm{ps}}_{G_2}$ pro-represents the product functor $D_{\overline{\Theta}_1} \times D_{\overline{\Theta}_2} : \mathfrak{A}_{\mathcal{O}} \to \mathrm{Set}$. If $A \in \mathfrak{A}_{\mathcal{O}}$ then the bijection between $\mathrm{cPC}^{\Gamma}_{G_1}(A) \times \mathrm{cPC}^{\Gamma}_{G_2}(A)$ and $\mathrm{cPC}^{\Gamma}_{G}(A)$ established in Lemma 8.8 induces a bijection between $D_{\overline{\Theta}_1}(A) \times D_{\overline{\Theta}_2}(A)$ and $D_{\overline{\Theta}}(A)$, which implies the assertion. □

The map from the usual tensor product into the completed tensor product induces a morphism of schemes $X^{\mathrm{ps}}_G \to X^{\mathrm{ps}}_{G_1} \times X^{\mathrm{ps}}_{G_2}$ over $\mathrm{Spec}\, \mathcal{O}$.

**Proposition 8.10.** *The natural maps $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{gen}}_{G_i,\overline{\rho}_i^{\mathrm{ss}}}$ constructed in Proposition 8.5 induce an isomorphism of schemes over $X^{\mathrm{ps}}_G$:*

$$X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} \xrightarrow{\sim} (X^{\mathrm{gen}}_{G_1,\overline{\rho}_1^{\mathrm{ss}}} \times X^{\mathrm{gen}}_{G_2,\overline{\rho}_2^{\mathrm{ss}}}) \times_{X^{\mathrm{ps}}_{G_1} \times X^{\mathrm{ps}}_{G_2}} X^{\mathrm{ps}}_G, \tag{38}$$

*where the fibre products without subscript are taken over* $\mathrm{Spec}\, \mathcal{O}$.

*Proof.* Let $A$ be an $R^{\mathrm{ps}}_G$-algebra. By Proposition 8.3 and Lemma 8.2 (3) applied with $R = R^{\mathrm{ps}}_G$, the $A$-valued points of the right hand side of (38) are pairs of $R^{\mathrm{ps}}_G$-condensed representations $\rho_1 : \Gamma \to G_1(A)$ and $\rho_2 : \Gamma \to G_2(A)$, such that $\Theta^u_{G_i} \otimes_{R^{\mathrm{ps}}_{G_i}} A = \Theta_{\rho_i}$ for $i = 1, 2$. Lemma 4.4 implies, that the representation $\rho := \rho_1 \times \rho_2 : \Gamma \to G(A)$ is $R^{\mathrm{ps}}_G$-condensed. It follows from Lemma 8.9 that the bijection

$$\mathrm{PC}^{\Gamma}_G(R^{\mathrm{ps}}_G) \xrightarrow{\sim} \mathrm{PC}^{\Gamma}_{G_1}(R^{\mathrm{ps}}_G) \times \mathrm{PC}^{\Gamma}_{G_2}(R^{\mathrm{ps}}_G)$$

maps $\Theta^u_G$ to $(\Theta^u_{G_1} \otimes_{R^{\mathrm{ps}}_{G_1}} R^{\mathrm{ps}}_G, \Theta^u_{G_2} \otimes_{R^{\mathrm{ps}}_{G_2}} R^{\mathrm{ps}}_G)$. Hence, $(\Theta^u_G)_{|A} \in \mathrm{PC}^{\Gamma}_G(A)$ is mapped to $(\Theta_{\rho_1}, \Theta_{\rho_2}) \in \mathrm{PC}^{\Gamma}_{G_1}(A) \times \mathrm{PC}^{\Gamma}_{G_2}(A)$ and hence $(\Theta^u_G)_{|A} = \Theta_\rho$ by Lemma 8.9. Proposition 8.3 implies that $\rho \in X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}(A)$ and hence (38) is surjective. Since $p_1 \circ \rho$ and $p_2 \circ \rho$ uniquely determine $\rho \in X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}(A)$, (38) is also injective. □

### *8.4. Functoriality in* $\Gamma$

Let $\Gamma'$ be an open subgroup of $\Gamma$. The restriction of $G$-pseudocharacters of $\Gamma$ to $\Gamma'$ defines a morphism $X^{\mathrm{ps},\Gamma}_G \to X^{\mathrm{ps},\Gamma'}_G$. Similarly, Lemma 4.13 implies that the restriction of representations of $\Gamma$ to $\Gamma'$ induces a morphism $X^{\mathrm{gen},\Gamma}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{gen},\Gamma'}_{G,\overline{\rho}^{\mathrm{ss}}}$.

**Lemma 8.11.** *The morphism $X^{\mathrm{ps},\Gamma}_G \to X^{\mathrm{ps},\Gamma'}_G$ is finite.*

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*Proof.* If $G = \mathrm{GL}_d$ then this is proved in [6, Lemma A.3]. In the general case, one could either use that $R^{\mathrm{ps}}_{\mathrm{GL}_d} \to R^{\mathrm{ps}}_G$ is finite, which follows from Propositions 6.4 and 7.4, or observe that the proof of [6, Lemma A.3] carries over to the more general setting. □

**Lemma 8.12.** *The morphism $X^{\mathrm{gen},\Gamma}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{gen},\Gamma'}_{G,\overline{\rho}^{\mathrm{ss}}}$ is of finite type.*

*Proof.* The assertion follows from Lemma 8.11 and the fact that $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is of finite type over $X^{\mathrm{ps}}_G$. □

## 9. Absolutely irreducible locus

Let $G$ be a generalised reductive $\mathcal{O}$-group scheme and let $\pi_G : G \to G/G^0$ be the quotient map. Naively, the absolutely irreducible locus should be an open subscheme $V$ of $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ such that a geometric point $x$ of $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ lies in $V$ if and only if the image of $\rho_x$ is not contained in any proper $R$-parabolic subgroup of $G_{\kappa(x)}$. It turns out that if $\pi_G \circ \overline{\rho}^{\mathrm{ss}} : \Gamma \to (G/G^0)(\overline{k})$ is not surjective then the naively defined absolutely irreducible locus might be empty, see Example 9.14. To overcome this problem we first show that there is a closed generalised reductive subgroup scheme $H$ of $G$, such that $H^0 = G^0$, $\overline{\rho}^{\mathrm{ss}}$ factors through $H$, $\pi_H \circ \overline{\rho}^{\mathrm{ss}} : \Gamma \to (H/H^0)(\overline{k})$ is surjective and $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} = X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}}$. In particular, we may always assume that $G/G^0$ is a finite constant group scheme.

We then define the absolutely irreducible locus in $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ under the assumption that $G$ is split and $\pi_G \circ \overline{\rho}^{\mathrm{ss}} : \Gamma \to (G/G^0)(\overline{k})$ is surjective, see Proposition 9.7 and Definition 9.12. It is convenient to define it as a subscheme of $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} \setminus Y_{\overline{\rho}^{\mathrm{ss}}}$, so that points in $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}(\overline{k})$ are never in the absolutely irreducible locus, even if the image of the corresponding representation $\rho_x : \Gamma \to G(\overline{k})$ is not contained in any proper R-parabolic subgroup of $G$.

The main result proved in Corollary 9.10 says that if the absolutely irreducible locus is non-empty then the difference between its dimension and the dimension of its GIT quotient by $G^0$ is equal to $\dim G_k - \dim Z(G_k)$. We also define schemes $V_{LG}$ and $U_{LG}$ for an R-Levi $L$, which along with Corollary 9.10 play an important role in the inductive argument in Section 13.

The results of this section apply to any profinite group $\Gamma$ satisfying Mazur's $p$-finiteness condition. Lemma 5.5 implies that we may replace $\Gamma$ with a topologically finitely generated quotient without changing $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$. Hence, without loss of generality we assume that $\Gamma$ is topologically finitely generated.

### *9.1. Trimming the component group*

Let $\rho_z : \Gamma \to G(\overline{k})$ be a representation corresponding to a point $z \in X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}(\overline{k})$. Then the composition $\pi_G \circ \rho_z : \Gamma \to (G/G^0)(\overline{k})$ depends only on the $G$-semisimplification of $\rho_z$. Since $\rho^{\mathrm{ss}}_z = \overline{\rho}^{\mathrm{ss}}$ by Lemma 6.13 the map $\pi_G \circ \overline{\rho}^{\mathrm{ss}}$ is well defined. Let $\Delta$ be the image of

$$\pi_G \circ \overline{\rho}^{\mathrm{ss}} : \Gamma \to (G/G^0)(\overline{k}).$$

Since $G$ is generalised reductive $G/G^0$ is finite étale over $\mathcal{O}$ by Remark 2.6. The map $\mathfrak{S} \mapsto \mathfrak{S}(\overline{k})$ induces an equivalence of categories between finite étale $\mathcal{O}$-groups schemes and the category of finite groups together with $\mathrm{Gal}(\overline{k}/k)$-action by group automorphisms. Since $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}(k)$ is non-empty, $\Delta$ is contained in $(G/G^0)(k)$ and hence the constant group scheme $\underline{\Delta}$ defined by $\Delta$ is a finite étale subgroup scheme of $G/G^0$.

By construction $X^{\mathrm{gen},\tau}_G$ is a closed subscheme of $G^N_{X^{\mathrm{ps}}_{\mathrm{GL}_d}}$. The map $\pi_G$ induces a $G^0$-equivariant morphism of schemes $\nu : X^{\mathrm{gen},\tau}_G \to (G/G^0)^N_{X^{\mathrm{ps}}_{\mathrm{GL}_d}}$ for the trivial action on the target. If $A$ is an $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebra then the map $\nu$ on $A$-points is given by

$$\nu : X^{\mathrm{gen},\tau}_G(A) \to (G/G^0)^N_{X^{\mathrm{ps}}_{\mathrm{GL}_d}}(A), \quad \rho \mapsto (\pi_G(\rho(\gamma_1)), \dots, \pi_G(\rho(\gamma_N))), \tag{39}$$

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where $\gamma_1, \dots, \gamma_N$ is a generating set for $\Gamma$ chosen before Definition 5.7. Since $\nu$ is $G^0$-equivariant it factors through $X_G^{\mathrm{git},\tau} \to (G/G^0)^N_{X^{\mathrm{ps}}_{\mathrm{GL}_d}}$. We view an $N$-tuple $\underline{\delta} = (\delta_1, \dots, \delta_N) \in \Delta^N$ as a closed constant subscheme of $\underline{\Delta}^N_{X^{\mathrm{ps}}_{\mathrm{GL}_d}}$ over $X^{\mathrm{ps}}_{\mathrm{GL}_d}$ and define

$$X^{\mathrm{git},\tau}_{G,\underline{\delta}} := X^{\mathrm{git},\tau}_G \times_{(G/G^0)^N_{X^{\mathrm{ps}}_{\mathrm{GL}_d}}} \underline{\delta}.$$

If $X$ is a scheme over $X_G^{\mathrm{git},\tau}$ we let

$$X_{\underline{\delta}} := X \times_{X_G^{\mathrm{git},\tau}} X^{\mathrm{git},\tau}_{G,\underline{\delta}}, \quad X_{\overline{\rho}^{\mathrm{ss}}} := X \times_{X_G^{\mathrm{git},\tau}} X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}. \tag{40}$$

**Lemma 9.1.** *If* $\underline{\delta} = (\pi_G(\overline{\rho}^{\mathrm{ss}}(\gamma_1)), \dots, \pi_G(\overline{\rho}^{\mathrm{ss}}(\gamma_N)))$ *then*

$$X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}} = ((X^{\mathrm{git},\tau}_G)_{\overline{\rho}^{\mathrm{ss}}})_{\underline{\delta}} = ((X^{\mathrm{git},\tau}_G)_{\underline{\delta}})_{\overline{\rho}^{\mathrm{ss}}}.$$

*In particular, for any scheme* $X$ *over* $X_G^{\mathrm{git},\tau}$ *we have* $(X_{\underline{\delta}})_{\overline{\rho}^{\mathrm{ss}}} = X_{\overline{\rho}^{\mathrm{ss}}}$.

*Proof.* If $R$ is a complete local noetherian $\mathcal{O}$-algebra with residue field $k$ then the map $R \to k$ induces a bijection $(G/G^0)(R) \xrightarrow{\sim} (G/G^0)(k)$ by [32, Exp. I, Cor. 6.2]. We apply this observation with $R = R^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and let $\underline{\delta} \in (G/G^0)^N(k)$ be the $N$-tuple corresponding to $\alpha : X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{git},\tau}_G \xrightarrow{\nu} (G/G^0)^N_{X^{\mathrm{ps}}_{\mathrm{GL}_d}}$. Since the closed point in $X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$ maps to $(\pi_G(\overline{\rho}^{\mathrm{ss}}(\gamma_1)), \dots, \pi_G(\overline{\rho}^{\mathrm{ss}}(\gamma_N))) \in (G/G^0)^N_{X^{\mathrm{git},\tau}_G}(k)$ under $\alpha$ we conclude that $\underline{\delta} = (\pi_G(\overline{\rho}^{\mathrm{ss}}(\gamma_1)), \dots, \pi_G(\overline{\rho}^{\mathrm{ss}}(\gamma_N)))$.

The image of $\alpha$ is contained in $\underline{\delta}$, which implies the first equality. The second equality follows from the symmetry of a tensor product. The last part follows from a manipulation with fibre products. □

**Lemma 9.2.** *Let* $H$ *be a closed subgroup scheme of* $G$, *such that* $H^0 = G^0$. *If* $(\pi_G \circ \overline{\rho}^{\mathrm{ss}})(\Gamma) \subseteq (H/H^0)(k)$ *then* $X^{\mathrm{git}}_{H,\overline{\rho}^{\mathrm{ss}}} = X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$ *and* $X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}} = X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$. *Moreover, there exists* $H$ *as above, such that* $\pi_H \circ \overline{\rho}^{\mathrm{ss}} : \Gamma \to (H/H^0)(\overline{k})$ *is surjective.*

*Proof.* We may write $X_H^{\mathrm{gen},\tau}$ as the fibre product of $X_G^{\mathrm{gen},\tau}$ and $(H/H^0)^N$ over $(G/G^0)^N$. The assumption implies that $X_H^{\mathrm{gen},\tau}$ contains $X^{\mathrm{gen},\tau}_{G,\underline{\delta}}$ as a closed subscheme with $\underline{\delta}$ as in Lemma 9.1. The assertion follows from Lemma 9.1. Since $H^0 = G^0$ by passing to GIT quotients we obtain $X^{\mathrm{git}}_{H,\overline{\rho}^{\mathrm{ss}}} = X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$. For the last part we may take $H$ to be the preimage of $\underline{\Delta}$ in $G$. □

If $H$ is a closed generalised reductive subgroup scheme of $G$ (resp. $G_k$) we let $X^{\mathrm{git},\tau}_{HG}$ (resp. $\overline{X}^{\mathrm{git},\tau}_{HG}$) be the scheme theoretic image of $X^{\mathrm{git},\tau}_H \to X^{\mathrm{git},\tau}_G$ (resp. $\overline{X}^{\mathrm{git},\tau}_H \to \overline{X}^{\mathrm{git},\tau}_G$). Since these maps are finite by Corollary 6.5 we have

$$\dim X^{\mathrm{git},\tau}_{HG} \le \dim X^{\mathrm{git},\tau}_H, \quad \dim \overline{X}^{\mathrm{git},\tau}_{HG} \le \dim \overline{X}^{\mathrm{git},\tau}_H. \tag{41}$$

**Remark 9.3.** We note that if $H$ is a closed generalised reductive subgroup of $G$ then the special fibre of $X^{\mathrm{git},\tau}_{HG}$ is homeomorphic to $\overline{X}^{\mathrm{git},\tau}_{H_kG}$, see Remark 6.1. We will denote both by $\overline{X}^{\mathrm{git},\tau}_{HG}$. This is harmless, since in our arguments we only care about dimensions.

We denote by $X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}}$ (resp. $X^{\mathrm{git}}_{H,\overline{\rho}^{\mathrm{ss}}}$, $X^{\mathrm{git}}_{HG,\overline{\rho}^{\mathrm{ss}}}$) the schemes $(X_H^{\mathrm{gen},\tau})_{\overline{\rho}^{\mathrm{ss}}}$ (resp. $(X_H^{\mathrm{git},\tau})_{\overline{\rho}^{\mathrm{ss}}}$, $(X^{\mathrm{git},\tau}_{HG})_{\overline{\rho}^{\mathrm{ss}}}$) with the notation introduced in (40). Lemma 9.1 implies that these schemes are independent of $\tau$ and only depend on the map $H \to G$.

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**Lemma 9.4.** *Let $H$ be a generalised reductive subgroup scheme of $G$ (resp. $G_k$). If $X^{\mathrm{git}}_{HG,\overline{\rho}^{\mathrm{ss}}}$ (resp. $\overline{X}^{\mathrm{git}}_{HG,\overline{\rho}^{\mathrm{ss}}}$) is non-empty then there exist*

(a) *a finite unramified extension $L'$ of $L$ with ring of integers $\mathcal{O}'$ and residue field $k'$;*
(b) *finitely many generalised reductive subgroup schemes $H_i$ of $H_{\mathcal{O}'}$;*
(c) *continuous representations $\overline{\rho}_i : \Gamma \to H_i(k')$;*

*such that the following hold:*

(1) *$H_i^0 = H^0_{\mathcal{O}'}$ and $\pi_{H_i} \circ \overline{\rho}_i : \Gamma \to (H_i/H_i^0)(\overline{k})$ is surjective;*
(2) *$(X^{\mathrm{git}}_{H,\overline{\rho}^{\mathrm{ss}}})_{\mathcal{O}'}$ (resp. $(\overline{X}^{\mathrm{git}}_{H,\overline{\rho}^{\mathrm{ss}}})_{k'}$) is a disjoint union of $X^{\mathrm{git}}_{H_i,\overline{\rho}_i^{\mathrm{ss}}}$ (resp. $\overline{X}^{\mathrm{git}}_{H_i,\overline{\rho}_i^{\mathrm{ss}}}$);*
(3) *$(X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}})_{\mathcal{O}'}$ (resp. $(\overline{X}^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}})_{k'}$) is a disjoint union of $X^{\mathrm{gen}}_{H_i,\overline{\rho}_i^{\mathrm{ss}}}$ (resp. $\overline{X}^{\mathrm{gen}}_{H_i,\overline{\rho}_i^{\mathrm{ss}}}$);*
(4) *$(X^{\mathrm{git}}_{HG,\overline{\rho}^{\mathrm{ss}}})_{\mathcal{O}'}$ (resp. $(\overline{X}^{\mathrm{git}}_{HG,\overline{\rho}^{\mathrm{ss}}})_{k'}$) is a union of scheme theoretic images*

$$X^{\mathrm{git}}_{H_i,\overline{\rho}_i^{\mathrm{ss}}} \to (X^{\mathrm{git},\tau}_G)_{\mathcal{O}'}, \quad (\text{resp. } \overline{X}^{\mathrm{git}}_{H_i,\overline{\rho}_i^{\mathrm{ss}}} \to (X^{\mathrm{git},\tau}_G)_{k'}).$$

*Proof.* The map $X^{\mathrm{git},\tau}_H \to X^{\mathrm{git},\tau}_G$ corresponds to the map between finite products of local rings $\prod_{i\in I} R_i \to \prod_{j\in J} R'_j$ by (31). The representation $\overline{\rho}^{\mathrm{ss}}$ defines an idempotent $e \in \prod_i R_i$, such that $e(\prod_i R_i) = R^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$. Then $X^{\mathrm{git}}_{H,\overline{\rho}^{\mathrm{ss}}}$ is the spectrum of $e(\prod_{j\in J} R'_j)$. Since $R'_j$ are local rings we have either $e(R'_j) = R'_j$ or $e(R'_j) = 0$. Thus $e(\prod_{j\in J} R'_j) = \prod_{j\in J'} R'_j$ for some subset $J' \subseteq J$. We may identify $J'$ with the set of $\mathrm{Gal}(\overline{k}/k)$-orbits in $X^{\mathrm{git}}_{H,\overline{\rho}^{\mathrm{ss}}}(\overline{k})$.

After replacing $L$ by a finite unramified extension we may ensure that every connected component of $X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}}$ has a $k$-rational point, see Remark 6.14. Then (31) yields decompositions:

$$X^{\mathrm{git}}_{H,\overline{\rho}^{\mathrm{ss}}} \cong \coprod_i X^{\mathrm{git}}_{H,\overline{\rho}_i^{\mathrm{ss}}}, \quad X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}} \cong \coprod_i X^{\mathrm{gen}}_{H,\overline{\rho}_i^{\mathrm{ss}}}$$

for finitely many continuous representations $\rho_i : \Gamma \to H(k)$. The existence of $H_i$ follows from Lemma 9.2. Parts (2), (3) and (4) then follows immediately. If $H$ is a closed subgroup of $G_k$ the argument is the same. □

### *9.2. Absolutely irreducible locus*

In this subsection we omit the fixed embedding $\tau : G \hookrightarrow \mathrm{GL}_d$ from the notation and write $X^{\mathrm{gen}}_G$, $X^{\mathrm{git}}_G$, $X^{\mathrm{gen}}_{HG}$ and $X^{\mathrm{git}}_{HG}$ for $X^{\mathrm{gen},\tau}_G$, $X^{\mathrm{git},\tau}_G$, $X^{\mathrm{gen},\tau}_{HG}$ and $X^{\mathrm{git},\tau}_{HG}$. We note that as a consequence of Corollary 8.4 $X^{\mathrm{gen}}_{HG,\overline{\rho}^{\mathrm{ss}}}$ and $X^{\mathrm{git}}_{HG,\overline{\rho}^{\mathrm{ss}}}$ are independent of $\tau$. In particular, the schemes $U_{LG,\overline{\rho}^{\mathrm{ss}}}$ and $V_{LG,\overline{\rho}^{\mathrm{ss}}}$ defined below are independent of $\tau$.

It follows from Proposition 6.4 that $X^{\mathrm{git}}_G$ has only finitely many closed points, we denote this subscheme by pts. We let $Y$ be the preimage of these closed points in $X^{\mathrm{gen}}_G$. Then $Y$ is a closed subscheme of $X^{\mathrm{gen}}_G$ contained in $\overline{X}^{\mathrm{gen}}_G$.

We assume for the rest of this subsection that $G^0$ is split and fix a maximal split torus $T$ defined over $\mathcal{O}$. Let $L$ be an R-Levi subgroup of $G$ containing $T$. Then we define

$$U_{LG} := X^{\mathrm{git}}_{LG} \setminus (\mathrm{pts} \cup \bigcup_M X^{\mathrm{git}}_{MG}), \quad \overline{U}_{LG} := \overline{X}^{\mathrm{git}}_{LG} \setminus (\mathrm{pts} \cup \bigcup_M \overline{X}^{\mathrm{git}}_{MG})$$

where the union is taken over all R-Levi subgroups $M$ of $G$, such that $T \subseteq M \subsetneq L$. Lemma 2.18 implies that there are only finitely many such $M$. Thus $U_{LG}$ is an open subscheme of $X^{\mathrm{git}}_{LG} \setminus \mathrm{pts}$ and $\overline{U}_{LG}$ is an

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open subscheme of $\overline{X}^{\mathrm{git}}_{LG} \setminus \mathrm{pts}$. Hence, $U_{LG}$ and $\overline{U}_{LG}$ are locally closed subschemes of $X^{\mathrm{git}}_G \setminus \mathrm{pts}$. We let $V_{LG}$ be the preimage of $U_{LG}$ in $X^{\mathrm{gen}}_G$ and $\overline{V}_{LG}$ be the preimage of $\overline{U}_{LG}$ in $X^{\mathrm{gen}}_G$. We then have

$$X^{\mathrm{gen}}_G = Y \cup \bigcup_L V_{LG}, \quad \overline{X}^{\mathrm{gen}}_G = Y \cup \bigcup_L \overline{V}_{LG}, \tag{42}$$

where the union is taken over all $R$-Levi subgroups $L$ of $G$ containing $T$.

**Definition 9.5.** We say that a geometric point $x \in X^{\mathrm{gen}}_G$ is *$G$-stable*, if the orbit $G \cdot x$ is closed and $\dim Z_G(x) = \dim Z(G)_{\kappa(x)}$, where $Z_G(x)$ is the $G_{\kappa(x)}$-centraliser of $x$.

**Remark 9.6.** The action of $G$ on $X^{\mathrm{gen}}_G$ factors through the action of $G/Z(G)$. The stability condition in the definition is equivalent to $G \cdot x$ closed and $\dim G \cdot x = \dim(G/Z(G))_{\kappa(x)}$. Hence, it is equivalent to $G/Z(G)$-stability in the sense of [57, Definition 1].

**Proposition 9.7.** *Let $x$ be a geometric point of $X^{\mathrm{gen}}_G \setminus Y$, and let $H$ be the Zariski closure of $\rho_x(\Gamma)$ in $G_{\kappa(x)}$. Then the following statements are equivalent:*

(1) *$x \in V_{GG}$;*
(2) *$\rho_x$ is $G$-irreducible;*
(3) *$H$ is $G$-irreducible;*
(4) *$x$ is $G$-stable.*

*Moreover, $H$ is reductive and* $\dim Z_G(H) = \dim Z(G_{\kappa(x)})$.

*Proof.* The equivalence of (1) and (2) follows from Corollary 6.11. The equivalence of (2) and (3) follows from Definition 2.26. The equivalence of (3) and (4) follows from [51, Proposition 16.7]. The last part follows from [42, Lemma 6.2]. □

**Corollary 9.8.** *Let $L$ be an $R$-Levi of $G$ and let $H$ be a closed generalised reductive subgroup of $L_{k'}$, where $k'$ is a finite extension of $k$. If $\overline{X}^{\mathrm{git}}_{HG} \cap U_{LG}$ is non-empty then* $\dim Z(H) \leq \dim Z(L_k)$.

*Proof.* After extending scalars we may assume that $k = k'$. Since $H$ is contained in $L$ the map $\overline{X}^{\mathrm{git}}_H \to \overline{X}^{\mathrm{git}}_G$ factors through $\overline{X}^{\mathrm{git}}_H \to \overline{X}^{\mathrm{git}}_L \to \overline{X}^{\mathrm{git}}_G$. The assumption that $\overline{X}^{\mathrm{git}}_{HG} \cap U_{LG}$ is non-empty implies that $\overline{X}^{\mathrm{git}}_{HL} \cap U_{LL}$ is non-empty. It follows from Proposition 9.7 that $H_{\kappa(x)}$ is not contained in any R-parabolic subgroup of $L_{\kappa(x)}$, where $x$ is any geometric point of $V_{LL}$, such that its image in $\overline{X}^{\mathrm{git}}_L$ is contained in $\overline{X}^{\mathrm{git}}_{HL}$. Moreover, we have

$$\dim Z(H) = \dim Z(H_{\kappa(x)}) \leq \dim Z(L_{\kappa(x)}) = \dim Z(L_k),$$

where the inequality in the middle follows from [42, Lemma 6.2], as the centre of $H_{\kappa(x)}$ is contained in the $L_{\kappa(x)}$-centraliser of $H_{\kappa(x)}$, and the other two follow from Lemma 2.2. □

**Corollary 9.9.** *Let $X^{\mathrm{gen}}_{G,y}$ be the fibre at a point $y \in U_{GG}$. Then*

$$\dim X^{\mathrm{gen}}_{G,y} = \dim G - \dim Z(G) = \dim G_k - \dim Z(G_k).$$

*Proof.* Let $\kappa$ be an algebraic closure of $\kappa(y)$. If $x \in X^{\mathrm{gen}}_{G,y}(\kappa)$ then the $G$-orbit $G \cdot x$ is closed, as $x$ is $G$-stable by Proposition 9.7. Proposition 6.7 implies that the $G^0$-orbit $G^0 \cdot x$ is also closed. Since $X^{\mathrm{gen}}_{G,y}$ contains a unique closed $G^0$-orbit by Proposition 6.9 we conclude that $G^0(\kappa)$ acts transitively on $X^{\mathrm{gen}}_{G,y}(\kappa)$. Thus the orbit map

$$G^0_\kappa / Z_{G^0_\kappa}(x) \to X^{\mathrm{gen}}_{G,y,\kappa}$$

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induces a bijection on $\kappa$-points. Since both schemes are of finite type over $\kappa$, we deduce that $\dim X^{\mathrm{gen}}_{G,y} = \dim G^0_\kappa/Z_{G^0_\kappa}(x) = \dim G^0_\kappa - \dim Z_{G^0_\kappa}(x)$. By passing to the underlying reduced subschemes of neutral components we obtain $(Z_{G^0_\kappa}(x)^0)^{\mathrm{red}} = (Z_{G_\kappa}(x)^0)^{\mathrm{red}} = (Z(G)^0_\kappa)^{\mathrm{red}}$, where the last equality holds as $x$ is $G$-stable. Thus $\dim Z_{G^0_\kappa}(x) = \dim Z(G)_\kappa$ and we conclude that

$$\dim X^{\mathrm{gen}}_{G,y} = \dim G_\kappa - \dim Z(G)_\kappa.$$

The other assertions follow from Proposition 2.9 and Lemma 2.1. □

**Corollary 9.10.** *Let $U$ be a non-empty locally closed subspace of $U_{GG}$ (resp. $\overline{U}_{GG}$) and let $V$ be the preimage of $U$ in $X^{\mathrm{gen}}_G$ (resp. $\overline{X}^{\mathrm{gen}}_G$). Then*

$$\dim V = \dim U + \dim G_k - \dim Z(G_k).$$

*Proof.* Corollary 9.9 says that $\dim V_y = d$, where $d := \dim G_k - \dim Z(G_k)$, for all $y \in U$. The inequality $\dim V \leq \dim U + d$ follows directly from [6, Lemma 3.18 (6)] (or alternatively [59, Tag 0BAG]).

To prove the reverse inequality we may assume that $U$ is irreducible and, since $V \to U$ is surjective by [57, Theorem 3 (ii)], we may replace $V$ by an irreducible component, whose image contains the generic point of $U$. Lemma 2.1 in [6] implies that $V$ is $G^0$-invariant and [57, Theorem 3 (iii)] implies that the image of $V$ in $U$ is closed, and hence $V \to U$ is surjective. It follows from [59, Tag 02JU] applied with $s \in U$ a closed point such that $\dim U = \dim \mathcal{O}_{U,s}$ and $x \in V$ a closed point, which maps to $s$, that $\dim V \geq \dim U + d$. □

**Lemma 9.11.** *Assume that $\pi_G \circ \overline{\rho}^{\mathrm{ss}} : \Gamma \to (G/G^0)(\overline{k})$ is surjective. Let $L$ be an $R$-Levi of $G$ containing $T$ and let $P$ be an $R$-parabolic subgroup of $G$ with $R$-Levi $L$ and unipotent radical $N$. Assume that one of the following is non-empty: $X^{\mathrm{gen}}_{L,\overline{\rho}^{\mathrm{ss}}}$, $X^{\mathrm{git}}_{L,\overline{\rho}^{\mathrm{ss}}}$, $V_{LG,\overline{\rho}^{\mathrm{ss}}}$, $U_{LG,\overline{\rho}^{\mathrm{ss}}}$. Then the following hold:*

(1) *$L/L^0 \to G/G^0$ is an isomorphism;*
(2) *if $\dim L = \dim G$ then $L = G$;*
(3) *If $L \neq G$ then $\dim N_k \geq 1$.*

*Proof.* It follows from Lemma 2.13 that $L^0 = L \cap G^0$, so that the map in (1) is injective. The assumption in all the cases implies that $X^{\mathrm{git}}_{L,\overline{\rho}^{\mathrm{ss}}}$ is non-empty. Lemma 9.1 implies that $X^{\mathrm{git}}_{L,\underline{\delta}}$ is non-empty, where $\underline{\delta}$ is as in Lemma 9.1. Thus $\underline{\delta} \in (L/L^0)(k)^N$ and thus $\pi_G(\overline{\rho}^{\mathrm{ss}}(\gamma_i)) \in (L/L^0)(k)$ for all $1 \leq i \leq N$. Since $\pi_G \circ \overline{\rho}^{\mathrm{ss}}$ is continuous and surjects onto $(G/G^0)(\overline{k})$ and $\gamma_1, \dots \gamma_N$ generate $\Gamma$ topologically, the entries of $\underline{\delta}$ generate $(G/G^0)(\overline{k})$ as a group. Thus $(L/L^0)(k) = (G/G^0)(k) = (G/G^0)(\overline{k})$. Thus $G/G^0$ is a constant group scheme and we deduce part (1).

As explained in Lemma 2.13, $P^0$ is a parabolic of $G^0$ with Levi $L^0$. Thus $\dim L = \dim G$ implies that $\dim L^0 = \dim G^0$ and hence $L^0 = G^0$. Combining this with part (1) we deduce part (2).

Part (3) follows from $\dim G_k - \dim L_k = \dim G^0_k - \dim L^0_k = 2 \dim N_k$. □

**Definition 9.12.** If $\pi_G \circ \overline{\rho}^{\mathrm{ss}} : \Gamma \to (G/G^0)(\overline{k})$ is surjective then we define the *absolutely $G$-irreducible* locus in $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ as $V_{GG,\overline{\rho}^{\mathrm{ss}}}$.

**Remark 9.13.** The justification for calling $V_{GG,\overline{\rho}^{\mathrm{ss}}}$ absolutely $G$-irreducible locus comes from Proposition 9.7. We are not sure if this is a good definition if $\pi_G \circ \overline{\rho}^{\mathrm{ss}}$ is not surjective, and we discuss an example below. However, one can get around the problem by replacing $G$ by a closed subgroup scheme $H$ such that $G^0 = H^0$ and $\pi_G \circ \overline{\rho}^{\mathrm{ss}}$ surjects onto $(H/H^0)(\overline{k})$ and using Lemma 9.2.

**Example 9.14.** Assume that $p > 2$ and let $G$ be the normaliser of the subgroup of diagonal matrices in $\mathrm{GL}_2$. Then $G \cong \mathbb{G}_m^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ as in the Example 2.14. Let $E$ be a quadratic Galois extension of $F$ and let

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$c \in \mathrm{Gal}(E/F)$ be non-trivial. Let $\overline{\rho}_1, \overline{\rho}_2 : \Gamma_F \to G(k)$ be representations, such that $\Gamma_E$ is mapped to identity and

$$\overline{\rho}_1(c) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \overline{\rho}_2(c) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

These representations are $G$-semisimple. Since $\overline{\rho}_1(c)$ and $\overline{\rho}_2(c)$ are conjugate in $\mathrm{GL}_2(k)$ the representations $\tau \circ \overline{\rho}_1$ and $\tau \circ \overline{\rho}_2$ will have the same $\mathrm{GL}_2$-pseudocharacter, where $\tau : G \hookrightarrow \mathrm{GL}_2$ is the natural inclusion. The image of $\overline{\rho}_1$ is contained in $G^0(k)$ and $G^0$ is a proper $R$-parabolic subgroup of $G$, as explained in Example 2.14, hence $V_{GG,\overline{\rho}_1}$ is empty, and hence the $G$-irreducible locus in $X^{\mathrm{gen}}_{G,\overline{\rho}_1}$ is also empty. On the other hand, $V_{GG,\overline{\rho}_2}$ is non-empty. We also would like to point out that in this case $Z(G^0)$ and $Z(G)$ have different dimensions, and this results in a difference between the dimensions of $V_{G^0G,\overline{\rho}_1} /\!\!/ G^0$ and $V_{GG,\overline{\rho}_2} /\!\!/ G^0$.

**Remark 9.15.** The referee suggested getting around the problem explained in Remark 9.13 and Example 9.14 by working only with $G$-normalisers of parabolic subgroups of $G^0$ (which are $R$-parabolic by [3, Proposition 6.1]) instead of all $R$-parabolic subgroups. This definition would save some work in Section 9.1, but would not match the standard definition for $G$-irreducibility in the literature such as [3], which we use. If $\pi_G \circ \overline{\rho}^{\mathrm{ss}}$ is surjective then both definitions of irreducibility coincide.

## 10. Bounds for the dimensions of reductive subgroups

Let $G$ be a generalised reductive group over an algebraically closed field $\kappa$. If $H$ is a closed reduced subgroup scheme of $G$, such that the unipotent radical of $H^0$ is trivial then $H$ is also generalised reductive by Lemma 2.7. Let $G'$ denote the derived subgroup of $G^0$. Then $G'$ is semisimple and $G^0 = Z(G^0)G'$, [9, 14.2]. Moreover, $G'$ is equal to its own derived subgroup, [16, Example 1.1.16].

**Proposition 10.1.** *Let $H$ be a closed reductive subgroup of $G$. Then one of the following holds:*

(i) *$H$ contains $G'$;*
(ii) $\dim G' - \dim(H \cap G') \geq 2$.

*Proof.* After replacing $G$ by $G'$ we may assume that $G$ is connected and semisimple. One may check that the unipotent radical of $(G' \cap H)^0$ is normal in $H^0$. Since $H^0$ is reductive, $(G' \cap H)^0$ is also reductive. Thus after replacing $H$ by $H \cap G'$ we may assume that $H$ is closed reductive subgroup of $G$.

Let $B_0 = T_0U_0$ be a Borel subgroup of $H$ with the unipotent radical $U_0$ and a maximal torus $T_0$. Let $B = TU$ be a Borel subgroup of $G$ with the unipotent radical $U$ and a maximal torus $T$, such that $U$ contains $U_0$ and $T$ contains $T_0$. We have $\dim G = 2\dim U + \dim T$ and $\dim H = 2\dim U_0 + \dim T_0$. If $U \neq U_0$ then $\dim U > \dim U_0$ and we deduce that $\dim G - \dim H \geq 2$.

Let us assume that $U = U_0$ and let $U_0^-$ be the unipotent radical of a Borel subgroup of $H$, which is opposite to $B_0$ with respect to $T_0$. Since $U_0 = U$, $U_0^-$ is a unipotent radical of a Borel subgroup of $G$. Since $U_0^- \cap U_0 = \{e\}$, [10, Proposition 4.10 e)] implies that $U_0^-$ is a unipotent radical of a parabolic subgroup of $G$, which is opposite to $B$ with respect to $T$. Since $G$ is by assumption semisimple, $U_0$ and $U_0^-$ generate it as a group by [58, Theorem 8.1.5(i)]. Thus $G = H$. □

**Corollary 10.2.** *Assume that we are in the situation of part (ii) of Proposition 10.1. Then* $\dim G - \dim H \geq 2$.

*Proof.* The $G'$-orbit of $H$ in $G/H$ is isomorphic to $G'/(G' \cap H)$ and hence

$$\dim G/H \geq \dim G'/(G' \cap H) \geq 2.$$ □

**Corollary 10.3.** *Let $V$ be a finite dimensional representation of $G$ and let $H$ be a closed reductive subgroup of $G$. If $V^{G'} = 0$ and $V^H \neq 0$ then* $\dim G - \dim H \geq 2$.

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*Proof.* If $H$ contains $G'$ then $V^H$ is contained in $V^{G'}$ and so $V^H \neq 0$ implies that $V^{G'} \neq 0$. The assertion follows from Corollary 10.2. □

Since $G'$ is a normal subgroup of $G$, $G$ acts on $\mathrm{Lie}(G')$ by conjugation.

**Lemma 10.4.** *If $\pi_1(G')$ is étale then $((\mathrm{Lie}\, G')^*)^{G'} = 0$.*

*Proof.* We may assume that $\mathrm{Lie}\, G'$ is non-zero. Let $G'_{\mathrm{sc}} \to G'$ be the universal simply connected cover of $G'$. Since $\pi_1(G')$ is étale the exact sequence:

$$0 \to \pi_1(G') \to G'_{\mathrm{sc}} \to G' \to 0$$

induces a $G'_{\mathrm{sc}}$-equivariant isomorphism of Lie algebras $\mathrm{Lie}\, G'_{\mathrm{sc}} \cong \mathrm{Lie}\, G'$. We thus may assume that $G'$ is simply connected. Let $T$ be a maximal torus in $G'$.

If $\dim T = 1$ then $G' = \mathrm{SL}_2$ by [58, 7.2.4], as the assumption that $G'$ is simply connected rules out $\mathrm{PGL}_2$. If $\mathrm{char}(\kappa) \neq 2$ then the action of $G'$ on $\mathrm{Lie}\, G'$ is an irreducible representation, isomorphic to $\mathrm{Sym}^2(\mathrm{Std})$, where Std is the standard 2-dimensional representation of $\mathrm{SL}_2$. If $\mathrm{char}(\kappa) = 2$ then $\mathrm{Lie}\, G'$ is a non-split extension $0 \to \mathbf{1} \to \mathrm{Lie}\, G' \to V \to 0$, with $\mathbf{1}$ equal to the subspace of scalar matrices and $V$ an irreducible 2-dimensional representation. In both cases we obtain the claim.

Let $\dim T$ be arbitrary and let $\chi : \mathrm{Lie}\, G' \to \mathbf{1}$ be $G'$-invariant. Let us abbreviate $\mathfrak{g} = \mathrm{Lie}\, G'$. We have $\mathfrak{g} = \mathfrak{g}^T \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, where $\Phi$ is the set of roots, and $\mathfrak{g}^T = \mathrm{Lie}(T)$, [9, 13.18(1)]. Let $\alpha \in \Phi$ and let $G_\alpha$ be the centraliser of $(\mathrm{Ker}\, \alpha)^0$ in $G'$. Then $G_\alpha$ is reductive of semisimple rank 1, [9, 13.18(4)]. Let $G'_\alpha$ be the derived subgroup of $G_\alpha$. The assumption that $G'$ is simply connected implies that $G'_\alpha \cong \mathrm{SL}_2$ by [16, 5.3.9]. It follows from [16, 1.2.7] that $\mathrm{Lie}\, G'_\alpha = \mathfrak{g}_{-\alpha} \oplus \mathrm{Lie}(\alpha^\vee(\mathbb{G}_m)) \oplus \mathfrak{g}_\alpha$. The restriction of $\chi$ to $\mathrm{Lie}\, G'_\alpha$ is $G'_\alpha$-invariant and hence zero by the previous part. Hence, $\chi$ is zero on $\mathrm{Lie}(\alpha^\vee(\mathbb{G}_m))$ and $\mathfrak{g}_\alpha$ for every coroot $\alpha^\vee$.

Let $T_1 = \prod_{\alpha \in \Delta} \mathbb{G}_m$, where $\Delta$ is the set of positive simple roots with respect to a Borel subgroup containing $T$. Since $G'$ is simply connected the map $\prod_{\alpha \in \Delta} \alpha^\vee : T_1 \to T$ induces an isomorphism on the character groups, [19, Example 1.3.10], and hence an isomorphism between tori. Thus the subspaces $\mathrm{Lie}(\alpha^\vee(\mathbb{G}_m))$ for $\alpha \in \Delta$ span $\mathrm{Lie}(T)$. Thus $\chi$ vanishes on $\mathfrak{g}^T$ and $\mathfrak{g}_\alpha$ for every $\alpha \in \Phi$ and hence $\chi = 0$. □

**Lemma 10.5.** *Let $G'_{\mathrm{sc}} \to G'$ be the simply connected central cover of $G'$. Then there is a natural action of $G$ on $\mathrm{Lie}(G'_{\mathrm{sc}})$ such that the map $\mathrm{Lie}(G'_{\mathrm{sc}}) \to \mathrm{Lie}\, G'$ is $G$-equivariant. Moreover, $G'$-invariants in $(\mathrm{Lie}(G'_{\mathrm{sc}}))^*$ are zero.*

*Proof.* The action has been defined over $\mathcal{O}$ in Lemma 2.11 and the same holds over $\kappa$. The $G'_{\mathrm{sc}}$-invariants in $(\mathrm{Lie}(G'_{\mathrm{sc}}))^*$ are zero by Lemma 10.4. The assertion follows as $G'_{\mathrm{sc}} \to G'$ is a central extension and therefore this action factors through the action of $G'$. □

Let $P$ be an $R$-parabolic subgroup of $G$, let $U$ its unipotent radical and let $L$ be an $R$-Levi subgroup of $P$. Then $P^0 = P \cap G^0$ is a parabolic subgroup of $G^0$ and $L^0 = L \cap G^0$ is a Levi subgroup of $P^0$, [42, Proposition 5.2]. In particular, $L^0$ is reductive. We let $L'$ be the derived subgroup of $L^0$.

**Lemma 10.6.** *Let $H$ be a closed reductive subgroup of $L$. If $((\mathrm{Lie}\, U)^*)^H \neq 0$ then one of the following holds:*

(i) $\dim L - \dim H \geq 2$;
(ii) $\dim L - \dim H = 1$ *and* $\dim_\kappa((\mathrm{Lie}\, U)^*)^H = 1$.

*Proof.* We may assume that $H$ contains $L'$ since otherwise (i) holds by Corollary 10.2. By replacing $H$ by $H \cap L^0$ and $L$ by $L^0$ we may assume that $L$ is connected. Then $L = Z(L)H$. Let $T$ be a maximal torus in $L$. Then $T$ contains $Z(L)$ and hence $L/H \cong T/(T \cap H)$. The action of $T$ on $\mathrm{Lie}\, U$ is semisimple, multiplicity free and $(\mathrm{Lie}\, U)^T = 0$, [9, 13.18]. Thus $\dim L > \dim H$. If $\dim_\kappa((\mathrm{Lie}\, U)^*)^H \geq 2$, then $\dim_\kappa((\mathrm{Lie}\, U)^*)^{T \cap H} \geq 2$. Let $\alpha, \beta \in \Phi$ be distinct roots such that the dual characters $\alpha^*$ and $\beta^*$ appear in $((\mathrm{Lie}\, U)^*)^{T \cap H}$. We note that both $\alpha$ and $\beta$ are positive with respect to a Borel subgroup of $G'$ containing

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$T$ and $U$. Thus $\alpha \neq -\beta$. Now $\{\alpha, -\alpha\}$ are the roots of the centraliser of $(\operatorname{Ker}\alpha)^0$ in $G'$, [9, 13.18(4d)]. Thus $\operatorname{Ker}\alpha \neq \operatorname{Ker}\beta$ and hence $\dim T - \dim(\operatorname{Ker}\alpha \cap \operatorname{Ker}\beta) = 2$. Thus $\dim T - \dim T \cap H \geq 2$, as $T \cap H$ is contained in $\operatorname{Ker}\alpha \cap \operatorname{Ker}\beta$. Thus part (i) holds. □

**Corollary 10.7.** $((\operatorname{Lie} U)^*)^{L^0} = 0$.

*Proof.* This follows from Lemma 10.6 as $\dim L = \dim L^0$. □

**Example 10.8.** Let $G = G' = G_1 \times \mathrm{SL}_2$, where $G_1$ is any semisimple group, $L = G_1 \times \mathbb{G}_m$ and $P = G_1 \times B$, where $B$ is the subgroup of upper triangular matrices in $\mathrm{SL}_2$. If $H = G_1 \times \{1\}$ then part (ii) of Lemma 10.6 holds.

## 11. R-Levi subgroups of codimension 2

Let $G$ be a generalised reductive group over an algebraically closed field $\kappa$. Let $P$ be an R-parabolic subgroup of $G$ with R-Levi $L$ and unipotent radical $U$. We have $\dim G - \dim L = 2\dim U$. In this section we assume that $\dim U = 1$ and the natural map $L/L^0 \to G/G^0$ is an isomorphism. The main result of this section is Proposition 11.5, which is a criterion for a representation $\rho : \Gamma_F \to L(\kappa)$ to have $h^0(\Gamma_F, (\operatorname{Lie} G'_{\mathrm{sc}})^*(1)) = 0$, where $G'_{\mathrm{sc}}$ is the simply connected central cover of $G'$ and the action of $G$ on $\operatorname{Lie} G'_{\mathrm{sc}}$ is given by Lemma 2.11. Proposition 11.5 is not needed to prove the complete intersection property of deformation rings, but is first used in Section 14 and is needed to compute their irreducible components.

Let $\varphi : G \to \operatorname{Aut}(G^0)$ be the group homomorphism induced by the action of $G$ on $G^0$ by conjugation. If $H$ is a subgroup of $G$ then we let $\overline{H}$ be the image of $H$ in $G/\operatorname{Ker}\varphi$. We have $G^0 \cap \operatorname{Ker}\varphi = Z(G^0)$ and $U \cong \overline{U}$. Since the image of any cocharacter $\lambda : \mathbb{G}_m \to G$ is contained in $G^0$, $\operatorname{Ker}\varphi$ is contained in every R-parabolic and R-Levi subgroup of $G$. Thus $L$ is the preimage of $\overline{L}$ and $P$ is the preimage of $\overline{P}$.

We will let $B_2, T_2$ and $U_2$ be subgroups of $\mathrm{PGL}_2$ consisting of upper-triangular, diagonal and unipotent upper-triangular matrices, respectively. We will identify $T_2$ with $\mathbb{G}_m$ via the character $\alpha$, which sends $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ to $ad^{-1}$.

**Proposition 11.1.** *There is a generalised reductive group $G_1$ such that $G_1^0$ is semisimple, $\overline{G} \cong G_1 \times \mathrm{PGL}_2$ and this isomorphism induces isomorphisms:*

$$\overline{P} \cong G_1 \times B_2, \quad \overline{L} \cong G_1 \times T_2, \quad \overline{U} \cong \{1\} \times U_2.$$

*Proof.* We fix a Borel subgroup $B^0$ of $G^0$ contained in $P$ and a maximal split torus $T$ contained in $B^0 \cap L$. Let $\Phi(G^0)$ be the set of roots of $G^0$ with respect to $T$, and let $\Phi^+(G^0)$ be the set of positive roots with respect to $(B^0, T)$. We define $\Phi(L^0), \Phi^+(L^0)$ in the same way with respect to $(B^0 \cap L^0, T)$. We then have

$$\dim G = \dim_\kappa \operatorname{Lie} G = \dim T + \sum_{\alpha \in \Phi(G^0)} \dim_\kappa \mathfrak{g}_\alpha, \tag{43}$$

$$\dim L = \dim_\kappa \operatorname{Lie} L = \dim T + \sum_{\alpha \in \Phi(L^0)} \dim_\kappa \mathfrak{g}_\alpha. \tag{44}$$

If $\Phi^+(L^0)$ contains all simple roots of $\Phi^+(G^0)$ then $\Phi(L^0) = \Phi(G^0)$ and hence $L^0 = G^0$. Thus there is a simple root $\beta \in \Phi^+(G^0)$, such that $\beta \notin \Phi^+(L^0)$. It follows from (43), (44) and the assumption $\dim G - \dim L = 2$ that $\Phi(G^0)$ is a disjoint union of $\Phi(L^0)$ and $\{\beta, -\beta\}$. Let $U_\alpha$ be the root subgroup of $G^0$ corresponding to $\alpha$. If $\alpha \in \Phi(L^0)$ then $r\alpha \pm s\beta$ is not a root of $G$ for any integers $r, s > 0$, it follows from assertion $(*)$ in the proof of [9, 14.5] that the commutator $(U_\alpha, U_{\pm\beta})$ vanishes.

We have $\overline{G}^0 \cong G^0/Z(G^0)$, and hence $\overline{G}^0$ is semisimple by [9, Corollary 14.2(a)]. Let $H_1$ be the closed subgroup of $\overline{G}^0$ generated by the subgroups $U_\alpha$ for $\alpha \in \Phi(L^0)$ and let $H_2$ be the closed subgroup of $\overline{G}^0$ generated by $U_\beta$ and $U_{-\beta}$. Arguing as in the proof of [9, Proposition 14.2 (2)] we deduce that $H_1$

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centralises $H_2$, $H_1$ and $H_2$ generate $\overline{G}^0$ as a group, and $H_1 \cap H_2$ is contained in the centre of $\overline{G}^0$. Since the centre of $\overline{G}^0$ is trivial, we deduce that $\overline{G}^0 \cong H_1 \times H_2$. Moreover, by [19, Lemma 1.2.3] there is a central isogeny $H_2 \to \mathrm{PGL}_2$. The centre of $H_2$ lies in the centre of $\overline{G}^0$ and hence is trivial, so that $H_2 \cong \mathrm{PGL}_2$.

For each simple root $\alpha$ in $\Phi^+(G^0)$ we fix an isomorphism $p_\alpha : \mathbb{G}_a \xrightarrow{\cong} U_\alpha$. Let $\Theta$ be the subgroup of $\mathrm{Aut}(G^0)$ consisting of automorphisms that preserve the pinning $((G^0, B^0, T), \{p_\alpha\}_\alpha)$. Theorem 7.1.9 in [19] gives an isomorphism of group schemes over $\kappa$:

$$\mathrm{Aut}(G^0) \cong \overline{G}^0 \rtimes \Theta.$$

If $\Theta'$ is a subgroup of $\Theta$ then there is a unique subgroup $H$ of $\mathrm{Aut}(G^0)$, such that $H$ contains $\overline{G}^0$, and the image of $H$ in $\mathrm{Aut}(G^0)/\overline{G}^0$ is equal to $\Theta'$, namely the inverse image of $\Theta'$. Uniqueness implies that $H \cong \overline{G}^0 \rtimes \Theta'$. Let $\Delta$ be the unique subgroup of $\Theta$, such that $\Delta$ maps isomorphically onto $\overline{G}/\overline{G}^0 \subseteq \mathrm{Aut}(G^0)/\overline{G}^0$. Then by the previous remark $\overline{G} \cong \overline{G}^0 \rtimes \Delta$. We claim that every $\delta \in \Delta$ maps $\Phi^+(L^0)$ to itself. The claim implies that $\Delta$ fixes $\beta$, and since it fixes the pinning $\Delta$ acts trivially on $H_2$. Thus $\overline{G} \cong (H_1 \rtimes \Delta) \times \mathrm{PGL}_2$ and the proposition follows.

To prove the claim we use the assumption that the map $L/L^0 \to G/G^0$ is an isomorphism. Thus given $\delta \in \Delta$ we may find $g \in L(\kappa)$, $h_1 \in H_1(\kappa)$ and $h_2 \in H_2(\kappa)$, such that for all $u \in U_\alpha$ we have $\delta(u) = (h_1, h_2) g u g^{-1} (h_1, h_2)^{-1}$. The image of $gug^{-1}$ in $\overline{G}^0$ lies in $H_1$ as $L$ normalises $H_1$. (It follows from the argument in the proof of [9, Proposition 14.10 (3)] that the closed subgroup of $L$ generated by $U_\alpha$ for all $\alpha \in \Phi(L^0)$ is the derived subgroup of $L^0$ and thus is normal in $L$.) Thus $\delta(u)$ lies in $H_1$, as $h_2$ centralises $H_1$. Hence $\delta(\alpha) \neq \pm\beta$ and this finishes the proof of the claim. □

**Corollary 11.2.** *The map $G' \to \overline{G}$ induces an isomorphism of $G$-representations:*

$$\mathrm{Lie}\, G'_{\mathrm{sc}} \cong \mathrm{Lie}\, \overline{G}_{\mathrm{sc}} \cong \mathrm{Lie}\, \mathrm{SL}_2 \oplus \mathrm{Lie}(G'_1)_{\mathrm{sc}}.$$

*Proof.* Since $\overline{G}^0 \cong G^0/Z(G^0)$, the map $G' \to \overline{G}^0$ is a central isogeny and hence they have the same simply connected central cover, which induces the first isomorphism. The second isomorphism follows from Proposition 11.1. □

Let $p_2 : G \to \mathrm{PGL}_2$ be the composition of the quotient map $G \to \overline{G}$ with the projection map $\overline{G} \to \mathrm{PGL}_2$ onto the second component. Let $H_0$ be the preimage of $\mu_{p-1} \subset T_2$ in $G$ under $p_2$.

**Lemma 11.3.** *$H_0$ is generalised reductive and* $\dim Z(H_0) < \dim Z(L)$.

*Proof.* If $H_0$ is contained in a proper R-parabolic of $L$ then the image of $H_0$ is contained in a proper R-parabolic of $\overline{L}$. This would imply that $Z(\overline{L})\overline{H}_0$ is contained in a proper R-parabolic of $\overline{L}$, which is not possible as Proposition 11.1 implies that $Z(\overline{L})\overline{H}_0 = \overline{L}$. Lemma 6.2 in [42] implies that $H_0$ is reductive and $Z_L(H_0)^0 = Z(L)^0$. Since $H_0 \subseteq L$ we have $Z(H_0) \subseteq Z_L(H_0)$ and hence $Z(H_0)^0 \subseteq Z(L)^0$. If $\dim Z(H_0) = \dim Z(L)$ then the quotient would be 0-dimensional and connected and hence $Z(H_0)^0 = Z(L)^0$. Thus it is enough to produce a cocharacter $\lambda : \mathbb{G}_m \to Z(L)$, such that its image is not contained in $H_0$.

We have an exact sequence $0 \to Z(G^0) \to L^0 \to \overline{L}^0 \to 0$. Since $L^0$ is reductive we have a surjection $Z(L^0) \twoheadrightarrow Z(\overline{L}^0)$. The group of components of $L$, which we denote by $\Delta$, acts on both groups and induces a finite map on invariants $Z(L^0)^\Delta \to Z(\overline{L}^0)^\Delta$. Hence, the quotient is 0-dimensional and hence every cocharacter $\mu : \mathbb{G}_m \to Z(\overline{L}^0)^\Delta$ can be lifted to a cocharacter $\tilde{\mu} : \mathbb{G}_m \to Z(L^0)^\Delta$. Let $\mu : \mathbb{G}_m \to \overline{L} = G_1 \times \mathbb{G}_m$ be the cocharacter $t \mapsto (1, t)$. The image of $\mu$ is contained in $Z^0(\overline{L})^\Delta$. Its lift $\tilde{\mu}$ takes values in $Z(L^0)^\Delta = Z(L) \cap L^0$, see the proof of Proposition 2.9. If the image of $\tilde{\mu}$ is contained in $H_0$, then the image of $\mu$ would be contained in $\overline{H}_0$, which is not the case. □

The following Lemma is essentially [6, Lemma 4.2], but made to work for all $F$ and not just $\mathbb{Q}_2$.

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**Lemma 11.4.** *Let $\rho : \Gamma_F \to B_2(\kappa)$ be a representation and let $\psi : \Gamma_F \to \kappa^\times$ be the character $\psi : \Gamma_F \xrightarrow{\rho} B_2(\kappa) \to B_2(\kappa)/U_2(\kappa) \xrightarrow{\alpha} \mathbb{G}_m(\kappa)$. If the following hold:*

(1) *$\psi \neq \omega^{\pm 1}$, where $\omega = \chi_{\mathrm{cyc}} \otimes_{\mathbb{Z}_p} \kappa$;*
(2) *$\rho(\Gamma_F)$ is not contained in any maximal split torus in $B_2$;*

*then $h^0(\Gamma_F, (\mathrm{Lie}\,\mathrm{SL}_2)^*(1)) = 0$, where the action of $\Gamma_F$ on $\mathrm{Lie}\,\mathrm{SL}_2$ is given by $\rho$ composed with the adjoint action of $\mathrm{PGL}_2$.*

*Proof.* Let $M_2(\kappa)$ be the space of $2 \times 2$-matrices over $\kappa$. We may identify $\mathrm{Lie}\,\mathrm{SL}_2$ with the subspace of trace zero matrices. The quadratic form $(A, B) \mapsto \mathrm{tr}(AB)$ identifies $(\mathrm{Lie}\,\mathrm{SL}_2)^*$ with the quotient of $M_2(\kappa)$ by the scalar matrices, and this isomorphism is $\mathrm{PGL}_2$-equivariant for the adjoint action on $M_2(\kappa)$. We denote the resulting representation of $\Gamma_F$ by $\overline{\mathrm{ad}}\rho$. For $i, j \in \{1, 2\}$ let $e_{ij} \in M_2(\kappa)$ be the matrix with $ij$-entry equal to 1 and the rest of entries equal to zero, and let $\bar{e}_{ij}$ be the image of $e_{ij}$ in $\overline{\mathrm{ad}}\rho$. Then $\bar{e}_{12}, \bar{e}_{11}, \bar{e}_{21}$ is a basis of $\overline{\mathrm{ad}}\rho$ as a $\kappa$-vector space.

The subgroup $\left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right) \subset \mathrm{GL}_2$ maps isomorphically onto $B_2$. Thus there is a representation $\tilde{\rho} : \Gamma_F \to \mathrm{GL}_2(\kappa)$ such that $\tilde{\rho}(\gamma) = \left(\begin{smallmatrix} 1 & b(\gamma) \\ 0 & \psi^{-1}(\gamma) \end{smallmatrix}\right)$, for all $\gamma \in \Gamma_F$, and $\rho(\gamma)$ is equal to $\tilde{\rho}(\gamma)$ modulo the scalar matrices. A direct computation of $\tilde{\rho}(\gamma) e_{ij} \tilde{\rho}(\gamma)^{-1}$, shows that the action of $\Gamma_F$ on $(\mathrm{Lie}\,\mathrm{SL}_2)^*(1)$ is given on the basis as follows:

$$\gamma \cdot \bar{e}_{12} = \omega(\gamma)\psi(\gamma)\bar{e}_{12}, \quad \gamma \cdot \bar{e}_{11} = \omega(\gamma)\bar{e}_{11} - \omega(\gamma)\psi(\gamma)b(\gamma)\bar{e}_{12},$$
$$\gamma \cdot \bar{e}_{21} = \omega(\gamma)\psi^{-1}(\gamma)\bar{e}_{21} - \omega(\gamma)\psi(\gamma)b(\gamma)^2\bar{e}_{12}.$$

Thus the flag $\langle \bar{e}_{12} \rangle \subset \langle \bar{e}_{12}, \bar{e}_{11} \rangle \subset \langle \bar{e}_{12}, \bar{e}_{11}, \bar{e}_{21} \rangle$ is $\Gamma_F$-stable and the action on the graded pieces is given by the characters $\omega\psi$, $\omega$, $\omega\psi^{-1}$. Since $\omega\psi^{-1}$ and $\omega\psi$ are non-trivial by assumption, if $h^0(\Gamma_F, (\mathrm{Lie}\,\mathrm{SL}_2)^*(1)) \neq 0$ then $\omega$ is trivial and the $\Gamma_F$-invariants are of the form $\langle \bar{e}_{11} + \lambda \bar{e}_{12} \rangle$ for some $\lambda \in \kappa$. A calculation shows that this is equivalent to $\lambda\psi(\gamma) = b(\gamma) + \lambda$ for all $\gamma \in \Gamma_F$, and a further calculation shows that this is equivalent to $\left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)^{-1} \tilde{\rho}(\gamma) \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$ being a diagonal matrix for all $\gamma \in \Gamma_F$, which would contradict assumption (2). □

**Proposition 11.5.** *Let $\rho : \Gamma_F \to P(\kappa)$ be a representation such that the following hold:*

(1) *$\rho(\Gamma_F)$ is not contained in an R-Levi of $P$;*
(2) *the $G$-semisimplification of $\rho$ is not contained in $H_0$;*
(3) *$h^0(\Gamma_F, (\mathrm{Lie}(G'_1)_{\mathrm{sc}})^*(1)) = 0$.*

*Then $h^0(\Gamma_F, (\mathrm{Lie}\,G'_{\mathrm{sc}})^*(1)) = 0$.*

*Proof.* Lemma 11.2 gives us an isomorphism of $G$-representations

$$(\mathrm{Lie}\,G'_{\mathrm{sc}})^* \cong (\mathrm{Lie}\,\mathrm{SL}_2)^* \oplus (\mathrm{Lie}(G'_1)_{\mathrm{sc}})^*,$$

and assumption (3) implies that it is enough to show that $h^0(\Gamma_F, (\mathrm{Lie}\,\mathrm{SL}_2)^*(1)) = 0$.

Let $\rho_2 : \Gamma_F \to \mathrm{PGL}_2(\kappa)$ be the composition of $\rho$ with $p_2$. The action of $\Gamma_F$ on $(\mathrm{Lie}\,\mathrm{SL}_2)^*$ factors through $\rho_2$. Thus it is enough to verify that $\rho_2$ satisfies the assumptions of Lemma 11.4.

If $\rho_2(\Gamma_F)$ is contained in a maximal split torus of $B_2$ then there exists $u_2 \in U_2(\kappa)$ such that $u_2\rho_2(\Gamma_F)u_2^{-1}$ is contained in $T_2(\kappa)$. Since $L = p_2^{-1}(T_2)$ and $p_2 : U \xrightarrow{\cong} U_2$, there is $u \in U(\kappa)$ such that $u\rho(\Gamma_F)u^{-1}$ is contained in $L(\kappa)$. Since $u^{-1}Lu$ is an R-Levi of $G$ we obtain a contradiction to assumption (1).

Let $\psi : \Gamma_F \to \kappa^\times$ be a character associated to $\rho_2$ in Lemma 11.4. If $\psi = \omega$ or $\psi = \omega^{-1}$ then $\psi^{p-1} = 1$. Thus $\rho_2(\Gamma_F)$ is contained in $\mu_{p-1}(\kappa)U_2(\kappa)$. In the proof of Lemma 11.3 we have constructed a cocharacter $\lambda : \mathbb{G}_m \to Z(L)$ such that $p_2 \circ \lambda : \mathbb{G}_m \to T_2$ is the unique cocharacter satisfying

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$\alpha \circ p_2 \circ \lambda = \mathrm{id}$. Then $\lim_{t\to 0} p_2(\lambda(t))U_2 p_2(\lambda(t))^{-1} = \{1\}$. Since $p_2 : G \to \mathrm{PGL}_2$ is a group homomorphism, which induces an isomorphism between $U$ and $U_2$, we conclude that

$$\lim_{t\to 0} \lambda(t)\rho(\Gamma_F)\lambda(t)^{-1} \subset \lim_{t\to 0} \lambda(t)H_0(\kappa)\lambda(t)^{-1} = H_0(\kappa),$$

as $H_0$ is a subgroup of $L$ and $\lambda$ takes values in $Z(L)$. But this contradicts the assumption (2). □

## 12. Dimension of fibres

Let $\mathfrak{p}$ be a prime ideal of $R^{\mathrm{ps}}_G$ such that $\dim R^{\mathrm{ps}}_G/\mathfrak{p} \leq 1$. Then $\kappa(\mathfrak{p})$ is either a finite or a local field, which we equip with its natural topology. We denote by $y$ the homomorphism $y : R^{\mathrm{ps}}_G \to \kappa(\mathfrak{p})$ and by $\bar{y}$ the composition of $y$ with an embedding of $\kappa(\mathfrak{p})$ into its algebraic closure $\overline{\kappa(\mathfrak{p})}$. We equip $\overline{\kappa(\mathfrak{p})}$ with its natural topology, extending the topology on $\kappa(\mathfrak{p})$. The goal of this section is to bound the dimension of

$$X^{\mathrm{gen}}_{G,y} := X^{\mathrm{gen}}_{G,\bar{\rho}^{\mathrm{ss}}} \times_{X^{\mathrm{ps}}_G} y = X^{\mathrm{gen},\tau}_{G} \times_{X^{\mathrm{ps}}_G} y$$

from above in Proposition 12.15.

**Lemma 12.1.** *There is a unique point $\bar{z} \in X^{\mathrm{git}}_{G,\bar{\rho}^{\mathrm{ss}}}(\overline{\kappa(\mathfrak{p})})$ mapping to $\bar{y} \in X^{\mathrm{ps}}_G(\overline{\kappa(\mathfrak{p})})$. Moreover, the map*

$$X^{\mathrm{gen}}_{G,\bar{\rho}^{\mathrm{ss}}} \times_{X^{\mathrm{git}}_{G,\bar{\rho}^{\mathrm{ss}}}} \bar{z} \to X^{\mathrm{gen}}_{G,\bar{\rho}^{\mathrm{ss}}} \times_{X^{\mathrm{ps}}_G} \bar{y}$$

*induces an isomorphism of underlying reduced subschemes.*

*Proof.* Proposition 7.4 implies that $X^{\mathrm{git}}_{G,\bar{\rho}^{\mathrm{ss}}} \times_{X^{\mathrm{ps}}_G} \bar{y}$ is finite over $\bar{y}$ and its underlying topological space is a point. This implies the assertion. □

We assume[9] that $G$ is split over $\mathcal{O}$ and let $T$ be a maximal split torus in $G$ defined over $\mathcal{O}$. Lemma 12.1 and Corollary 6.11 imply that there is $\bar{x} \in X^{\mathrm{gen}}_{G,y}(\overline{\kappa(\mathfrak{p})})$, an R-Levi subgroup $L$ of $G$ containing $T$ and a continuous $L_{\bar{y}}$-irreducible representation $\rho : \Gamma_F \to L(\kappa(\bar{y}))$ such that $\rho = \rho_{\bar{x}}$ and the $G^0$-orbit of $\bar{x}$ is closed in $X^{\mathrm{gen}}_{G,\bar{y}}$.

**Remark 12.2.** Proposition 7.4 implies that there is a unique point $y' \in X^{\mathrm{git}}_{G,\bar{\rho}^{\mathrm{ss}}}$ mapping to $y$. It follows from Lemma 12.1 that the fibres of $X^{\mathrm{gen}}_{G,\bar{\rho}^{\mathrm{ss}}}$ at $y$ and $y'$ have the same dimension. Moreover, if $\kappa(\mathfrak{p})$ is a local field then $y'$ is a closed point of the subscheme $U_{LG,\bar{\rho}^{\mathrm{ss}}}$ defined in section 9. Conversely, if $y$ is the image of a closed point of $U_{LG,\bar{\rho}^{\mathrm{ss}}}$ then $\kappa(y)$ is a local field and we may take the R-Levi in the above discussion to be equal to $L$.

Let $\mathfrak{q}$ be the kernel of $\bar{x} : A^{\mathrm{gen}}_{G,\bar{\rho}^{\mathrm{ss}}} \to \overline{\kappa(\mathfrak{p})}$. We let $\kappa := \kappa(\mathfrak{q})$, let $x$ be the homomorphism $x : A^{\mathrm{gen}}_G \to \kappa$. Then the image of $\rho$ is contained in $L(\kappa)$. We write $\overline{\kappa}$ for $\overline{\kappa(\mathfrak{p})}$. We consider $X^{\mathrm{gen}}_{G,y}$ as a scheme over $\kappa$ by considering $y : R^{\mathrm{ps}}_G \to \kappa(\mathfrak{p}) \hookrightarrow \kappa$.

Let $P$ be any R-parabolic of $G$ which admits $L$ as its R-Levi and let $U$ be the unipotent radical of $P$.

**Definition 12.3.** Let $X^{\mathrm{gen}}_{P,\rho} : \kappa\text{-alg} \to \mathrm{Set}$ be the functor which sends $A$ to the set of $R^{\mathrm{ps}}_G$-condensed representations $\rho' : \Gamma_F \to P(A)$ such that

$$\rho'(\gamma) \equiv \rho(\gamma) \otimes_\kappa A (\mathrm{mod}\ U(A)), \quad \forall \gamma \in \Gamma. \tag{45}$$

**Lemma 12.4.** *The functor $X^{\mathrm{gen}}_{P,\rho}$ is represented by a closed subscheme of $X^{\mathrm{gen}}_{G,y}$.*

[9]The assumption is for convenience only as in Proposition 12.15 we consider geometric fibres.

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*Proof.* It follows from Proposition 8.3 that for a $\kappa$-algebra $A$, $X^{\mathrm{gen}}_{G,y}(A)$ is the set of $R^{\mathrm{ps}}_G$-condensed representations $\rho' : \Gamma_F \to G(A)$ such that $\Theta_{\rho'} = \Theta_\rho \otimes_\kappa A$. If $\rho' \in X^{\mathrm{gen}}_{P,\rho}(A)$ then (45) implies that $\Theta_{\rho'} = \Theta_\rho \otimes_\kappa A$, and hence $\rho' \in X^{\mathrm{gen}}_{G,y}(A)$.

Since $X^{\mathrm{gen},\tau}_G$ is a closed subscheme of $X^{\mathrm{ps}}_{\mathrm{GL}_d} \times G^N$ by Remark 5.8, the map $\rho' \mapsto (\rho'(\gamma_1), \dots, \rho'(\gamma_N))$ identifies $X^{\mathrm{gen}}_{G,y}$ with a closed subscheme of $G^N_\kappa$. Lemma 4.15 implies that $(X^{\mathrm{gen}}_{G,y} \times_{G^N_\kappa} P^N_\kappa)(A)$ is the set of $R^{\mathrm{ps}}_G$-condensed representations $\rho' : \Gamma_F \to P(A)$ such that $\Theta_{\rho'} = \Theta_\rho \otimes_\kappa A$. The proof of Lemma 4.15 shows that an $R^{\mathrm{ps}}_G$-condensed representation $\rho' : \Gamma_F \to P(A)$ satisfies (45) if and only if

$$\rho'(\gamma_i) \equiv \rho(\gamma_i) \pmod{U(A)}, \quad \forall 1 \le i \le N.$$

We thus may identify $X^{\mathrm{gen}}_{P,\rho}$ with $X^{\mathrm{gen}}_{G,y} \times_{G^N_\kappa} P^N_\kappa \times_{(P/U)^N_\kappa} \operatorname{Spec}\kappa$, where $\operatorname{Spec}\kappa \to (P/U)^N_\kappa$ corresponds to a closed point $(\rho(\gamma_1), \dots, \rho(\gamma_N)) \in (P/U)^N_\kappa(\kappa)$. The claim follows as the fibre products are taken along closed immersions. □

**Lemma 12.5.** *Let $x \in X^{\mathrm{gen}}_{P,\rho}(\kappa)$ be the point corresponding to $\rho$ and let $T_x X^{\mathrm{gen}}_{P,\rho}$ be the tangent space at $x$. There is a natural isomorphism*

$$T_x X^{\mathrm{gen}}_{P,\rho} \xrightarrow{\cong} Z^1(\Gamma_F, (\operatorname{Lie} U)_\kappa) \tag{46}$$

*of $\kappa$-vector spaces, where the action of $\Gamma_F$ on $(\operatorname{Lie} U)_\kappa$ is given via the homomorphism $\rho : \Gamma_F \to L(\kappa)$ composed with the adjoint action of $L$ on $\operatorname{Lie} U$.*

*Proof.* We may identify $T_x X^{\mathrm{gen}}_{P,\rho}$ with the set of $\rho' \in X^{\mathrm{gen}}_{P,\rho}(\kappa[\varepsilon])$ such that $\rho' \equiv \rho \pmod{\varepsilon}$, which implies that we may write $\rho'(\gamma) = (1 + \varepsilon\beta(\gamma))\rho(\gamma)$ for a unique function $\beta : \Gamma_F \to (\operatorname{Lie} U)_\kappa$. Lemma 4.14 implies that $\rho' : \Gamma_F \to P(A)$ is $R^{\mathrm{ps}}_G$-condensed if and only if $\rho'$ is continuous for the topology on $\kappa[\varepsilon]$ induced by $\kappa$, which is equivalent to continuity of $\beta$. A standard calculation shows that $\rho'$ is a representation if and only if $\beta$ defines a 1-cocycle. Mapping $\rho'$ to $\beta \in Z^1(\Gamma_F, (\operatorname{Lie} U)_\kappa)$ induces the claimed isomorphism. □

**Lemma 12.6.** *The set $X^{\mathrm{gen}}_{P,\rho}(\overline{\kappa})$ is naturally in bijection with the set of continuous representations $\rho' : \Gamma_F \to P(\overline{\kappa})$, such that $\rho' \equiv \rho \pmod{U(\overline{\kappa})}$.*

*Proof.* The assertion follows from Lemma 4.14. □

Let $\lambda : \mathbb{G}_m \to G$ be a cocharacter such that $P = P_\lambda$ and $L = L_\lambda$. Since $\lambda$ centralises $L$ and normalises $P$ the adjoint action of $\mathbb{G}_m$ on $X^{\mathrm{gen}}_G$ induces an action of $\mathbb{G}_m$ on $X^{\mathrm{gen}}_{P,\rho}$.

**Lemma 12.7.** *The unique closed $\mathbb{G}_m$-orbit in $X^{\mathrm{gen}}_{P,\rho}(\overline{\kappa})$ is the singleton $\{\rho\}$.*

*Proof.* Since $\lim_{t\to 0} \lambda(t) U \lambda(t)^{-1} = 1$, using Lemma 12.6, we deduce that the closure of any $\mathbb{G}_m$-orbit in $X^{\mathrm{gen}}_{P,\rho}(\overline{\kappa})$ will contain $\{\rho\}$, which is a closed $\mathbb{G}_m$-orbit, as it is zero dimensional and $\rho(\Gamma_F) \subseteq L(\kappa)$. □

**Proposition 12.8.** *Let $x \in X^{\mathrm{gen}}_{P,\rho}$ be the closed point corresponding to $\rho$. Then*

$$\dim X^{\mathrm{gen}}_{P,\rho} \le \dim_\kappa T_x X^{\mathrm{gen}}_{P,\rho} = \dim U_\kappa + \dim U_\kappa [F : \mathbb{Q}_p] + h^0(\Gamma_F, (\operatorname{Lie} U)^*_\kappa(1)),$$

*where the action of $\Gamma_F$ on $\operatorname{Lie} U$ is given by the composition of $\rho$ with the adjoint action of $L$ on $\operatorname{Lie} U$.*

*Proof.* The first inequality follows from [6, Lemma 2.2] together with Lemma 12.7. (Its proof shows that $x$ lies on every irreducible component of $X^{\mathrm{gen}}_{P,\rho}$.)

Lemma 12.5 identifies $T_x X^{\mathrm{gen}}_{P,\rho}$ with $Z^1(\Gamma_F, (\operatorname{Lie} U)_\kappa)$. The assertion follows from Lemma 3.1, noting that $\dim U_\kappa$ is equal to $\dim_\kappa (\operatorname{Lie} U)_\kappa$. □

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**Lemma 12.9.** *Let $x' \in X^{\mathrm{gen}}_{G,y}(\overline{\kappa})$. Then there is an R-parabolic subgroup $P$ of $G$ with R-Levi equal to $L$, $g \in G^0(\overline{\kappa})$ and $x'' \in X^{\mathrm{gen}}_{P,\rho}(\overline{\kappa})$ such that*

$$\rho_{x'}(\gamma) = g\rho_{x''}(\gamma)g^{-1}, \quad \forall \gamma \in \Gamma_F.$$

*Proof.* The assertion follows from Lemma 12.1, Corollary 6.11 and Lemma 2.17. □

**Definition 12.10.** We define the defect of the fibre $X^{\mathrm{gen}}_{G,y}$ to be

$$\delta(y) := \max_P h^0(\Gamma_F, (\mathrm{Lie}\, U)^*_\kappa(1)), \tag{47}$$

where $P$ ranges over all R-parabolic subgroups of $G$ with R-Levi $L$, and $U$ is the unipotent radical of $P$ and $\Gamma_F$ acts on $(\mathrm{Lie}\, U)^*_\kappa$ via $\rho$ composed with the adjoint action of $L$.

**Remark 12.11.** Lemma 2.18 implies that only finitely many $P$ appear in (47).

**Remark 12.12.** If we replace $(L, \rho)$ in (47) by a conjugate then $\delta(y)$ does not change. Since Lemma 12.9 implies that the pair $(L, \rho)$ is uniquely determined by $y$ up to $G^0$-conjugation the defect $\delta(y)$ depends only on $y$.

**Remark 12.13.** Let $\lambda : \mathbb{G}_m \to G_\kappa$ be a cocharacter. Let $M = L_\lambda$, $U = U_\lambda$ and $U^- = U_{-\lambda}$. Then $\mathrm{Lie}\, G_\kappa = \mathrm{Lie}\, U^- \oplus \mathrm{Lie}\, M \oplus \mathrm{Lie}\, U$. Moreover, the theory of root systems gives us $\dim_\kappa \mathrm{Lie}\, U^- = \dim_\kappa \mathrm{Lie}\, U$. Hence, $\dim_\kappa \mathrm{Lie}\, U = \dim U = \frac{1}{2}(\dim G_\kappa - \dim M)$. Thus for all the unipotent radicals $U$ appearing in (47) we have $\dim_\kappa \mathrm{Lie}\, U_\kappa = \frac{1}{2}(\dim G_\kappa - \dim L_\kappa)$.

**Lemma 12.14.** $\delta(y) \leq \frac{1}{2}(\dim G_{\overline{\kappa}} - \dim L_{\overline{\kappa}})$

*Proof.* Since the dimension of $\Gamma_F$-invariants is less or equal to the dimension of the vector space, we have $h^0(\Gamma_F, (\mathrm{Lie}\, U)^*_\kappa(1)) \leq \dim_\kappa \mathrm{Lie}\, U_\kappa$. The bound follows from Remark 12.13 by observing that extending scalars to the algebraic closure does not change the dimension. □

**Proposition 12.15.**

$$\dim X^{\mathrm{gen}}_{G,y} \leq \dim G_{\overline{\kappa}} - \dim Z(L_{\overline{\kappa}}) + \tfrac{1}{2}(\dim G_{\overline{\kappa}} - \dim L_{\overline{\kappa}})[F : \mathbb{Q}_p] + \delta(y).$$

*Proof.* The dimension does not change if we replace $X^{\mathrm{gen}}_{G,y}$ by the geometric fibre $X^{\mathrm{gen}}_{G,\bar{y}}$. The action of $G^0$ on $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ induces an action of $G^0_{\overline{\kappa}}$ on $X^{\mathrm{gen}}_{G,\bar{y}}$. Moreover, $X^{\mathrm{gen}}_{P,\rho} \times_y \bar{y}$ is invariant under the action of the subgroup $Z(L_{\overline{\kappa}})^0 U_{\overline{\kappa}}$. Thus we obtain a map of schemes

$$\coprod_P G^0_{\overline{\kappa}} \times^{Z(L_{\overline{\kappa}})^0 U_{\overline{\kappa}}} (X^{\mathrm{gen}}_{P,\rho} \times_y \bar{y}) \to X^{\mathrm{gen}}_{G,\bar{y}}, \tag{48}$$

where the disjoint union is taken over all the R-parabolics $P$ of $G$ with R-Levi $L$. We note that there are only finitely many such $P$ by Lemma 2.18. It follows from Lemma 12.9 that (48) induces a surjection on $\overline{\kappa}$-points. Since both the source of and the target of (48) are of finite type over $\overline{\kappa}$, [6, Lemma 3.14] implies that

$$\begin{aligned}
\dim X^{\mathrm{gen}}_{G,\bar{y}} &\leq \max_P \dim(G^0_{\overline{\kappa}} \times^{Z(L_{\overline{\kappa}})^0 U_{\overline{\kappa}}} (X^{\mathrm{gen}}_P \times_y \bar{y})^{\mathrm{red}}) \\
&= \dim G^0_{\overline{\kappa}} - \dim Z(L_{\overline{\kappa}})^0 - \dim U_{\overline{\kappa}} + \max_P \dim(X^{\mathrm{gen}}_{P,\rho} \times_y \bar{y}) \\
&\leq \dim G_{\overline{\kappa}} - \dim Z(L_{\overline{\kappa}}) + \tfrac{1}{2}(\dim G_{\overline{\kappa}} - \dim L_{\overline{\kappa}})[F : \mathbb{Q}_p] + \delta(y),
\end{aligned}$$

where in the last two lines we have used Remark 12.13 and in the last inequality we have used Proposition 12.8. □

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**Remark 12.16.** If $y'$ is a closed point of $U_{LG,\overline{\rho}^{\mathrm{ss}}}$ then Remark 12.2 implies that the bound in Proposition 12.15 also applies to the fibre of $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ at $y'$. We will use this without further comment in section 13.

**Lemma 12.17.** *Let $M$ be an R-Levi of $G_{\overline{\kappa}}$. Assume that* $\dim G_{\overline{\kappa}} > 1$. *If* $\dim M = 1$ *then* $Z(M)^0 = M^0$. *In particular,* $\dim Z(M) = 1$. *Moreover,* $\dim G_{\overline{\kappa}} = 3$.

*Proof.* The assumption on $M$ implies that the semisimple rank of $G^0_{\overline{\kappa}}$ is 1 and hence $G^0_{\overline{\kappa}}$ is isomorphic to either $\mathrm{SL}_2$ or $\mathrm{PGL}_2$, [44, Theorem 20.33], which implies the Lemma. □

**Corollary 12.18.** *Let $y : \operatorname{Spec}\overline{k} \to X^{\mathrm{ps}}_G$ be a geometric point above the closed point of $X^{\mathrm{ps}}_G$ and let $Y_{\overline{\rho}^{\mathrm{ss}}} := X^{\mathrm{gen}}_{G,y}$. If* $\dim G_{\overline{k}} > 1$ *then*

$$\dim G_{\overline{k}}(1 + [F : \mathbb{Q}_p]) - \dim Y_{\overline{\rho}^{\mathrm{ss}}} \geq [F : \mathbb{Q}_p] \dim L_{\overline{k}} + \dim Z(L_{\overline{k}}) \geq [F : \mathbb{Q}_p] + 1. \tag{49}$$

*Moreover, if (49) is an equality*[10] *then* $\dim G_{\overline{k}} - \dim L_{\overline{k}} = 2$ *and $L$ is $\overline{\rho}^{\mathrm{ss}}$-compatible in the sense of Section 13.1 below. Otherwise,*

$$\dim G_{\overline{\kappa}}(1 + [F : \mathbb{Q}_p]) - \dim Y_{\overline{\rho}^{\mathrm{ss}}} \geq 2[F : \mathbb{Q}_p] + 1. \tag{50}$$

*Proof.* The first inequality follows from Proposition 12.15 and Lemma 12.14. The second inequality follows from Lemma 12.17.

If (49) is an equality then $\dim L_{\overline{k}} = 1$. The last part of Lemma 12.17 says that $\dim G_{\overline{\kappa}} = 3$. So the difference of dimensions is 2. The compatibility of $L_k$ with $\overline{\rho}^{\mathrm{ss}}$ follows Lemma 13.1 below, since $\overline{X}^{\mathrm{git}}_{L,\overline{\rho}^{\mathrm{ss}}}$ is non-empty by construction of $L$.

If $\dim L_{\overline{k}} \neq 1$ then $\dim L_{\overline{k}} \geq 2$ and (49) implies (50). □

## 13. Complete intersection

In this section we will show that the deformation rings $R^{\square}_{G,\overline{\rho}}$ are complete intersections of expected dimension. As explained in the introduction it is enough to bound the dimension of their special fibres from above. We will achieve this by bounding the dimension of $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$.

### *13.1. Induction hypothesis*

Let us recall the setup. We start with a continuous representation $\overline{\rho} : \Gamma_F \to G(k)$, such that the group homomorphism $\pi_G \circ \overline{\rho}^{\mathrm{ss}} : \Gamma_F \to (G/G^0)(\overline{k})$ is surjective, where $\pi_G : G \to G/G^0$ is the quotient map.

We will bound the dimension of $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ arguing by induction that the assertion

$$\dim \overline{X}^{\mathrm{git}}_{H,\overline{\rho}_1^{\mathrm{ss}}} \leq \dim Z(H) + [F : \mathbb{Q}_p] \dim H. \tag{DIM}$$

holds for all *pairs* $(H, \overline{\rho}_1)$, where $H$ is a generalised reductive group scheme defined over a finite extension of $k$ and $\overline{\rho}_1 : \Gamma_F \to H(\overline{k})$ is a continuous $H$-semisimple representation such that $\pi_H \circ \overline{\rho}_1 : \Gamma_F \to (H/H^0)(\overline{k})$ is surjective. We refer the reader to Section 1.4.2 for the sketch of the proof strategy. Instead of reading this section linearly we suggest to first look at Proposition 13.25, where the induction step is carried out, and then look up the statements needed for its proof.

The induction argument takes place entirely in the special fibre and the group schemes $H$ appearing in the induction need not be special fibres of generalised reductive group schemes defined over $\mathcal{O}$. We note that the schemes $\overline{X}^{\mathrm{gen}}_{H,\overline{\rho}_1^{\mathrm{ss}}}$, $\overline{X}^{\mathrm{git}}_{H,\overline{\rho}_1^{\mathrm{ss}}}$ and $\overline{X}^{\mathrm{ps}}_H$ are still defined for such $H$ by replacing $\mathcal{O}$ with $k$ in all the definitions. All the statements proved for $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$, $\overline{X}^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and $\overline{X}^{\mathrm{ps}}_G$ when $G$ is a generalised reductive group scheme over $\mathcal{O}$, also hold for $\overline{X}^{\mathrm{gen}}_{H,\overline{\rho}_1^{\mathrm{ss}}}$, $\overline{X}^{\mathrm{git}}_{H,\overline{\rho}_1^{\mathrm{ss}}}$ and $\overline{X}^{\mathrm{ps}}_H$. In particular, if $x$ is a closed point of

[10] By which we mean both inequalities are equalities.

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$Y_{\overline{\rho}_1^{\mathrm{ss}}}$ or a closed point of $\overline{X}^{\mathrm{gen}}_{H,\overline{\rho}_1^{\mathrm{ss}}} \setminus Y_{\overline{\rho}_1^{\mathrm{ss}}}$ then the arguments of Proposition 5.16, Lemma 5.17 carry over and relate the completed local ring of $\overline{X}^{\mathrm{git}}_{H,\overline{\rho}_1^{\mathrm{ss}}}$ at $x$ to the deformation ring $\overline{R}^{\square}_{\rho_x,H}$, which parameterises deformations $\rho : \Gamma_F \to H(A)$ of $\rho_x$ for $A \in \mathfrak{A}_{\kappa(x)}$. If $H$ is a generalised reductive group scheme over $\mathcal{O}$ then $\overline{R}^{\square}_{\rho_x,H} = R^{\square}_{\rho_x,H}/\varpi$.

We say that a pair $(H, \overline{\rho}_1)$ is $\overline{\rho}^{\mathrm{ss}}$-*compatible* if $H$ is a closed subgroup scheme of $G_{\overline{k}}$ and the $G$-semisimplification of $\overline{\rho}_1$ is $G^0(\overline{k})$-conjugate to $\overline{\rho}^{\mathrm{ss}}$. We say that a closed generalised reductive subgroup $H$ of $G_{\overline{k}}$ is $\overline{\rho}^{\mathrm{ss}}$-compatible if it appears in some $\overline{\rho}^{\mathrm{ss}}$-compatible pair $(H, \overline{\rho}_1)$.

**Lemma 13.1.** *Let $L$ be an $R$-Levi of $G$. If $\overline{X}^{\mathrm{git}}_{LG,\overline{\rho}^{\mathrm{ss}}}$ is non-empty then $L_k$ occurs in a $\overline{\rho}^{\mathrm{ss}}$-compatible pair. Moreover, if $L \neq G$ then* $\dim L_k < \dim G_k$.

*Proof.* This follows from Lemma 9.11. □

**Lemma 13.2.** *Let $U$ be a non-empty locally closed subscheme of $X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}} \setminus \{*\}$, where $*$ is the closed point, and let $Z$ be the closure of $U$ in $X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$. Then* $\dim Z = \dim U + 1$.

*Proof.* This follows from [6, Lemma 3.5 (5)] applied to $\operatorname{Spec} R = \operatorname{Spec} S = Z$. □

**Lemma 13.3.** *Let $Y_{\overline{\rho}^{\mathrm{ss}}}$ be the preimage in $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ of the closed point in $X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and let $V$ be an open non-empty $G^0$-invariant subscheme of $X \setminus Y_{\overline{\rho}^{\mathrm{ss}}}$, where $X$ is either $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ or $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$. Let $Z$ be the closure of $V$ in $X$. Then* $\dim Z = \dim V + 1$.

*Proof.* Since $V$ is $G^0$-invariant, its closure $Z$ is also $G^0$-invariant. It follows from the proof of [6, Lemma 3.21] that every irreducible component of $Z$ meets $Y_{\overline{\rho}^{\mathrm{ss}}}$ non-trivially. The assertion follows from [6, Lemma 3.18 (5)]. □

**Lemma 13.4.** $\dim X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} = \max_x \dim R^{\square}_{\rho_x,G}$, $\dim \overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} = \max_x \dim R^{\square}_{\rho_x,G}/\varpi$, *where the maximum is taken over all the closed points $x \in Y$.*

*Proof.* Let $X$ be either $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ or $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and let $x$ be a closed point of $X$. If $x \notin Y$ then $x$ is a closed point in $X \setminus Y$ and since $\dim X \setminus Y = \dim X - 1$ by Lemma 13.3 we have $\dim \mathcal{O}_{X,x} < \dim X$. Thus $\dim X = \max_x \dim \mathcal{O}_{X,x}$, where the maximum is taken over the closed points $x \in Y$. Since $\dim \mathcal{O}_{X,x} = \dim \hat{\mathcal{O}}_{X,x}$ the assertion follows from Lemma 5.17. □

**Proposition 13.5.** *Let $L$ be an $R$-Levi of $G$ and let $H$ be a closed reductive subgroup of $L_{k'}$, where $k'$ is a finite extension of $k$. If $L = G$ then we assume that $H \neq G_{k'}$. Let $U$ be an open subscheme of $\overline{X}^{\mathrm{git}}_{HG} \cap U_{LG,\overline{\rho}^{\mathrm{ss}}}$, let $V$ be the preimage of $U$ in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$, and let $Z$ be the closure of $V$. If (DIM) holds for all $\overline{\rho}^{\mathrm{ss}}$-compatible pairs $(H_1, \overline{\rho}_1)$ with $\dim H_1 < \dim G_k$ then*

$$\begin{aligned} \dim G_k([F : \mathbb{Q}_p] + 1) - \dim Z \geq \\ (\dim L_k - \dim H)[F : \mathbb{Q}_p] + \dim N_k[F : \mathbb{Q}_p] - \delta(U), \end{aligned} \tag{51}$$

*where $\delta(U) = \max_y \delta(y)$, where the maximum is taken over all the closed points $y \in U$ and $\delta(y)$ denotes the defect of the fibre at $y$ and $N$ is a unipotent radical of any $R$-parabolic subgroup of $G$ with $R$-Levi $L$.*

*Proof.* After extending scalars we may assume that $k = k'$. We may assume that $\overline{X}^{\mathrm{git}}_{H,\overline{\rho}^{\mathrm{ss}}}$ is non-empty, since otherwise both $V$ and $Z$ are empty and the claim holds trivially. Using Lemma 9.4 we may further assume that the map $\overline{\rho}^{\mathrm{ss}} : \Gamma_F \to (H/H^0)(k)$ is surjective. If $\dim H = \dim G_k$ then Lemma 9.11 implies that $H = G_k$ contradicting our assumption. Thus $\dim H < \dim G_k$ and by hypothesis we have

$$\dim \overline{X}^{\mathrm{git}}_{H} \leq \dim H[F : \mathbb{Q}_p] + \dim Z(H). \tag{52}$$

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Since $V$ is a preimage of $U$, it is $G^0$-invariant and Lemma 13.3 implies that

$$\dim Z = \dim V + 1. \tag{53}$$

From [6, Lemma 3.18] we deduce that

$$\dim V \leq \dim U + \max_y \dim X^{\mathrm{gen}}_{G,y}, \tag{54}$$

where the maximum is taken over all closed points $y$ of $U$. Since $U$ is open in the punctured spectrum of a local ring, and $\overline{X}^{\mathrm{git}}_H \to \overline{X}^{\mathrm{git}}_G$ is finite by Corollary 6.5, we obtain

$$\dim U + 1 \leq \dim \overline{X}^{\mathrm{git}}_{HG,\overline{\rho}^{\mathrm{ss}}} \leq \dim \overline{X}^{\mathrm{git}}_{H,\overline{\rho}^{\mathrm{ss}}} \leq \dim H[F:\mathbb{Q}_p] + \dim Z(H), \tag{55}$$

where the last inequality is given by (52). Proposition 12.15 implies that

$$\dim X^{\mathrm{gen}}_{G,y} \leq \dim G_k + \dim N_k[F:\mathbb{Q}_p] + \delta(y) - \dim Z(L_k), \tag{56}$$

where $\delta(y)$ is the defect of the fibre. Since $\dim Z(H) \leq \dim Z(L_k)$ by Corollary 9.8 putting all the inequalities together we obtain

$$\dim Z \leq \dim H[F:\mathbb{Q}_p] + \dim G_k + \dim N_k[F:\mathbb{Q}_p] + \delta(U). \tag{57}$$

where $\delta(U) = \max_y \delta(y)$. Since $\dim G_k = \dim L_k + 2\dim N_k$ we obtain (51). □

**Corollary 13.6.** *Let $L$ be an $R$-Levi of $G$ and let $H$ be a closed reductive subgroup of $L_{k'}$, where $k'$ is a finite extension of $k$. If $L = G$ then we assume that $H \neq G_{k'}$. Let $V$ be the preimage of $U = \overline{X}^{\mathrm{git}}_{HG} \cap U_{LG,\overline{\rho}^{\mathrm{ss}}}$ in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$, and let $Z$ be the closure of $V$ in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$. If (DIM) holds for all $\overline{\rho}^{\mathrm{ss}}$-compatible pairs $(H_1,\overline{\rho}_1)$ with $\dim H_1 < \dim G_k$ then*

$$\dim G_k([F:\mathbb{Q}_p]+1) - \dim Z \geq (\dim L_k - \dim H)[F:\mathbb{Q}_p].$$

*Proof.* Remark 12.2 and Lemma 12.14 imply that

$$\delta(U) \leq \frac{1}{2}(\dim G_k - \dim L_k) = \dim N_k. \tag{58}$$

The assertion follows from Proposition 13.5. □

**Corollary 13.7.** *Let $L$ be an $R$-Levi of $G$, such that $L \neq G$, let $H$ be a closed reductive subgroup of $L_{k'}$, where $k'$ is a finite extension of $k$. Let $U$ be an open subscheme of $U_{LG,\overline{\rho}^{\mathrm{ss}}} \cap \overline{X}^{\mathrm{git}}_{HG}$, let $V$ be a preimage of $U$ in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and let $Z$ be its closure. If*

$$\delta(U) \leq \dim L_k - \dim H. \tag{59}$$

*and (DIM) holds for all $\overline{\rho}^{\mathrm{ss}}$-compatible pairs $(H_1,\overline{\rho}_1)$ with $\dim H_1 < \dim G_k$ then*

$$\dim G_k([F:\mathbb{Q}_p]+1) - \dim Z \geq \dim N_k[F:\mathbb{Q}_p] \geq 1,$$

*where $N$ is the unipotent radical of any $R$-parabolic of $G$ with $R$-Levi $L$. Moreover, if either $F \neq \mathbb{Q}_p$ or $\dim N_k \neq 1$ then the lower bound can be changed to 2.*

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*Proof.* We may assume that $\overline{X}^{\mathrm{git}}_{LG,\overline{\rho}^{\mathrm{ss}}}$ is non-empty. Lemma 9.11 implies that $\dim L < \dim G$ and $\dim N_k \geq 1$. As in the proof of Corollary 13.6 the assertion follows from Proposition 13.5, but using (59) instead of (58):

$$\begin{aligned}\dim G_k([F:\mathbb{Q}_p]+1)-\dim Z \geq \\ (\dim L_k - \dim H)([F:\mathbb{Q}_p]-1)+\dim N_k[F:\mathbb{Q}_p].\end{aligned} \tag{60}$$

Since, as explained above, $\dim N_k \geq 1$ this implies the assertion. □

### *13.2. W-special locus*

In this subsection we define the $W$-special locus, where $W$ is a representation of $G$ on a free $\mathcal{O}$-module of finite rank. Using it we will show that if $L$ is an $R$-Levi of $G$ then the locus of $\overline{U}_{LG,\overline{\rho}^{\mathrm{ss}}}$, where the defect of the fibre is non-zero is contained in a finite union of subschemes $\overline{X}^{\mathrm{git}}_{HG} \cap U_{LG,\overline{\rho}^{\mathrm{ss}}}$, where $H$ is a closed reductive subgroup of $L_k$ of smaller dimension. We will combine this with Corollaries 13.6 and 13.7 to carry out the induction argument in Section 13.3.

**Lemma 13.8.** *Let $R$ be a noetherian $\mathcal{O}$-algebra, let $G$ be an affine group scheme over* Spec $R$ *and let $W$ be a representation of $G$ on a free $R$-module of finite rank. Let $A$ be a noetherian $R$-algebra and let $\rho : \Gamma_F \to G(A)$ be a group homomorphism. Then for each $j \geq 0$ there is a closed reduced subscheme $X_{W,j}$ of $X :=$* Spec $A$ *such that for $x \in X$ we have*

$$x \in X_{W,j} \iff \dim_{\kappa(x)} W_x(1)^{\Gamma_F} \geq j+1,$$

*where $W_x = W \otimes_R \kappa(x)$ with the $\Gamma_F$ action given by the specialisation of $\rho$ at $x$.*

*Proof.* Let $\check{W} := \mathrm{Hom}_R(W,R)$ equipped with the contragredient left action of $G$. Then

$$\check{W}_x := \check{W} \otimes_R \kappa(x) \cong \mathrm{Hom}_{\kappa(x)}(W_x, \kappa(x)), \quad \forall x \in X.$$

In particular, $\Gamma_F$-invariants in $W_x(1)$ are non-zero if and only if $\Gamma_F$-coinvariants in $\check{W}_x(-1)$ are non-zero.

Evaluating $\check{W}$ at $A$ we get a free finite $A$-module $\check{W}(A) = \mathrm{Hom}_A(W(A), A)$. The action of $\Gamma_F$ on $\check{W}(A)$ is given by $g \cdot \ell := \ell \circ \rho(g)^{-1}$. Let $M$ be the $A$-submodule of $\check{W}(A)$ generated by elements of the form $\chi_{\mathrm{cyc}}(g)^{-1} g \cdot \ell - \ell$ for all $g \in \Gamma_F$ and all $\ell \in \check{W}(A)$. Let $X_W$ be the support of $\check{W}(A)/M$ in $X$. Since $A$ is noetherian $X_W$ is closed in $X$ and we equip it with the reduced subscheme structure.

For all $x \in X$ we have an exact sequence

$$M \otimes_A \kappa(x) \to \check{W}_x \to (\check{W}(A)/M) \otimes_A \kappa(x) \to 0.$$

Thus $(\check{W}(A)/M) \otimes_A \kappa(x)$ is isomorphic to $\Gamma_F$-coinvariants of $\check{W}_x(-1)$. We conclude that $x \in X_W$ if and only if $\Gamma_F$-coinvariants of $\check{W}_x(-1)$ are non-zero, and hence $X_{W,0} = X_W$. It follows from upper-semicontinuity of the dimension function, [33, Example III.12.7.2] that $X_{W,j}$ is a closed subscheme of $X_W$ for $j \geq 1$. □

**Definition 13.9.** In the situation of Lemma 13.8 we call $X_{W,0}$ the *$W$-special locus* in $X$ and its complement $X \setminus X_{W,0}$ the *$W$-nonspecial locus* in $X$. We will refer to $X_{W,j}$ as the *$W$-special locus of level $j$*.

**Lemma 13.10.** *Let us assume the setup of Lemma 13.8 and let us further assume that* $\mathrm{char}(R) = p$. *Let $x \in X_{W,j}$ and let $H$ be the Zariski closure of $\rho_x(\Gamma_F)$ in $G_{\kappa(x)}$. Then* $\dim_{\kappa(x)} W_x^{H^0} \geq j+1$.

*Proof.* Since $\mathrm{char}(R) = p$ the restriction of $W_x(1)$ to $\Gamma_{F(\zeta_p)}$ is isomorphic to the restriction of $W_x$ to $\Gamma_{F(\zeta_p)}$. Since $x \in X_{W,j}$ we have

$$\dim_{\kappa(x)} W_x^{\Gamma_{F(\zeta_p)}} \geq \dim_{\kappa(x)} W_x(1)^{\Gamma_F} \geq j+1$$

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and thus $\dim_{\kappa(x)} W_x^{H_1} \geq j+1$, where $H_1$ is the closure of $\rho(\Gamma_{F(\zeta_p)})$ in $G_{\kappa(x)}$. Since $\Gamma_{F(\zeta_p)}$ is of finite index in $\Gamma_F$, we get that $H_1$ is of finite index in $H$, and thus $H_1^0 = H^0$. □

From now on let $G$ be a generalised reductive $\mathcal{O}$-group scheme as in the previous sections. Then its Lie algebra $\mathrm{Lie}\, G$ is a finite free $\mathcal{O}$-module equipped with a $G$-action. If $P$ is an R-parabolic with R-Levi $L$ and unipotent radical $U$, then $\mathrm{Lie}\, L$ and $\mathrm{Lie}\, U$ are $\mathcal{O}$-submodules of $\mathrm{Lie}\, G$ by [16, Theorem 4.1.7, part 4]. Thus they are free $\mathcal{O}$-modules of finite rank. Moreover, they are equipped with the adjoint action of the R-Levi $L$.

**Proposition 13.11.** *Let us assume the setup of Lemma 13.8 in one of the following cases:*

(1) $R = \mathcal{O}$, $A = A^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$, $X = X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$;
(2) $R = k$, $A = A^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}/\varpi$, $X = \overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$;
(3) $R = L$, $A = A^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$, $X = X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$.

*and let $\rho : \Gamma_F \to G(A)$ be the universal representation. Then the W-special locus of level $j$ in $X$ is $G^0$-invariant.*

*Proof.* It follows from the universal property of $A^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ that if $g \in G^0(A)$ then there is an isomorphism of $R^{\mathrm{ps}}_G$-algebras $\varphi_g : A \to A$ such that for all $\gamma \in \Gamma_F$ we have

$$g\rho(\gamma)g^{-1} = G(\varphi_g)(\rho(\gamma)), \tag{61}$$

where $G(\varphi_g) : G(A) \to G(A)$ is the group homomorphism induced by $\varphi_g : A \to A$.

Let $x : A \to \kappa$ be a homomorphism of $R^{\mathrm{ps}}_{\mathrm{GL}_d}$-algebras, where $\kappa$ is a field. Then $y := g(x) = x \circ \varphi_g^{-1} : A \to \kappa$. In particular, the residue fields of $x$ and $y$ are the same and we may assume that they are equal to $\kappa$. If $h \in G(A)$ then let $h_x = G(x)(h), h_y = G(y)(h) \in G(\kappa)$ be the specialisations of $h$ at $x$ and $y$ respectively. It follows from (61) that

$$\rho_x(\gamma) = G(x)(\rho(\gamma)) = G(y)(G(\varphi_g)(\rho(\gamma))) = G(y)(g\rho(\gamma)g^{-1}) = g_y\rho_y(\gamma)g_y^{-1}. \tag{62}$$

We may identify $W_x = W_y = W_\kappa$ as $\kappa$-vector spaces. If $v \in W_\kappa$ such that $\rho_x(\gamma)v = \chi_{\mathrm{cyc}}(\gamma)^{-1}v$ for all $\gamma \in \Gamma_F$ then it follows from (62) that $\rho_y(\gamma)(g_y^{-1}v) = \chi_{\mathrm{cyc}}(\gamma)^{-1}g_y^{-1}v$ for all $\gamma \in \Gamma_F$. Hence, $g_y^{-1} : W_\kappa \to W_\kappa$ induces an isomorphism between $W_x(1)^{\Gamma_F}$ and $W_y(1)^{\Gamma_F}$. Thus $x \in X_{W,j}$ if and only if $y \in X_{W,j}$. □

**Corollary 13.12.** *Assume the setup of Proposition 13.11 and let $X'$ be a closed $G^0$-invariant subscheme of $X$. Then the W-special locus of level $j$ in $X'$ is $G^0$-invariant.*

*Proof.* The fibre product $X_{W,j} \times_X X'$ is a closed $G^0$-invariant subscheme of $X'$ and its underlying reduced subscheme coincides with $X'_{W,j}$. Thus $X'_{W,j}$ is also $G^0$-invariant. □

**Lemma 13.13.** *Let $Z$ be a closed non-empty $G^0$-invariant subscheme of $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and let $Z'$ be an irreducible component of $Z$. Then $Z'$ is $G^0$-invariant and $Z' \cap Y_{\overline{\rho}^{\mathrm{ss}}}$ is non-empty. Moreover, if $x$ is a closed point of $Z'$ then one of the following holds:*

(1) *if $x \in Y_{\overline{\rho}^{\mathrm{ss}}}$ then $\dim \mathcal{O}_{Z',x} = \dim Z'$;*
(2) *if $x \notin Y_{\overline{\rho}^{\mathrm{ss}}}$ then $\dim \mathcal{O}_{Z',x} = \dim Z' - 1$.*

*Proof.* The proof is the same as the proof of [6, Lemma 3.21]. □

**Proposition 13.14.** *If $Z$ is a closed $G^0$-invariant subscheme of $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ then*

$$\dim Z[1/p] \leq \dim \overline{Z}, \quad \dim Z \leq \dim \overline{Z} + 1,$$

*where $\overline{Z}$ is its special fibre and $Z[1/p]$ is its generic fibre.*

*Proof.* The proof is the same as the proof of [6, Lemma 3.23]. □

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**Corollary 13.15.** *Let $X'$ be a closed $G$-invariant subscheme of $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$, let $W$ be a representation of $G$ on a free $\mathcal{O}$-module of finite rank. Then*

$$\dim X'_{W_L} \leq \dim X'_{W_k}, \quad \dim X'_{W} \leq \dim X'_{W_k} + 1$$

*where $X'_{W_L}$ is the $W_L$-special locus in $X'[1/p]$ and $X'_{W_k}$ is the $W_k$-special locus in $\overline{X}'$ and $X'_W$ is the $W$-special locus in $X'$.*

*Proof.* Let $X'_W$ be the $W$-special locus in $X'$. Then it follows from the definition of the $W$-special locus that $X'_{W_L} = X'_W[1/p]$ and $X'_{W_k}$ is the underlying reduced subscheme of the special fibre of $X'_W$. The assertion follows from Corollary 13.12 and Proposition 13.14. □

**Proposition 13.16.** *Assume the setup of Proposition 13.11 (2) so that $R = k$ and $A = A^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}/\varpi$. If the $W$-special locus of level $j$ in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is nonempty then the following assertions hold:*

(1) *The image $C_{W,j}$ in $\overline{X}^{\mathrm{git}}_{G}$ of the $W$-special locus of level $j$ in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is closed;*
(2) *there exists a finite extension $k'$ of $k$ and finitely many reductive subgroups $H_1, \dots, H_r$ of $G_{k'}$ such that $C_{W,j}$ is contained in the union of $\overline{X}^{\mathrm{git}}_{H_iG,\overline{\rho}^{\mathrm{ss}}}$ and*

$$\dim_{k'} W^{H_i^0}_{k'} \geq j+1, \quad \forall 1 \leq i \leq r.$$

(3) *if $W^{G^0} = 0$ then $H_i$ in (2) satisfy $\dim H_i \leq \dim G_k - 1$;*
(4) *if $W^{G'} = 0$ then $H_i$ in (2) satisfy $\dim H_i \leq \dim G_k - 2$.*

*Proof.* Since the $W$-special locus of level $j$ is closed in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ by Lemma 13.8 and $G^0$-invariant by Proposition 13.11, its image in $\overline{X}^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is also closed by [57, Theorem 3 (iii)]. To prove part (2), let $\eta_1, \dots, \eta_r$ be the generic points of $C_{W,j}$. It is enough to find $H_i$ satisfying the required bounds on the dimension such that $\eta_i \in \overline{X}^{\mathrm{git}}_{H_iG,\overline{\rho}^{\mathrm{ss}}}$.

To ease the notation let $X$ be the $W$-special locus of level $j$ in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$, let $\eta = \eta_1$ and let $\bar{\eta}$ be a geometric point above $\eta$. It follows from [57, Theorem 3] that there is $x \in X(\kappa(\bar{\eta}))$ above $\bar{\eta}$ such that the orbit $G^0 \cdot x$ is closed in $X_{\bar{\eta}}$. Since $X$ is closed in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$, Lemma 5.6 together with (28) imply that $X_{\bar{\eta}}$ is a closed $G^0$-invariant subscheme of $G^N_{\bar{\eta}}$, where $x \in X_{\bar{\eta}}$ is mapped to the $N$-tuple $y = (\rho_x(\gamma_1), \dots, \rho_x(\gamma_N))$, where $\gamma_i$ are elements of $\Gamma_F$ chosen before Lemma 5.6. Hence, the orbit $G^0 \cdot y$ is closed in $G^N_{\bar{\eta}}$ under the conjugation action. Let $H_x$ be the smallest closed subgroup of $G_{\bar{\eta}}$ containing the entries of $y$. It follows from [3, Theorem 3.1] that $H_x$ is a strongly reductive subgroup of $G_{\bar{\eta}}$, and thus is reductive by [42, Section 6]. It follows from Lemma 5.13 that $H_x$ is equal to the Zariski closure of $\rho_x(\Gamma_F)$ in $G_{\bar{\eta}}$. Since $x \in X$, the dimension of the $H^0_x$-invariants of $W_x$ is at least $j+1$ by Lemma 13.10. If $\dim H_x = \dim G_k$ then $H^0_x = G^0_{\bar{\eta}}$ and hence $W^{G^0} \neq 0$. If $W^{G'} = 0$ then $\dim H_x \leq \dim G_k - 2$ by Corollary 10.3.

By [42, Theorem 10.3] there is a closed reductive subgroup $H$ of $G$ defined over $\bar{k}$ and $g \in G(\kappa(\bar{\eta}))$ such that $H_x = g(H_{\bar{\eta}})g^{-1}$. Since $H$ is defined by finitely many polynomials with coefficients in $\bar{k}$, $H$ is defined over a finite extension of $k$. The point $x' := g(x) \in \overline{X}^{\mathrm{gen}}_{H}(\kappa(\bar{\eta}))$ maps to $\eta$ in $\overline{X}^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$, thus $\eta$ lies in $\overline{X}^{\mathrm{git}}_{HG,\overline{\rho}^{\mathrm{ss}}}$. □

### *13.3. Bounding the dimension of the space*

In this subsection we carry out the induction to prove (DIM). It follows from [48, Theorem 9.3 (3)] that (DIM) holds if $H^0$ is a torus. Thus we may assume that $\dim H \geq 3$.

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**Proposition 13.17.** *Let $L$ be an R-Levi of $G$. Let $\overline{V}_{LG,\overline{\rho}^{\mathrm{ss}}}$ be the preimage of $\overline{U}_{LG,\overline{\rho}^{\mathrm{ss}}}$ in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and let $\overline{Z}_{LG,\overline{\rho}^{\mathrm{ss}}}$ be its closure. Assume that $L \neq G$ and $\overline{U}_{LG,\overline{\rho}^{\mathrm{ss}}}$ is non-empty. If (DIM) holds for all $\overline{\rho}^{\mathrm{ss}}$-compatible pairs $(H_1, \overline{\rho}_1)$ with $\dim H_1 < \dim G_k$ then*

$$\dim G_k([F:\mathbb{Q}_p]+1) - \dim \overline{Z}_{LG,\overline{\rho}^{\mathrm{ss}}} \geq \min(2, \dim N_k)[F:\mathbb{Q}_p] \geq 1, \tag{63}$$

*where $N$ is the unipotent radical of an R-parabolic subgroup with R-Levi $L$. Moreover, if either $F \neq \mathbb{Q}_p$ or $\dim N_k \neq 1$ then the lower bound can be replaced by 2.*

*Proof.* Lemma 9.11 implies that $\dim N_k \geq 1$. Hence, $\min(2, \dim N_k)[F:\mathbb{Q}_p] \geq 1$ with equality if and only if $F = \mathbb{Q}_p$ and $\dim N_k = 1$.

It follows from Lemma 2.18 that there are only finitely many R-parabolic subgroups $P$ with R-Levi $L$. For each such $P$ the Lie algebra of its unipotent radical $\mathrm{Lie}(\mathrm{rad}(P))$ is a free $\mathcal{O}$-module of finite rank with the adjoint action of $L$. Moreover, $\mathrm{Lie}(\mathrm{rad}(P)) \otimes_{\mathcal{O}} k = \mathrm{Lie}(\mathrm{rad}(P_k))$. Let $W_P$ denote the $k$-linear dual of $\mathrm{Lie}(\mathrm{rad}(P_k))$ and for each $j \geq 0$ let $C_{W_P,j}$ be the image in $\overline{X}^{\mathrm{git}}_{L,\overline{\rho}^{\mathrm{ss}}}$ of the $W_P$-special locus of level $j$ in $\overline{X}^{\mathrm{gen}}_{L,\overline{\rho}^{\mathrm{ss}}}$. If $C_{W_P,j}$ is nonempty then it follows from Proposition 13.16, Corollary 10.7 and Lemma 10.6 that (after possibly replacing $k$ by a finite extension) there exist finitely many closed reductive subgroups $H_{ij}$, for $1 \leq i \leq r_{P,j}$, of $L_k$ defined over $k$, such that $C_{W_P,j}$ is contained in the union of $\overline{X}^{\mathrm{git}}_{H_{ij}L,\overline{\rho}^{\mathrm{ss}}}$ and

$$\dim_k W_P^{H_{ij}^0} \geq j+1 \tag{64}$$

and one of the following holds:

(a) $\dim L_k - \dim H_{ij} \geq 2$, or

(b) $\dim L_k - \dim H_{ij} = 1$ and $\dim_k W_P^{H_{ij}^0} = 1$.

We note that if $j \geq 1$ then (64) rules out (b), so that (a) holds.

Let $\overline{V}_{P,ij}$ be the preimage of $\overline{U}_{P,ij} := \overline{U}_{LG,\overline{\rho}^{\mathrm{ss}}} \cap \overline{X}^{\mathrm{git}}_{H_{ij}L,\overline{\rho}^{\mathrm{ss}}}$ in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and let $\overline{Z}_{P,ij}$ be its closure. If part (a) holds then Corollary 13.6 gives us

$$\dim G_k([F:\mathbb{Q}_p]+1) - \dim \overline{Z}_{P,ij} \geq 2[F:\mathbb{Q}_p]. \tag{65}$$

As remarked above part (a) holds for $j = 1$. If part (b) holds for $j = 0$ then we let $\overline{V}'_{P,i}$ be the preimage in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ of

$$\overline{U}'_{P,i} := \overline{U}_{P,i0} \setminus \bigcup_{Q} (\overline{U}_{P,i0} \cap C_{W_Q,1}),$$

where the union is taken over all R-parabolic subgroups $Q$ of $G$ with R-Levi $L$. Let $\overline{Z}'_{P,i}$ be the closure of $\overline{V}'_{P,i}$ in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$.

Let $y : \mathrm{Spec}\,\kappa \to U'_{P,i}$ be a closed geometric point and let $x \in \overline{X}^{\mathrm{gen}}_{L,\overline{\rho}^{\mathrm{ss}}}(\kappa)$ lie in the preimage of $y$, such that $L \cdot x$ is a closed orbit in $\overline{X}^{\mathrm{gen}}_{L,\overline{\rho}^{\mathrm{ss}}}$. Let $\rho_x : \Gamma_F \to L(\kappa)$ be the specialisation of the universal representation of $\Gamma_F$ over $\overline{X}^{\mathrm{gen}}_{L,\overline{\rho}^{\mathrm{ss}}}$ at $x$. Let $x'$ be the image of $x$ in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and let $\rho_{x'}$ be the specialisation of the universal representation over $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ at $x'$. Then $\rho_{x'}$ is equal to the composition of $\rho_x$ with the embedding $L(\kappa) \hookrightarrow G(\kappa)$. Since the orbit $L \cdot x$ is closed $\rho_x$ is $L$-semisimple and hence $\rho_{x'}$ is $G$-semisimple and the orbit $G \cdot x'$ is closed in $X^{\mathrm{gen}}_{G,y}$. We let $\Gamma_F$ act on $W_{Q,\kappa}$ via $\rho_x$. If $h^0(\Gamma_F, W_{Q,\kappa}(1)) \geq 2$ then $x'$ is in $W_Q$-special locus of level 1. Since $x'$ maps to $y$, we would obtain that $y \in C_{W_Q,1}$, which yields a contradiction. Hence $h^0(\Gamma_F, W_{Q,\kappa}(1)) \leq 1$, which implies that $\delta(y) \leq 1$ and hence $\delta(U'_{P,i}) \leq 1$.

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Since part (b) holds by assumption we deduce that $\delta(U'_{P,i}) \leq \dim L_k - \dim H_{i0}$. Corollary 13.7 implies that

$$\dim G_k([F:\mathbb{Q}_p]+1) - \dim \overline{Z}'_{P,i} \geq \dim N_k[F:\mathbb{Q}_p]. \tag{66}$$

Since $Z_{P,i0} \subseteq \overline{Z}'_{P,i} \cup \bigcup_{Q,k} \overline{Z}_{Q,k1}$ we deduce from (65) and (66) that

$$\dim G_k([F:\mathbb{Q}_p]+1) - \dim \overline{Z}_{P,i0} \geq \min(2, \dim N_k)[F:\mathbb{Q}_p]. \tag{67}$$

Let $\overline{V}''$ be the preimage of $\overline{U}'' := \overline{U}_{LG,\overline{\rho}^{\mathrm{ss}}} \setminus \bigcup_P (\overline{U}_{LG,\overline{\rho}^{\mathrm{ss}}} \cap C_{W_P,0})$ and let $\overline{Z}''$ be its closure. The same argument as above shows that $\delta(\overline{U}'') = 0$. Corollary 13.7 applied with $H = L$ gives

$$\dim G_k([F:\mathbb{Q}_p]+1) - \dim \overline{Z}'' \geq \dim N_k[F:\mathbb{Q}_p]. \tag{68}$$

Since $\overline{Z}_{LG,\overline{\rho}^{\mathrm{ss}}} = \overline{Z}'' \cup \bigcup_{P,i} \overline{Z}_{P,i0}$ we conclude from (67) and (68) that (63) holds. □

Let $G'_{\mathrm{sc}} \to G'$ be the simply connected central cover of $G'$. We have shown in Lemma 2.11 that $\mathrm{Lie}\, G'_{\mathrm{sc}}$ is naturally a $G$-representation.

**Proposition 13.18.** *Let $\overline{X}^{\mathrm{spcl}}_{G,\overline{\rho}^{\mathrm{ss}}}$ be the $(\mathrm{Lie}\, G'_{\mathrm{sc}})^*$-special locus in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$. Let $\overline{V}^{\mathrm{spcl}}_{GG,\overline{\rho}^{\mathrm{ss}}} = \overline{V}_{GG,\overline{\rho}^{\mathrm{ss}}} \cap \overline{X}^{\mathrm{spcl}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and let $\overline{Z}^{\mathrm{spcl}}_{GG,\overline{\rho}^{\mathrm{ss}}}$ be its closure. If (DIM) holds for all $\overline{\rho}^{\mathrm{ss}}$-compatible pairs $(H_1, \overline{\rho}_1)$ with $\dim H_1 < \dim G_k$ then*

$$\dim G_k([F:\mathbb{Q}_p]+1) - \dim \overline{Z}^{\mathrm{spcl}}_{GG,\overline{\rho}^{\mathrm{ss}}} \geq 2[F:\mathbb{Q}_p]. \tag{69}$$

*Proof.* We may assume that $\overline{V}^{\mathrm{spcl}}_{GG,\overline{\rho}^{\mathrm{ss}}}$ is non-empty. Let $W = (\mathrm{Lie}\, G'_{\mathrm{sc}})^*$ and let $C_W$ be the image of $\overline{X}^{\mathrm{spcl}}_{G,\overline{\rho}^{\mathrm{ss}}}$ in $\overline{X}^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$. Since $W^{G'} = 0$ by Lemma 10.5, Proposition 13.16 implies that there exist finitely many reductive subgroups $H_i$ of $G_k$, such that $C_W$ is contained in the union of $\overline{X}^{\mathrm{git}}_{H_i G,\overline{\rho}^{\mathrm{ss}}}$ and $\dim G_k - \dim H_i \geq 2$. The assertion follows from Corollary 13.6. □

**Definition 13.19.** We define the *absolutely irreducible non-special locus* $V^{\text{n-spcl}}_{GG,\overline{\rho}^{\mathrm{ss}}}$ in $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ as the complement of the $(\mathrm{Lie}\, G'_{\mathrm{sc}})^*$-special locus in $V_{GG,\overline{\rho}^{\mathrm{ss}}}$. We define the *absolutely irreducible non-special locus* $\overline{V}^{\text{n-spcl}}_{GG,\overline{\rho}^{\mathrm{ss}}}$ in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ as the complement of the $(\mathrm{Lie}\, G'_{\mathrm{sc}})^*_k$-special locus in $\overline{V}_{GG,\overline{\rho}^{\mathrm{ss}}}$.

**Remark 13.20.** $\overline{V}^{\text{n-spcl}}_{GG,\overline{\rho}^{\mathrm{ss}}}$ coincides with the reduced special fibre of $V^{\text{n-spcl}}_{GG,\overline{\rho}^{\mathrm{ss}}}$.

Recall that $Y_{\overline{\rho}^{\mathrm{ss}}}$ is the preimage in $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ of the closed point in $X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$.

**Proposition 13.21.** *If (DIM) holds for all $\overline{\rho}^{\mathrm{ss}}$-compatible pairs $(H_1, \overline{\rho}_1)$ with $\dim H_1 < \dim G_k$ then the complement of $\overline{V}^{\text{n-spcl}}_{GG,\overline{\rho}^{\mathrm{ss}}}$ in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ has positive codimension. Moreover,*

$$\dim \overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} = \dim \overline{V}^{\text{n-spcl}}_{GG,\overline{\rho}^{\mathrm{ss}}} + 1 = \dim \overline{X}^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}} + \dim G_k - \dim Z(G_k). \tag{70}$$

*In particular, $\overline{V}^{\text{n-spcl}}_{GG,\overline{\rho}^{\mathrm{ss}}}$ is non-empty.*

*Proof.* We have $\overline{V}^{\text{n-spcl}}_{GG,\overline{\rho}^{\mathrm{ss}}}$ is open in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and its complement $C$ is contained in $Y_{\overline{\rho}^{\mathrm{ss}}} \cup \overline{Z}^{\mathrm{spcl}}_{GG,\overline{\rho}^{\mathrm{ss}}} \cup \bigcup_L \overline{Z}_{LG,\overline{\rho}^{\mathrm{ss}}}$, where $L$ runs over R-Levi subgroups of $G$, such that $L \neq G$ and $L$ contains a fixed maximal split torus. It follows from Propositions 13.17, 13.18 and 12.18 that

$$\dim G_k([F:\mathbb{Q}_p]+1) - \dim C \geq 1. \tag{71}$$

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If $\overline{V}^{\text{n-spcl}}_{GG,\overline{\rho}^{\text{ss}}}$ is empty then $C = \overline{X}^{\text{gen}}_{G,\overline{\rho}^{\text{ss}}}$ and we deduce that

$$\dim G_k([F:\mathbb{Q}_p]+1) > \dim \overline{X}^{\text{gen}}_{G,\overline{\rho}^{\text{ss}}}. \tag{72}$$

The representation $\overline{\rho}:\Gamma_F \to G(k)$ defines a closed point $x \in Y_{\overline{\rho}^{\text{ss}}}(k)$. The completion of the local ring at $x$ with respect to the maximal ideal is isomorphic to $R^{\square}_{G,\overline{\rho}}/\varpi$ by Lemma 5.17. Since $\dim R^{\square}_{G,\overline{\rho}}/\varpi \geq \dim G_k([F:\mathbb{Q}_p]+1)$ by Corollary 3.8, we conclude that

$$d := \dim \overline{X}^{\text{gen}}_{G,\overline{\rho}^{\text{ss}}} \geq \dim G_k([F:\mathbb{Q}_p]+1) \tag{73}$$

contradicting (72). Thus $\overline{V}^{\text{n-spcl}}_{GG,\overline{\rho}^{\text{ss}}}$ is non-empty and it follows from (71) and (72) that $C$ has positive codimension.

We may write $\overline{X}^{\text{gen}}_{G,\overline{\rho}^{\text{ss}}} = \overline{Z}^{\text{n-spcl}}_{GG,\overline{\rho}^{\text{ss}}} \cup C$, where $\overline{Z}^{\text{n-spcl}}_{GG,\overline{\rho}^{\text{ss}}}$ is the closure of $\overline{V}^{\text{n-spcl}}_{GG,\overline{\rho}^{\text{ss}}}$ in $\overline{X}^{\text{gen}}_{G,\overline{\rho}^{\text{ss}}}$. We deduce that $\dim \overline{Z}^{\text{n-spcl}}_{GG,\overline{\rho}^{\text{ss}}} = d$, as $C$ has positive codimension. Lemma 13.3 implies the first equality in (70). We have $d-1 = \dim \overline{V}^{\text{n-spcl}}_{GG,\overline{\rho}^{\text{ss}}} \leq \dim \overline{V}_{GG,\overline{\rho}^{\text{ss}}} \leq d-1$ by Lemma 13.3. Thus $\dim \overline{V}_{GG,\overline{\rho}^{\text{ss}}} = d-1$. It follows from Corollary 9.10 that $\dim \overline{U}_{GG,\overline{\rho}^{\text{ss}}} = d-1-\dim G_k + \dim Z(G_k)$. Let $\overline{Z}$ denote the closure of $\overline{U}_{GG,\overline{\rho}^{\text{ss}}}$ in $\overline{X}^{\text{git}}_{G,\overline{\rho}^{\text{ss}}}$. Lemma 13.2 implies

$$\dim \overline{Z} = \dim \overline{U}_{GG,\overline{\rho}^{\text{ss}}} + 1 = d - \dim G_k + \dim Z(G)_k. \tag{74}$$

We have $\overline{X}^{\text{git}}_{G,\overline{\rho}^{\text{ss}}} = \overline{Z} \cup \bigcup_L \overline{X}^{\text{git}}_{LG,\overline{\rho}^{\text{ss}}}$, where the union is taken over all R-Levi subgroups $L$ of $G$, containing a fixed maximal split torus in $G$ and not equal to $G$. If $\overline{X}^{\text{git}}_{LG,\overline{\rho}^{\text{ss}}}$ is non-empty then $L_k$ is $\overline{\rho}^{\text{ss}}$-compatible by Lemma 13.1 and $\dim L_k < \dim G_k$ as $L \neq G$. Thus we may apply (DIM) to deduce that

$$\dim \overline{X}^{\text{git}}_{LG,\overline{\rho}^{\text{ss}}} \leq \dim L_k[F:\mathbb{Q}_p] + \dim Z(L_k). \tag{75}$$

Lemma 9.11 implies that $\dim N_k \geq 1$, where $N$ is the unipotent radical of any R-parabolic with R-Levi $L$. It follows from Lemma 2.16, (73), (74) and (75) that $\dim \overline{Z} > \dim \overline{X}^{\text{git}}_{LG,\overline{\rho}^{\text{ss}}}$ and thus $\dim \overline{X}^{\text{git}}_{G,\overline{\rho}^{\text{ss}}} = \dim \overline{Z}$, which implies the last equality in (70). □

**Lemma 13.22.** *Let $x$ be a closed point of $X^{\text{gen}}_{G,\overline{\rho}^{\text{ss}}} \setminus Y_{\overline{\rho}^{\text{ss}}}$ and let $\rho_x : \Gamma_F \to G(\kappa(x))$ be the specialisation of the universal Galois representation over $X^{\text{gen}}_{G,\overline{\rho}^{\text{ss}}}$ at $x$. If $\pi_1(G')$ is étale or $x \in X^{\text{gen}}_{G,\overline{\rho}^{\text{ss}}}[1/p]$ then $x \in V^{\text{n-spcl}}_{GG,\overline{\rho}^{\text{ss}}}$ if and only if $\rho_x$ is absolutely $G$-irreducible and $H^2(\Gamma_F, \mathrm{ad}^0\,\rho_x) = 0$.*

*Proof.* If $\pi_1(G')$ is étale or $x \in X^{\text{gen}}_{G,\overline{\rho}^{\text{ss}}}[1/p]$ then $(\mathrm{Lie}\, G'_{\text{sc}})_x = (\mathrm{Lie}\, G')_x$. It follows from Proposition 9.7 and Lemma 13.8 that $x \in V^{\text{n-spcl}}_{GG,\overline{\rho}^{\text{ss}}}$ if and only if $\rho_x$ is absolutely $G$-irreducible and $H^0(\Gamma_F, (\mathrm{ad}^0\,\rho_x)^*(1)) = 0$. Since $x$ is closed in $X^{\text{gen}}_{G,\overline{\rho}^{\text{ss}}} \setminus Y_{\overline{\rho}^{\text{ss}}}$ its residue field is a local field by [6, Lemmas 3.17, 3.18]. Local Tate duality gives us an isomorphism $H^0(\Gamma_F, (\mathrm{ad}^0\,\rho_x)^*(1)) \cong H^2(\Gamma_F, \mathrm{ad}^0\,\rho_x)^*$, which implies the assertion. □

**Lemma 13.23.** *Assume that $\pi_1(G')$ is étale. If $V^{\text{n-spcl}}_{GG,\overline{\rho}^{\text{ss}}}$ is non-empty then it is flat over* Spec $\mathcal{O}$ *of relative dimension* $\dim G_k([F:\mathbb{Q}_p]+1) - 1$, *which is also equal to the dimension of $\overline{V}^{\text{n-spcl}}_{GG,\overline{\rho}^{\text{ss}}}$.*

*Proof.* Let $x$ be a closed point of $V := V^{\text{n-spcl}}_{GG,\overline{\rho}^{\text{ss}}}$. Since $H^2(\Gamma_F, \mathrm{ad}^0\,\rho_x) = 0$ by Lemma 13.22, Proposition 3.6 implies that

$$R^{\square}_{G,\rho_x} \cong R^{\square}_{G/G',\varphi\circ\rho_x}[\![x_1,\ldots,x_r]\!], \tag{76}$$

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where $\varphi : G \to G/G'$ is the quotient map and $r = \dim G'_k([F : \mathbb{Q}_p] + 1)$. It is proved in [48, Corollary 9.5] that $X^{\mathrm{gen}}_{G/G',\varphi\circ\overline{\rho}^{\mathrm{ss}}}$ is $\mathcal{O}$-flat of relative dimension $\dim(G/G')_k([F : \mathbb{Q}_p]+1)$. It follows from Proposition 5.16 that $R^{\square}_{G/G',\varphi\circ\rho_x}$ is flat over the coefficient ring $\Lambda$ for $\kappa(x)$ of relative dimension $\dim(G/G')_k([F : \mathbb{Q}_p] + 1)$. Thus $R^{\square}_{G,\rho_x}$ is flat over $\Lambda$ of relative dimension $\dim G_k([F : \mathbb{Q}_p] + 1)$.

If the characteristic of $\kappa(x)$ is zero then $\Lambda$ is a finite field extension of $L$ and $R^{\square}_{G,\rho_x} \cong \hat{\mathcal{O}}_{V,x}$ by Lemma 5.17. Thus $\mathcal{O}_{V,x}$ is flat over $\mathcal{O}$ and $\dim \mathcal{O}_{V,x} = \dim \hat{\mathcal{O}}_{V,x} = \dim G_k([F : \mathbb{Q}_p] + 1)$.

If the characteristic of $\kappa(x)$ is $p$ then $\Lambda$ is a DVR, which is flat over $\mathcal{O}$ and $R^{\square}_{G,\rho_x} \cong \hat{\mathcal{O}}_{V,x}[\![T]\!]$ by Lemma 5.17. Thus $\mathcal{O}_{V,x}$ is flat over $\mathcal{O}$ and $\dim \mathcal{O}_{V,x} = \dim \hat{\mathcal{O}}_{V,x} = \dim G_k([F : \mathbb{Q}_p] + 1)$.

Hence, $V$ is flat over $\operatorname{Spec}\mathcal{O}$ and $\dim V = \max_x \dim \mathcal{O}_{V,x} = \dim G_k([F : \mathbb{Q}_p] + 1)$, where the maximum is taken over all the closed points $x$ in $V$. We have to subtract 1 to get the relative dimension over $\mathcal{O}$, which is equal to the dimension of the special fibre. □

**Corollary 13.24.** *Assume that $\pi_1(G')$ is étale. If* (DIM) *holds for all $\overline{\rho}^{\mathrm{ss}}$-compatible pairs $(H_1, \overline{\rho}_1)$ with $\dim H_1 < \dim G_k$ then the following hold:*

(1) $\dim \overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} = \dim G_k([F : \mathbb{Q}_p] + 1)$;

(2) $\dim \overline{X}^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}} = \dim G_k[F : \mathbb{Q}_p] + \dim Z(G_k)$.

*Proof.* The assertion follows from Proposition 13.21 and Lemma 13.23. □

**Proposition 13.25.** *If* (DIM) *holds for all pairs $(H_1, \overline{\rho}_1)$ with $\dim H_1 < \dim G_k$, then* (DIM) *holds for $(G_k, \overline{\rho}^{\mathrm{ss}})$.*

*Proof.* For the purpose of establishing (DIM), after passing to a finite extension of $\mathcal{O}$ we may assume that $G^0$ is split, see Remark 5.9. Let $\mu := Z(G^0) \cap G' = Z(G')$, $G_1 := G/\mu$ and let $\varphi_1 : G \to G_1$ be the projection map. Since $Z(G^0)$ is a split torus, $\mu$ is a finite diagonalisable group scheme and we may decompose $\mu$ canonically into a $p$-part and a prime-to-$p$ part, which are both normal in $G$. Let $x$ be a closed point of $Y_{\overline{\rho}^{\mathrm{ss}}}$. Corollary 3.12 and Proposition 3.16 imply that $\dim \overline{R}^{\square}_{G,\rho_x} = \dim \overline{R}^{\square}_{G_1,\varphi_1\circ\rho_x}$.

As $G'$ is the fppf sheafification of $A \mapsto [G^0(A), G^0(A)]$, the map $G' \to G'_1$ is surjective with kernel $Z(G')$. So $G'_1$ is semisimple of adjoint type and we have $Z(G^0_1) \cap G'_1 = Z(G'_1) = 1$; this can be checked on geometric fibres using that semisimple groups of adjoint type over algebraically closed fields are centreless. We can apply Corollary 3.14 to $Z(G^0_1)$ in $G_1$ and $G_2 := G_1/Z(G^0_1) = G/Z(G^0)$ and obtain

$$\dim \overline{R}^{\square}_{G_1,\varphi_1\circ\rho_x} = \dim \overline{R}^{\square}_{G_2,\varphi\circ\rho_x} + ([F : \mathbb{Q}_p] + 1) \dim Z(G^0)_k, \tag{77}$$

where $\varphi : G \to G/Z(G^0)$ is the projection map.

By Proposition 2.30 we find a surjection of generalised reductive $k$-group schemes $G_3 \to (G_2)_k$ with finite connected kernel, such that $\pi_1(G^0_3)$ is étale. Since $G^0_3$ is semisimple, we have $G'_3 = G^0_3$. Thus Corollary 13.24 can be applied to $G_3$.

Let $\kappa$ be the residue field of $x$. Then $G_2(\kappa) = G_3(\kappa)$ and we can lift $\varphi \circ \rho_x$ to a representation $\rho_3 : \Gamma_F \to G_3(\kappa)$. Using Proposition 3.16, we get $\dim \overline{R}^{\square}_{G_2,\varphi\circ\rho_x} = \dim \overline{R}^{\square}_{G_3,\rho_3}$. Since $\kappa$ is finite $\rho_3$ defines a closed point in $\overline{X}^{\mathrm{gen}}_{G_3,\rho_3^{\mathrm{ss}}}$ with a finite residue field. Using Lemma 13.4 for the inequality, we get

$$\begin{aligned} \dim \overline{R}^{\square}_{G_1,\varphi_1\circ\rho_x} &\overset{(77)}{=} \dim \overline{R}^{\square}_{G_3,\rho_3} + ([F : \mathbb{Q}_p] + 1) \dim Z(G^0)_k \\ &\leq \dim \overline{X}^{\mathrm{gen}}_{G_3,\rho_3^{\mathrm{ss}}} + ([F : \mathbb{Q}_p] + 1) \dim Z(G^0)_k. \end{aligned} \tag{78}$$

Corollary 13.24 applied to $G_3$ gives $\dim \overline{X}^{\mathrm{gen}}_{G_3,\rho_3^{\mathrm{ss}}} = \dim G_3([F : \mathbb{Q}_p] + 1)$. Since $\dim G_k = \dim(G_2)_k + \dim Z(G^0)_k = \dim G_3 + \dim Z(G^0)_k$, we obtain

$$\dim \overline{R}^{\square}_{G,\rho_x} \leq \dim G_k([F : \mathbb{Q}_p] + 1), \quad \forall x \in Y_{\overline{\rho}^{\mathrm{ss}}}. \tag{79}$$

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Lemma 13.4 implies that $\dim \overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} \leq \dim G_k([F:\mathbb{Q}_p]+1)$. It follows from (73) that this is an equality. The assertion follows from Proposition 13.21. □

**Theorem 13.26.** *The hypothesis* (DIM) *holds for all pairs* $(H_1, \overline{\rho}_1)$ *with* $\dim H_1 < \dim G_k$. *In particular, the following hold:*

(1) *the complement of* $\overline{V}^{\text{n-spcl}}_{GG,\overline{\rho}^{\mathrm{ss}}}$ *in* $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ *has positive codimension;*
(2) $\dim \overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} = \dim G_k([F:\mathbb{Q}_p]+1)$;
(3) $\dim \overline{X}^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}} = \dim G_k[F:\mathbb{Q}_p] + \dim Z(G_k)$.

*Proof.* Let $S$ be the set of pairs $(H_1, \overline{\rho}_1)$ such that $\dim H_1 < \dim G_k$, and let $S'$ be the subset of $S$ consisting of those pairs for which (DIM) does not hold. If $S'$ is non-empty then let $(H_1, \overline{\rho}_1) \in S'$ be such that $\dim H_1$ is minimal. Then (DIM) holds for all pairs of dimension less than $\dim H_1$, so Proposition 13.25 applies to $(H_1, \overline{\rho}_1)$, yielding a contradiction. So $S' = \emptyset$ and Proposition 13.25 implies that (DIM) holds for $(G_k, \overline{\rho}^{\mathrm{ss}})$. Moreover, Proposition 13.21 implies that $\dim \overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} \leq \dim G_k([F:\mathbb{Q}_p]+1)$, and (73) implies that part (2) of the Theorem holds. Parts (1) and (3) follow from Proposition 13.21. □

### *13.4. Consequences*

We record some consequences of Theorem 13.26. Most of the results follow the proof of analogous results for $\mathrm{GL}_d$ in [6] with the exception of Corollaries 13.31 and 13.32, where we prove the analogues of the main results of [7] and the argument is new even when $G = \mathrm{GL}_d$.

**Corollary 13.27.** *Let* $x$ *be either a closed point of* $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} \setminus Y_{\overline{\rho}^{\mathrm{ss}}}$ *or a closed point of* $Y_{\overline{\rho}^{\mathrm{ss}}}$ *and let* $\Lambda$ *be a coefficient ring for* $\kappa(x)$ *introduced in Section 3. Then the following hold:*

(1) $R^{\square}_{G,\rho_x}$ *is a flat* $\Lambda$*-algebra of relative dimension* $\dim G_k([F:\mathbb{Q}_p]+1)$ *and is a complete intersection;*
(2) *if* $\mathrm{char}(\kappa(x)) = p$ *then* $R^{\square}_{G,\rho_x}/\varpi$ *is a complete intersection of dimension* $\dim G_k([F:\mathbb{Q}_p]+1)$.

*Proof.* The proof is essentially the same as the proof of [6, Corollary 3.38], so we only give a sketch. Corollary 3.8 and Lemma 3.9 applied with $A = \Lambda$ and $B = R^{\square}_{G,\rho_x}$ imply that to prove part (1) it is enough to show that

$$\dim \kappa(x) \otimes_{\Lambda} R^{\square}_{G,\rho_x} \leq \dim G_k([F:\mathbb{Q}_p]+1). \tag{80}$$

If $\mathrm{char}(\kappa(x)) = 0$ then $\Lambda = \kappa(x)$. If $\mathrm{char}(\kappa(x)) = p$ then $\Lambda$ is a DVR with uniformiser $\varpi$ and the fibre ring in (80) is $R^{\square}_{G,\rho_x}/\varpi$. Thus, if we can prove (80) then Lemma 3.9 applied with $A = \kappa(x)$ and $B = R^{\square}_{\rho_x}/\varpi$ will imply part (2).

Let $X := X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and let $\overline{X} := \overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$. If $x \in Y_{\overline{\rho}^{\mathrm{ss}}}$ then Lemma 13.13 applied to $Z = \overline{X}$ gives us $\dim \mathcal{O}_{\overline{X},x} \leq \dim \overline{X} = \dim G_k([F:\mathbb{Q}_p]+1)$, where the last equality follows from Theorem 13.26 (2). Lemma 5.17 implies that $R^{\square}_{\rho_x}/\varpi \cong \hat{\mathcal{O}}_{\overline{X},x}$ and this implies (80). If $x \in \overline{X} \setminus Y_{\overline{\rho}^{\mathrm{ss}}}$ then $R^{\square}_{\rho_x}/\varpi \cong \hat{\mathcal{O}}_{\overline{X},x}[\![T]\!]$ by Lemma 5.17 and Lemma 13.13 gives $\dim \mathcal{O}_{\overline{X},x} + 1 \leq \dim \overline{X}$. If $\mathrm{char}(\kappa(x)) = 0$ then $x \in X[1/p]$, $R^{\square}_{G,\rho_x} \cong \hat{\mathcal{O}}_{X,x}$ by Lemma 5.17 and $\dim \mathcal{O}_{X,x} \leq \dim X - 1 \leq \dim \overline{X}$ by Lemma 13.13 applied to $Z = X$ and Proposition 13.14. Hence, (80) holds also in this case. □

**Corollary 13.28.** *Let* $d = \dim G_k([F:\mathbb{Q}_p]+1)$. *Then the following hold:*

(1) $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ *is flat over* $\mathrm{Spec}\,\mathcal{O}$, *equidimensional of relative dimension* $d$ *and is locally complete intersection;*
(2) $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ *is equidimensional of dimension* $d$ *and is locally complete intersection.*

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*Proof.* The proof is essentially the same as the proof of [6, Corollary 3.40] so we only give a sketch. Lemma 5.17 relates $R^{\square}_{G,\rho_x}$ and the local ring $\mathcal{O}_{X,x}$, where $X := X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and $x$ is a closed point of $X$. Using Corollary 13.27 we deduce that $\mathcal{O}_{X,x}$ are complete intersection and $\mathcal{O}$-flat for all closed points $x$. Thus $X$ is flat over Spec $\mathcal{O}$ and locally complete intersection. The last property implies that $X$ is equidimensional. Part (1) implies part (2). □

**Corollary 13.29.** *Let $H$ be a closed generalised reductive subgroup of $G$ (resp. $G_k$) such that $X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}}$ (resp. $\overline{X}^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}}$) is non-empty. Let $d = \dim H_k([F:\mathbb{Q}_p]+1)$ then $X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}}$ is $\mathcal{O}$-flat of relative dimension $d$ and is locally complete intersection (resp. $\overline{X}^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}}$ is equidimensional of dimension $d$ and is locally complete intersection).*

*Proof.* This follows from Corollary 13.28 applied to $X^{\mathrm{gen}}_{H_j,\overline{\rho}_j^{\mathrm{ss}}}$ (resp. $\overline{X}^{\mathrm{gen}}_{H_j,\overline{\rho}_j^{\mathrm{ss}}}$) appearing in Corollary 9.4 (1). □

**Corollary 13.30.** *The non-special absolutely irreducible locus is Zariski dense in $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$.*

*Proof.* Theorem 13.26 (1) implies that the complement $\overline{C}$ of $\overline{V}^{\text{n-spcl}}_{GG,\overline{\rho}^{\mathrm{ss}}}$ in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is closed and has positive codimension. Since $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is equidimensional by Corollary 13.28 we conclude that $\overline{V}^{\text{n-spcl}}_{GG,\overline{\rho}^{\mathrm{ss}}}$ contains all the generic points of $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and hence is dense. Let $C$ be the complement of $V^{\text{n-spcl}}_{GG,\overline{\rho}^{\mathrm{ss}}}$ in $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$. It follows from Lemma 13.22 that the reduced special fibre of $C$ is equal to $\overline{C}$. Since $C$ is $G^0$-invariant we have $\dim C \leq \dim \overline{C} + 1$ by Proposition 13.14. Since $\dim X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} = \dim \overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} + 1$ by Corollary 13.28, $C$ has positive codimension in $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and the same argument shows that $V^{\text{n-spcl}}_{GG,\overline{\rho}^{\mathrm{ss}}}$ is dense in $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$. □

**Corollary 13.31.** *Let $d = \dim G_k[F:\mathbb{Q}_p] + \dim Z(G_k)$. Then the following hold:*

(1) *$X^{\mathrm{ps}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is equidimensional of dimension $d+1$;*
(2) *$\overline{X}^{\mathrm{ps}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is equidimensional of dimension $d$.*

*Proof.* Since the claim is about the underlying topological spaces, it is enough to prove the Corollary after replacing $X^{\mathrm{ps}}_{G,\overline{\rho}^{\mathrm{ss}}}$ with $X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and $\overline{X}^{\mathrm{ps}}_{G,\overline{\rho}^{\mathrm{ss}}}$ with $\overline{X}^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$, as these spaces are homeomorphic by Proposition 7.4.

We will prove (1); the proof of (2) is identical. Let $\eta$ be a generic point of $X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$. Let $\bar{\eta}$ be a geometric point above $\eta$. It follows from [57, Theorem 3 (ii)] that there is a geometric point $\xi : \mathrm{Spec}\,\kappa(\bar{\eta}) \to X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ above $\bar{\eta}$. Let $X$ be an irreducible component of $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ containing $\xi$ and let $X^{\mathrm{git}}$ be the image of $X$ in $X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$. Then $X^{\mathrm{git}}$ contains $\eta$. It follows from [6, Lemma 2.1] that $X$ is $G^0$-invariant. Since $X$ is closed in $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$, [57, Theorem 3 (iii)] implies that $X^{\mathrm{git}}$ is closed in $X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$. Since $X$ is irreducible, $X^{\mathrm{git}}$ is also irreducible, [59, Tag 0379]. We conclude that $X^{\mathrm{git}}$ is the irreducible component of $X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$ containing $\eta$.

Corollary 13.30 implies that $V := X \cap V_{GG,\overline{\rho}^{\mathrm{ss}}}$ is non-empty open dense subset of $X$, thus $U := X^{\mathrm{git}} \cap U_{GG,\overline{\rho}^{\mathrm{ss}}}$ is a non-empty open dense subset of $X^{\mathrm{git}}$. Lemmas 13.2 and 13.3 give us $\dim X = \dim V + 1$, $\dim X^{\mathrm{git}} = \dim U + 1$. Corollary 9.10 implies that $\dim V = \dim U + \dim G_k - \dim Z(G_k)$, which together with Corollary 13.28 implies that $\dim X^{\mathrm{git}} = \dim G_k[F:\mathbb{Q}_p] + \dim Z(G_k) + 1$. □

**Corollary 13.32.** *The image of the non-special absolutely irreducible locus is Zariski dense in $X^{\mathrm{ps}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and in $\overline{X}^{\mathrm{ps}}_{G,\overline{\rho}^{\mathrm{ss}}}$.*

*Proof.* As explained in the proof of Corollary 13.31 it is enough to prove the statement for $X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and $\overline{X}^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$. Let $U^{\text{n-spcl}}_{GG,\overline{\rho}^{\mathrm{ss}}}$ be the image of $V^{\text{n-spcl}}_{G,\overline{\rho}^{\mathrm{ss}}}$ in $X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and let $U$ be a non-empty open subset of $X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$. Then its preimage $V$ in $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is open and non-empty. Corollary 13.30 implies that $V \cap V^{\text{n-spcl}}_{GG,\overline{\rho}^{\mathrm{ss}}}$ is

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non-trivial, which implies that $U \cap U^{\text{n-spcl}}_{GG,\overline{\rho}^{\text{ss}}}$ is non-trivial, and hence $U^{\text{n-spcl}}_{GG,\overline{\rho}^{\text{ss}}}$ is dense in $X^{\text{git}}_{G,\overline{\rho}^{\text{ss}}}$. The argument with $\overline{X}^{\text{git}}_{G,\overline{\rho}^{\text{ss}}}$ is the same. □

**Remark 13.33.** Corollaries 13.31 and 13.32 are the analogues of the main results of [7]; compare [7, Theorem 5.5.1]. Even if $G = \mathrm{GL}_d$ the proof given here is new: although we use many techniques introduced by Böckle and Juschka in [7], we never appeal to [7, Section 5].

**Corollary 13.34.** *Let $\rho : \Gamma_F \to G(\kappa)$ be a continuous representation with $\kappa$ a local field. Then the conclusions of Corollary 13.27 apply to $R^{\square}_{G,\rho}$.*

*Proof.* Using Lemma 3.10 it is enough to prove the statement after replacing $\kappa$ by a finite extension. Using Proposition 6.6 we may assume that there exist $g \in G^0(\kappa)$ such that $\rho'(\gamma) := g\rho(\gamma)g^{-1} \in G(\mathcal{O}_\kappa)$ for all $\gamma \in \Gamma_F$. Since $G$ is smooth over $\mathcal{O}$ we may pick $h \in G(\Lambda)$ which maps to $g$ in $G(\kappa)$. Conjugation by $h$ induces an isomorphism of $\Lambda$-algebras between $R^{\square}_{G,\rho}$ and $R^{\square}_{G,\rho'}$, so we may assume that $\rho$ takes values in $G(\mathcal{O}_\kappa)$. Let $\overline{\rho}$ be the reduction of $\rho$ modulo the uniformiser of $\mathcal{O}_\kappa$. Let $\mathcal{O}$ be the Witt ring of residue field of $\mathcal{O}_\kappa$. Since $\rho$ is a deformation of $\overline{\rho}$ to $\mathcal{O}_\kappa$ we obtain a map of local $\mathcal{O}$-algebras $R^{\square}_{G,\overline{\rho}} \to \mathcal{O}_\kappa$. Lemma 3.11 says that $R^{\square}_{G,\overline{\rho}}$ is naturally isomorphic to the completion of $\Lambda \otimes_{\mathcal{O}} R^{\square}_{G,\overline{\rho}}$ with respect to the kernel $\mathfrak{q}$ of the natural map to $\kappa(x)$. Corollary 13.27 together with [6, Lemmas 3.36, 3.37] allow us to bound the dimension of the completion by $\dim \Lambda + \dim G_k([F : \mathbb{Q}_p] + 1)$, and if $\mathrm{char}(\kappa(x)) = p$ then the dimension of the special fibre of the completion by $\dim G_k([F : \mathbb{Q}_p] + 1)$. We obtain $\dim \kappa \otimes_\Lambda R^{\square}_{G,\rho} \leq \dim G_k([F : \mathbb{Q}_p] + 1)$ and the proof of Corollary 13.27 goes through. □

**Corollary 13.35.** *Every continuous representation $\overline{\rho} : \Gamma_F \to G(k)$ can be lifted to characteristic zero. More precisely, there is a finite extension $L'$ of $L$ with a ring of integers $\mathcal{O}'$ and uniformiser $\varpi'$ and a continuous representation $\rho : \Gamma_F \to G(\mathcal{O}')$ such that $\rho \equiv \overline{\rho} \pmod{\varpi'}$.*

*Proof.* It follows from Corollary 13.27 that $R^{\square}_{G,\overline{\rho}}[1/p]$ is non-zero. If $x$ is a closed point of $\mathrm{Spec}\, R^{\square}_{G,\overline{\rho}}[1/p]$ then its residue field is a finite extension of $L$ and specialising the universal deformation along $x$ gives the required lift. □

**Corollary 13.36.** *Let $\overline{\rho} : \Gamma_F \to G(k)$ be a continuous representation, $X^{\square}_{G,\overline{\rho}} := \mathrm{Spec}\, R^{\square}_{G,\overline{\rho}}$ and let $*$ be the closed point in $X^{\square}_{G,\overline{\rho}}$. Let $\Sigma^{\text{n-spcl}}$ be the subset of closed points of $X^{\square}_{G,\overline{\rho}} \setminus \{*\}$ such that $x \in \Sigma^{\text{n-spcl}}$ if and only if $\rho_x$ is absolutely $G$-irreducible and $H^2(\Gamma_F, (\mathrm{Lie}\, G'_{\mathrm{sc}})_x) = 0$. Then $\Sigma^{\text{n-spcl}}$ is Zariski dense in $X^{\square}_{G,\overline{\rho}}$ and $\Sigma^{\text{n-spcl}} \cap \overline{X}^{\square}_{G,\overline{\rho}}$ is Zariski dense in the special fibre $\overline{X}^{\square}_{G,\overline{\rho}}$.*

*Proof.* The proof is essentially the same as the proof of [6, Proposition 3.55]. The map $X^{\square}_{G,\overline{\rho}} \to X^{\text{gen}}_{G,\overline{\rho}^{\text{ss}}}$ is flat, since it is a localisation followed by completion. Let $V$ be a preimage of $V^{\text{n-spcl}}_{GG,\overline{\rho}^{\text{ss}}}$ in $X^{\square}_{G,\overline{\rho}}$. Then $V$ is contained in $X^{\square}_{G,\overline{\rho}} \setminus \{*\}$ and Lemma 3.54 in [6] together with Corollary 13.30 implies that $V$ is Zariski dense in $X^{\square}_{G,\overline{\rho}}$. The set of closed points in $V$ is equal to $\Sigma^{\text{n-spcl}}$. Since $V$ is open $X^{\square}_{G,\overline{\rho}} \setminus \{*\}$, it is Jacobson, and hence $\Sigma^{\text{n-spcl}}$ is dense in $V$ and also in $X^{\square}_{G,\overline{\rho}}$. The same argument works for the special fibre. □

**Remark 13.37.** Corollary 13.36 implies that $\Sigma^{\text{n-spcl}} \cap U$ is dense in every open subscheme $U$ of $X^{\square}_{G,\overline{\rho}}$. In particular, this applies when $U$ is the generic fibre $X^{\square}_{G,\overline{\rho}}[1/p]$.

**Corollary 13.38.** *One may choose a characteristic zero lift $\rho$ of $\overline{\rho}$ in Corollary 13.35 to be absolutely irreducible with $H^2(\Gamma_F, \mathrm{ad}^0\, \rho) = 0$.*

*Proof.* Since $X^{\square}_{G,\overline{\rho}}[1/p]$ is non-empty and open in $X^{\square}_{G,\overline{\rho}}$, and $\Sigma^{\text{n-spcl}}$ is dense by Corollary 13.36, we conclude that $X^{\square}_{G,\overline{\rho}}[1/p] \cap \Sigma^{\text{n-spcl}}$ is non-empty. Lemma 13.22 implies that the specialisation of the universal deformation at $x \in X^{\square}_{G,\overline{\rho}}[1/p] \cap \Sigma^{\text{n-spcl}}$ gives the required lift. □

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## 14. Non-special locus

Let $G$ be a generalised reductive $\mathcal{O}$-group scheme. By Lemma 2.11 we have a $G$-equivariant map $\operatorname{Lie} G'_{\mathrm{sc}} \to \operatorname{Lie} G'$, where $G'_{\mathrm{sc}} \to G'$ is the simply connected central cover of $G'$. This map becomes an isomorphism after inverting $p$.

Let $X^{\mathrm{spcl}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and $\overline{X}^{\mathrm{spcl}}_{G,\overline{\rho}^{\mathrm{ss}}}$ be the $(\operatorname{Lie} G'_{\mathrm{sc}})^*$-special locus in $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$, respectively. Then $\overline{X}^{\mathrm{spcl}}_{G,\overline{\rho}^{\mathrm{ss}}}$ coincides with the reduced special fibre of $X^{\mathrm{spcl}}_{G,\overline{\rho}^{\mathrm{ss}}}$. In this subsection we bound the codimension of $X^{\mathrm{spcl}}_{G,\overline{\rho}^{\mathrm{ss}}}$ in $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ by $1+[F:\mathbb{Q}_p]$ from below. Recall via Definition 13.9 that $x \in X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ lies in $X^{\mathrm{spcl}}_{G,\overline{\rho}^{\mathrm{ss}}}$ if and only if $h^0(\Gamma_F, (\operatorname{Lie} G'_{\mathrm{sc}})^*_x(1)) \neq 0$. If $x$ is a closed point of $Y_{\overline{\rho}^{\mathrm{ss}}}$ or a closed point of $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} \setminus Y_{\overline{\rho}^{\mathrm{ss}}}$ then $\kappa(x)$ is either a finite or a local field and local Tate duality implies that $x$ is $(\operatorname{Lie} G'_{\mathrm{sc}})^*$-non-special if and only if $H^2(\Gamma_F, (\operatorname{Lie} G'_{\mathrm{sc}})_x) = 0$.

If $\pi_1(G')$ is étale or $x \in X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$ then $(\operatorname{Lie} G'_{\mathrm{sc}})_x = (\operatorname{Lie} G')_x$ and the above is equivalent to vanishing of $H^2(\Gamma_F, \mathrm{ad}^0\,\rho_x)$. Proposition 3.6 implies that in this case $R^{\square}_{G,\rho_x}$ is formally smooth over $R^{\square}_{G/G',\varphi\circ\rho_x}$, where $\varphi : G \to G/G'$ is the quotient map. In the next section we will use this to compute the set of irreducible components of $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$. It will be important for us that $1+[F:\mathbb{Q}_p] \geq 2$ as this will be used to verify Serre's criterion for normality.

If $G^0$ is a torus then $G'$ is trivial and there is nothing to verify. In particular, we assume that $\dim G_k > 1$ in this section.

Let $H$ be a closed generalised reductive subgroup of $G$ such that $X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}}$ is non-empty. Then $H^0$ acts on $G^0 \times X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}}$ via $h\cdot(g,x) = (gh^{-1}, h\cdot x)$. We let $G^0 \times^{H^0} X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}}$ be the GIT quotient for this action. Since the action is free the fibre at a geometric point $x$ is isomorphic to $H^0_{\kappa(x)}$. This implies that

$$\dim(G^0 \times^{H^0} X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}}) = \dim G^0 + \dim X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}} - \dim H^0. \tag{81}$$

It follows from Corollaries 13.28, 13.29 that if $X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}}$ is non-empty then

$$\dim X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} - \dim X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}} = (\dim G - \dim H)([F:\mathbb{Q}_p]+1). \tag{82}$$

A similar calculation can be made over $\operatorname{Spec} k$.

**Definition 14.1.** If $H$ is a closed generalised reductive subgroup of $G$ (resp. $G_k$) we denote by $G^0\cdot X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}}$ (resp. $G^0\cdot \overline{X}^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}}$) the scheme theoretic image of $G^0 \times^{H^0} X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}}$ (resp. $G^0 \times^{H^0} \overline{X}^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}}$) in $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$.

The following Lemma formalises an argument made in [6, Proposition 3.8].

**Lemma 14.2.** *Let $f : X \to Y$ be a morphism of affine schemes of finite type over $S = \operatorname{Spec} R$, where $R$ is a complete local noetherian $\mathcal{O}$-algebra with residue field $k$. Let $s \in S$ be the closed point and let $X_s$ and $Y_s$ denote the fibres of $X$ and $Y$ at $s$ respectively. Suppose that the following hold:*

(1) *$f$ is dominant;*
(2) *every irreducible component of $X$ intersects $X_s$ non-trivially;*

*Then* $\dim Y \leq \dim X$.

*Proof.* Let $U = X \setminus X_s$ and let $V = Y \setminus Y_s$. Since $Y_s$ is closed in $Y$ and $f(X_s) \subseteq Y_s$, the closure of $f(X_s)$ cannot contain the generic points of $V$. Since $f$ is dominant its restriction $f : U \to V$ is still dominant. Now both $U$ and $V$ are Jacobson by [6, Lemma 3.18 (1)] and [6, Lemma 3.14] implies that $\dim V \leq \dim U$. Let $Z$ be the closure of $U$ in $X$ and let $W$ be the closure of $V$ in $Y$. The assumption (2) and [6, Lemma 3.18 (5)] imply that $\dim Z = \dim U + 1$. The proof of [6, Lemma 3.18 (5)], namely the inequality (14) in [6], implies that $\dim W \leq \dim V + 1$. Thus $\dim W \leq \dim Z$.

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Let $W'$ be the closure of $f(X_s)$ in $Y_s$. Since both $X_s$ and $Y_s$ are of finite type over $\operatorname{Spec} k$ they are Jacobson, and hence $W'$ is Jacobson. Lemma 3.14 in [6] implies that $\dim W' \leq \dim X_s$. Since $Y_s$ is closed in $Y$, $W'$ is also closed in $Y$.

Since $f$ is dominant, we have $Y = W \cup W'$. Thus $\dim Y = \max(\dim W, \dim W') \leq \max(\dim Z, \dim X_s) = \dim X$. □

**Remark 14.3.** We note that $\mathbb{Q}_p = \mathbb{Z}_p[x]/(px-1)$ is of finite type over $\mathbb{Z}_p$ and $\operatorname{Spec} \mathbb{Q}_p \to \operatorname{Spec} \mathbb{Z}_p$ is dominant, and so that the assumption (2) in Lemma 14.2 is necessary.

**Proposition 14.4.** *If $H$ is a closed generalised reductive subgroup of $G$ such that $X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}}$ is non-empty then*

$$\dim X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} - \dim G^0 \cdot X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}} \geq (\dim G - \dim H)[F:\mathbb{Q}_p]. \tag{83}$$

*If $H$ is a closed generalised reductive subgroup of $G_k$ such that $\overline{X}^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}}$ is non-empty then*

$$\dim \overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} - \dim G^0 \cdot \overline{X}^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}} \geq (\dim G_k - \dim H)[F:\mathbb{Q}_p]. \tag{84}$$

*Proof.* Let $H$ be a closed generalised reductive subgroup of $G$. We claim that the assumptions of Lemma 14.2 are satisfied with $X := G^0 \times^{H^0} X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}}$ and $Y := G^0 \cdot X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}}$. Since $\dim G = \dim G^0$ and $\dim H = \dim H^0$, the claim together with Lemma 14.2, (81) and (82) imply (83).

To prove the claim we observe that the map $X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}} \to G^0 \times X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}}$, $x \mapsto (1,x)$ induces an isomorphism $X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}} /\!\!/ H^0 \xrightarrow{\cong} (G^0 \times^{H^0} X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}}) /\!\!/ G^0$ with the inverse map induced by $(g,x) \mapsto x$. Now $X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}} /\!\!/ H^0 = X^{\mathrm{git}}_{H,\overline{\rho}^{\mathrm{ss}}}$ is a spectrum of a finite product of local noetherian $\mathcal{O}$-algebras by (31). Since $G^0$ is connected every irreducible component $X'$ of $X$ is $G^0$-invariant by [6, Lemma 2.1] and hence its image in $X /\!\!/ G^0$ is closed by Corollary 2 (ii) to [57, Proposition 9]. Thus the image of $X'$ in $X /\!\!/ G^0 = X^{\mathrm{git}}_{H,\overline{\rho}^{\mathrm{ss}}}$ will contain a closed point. Since $X^{\mathrm{git}}_{H,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is finite, it sends closed points to the closed point of $X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$. This implies that $X$ satisfies the assumption (2) of Lemma 14.2. The assumption (1) holds as $Y$ is by definition the closure of $f(X)$. The argument in the special fibre is the same. □

**Proposition 14.5.** *Let $L$ be a $\overline{\rho}^{\mathrm{ss}}$-compatible R-Levi of $G$ such that $\dim G - \dim L = 2$. There is a closed subscheme $\overline{D}_L$ of $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ such that the following hold:*

(1) $\dim \overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} - \dim \overline{D}_L \geq 1 + [F:\mathbb{Q}_p]$;

(2) *if $x \in \overline{V}_{LG}$ and $x \notin \overline{D}_L$ then $h^0(\Gamma_F, (\operatorname{Lie} G'_{\mathrm{sc}})^*_x(1)) = 0$.*

*Proof.* We want to define $\overline{D}_L$, so that for every geometric point $x \in \overline{V}_{LG,\overline{\rho}^{\mathrm{ss}}}$ such that $x \notin \overline{D}_L$ the assumptions of Proposition 11.5 hold for $\rho_x : \Gamma_F \to G(\kappa(x))$, since then Proposition 11.5 implies that part (2) holds.

If $x \in \overline{V}_{LG,\overline{\rho}^{\mathrm{ss}}}$ then $\rho_x(\Gamma_F)$ is contained in $P(\kappa(x))$ for some R-parabolic subgroup $P$ of $G_{\kappa(x)}$ such that some R-Levi of $P$ is conjugate to $L_{\kappa(x)}$. Thus if $x \notin G^0 \cdot \overline{X}^{\mathrm{gen}}_{L,\overline{\rho}^{\mathrm{ss}}}$ then $\rho_x(\Gamma_F)$ is not contained in any R-Levi subgroup of $P$, and so part (1) of Proposition 11.5 holds for $\rho_x$. Moreover, Proposition 14.4 implies that

$$\dim \overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} - \dim G^0 \cdot \overline{X}^{\mathrm{gen}}_{L,\overline{\rho}^{\mathrm{ss}}} \geq 2[F:\mathbb{Q}_p]. \tag{85}$$

Let $\varphi : G_k \to \operatorname{Aut}(G^0_k)$ be the map induced by the conjugation action of $G_k$ on $G^0_k$. Let $\overline{G} = G_k/\operatorname{Ker}\varphi$ and $\overline{L} = L_k/\ker\varphi$. It follows from [19, Theorem 7.1.9] and Proposition 11.1 that $\overline{G}_{\overline{k}} \cong G_1 \times \mathrm{PGL}_2$ and $\overline{L}_{\overline{k}} \cong G_1 \times T_2$, where $G_1$ is a generalised reductive group and $T_2$ is the subgroup of diagonal matrices in $\mathrm{PGL}_2$. The isomorphisms are defined over some finite extension of $k$, and after

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extending scalars we may assume that it is defined over $k$. Let $p_2 : G \to \mathrm{PGL}_2$ be the quotient map composed with the projection onto the second factor.

Let $H_0$ be the preimage of $\mu_{p-1} \subset T_2$ in $G$ under $p_2$. It follows from Lemma 11.3 applied to the base change of $H_0$ over $\overline{k}$ that $H_0$ is reductive and $\dim Z(H_0) < \dim Z(L_k)$. Since the inequality is strict we can improve the codimension bound in Corollary 13.6 by 1:

$$\dim \overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} - \dim \overline{Z}_{H_0G,\overline{\rho}^{\mathrm{ss}}} \geq 1 + (\dim L_k - \dim H_0)[F : \mathbb{Q}_p] = 1 + [F : \mathbb{Q}_p], \tag{86}$$

where $\overline{Z}_{H_0G,\overline{\rho}^{\mathrm{ss}}}$ is the closure of the preimage of $\overline{X}^{\mathrm{git}}_{H_0G} \cap U_{LG,\overline{\rho}^{\mathrm{ss}}}$ in $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$, and the extra 1 comes as a difference between the dimensions of the centres in (55) and (56).

If $x$ is a geometric point of $\overline{V}_{LG,\overline{\rho}^{\mathrm{ss}}}$ then the $G$-semisimplification of $\rho_x$ is contained in $H_0$ if and only if $x \in \overline{Z}_{H_0G,\overline{\rho}^{\mathrm{ss}}}$. Thus if $x \notin \overline{Z}_{H_0G,\overline{\rho}^{\mathrm{ss}}}$ then part (1) of Proposition 11.5 holds for $\rho_x$.

If $G_1^0$ is a torus then $G_1'$ is trivial and part (3) of Proposition 11.5 holds for $\rho_x$. Proposition 11.5 implies that if $x \notin \overline{D}_L := G^0 \cdot \overline{X}^{\mathrm{gen}}_{L,\overline{\rho}^{\mathrm{ss}}} \cup \overline{Z}_{H_0G,\overline{\rho}^{\mathrm{ss}}}$ then part (2) holds, and (85) and (86) imply that part (1) holds.

If $G_1^0$ is not a torus then we have to further cut out some pieces to make sure that $h^0(\Gamma_F, (\mathrm{Lie}\, G_1')^*_x(1)) = 0$, so that part (3) of Proposition 11.5 is satisfied. Let $p_1 : G \to G_1$ be the composition of the quotient map with the projection onto the first component. Let $\overline{\rho}_1 = p_1 \circ \rho : \Gamma_F \to G_1(k)$. As in the proof of Proposition 13.18 let $J_1, \dots, J_r$ be reductive subgroups of $G_1$, such that $\dim G_1 - \dim J_i \geq 2$ and the image of the special $G_1$-absolutely irreducible locus in $\overline{X}^{\mathrm{git}}_{G_1,\overline{\rho}_1^{\mathrm{ss}}}$ is contained in the union of $\overline{X}^{\mathrm{git}}_{J_iG_1,\overline{\rho}_1^{\mathrm{ss}}}$. Let $H_i$ be the preimage of $J_i \times T_2$ in $L_k$ for $1 \leq i \leq r$. Then $\dim L_k - \dim H_i = \dim G_1 - \dim J_i \geq 2$. It follows from Corollary 13.6 that

$$\dim \overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} - \dim \overline{Z}_{H_iG,\overline{\rho}^{\mathrm{ss}}} \geq (\dim L_k - \dim H_i)[F : \mathbb{Q}_p] \geq 2[F : \mathbb{Q}_p]. \tag{87}$$

If $x \in \overline{V}_{LG,\overline{\rho}^{\mathrm{ss}}}$ then the action of $\Gamma_F$ on $(\mathrm{Lie}(G_1')_{\mathrm{sc}})^*_x$ factors through $p_1 \circ \rho_x : \Gamma_F \to G_1(\kappa(x))$. If $p_1 \circ \rho_x$ is not $G_1$-irreducible, then its image would be contained in a proper R-parabolic of $G_1$, and then the image of $\rho_x$ would be contained in a proper R-parabolic of $L_k$, which would contradict $x \in \overline{V}_{LG}$. Thus $p_1 \circ \rho_x$ is $G_1$-irreducible and $h^0(\Gamma_F, (\mathrm{Lie}(G_1')_{\mathrm{sc}})^*_x(1)) = 0$ if and only if $p_1 \circ \rho_x$ corresponds to a non-special point in $\overline{X}^{\mathrm{gen}}_{G_1,\overline{\rho}_1^{\mathrm{ss}}}$. The map $p_1 : L_k \to G_1$ induces a map $\overline{V}_{LG,\overline{\rho}^{\mathrm{ss}}} \to \overline{U}_{LG,\overline{\rho}^{\mathrm{ss}}} \to \overline{X}^{\mathrm{git}}_{G_1,\overline{\rho}_1^{\mathrm{ss}}}$ and the preimage of $\overline{X}^{\mathrm{git}}_{J_iG_1,\overline{\rho}_1^{\mathrm{ss}}}$ under this map is equal to $\overline{V}_{LG,\overline{\rho}^{\mathrm{ss}}} \cap \overline{Z}_{H_iG,\overline{\rho}^{\mathrm{ss}}}$. We conclude that if $x \notin \overline{Z}_{H_iG,\overline{\rho}^{\mathrm{ss}}}$ for $1 \leq i \leq r$ then $h^0(\Gamma_F, (\mathrm{Lie}(G_1')_{\mathrm{sc}})^*_x(1)) = 0$. Thus part (3) of Proposition 11.5 holds for $\rho_x$. Hence, if $x \notin \overline{D}_L := G^0 \cdot \overline{X}^{\mathrm{gen}}_{L,\overline{\rho}^{\mathrm{ss}}} \cup \bigcup_{i=0}^{r} \overline{Z}_{H_iG,\overline{\rho}^{\mathrm{ss}}}$ then Proposition 11.5 implies that part (2) holds. Moreover, (85), (86) and (87) imply that part (1) holds. □

**Theorem 14.6.** *The following hold:*

(1) $X^{\mathrm{spcl}}_{G,\overline{\rho}^{\mathrm{ss}}}$ *has codimension at least* $1 + [F : \mathbb{Q}_p]$ *in* $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$;
(2) $\overline{X}^{\mathrm{spcl}}_{G,\overline{\rho}^{\mathrm{ss}}}$ *has codimension at least* $1 + [F : \mathbb{Q}_p]$ *in* $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$;
(3) $X^{\mathrm{spcl}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$ *has codimension at least* $1 + [F : \mathbb{Q}_p]$ *in* $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$.

*Moreover, if there is no* $\overline{\rho}^{\mathrm{ss}}$*-compatible R-Levi subgroup* $L$ *of* $G$ *such that* $\dim G - \dim L = 2$ *then the bounds can be improved to* $2[F : \mathbb{Q}_p]$.

*Proof.* Since $\dim X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} = \dim \overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} + 1$ by Corollary 13.28, Corollary 13.15 applied with $W = (\mathrm{Lie}\, G'_{\mathrm{sc}})^*$ implies that it is enough to prove the assertion about the special fibre.

We claim that $\overline{X}^{\mathrm{spcl}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is contained in the union of $Y_{\overline{\rho}^{\mathrm{ss}}}$, $\overline{Z}^{\mathrm{spcl}}_{GG,\overline{\rho}^{\mathrm{ss}}}$, $\overline{Z}_{LG,\overline{\rho}^{\mathrm{ss}}}$, for $\overline{\rho}^{\mathrm{ss}}$-compatible R-Levi subgroups $L$ of $G$ containing a fixed maximal split torus $T$ and satisfying $\dim G - \dim L > 2$, and $\overline{D}_L$ for $\overline{\rho}^{\mathrm{ss}}$-compatible R-Levi subgroups $L$ of $G$ containing $T$ and satisfying $\dim G - \dim L = 2$. Since this is

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a finite union and each piece has codimension of at least $1 + [F : \mathbb{Q}_p]$ by Corollary 12.18, Propositions 13.18, 13.17, 14.5 we obtain the assertion. Moreover, if there are no $\overline{\rho}^{\mathrm{ss}}$-compatible R-Levi subgroups $L$ with $\dim G - \dim L = 2$ then the same references imply that each piece has codimension at least $2[F : \mathbb{Q}_p]$.

We will now prove the claim. Let $x \in \overline{X}^{\mathrm{spcl}}_{G,\overline{\rho}^{\mathrm{ss}}}$ be a geometric point. If $x$ is not contained in $Y_{\overline{\rho}^{\mathrm{ss}}}$ then it is contained in $\overline{V}_{LG,\overline{\rho}^{\mathrm{ss}}}$ for some $\overline{\rho}^{\mathrm{ss}}$-compatible R-Levi $L$ of $G$ containing $T$. If $L = G$ then $x \in \overline{Z}^{\mathrm{spcl}}_{GG,\overline{\rho}^{\mathrm{ss}}}$ by definition of the absolutely irreducible special locus. If $L \neq G$ then $\dim G - \dim L \geq 2$ by part (3) of Lemma 9.11. If $\dim G - \dim L > 2$ then $x \in \overline{Z}_{LG,\overline{\rho}^{\mathrm{ss}}}$, as this locus contains $\overline{V}_{LG,\overline{\rho}^{\mathrm{ss}}}$. If $\dim G - \dim L = 2$ then Proposition 14.5 implies that $x \in \overline{D}_L$. □

**Example 14.7.** Let $G = \mathrm{GL}_1 \times \mathrm{GL}_2$ and $\overline{\rho} = \overline{\rho}_1 \times \overline{\rho}_2$. If $\overline{\rho}_2 : \Gamma_F \to \mathrm{GL}_2(k)$ is absolutely irreducible then the only $\overline{\rho}^{\mathrm{ss}}$-compatible Levi of $G$ is $G$ itself and so Theorem 14.6 implies that the codimension of $(\mathrm{Lie}\, G'_{\mathrm{sc}})^*$-special locus is at least $2[F : \mathbb{Q}_p]$. On the other hand if $\overline{\rho}_2$ is reducible then the subgroup of diagonal matrices in $G$ is also $\overline{\rho}^{\mathrm{ss}}$-compatible and the bound given by the theorem is $1 + [F : \mathbb{Q}_p]$.

## 15. Irreducible components and normality

We assume that $G^0$ is split over $\mathcal{O}$ and the map $\pi_G \circ \overline{\rho}^{\mathrm{ss}} : \Gamma_F \to (G/G^0)(\overline{k})$ is surjective. It follows from Remark 5.9 and Lemma 9.2 that we can always achieve this after shrinking the group of connected components of $G$ and after replacing $L$ by a finite unramified extension. Let $G'$ be the derived subgroup scheme of $G^0$ and let $\varphi : G \to G/G'$ be the quotient map.

Let $\Gamma_E$ be the kernel of $\pi_G \circ \overline{\rho}^{\mathrm{ss}}$. Our assumptions on $G$ imply that $G^0/G'$ is a torus over $\mathcal{O}$. We thus may write $G^0/G' = \mathrm{Spec}\, \mathcal{O}[M]$, where $M$ is the character lattice of $G^0/G'$. Let $\Delta := (G/G^0)(\overline{k})$. Our assumptions on $\pi_G \circ \overline{\rho}^{\mathrm{ss}}$ imply that

$$\Delta = (G/G^0)(k) = \mathrm{Gal}(E/F).$$

The action of $G$ by conjugation on $\mathrm{Hom}_{\mathcal{O}\text{-GrpSch}}(G^0/G', \mathbb{G}_m) = M$ induces an action of $\Delta$ on $M$. We assume that we are given a $\Delta$-invariant decomposition

$$M = M_1 \oplus M_2. \tag{88}$$

The projections $M \to M_i$ identify $K_i = \mathrm{Spec}\, \mathcal{O}[M_i]$ with a normal subgroup scheme of $G/G'$. We let

$$H_1 = (G/G')/K_2, \quad H_2 := (G/G')/K_1.$$

We fix $\overline{\rho} : \Gamma_F \to G(k)$ corresponding to a point in $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}(k)$. Let $\varphi_i : G \to H_i$ denote the quotient map and let $\overline{\psi}_i = \varphi_i \circ \overline{\rho}$ and $\overline{\psi} = \varphi \circ \overline{\rho}$. We fix a continuous representation $\psi_1 : \Gamma_F \to H_1(\mathcal{O})$ lifting $\overline{\psi}_1 : \Gamma_F \to H_1(k)$. The lift $\psi_1$ induces a morphism $\mathrm{Spec}\, \mathcal{O} \to X^{\mathrm{gen}}_{H_1,\overline{\psi}_1^{\mathrm{ss}}}$. Whenever $X = \mathrm{Spec}\, A$ is a scheme over $X^{\mathrm{gen}}_{H_1,\overline{\psi}_1^{\mathrm{ss}}}$, we denote by $X^{\psi_1}$ the fibre product along this morphism, and $A^{\psi_1}$ the ring of global sections of $X^{\psi_1}$. By functoriality proved in Proposition 8.5 the quotient morphism $\varphi_1 : G \to H_1$ induces a morphism

$$X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{gen}}_{H_1,\overline{\psi}_1^{\mathrm{ss}}}. \tag{89}$$

We obtain closed subschemes $X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}}$ of $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and $\overline{X}^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}}$ of $\overline{X}^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$, respectively.

In this section we study the irreducible components of $X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}}$ and the corresponding deformation rings and show that they are normal in many cases. The main results are Theorem 15.18 and its

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Corollaries 15.20, 15.26 and 15.30, where we prove an analogue of a conjecture of Böckle–Juschka posed originally for $G = \mathrm{GL}_d$.

Before delving into the details let us point out two interesting cases: if $M_1 = 0$ then $X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}} = X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$, if $M_1 = M$ then $X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}}$ parameterises representations with 'a fixed determinant'. If $G = \mathrm{GL}_d$ then $G/G' = \mathbb{G}_m$, $M = \mathbb{Z}$ and these are the only two cases that can occur. A further interesting case occurs, when $G$ is a $C$-group; we discuss it in Theorem 16.5.

Let $\kappa$ be either a finite or a local field, which is an $\mathcal{O}$-algebra, and let $\Lambda$ be a coefficient ring for $\kappa$ in the sense of section 5.2. Let $\rho : \Gamma_F \to G(\kappa)$ be a continuous representation. Let $D^{\square,\psi_1}_{G,\rho} : \mathfrak{A}_\Lambda \to \mathrm{Set}$ be the functor

$$D^{\square,\psi_1}_{G,\rho}(A) := \{\rho_A \in D^{\square}_{G,\rho}(A) : \varphi_1 \circ \rho_A = \psi_1 \otimes_{\mathcal{O}} A\}.$$

It is easy to see that $D^{\square,\psi_1}_{G,\rho}$ is pro-represented by a quotient of $R^{\square}_{G,\rho}$, which we denote by $R^{\square,\psi_1}_{G,\rho}$.

Let $N$ be the character lattice of $Z(G^0)^0$. The map $Z(G^0) \to G/G'$ is a central isogeny by [19, Corollary 5.3.3]. We thus obtain a homomorphism $M \to N$ of $\Delta$-modules, which induces an isomorphism $M \otimes \mathbb{Q} \cong N \otimes \mathbb{Q}$. Let $N_1$ be the image of $N$ in $(N \otimes \mathbb{Q})/(M_2 \otimes \mathbb{Q})$ and let $N_2$ be the image of $N$ in $(N \otimes \mathbb{Q})/(M_1 \otimes \mathbb{Q})$. Then $Z_i := \mathrm{Spec}\, \mathcal{O}[N_i]$ is a $\Delta$-invariant subtorus of $Z(G^0)$, and hence a normal subgroup of $G$. Moreover, the map $Z_i \to H_i$ is an isogeny as $M_i \otimes \mathbb{Q} \cong N_i \otimes \mathbb{Q}$ by construction.

### *15.1. Dimension*

In this section we bound the dimension of $\overline{X}^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}}$. We obtain the bound from the results of the previous section applied to the group $\overline{G} := G/Z_1$ via the map $X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{gen}}_{\overline{G},q\circ\overline{\rho}^{\mathrm{ss}}}$, where the second arrow is induced by the quotient map $q : G \to \overline{G}$ and functoriality proved in Proposition 8.5. For example, if $G = \mathrm{GL}_d$ then we deduce the bound in the fixed determinant case from our results on $\mathrm{PGL}_d$. This differs from the argument in [6], where the authors twist the universal deformation by characters to unfix the determinant and get back to the $\mathrm{GL}_d$-case. To ease the notation we will drop $q$ and write $X^{\mathrm{gen}}_{\overline{G},\overline{\rho}^{\mathrm{ss}}}$ instead of $X^{\mathrm{gen}}_{\overline{G},q\circ\overline{\rho}^{\mathrm{ss}}}$.

**Proposition 15.1.** *The morphism $X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{gen}}_{\overline{G},\overline{\rho}^{\mathrm{ss}}}$ is finite.*

*Proof.* Let $H := H_1 \times \overline{G}$. Then the natural map $q : G \to H$ is a central isogeny as $Z_1 \to H_1$ is an isogeny by construction. Thus the induced map $f : X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{gen}}_{H,q\circ\overline{\rho}^{\mathrm{ss}}}$ is finite by Proposition 8.7. The quotient maps $G \to H_1$, $H \to H_1$ induce maps $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{gen}}_{H_1,\overline{\psi}_1^{\mathrm{ss}}}$ and $X^{\mathrm{gen}}_{H,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{gen}}_{H_1,\overline{\psi}_1^{\mathrm{ss}}}$ which make $f$ into a morphism of $X^{\mathrm{gen}}_{H_1,\overline{\psi}_1^{\mathrm{ss}}}$-schemes. We thus obtain a finite map $X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{gen},\psi_1}_{H,q\circ\overline{\rho}^{\mathrm{ss}}}$. It follows from Proposition 8.10 that the projection onto the second factor $H \to \overline{G}$ induces an isomorphism $X^{\mathrm{gen},\psi_1}_{H,q\circ\overline{\rho}^{\mathrm{ss}}} \cong X^{\mathrm{gen}}_{\overline{G},\overline{\rho}^{\mathrm{ss}}}$. □

**Corollary 15.2.** *Let $Z$ be a closed subscheme of $X^{\mathrm{gen}}_{\overline{G},\overline{\rho}^{\mathrm{ss}}}$ and let $Z'$ be its preimage in $X^{\mathrm{gen},\psi}_{G,\overline{\rho}^{\mathrm{ss}}}$. Then* $\dim Z' \leq \dim Z$. *In particular,*

$$\dim \overline{X}^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}} \leq \dim \overline{G}_k([F : \mathbb{Q}_p] + 1).$$

*Proof.* Proposition 15.1 implies that $Z' \to Z$ is finite, which implies the assertion about the dimension. The last assertion follows from Corollary 13.28. □

Since $\dim(H_1)_k + \dim(H_2)_k = \dim(Z_1)_k + \dim(Z_2)_k = \dim Z(G^0)_k$ we obtain

$$\dim \overline{G}_k = \dim G_k - \dim(H_1)_k = \dim(H_2)_k + \dim G'_k. \tag{90}$$

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**Corollary 15.3.** *The morphism* $X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}} /\!\!/ Z_2G' \to X^{\mathrm{git}}_{\overline{G},\overline{\rho}^{\mathrm{ss}}}$ *is finite.*

*Proof.* Proposition 15.1 together with [57, Theorem 2 (ii)] imply that

$$X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}} /\!\!/ Z_2G' \to X^{\mathrm{gen}}_{\overline{G},\overline{\rho}^{\mathrm{ss}}} /\!\!/ Z_2G'$$

is finite. Since the action of $G'$ on $X^{\mathrm{gen}}_{\overline{G},\overline{\rho}^{\mathrm{ss}}}$ factors through the surjective morphism $Z_2G' \to \overline{G}^0$ we obtain $X^{\mathrm{gen}}_{\overline{G},\overline{\rho}^{\mathrm{ss}}} /\!\!/ Z_2G' = X^{\mathrm{git}}_{\overline{G},\overline{\rho}^{\mathrm{ss}}}$. □

**Corollary 15.4.** *Let R be the ring of global sections of* $X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}} /\!\!/ Z_2G'$*, then R is a finite product of local noetherian* $\mathcal{O}$*-algebras with residue fields finite extensions of k. Moreover, if the orders of* $(G/G^0)(\overline{k})$ *and* $(Z_1 \cap Z_2G')(\overline{k})$ *are coprime then R is a complete local noetherian* $\mathcal{O}$*-algebra with residue field k.*

*Proof.* Since $R^{\mathrm{git}}_{\overline{G},\overline{\rho}^{\mathrm{ss}}}$ is a complete noetherian $\mathcal{O}$-algebra with residue field $k$, and $R^{\mathrm{git}}_{\overline{G},\overline{\rho}^{\mathrm{ss}}} \to R$ is a finite map by Corollary 15.3, $R$ is a finite product of complete local noetherian $\mathcal{O}$-algebras with residue field a finite extension of $k$. Moreover, $R$ is a local noetherian $\mathcal{O}$-algebra with residue field $k$ if and only if $(X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}} /\!\!/ Z_2G')(\overline{k})$ consists of one point.

Let $x \in X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}}(\overline{k})$ be such that the orbit $Z_2G' \cdot x$ is closed. We claim that the orbit $G^0 \cdot x$ is also closed in $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$. Let $P$ be a minimal R-parabolic of $G_{\overline{k}}$ containing the image of $\rho_x$, and let $U$ be its unipotent radical. Then $P \cap G^0 = P^0$ is a parabolic of $G^0$ by Lemma 2.13 and thus $P \cap Z_2G'$ is a parabolic of $Z_2G'$ with unipotent radical $U$. We may find a cocharacter $\lambda : \mathbb{G}_m \to Z_2G' \cap P$ such that $\lim_{t\to 0} \lambda(t)U\lambda(t)^{-1} = 1$. Let $x' := \lim_{t\to 0} \lambda(t) \cdot x$. Then $\rho_{x'} \equiv \rho_x \pmod U$ and the minimality of $P$ implies that $\rho_{x'}$ is the $G$-semisimplification of $\rho_x$. Thus $G^0 \cdot x'$ is the unique closed $G^0$-orbit contained in the closure of $G^0 \cdot x$ by Proposition 6.7. On the other hand, $x'$ lies in the closure of $Z_2G' \cdot x$. Since $Z_2G' \cdot x$ is closed by assumption we deduce that $x \in Z_2G' \cdot x'$ and hence $G^0 \cdot x = G^0 \cdot x'$ is closed.

Let $y, y' \in (X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}} /\!\!/ Z_2G')(\overline{k})$ and let $Z_2G' \cdot x$, $Z_2G' \cdot x'$ be the corresponding closed orbits in $X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}}(\overline{k})$. As discussed above $G^0 \cdot x$ and $G^0 \cdot x'$ are closed in $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}(\overline{k})$. Since $R^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is a local ring with residue field $k$ there is a unique such closed $G^0$-orbit. Thus there is $g \in G^0(\overline{k})$ such that

$$g\rho_x(\gamma)g^{-1} = \rho_{x'}(\gamma), \quad \forall \gamma \in \Gamma_F. \tag{91}$$

We may write $g = z_1z_2h$, where $z_1 \in Z_1(\overline{k})$, $z_2 \in Z_2(\overline{k})$ and $h \in G'(\overline{k})$. After replacing $x$ by a translate with $z_2h$ we may assume that $g \in Z_1(\overline{k})$. By applying $\varphi_1$ to (91) we get

$$\varphi_1(g)\psi_1(\gamma)\varphi_1(g)^{-1}\psi_1(\gamma)^{-1} = 1, \quad \forall \gamma \in \Gamma_F \tag{92}$$

in $H_1(\overline{k})$. Since by assumption the map $\pi_G \circ \overline{\rho} : \Gamma_F \to \Delta$ is surjective, for every $\delta \in \Delta$ there is $\gamma \in \Gamma_F$ such that $\delta = \psi_1(\gamma)$. We deduce from (92) that $\varphi_1(g) \in H_1^{\Delta}(\overline{k})$. Since $g \in G^0(\overline{k})$ we have $\varphi_1(g) \in (H_1^0)^{\Delta}(\overline{k})$.

The isogeny $Z_1 \times Z_2 \to G^0/G'$ induces an isomorphism $\varphi_1 : G^0/Z_2G' \cong H_1^0$ and an exact sequence of diagonalisable groups

$$0 \to Z_1 \cap Z_2G' \to Z_1 \to H_1^0 \to 0.$$

This induces an exact sequence on $\overline{k}$-points, and after taking $\Delta$-invariants we obtain an exact sequence of abelian groups:

$$0 \to (Z_1 \cap Z_2G')(\overline{k})^{\Delta} \to Z_1^{\Delta}(\overline{k}) \to (H_1^0)^{\Delta}(\overline{k}) \to H^1(\Delta, (Z_1 \cap Z_2G')(\overline{k})).$$

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If the orders of $\Delta$ and $(Z_1 \cap Z_2G')(\overline{k})$ are coprime then the $H^1$-term vanishes and there is $g' \in Z_1^{\Delta}(\overline{k})$ which maps to $g \in (H_1^0)^{\Delta}(\overline{k})$. Since $Z_1^{\Delta}$ is contained in $Z(G)$, $g'$ acts trivially on $x$, and hence $x$ and $x'$ lie in the same $Z_2G'$-orbit and thus $y = y'$. □

**Corollary 15.5.** *Let $Z$ be an irreducible component of a closed $Z_2G'$-invariant subscheme of $X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}}$. Then $Z$ is $Z_2G'$-invariant and intersects $Y^{\psi_1}_{\overline{\rho}^{\mathrm{ss}}}$ non-trivially. Moreover, if $x$ is a closed point of $Z$ then following hold:*

(1) *if $x$ is a closed point of $Z \cap Y^{\psi_1}_{\overline{\rho}^{\mathrm{ss}}}$ then $\dim \mathcal{O}_{Z,x} = \dim Z$;*

(2) *if $x$ is a closed point of $Z \setminus (Z \cap Y^{\psi_1}_{\overline{\rho}^{\mathrm{ss}}})$ then $\dim \mathcal{O}_{Z,x} = \dim Z - 1$.*

*Proof.* It follows from Corollary 15.4 that $Y^{\psi_1}_{\overline{\rho}^{\mathrm{ss}}}$ is the preimage of closed points in $X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}} /\!\!/ Z_2G'$. Given this the proof is the same as the proof of [6, Lemma 3.21]. □

### *15.2. Generalised tori*

We summarise some of the results of [48]. The group

$$\mu := (\mu_{p^\infty}(E) \otimes M_2)^{\Delta}$$

is a finite abelian group of order $p^m$. We assume that $L$ contains all the $p^m$-th roots of unity. We further assume that $L$ is large enough so that $X^{\mathrm{gen}}_{H_2,\overline{\psi}_2}(\mathcal{O})$ is non-empty. This is possible by [48, Theorem 9.3]. Under these assumptions we prove in [48, Theorem 9.3] that there is an isomorphism of complete local noetherian $\mathcal{O}$-algebras:

$$\alpha : \mathcal{O}[\![(\Gamma_E^{\mathrm{ab},p} \otimes M_2)^{\Delta}]\!] \xrightarrow{\cong} R^{\mathrm{ps}}_{H_2,\overline{\psi}_2} \tag{93}$$

where $\Gamma_E^{\mathrm{ab},p}$ is the maximal abelian pro-$p$ quotient of $\Gamma_E$, and an isomorphism of $\mathcal{O}$-algebras

$$A^{\mathrm{gen}}_{H_2,\overline{\psi}_2^{\mathrm{ss}}} \cong R^{\mathrm{ps}}_{H_2,\overline{\psi}_2^{\mathrm{ss}}}[t_1^{\pm 1},\dots,t_s^{\pm 1}], \tag{94}$$

where $s = \mathrm{rank}_{\mathbb{Z}} M_2 - \mathrm{rank}_{\mathbb{Z}}(M_2)_{\Delta}$. The Artin map of local class field theory $\mathrm{Art}_E : E^{\times} \to \Gamma_E^{\mathrm{ab}}$ identifies $\mu$ with the torsion subgroup of $(\Gamma_E^{\mathrm{ab},p} \otimes M_2)^{\Delta}$. We show in [48, Corollary 8.8] that

$$\mathcal{O}[\![(\Gamma_E^{\mathrm{ab},p} \otimes M_2)^{\Delta}]\!] \cong \mathcal{O}[\mu][\![x_1,\dots,x_r]\!], \tag{95}$$

where $r = \mathrm{rank}_{\mathbb{Z}} M_2 \cdot [F : \mathbb{Q}_p] + \mathrm{rank}_{\mathbb{Z}}(M_2)_{\Delta}$ is the rank of $(\Gamma_E^{\mathrm{ab},p} \otimes M_2)^{\Delta}$ as a $\mathbb{Z}_p$-module. It follows from (93), (94) and (95) that

$$A^{\mathrm{gen}}_{H_2,\overline{\psi}_2^{\mathrm{ss}}} \cong \mathcal{O}[\mu][\![x_1,\dots,x_r]\!][t_1^{\pm 1},\dots,t_s^{\pm 1}], \tag{96}$$

where $r + s = \mathrm{rank}_{\mathbb{Z}} M_2([F : \mathbb{Q}_p] + 1) = \dim(H_2)_k([F : \mathbb{Q}_p] + 1)$.

Let $\mathrm{X}(\mu)$ be the group of characters $\chi : \mu \to \mathcal{O}^{\times}$. We may interpret $\chi \in \mathrm{X}(\mu)$ as $\mathcal{O}$-algebra homomorphisms $\chi : \mathcal{O}[\mu] \to \mathcal{O}$. If $X = \mathrm{Spec}\, A$ is a scheme over $X^{\mathrm{ps}}_{H,\overline{\psi}_2^{\mathrm{ss}}}$ then (93) induces a morphism $\mathcal{O}[\mu] \to A$ and we define

$$A^{\chi} := A \otimes_{\mathcal{O}[\mu],\chi} \mathcal{O}, \quad X^{\chi} := \mathrm{Spec}\, A^{\chi}. \tag{97}$$

**Lemma 15.6.** *Let $Z$ be an affine scheme over $\overline{X}^{\mathrm{ps}}_{H,\overline{\psi}_2^{\mathrm{ss}}}$ Then the underlying reduced subschemes of $Z$ and $Z^{\chi}$ coincide. In particular,* $\dim Z = \dim Z^{\chi}$.

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*Proof.* Since $\mu$ is a finite $p$-group $k[\mu]$ is a local $k$-algebra with a nilpotent maximal ideal. This implies the first assertion. Since dimension is a topological invariant the second assertion follows. □

**Lemma 15.7.** *Let $\kappa$ be either a finite or a local $\mathcal{O}$-field and let $\psi_2 : \Gamma_F \to H_2(\kappa)$ be a continuous representation corresponding to $x \in X^{\mathrm{gen},\chi}_{H_2,\overline{\psi}_2^{\mathrm{ss}}}(\kappa)$. Then*

$$R^{\square,\chi}_{H_2,\psi_2} \cong \Lambda[\![z_1,\dots,z_t]\!],$$

*where $t = \dim(H_2)_k([F:\mathbb{Q}_p]+1)$ and $\Lambda$ is a coefficient ring for $\kappa$.*

*Proof.* It follows from the proof of Proposition 5.16 that $R^{\square,\chi}_{H_2,\psi_2}$ is the completion of $\Lambda \otimes_{\mathcal{O}} A^{\mathrm{gen},\chi}_{H_2,\overline{\psi}_2^{\mathrm{ss}}}$ with respect to the kernel of the natural map $\Lambda \otimes_{\mathcal{O}} A^{\mathrm{gen},\chi}_{H_2,\overline{\psi}_2^{\mathrm{ss}}} \twoheadrightarrow \kappa$. It follows from (96) that

$$A^{\mathrm{gen},\chi}_{H_2,\overline{\psi}_2^{\mathrm{ss}}} \cong \mathcal{O}[\![x_1,\dots,x_r]\!][t_1^{\pm 1},\dots,t_s^{\pm 1}], \tag{98}$$

with $r+s = \dim(H_2)_k([F:\mathbb{Q}_p]+1)$. The assertion follows from Lemma 5.17. □

### *15.3. Serre's criterion for normality*

In this subsection we show that the deformation rings $R^{\square,\psi_1,\chi}_{G,\rho_x}$ (defined below) and their special fibres are integral domains by verifying Serre's criterion for normality for $X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$ and its special fibre, when $\pi_1(G')$ is étale. We also show that $X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$ is normal without making any assumptions on $\pi_1(G')$.

**Lemma 15.8.** *The composition with $\varphi_2 : G/G' \to H_2$ induces an isomorphism*

$$X^{\mathrm{gen},\psi_1}_{G/G',\overline{\psi}^{\mathrm{ss}}} \xrightarrow{\cong} X^{\mathrm{gen}}_{H_2,\overline{\psi}_2^{\mathrm{ss}}}. \tag{99}$$

*Proof.* The natural map $G/G' \to H := H_1 \times H_2$ induces an isomorphism between neutral components. Lemma 9.2 implies that $X^{\mathrm{gen}}_{G/G',\overline{\psi}^{\mathrm{ss}}} \cong X^{\mathrm{gen}}_{H,\overline{\psi}^{\mathrm{ss}}}$. The assertion then follows from Proposition 8.10 and Lemma 8.9. □

If $X = \operatorname{Spec} A$ is a scheme over $X^{\mathrm{gen}}_{G/G',\overline{\psi}^{\mathrm{ss}}}$ then $X^{\psi_1}$ is a scheme over $X^{\mathrm{gen},\psi_1}_{G/G',\overline{\psi}^{\mathrm{ss}}}$. The isomorphism (99) allows us to consider $X^{\psi_1,\chi}$ for every character $\chi \in \mathrm{X}(\mu)$ with the notation introduced in (97). We will denote its ring of global sections by $A^{\psi_1,\chi}$.

**Lemma 15.9.** *$X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$ and $\overline{X}^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$ are non-empty for every $\chi \in \mathrm{X}(\mu)$.*

*Proof.* The underlying reduced schemes of $\overline{X}^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$ and $\overline{X}^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}}$ coincide by Lemma 15.6. Moreover, $\overline{X}^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}}(k)$ is non-empty as it contains a point corresponding to the representation $\overline{\rho}$ fixed at the beginning of the section. □

**Lemma 15.10.** *Let $\kappa$ be a finite or a local $\mathcal{O}$-field and let $\psi : \Gamma_F \to (G/G')(\kappa)$ be a representation corresponding to $x \in X^{\mathrm{gen},\psi_1,\chi}_{G/G',\overline{\psi}^{\mathrm{ss}}}(\kappa)$ and let $\psi_2 = \varphi_2 \circ \psi$. Then*

$$R^{\square,\psi_1,\chi}_{G/G',\psi} \cong R^{\square,\chi}_{H_2,\psi_2}.$$

*In particular, $R^{\square,\psi_1,\chi}_{G/G',\overline{\psi}} \cong \Lambda[\![z_1,\dots,z_t]\!]$, where $t = \dim(H_2)_k([F:\mathbb{Q}_p]+1)$.*

*Proof.* The isomorphism follows from Lemma 15.8 and Proposition 5.16. Formal smoothness follows from Lemma 15.7. □

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**Lemma 15.11.** *Let $X = X^{\mathrm{gen},\psi_1,\mathcal{X}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and let $\overline{X} = \overline{X}^{\mathrm{gen},\psi_1,\mathcal{X}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and let $x$ be a closed point of $Y^{\psi_1,\mathcal{X}}_{\overline{\rho}^{\mathrm{ss}}}$ or a closed point $X \setminus Y^{\psi_1,\mathcal{X}}_{\overline{\rho}^{\mathrm{ss}}}$. Then we have the following isomorphisms of local rings:*

(1) *if $x \in Y^{\psi_1,\mathcal{X}}_{\overline{\rho}^{\mathrm{ss}}}$ then $R^{\square,\psi_1,\mathcal{X}}_{G,\rho_x} \cong \hat{\mathcal{O}}_{X,x}$, $R^{\square,\psi_1,\mathcal{X}}_{G,\rho_x}/\varpi \cong \hat{\mathcal{O}}_{\overline{X},x}$;*
(2) *if $x \in \overline{X} \setminus Y^{\psi_1,\mathcal{X}}_{\overline{\rho}^{\mathrm{ss}}}$ then $R^{\square,\psi_1,\mathcal{X}}_{G,\rho_x} \cong \hat{\mathcal{O}}_{X,x}[\![T]\!]$, $R^{\square,\psi_1,\mathcal{X}}_{G,\rho_x}/\varpi \cong \hat{\mathcal{O}}_{\overline{X},x}[\![T]\!]$;*
(3) *if $x \in X[1/p]$ then $R^{\square,\psi_1,\mathcal{X}}_{G,\rho_x} \cong \hat{\mathcal{O}}_{X,x}$.*

*Proof.* Since $X^{\mathrm{gen},\psi_1,\mathcal{X}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is a closed subscheme of $X^{\mathrm{gen},\tau}_{G}$ the assertion follows from Proposition 5.16 and Lemma 5.17. □

**Corollary 15.12.** *Let $x$ be either a closed point of $X^{\mathrm{gen},\psi_1,\mathcal{X}}_{G,\overline{\rho}^{\mathrm{ss}}} \setminus Y^{\psi_1,\mathcal{X}}_{\overline{\rho}^{\mathrm{ss}}}$ or a closed point of $Y^{\psi_1,\mathcal{X}}_{\overline{\rho}^{\mathrm{ss}}}$. Then the following hold:*

(1) *$R^{\square,\psi_1,\mathcal{X}}_{G,\rho_x}$ is a flat $\Lambda$-algebra of relative dimension $\dim \overline{G}_k([F:\mathbb{Q}_p]+1)$ and is complete intersection;*
(2) *if $\mathrm{char}(\kappa(x)) = p$ then $R^{\square,\psi_1,\mathcal{X}}_{G,\rho_x}/\varpi$ is complete intersection of dimension $\dim \overline{G}_k([F:\mathbb{Q}_p]+1)$.*

*Proof.* Proposition 3.6 and Lemma 15.10 give us a presentation

$$R^{\square,\psi_1,\mathcal{X}}_{G,\rho_x} \cong \frac{R^{\square,\psi_1,\mathcal{X}}_{G,\varphi\circ\rho_x}[\![x_1,\dots,x_r]\!]}{(f_1,\dots,f_s)} \cong \frac{\Lambda[\![z_1,\dots,z_t,x_1,\dots,x_r]\!]}{(f_1,\dots,f_s)} \tag{100}$$

with $r-s = \dim G'_k([F:\mathbb{Q}_p]+1)$ and $t = \dim(H_2)_k([F:\mathbb{Q}_p]+1)$. Lemma 3.9 implies that it is enough to show that

$$\dim(\kappa(x) \otimes_\Lambda R^{\square,\psi_1,\mathcal{X}}_{G,\rho_x}) \le r-s+t \overset{(90)}{=} \dim \overline{G}_k([F:\mathbb{Q}_p]+1). \tag{101}$$

Lemma 15.6 and Corollary 15.2 give us

$$\dim \overline{X}^{\mathrm{gen},\psi_1,\mathcal{X}}_{G,\overline{\rho}^{\mathrm{ss}}} \le \dim \overline{G}_k([F:\mathbb{Q}_p]+1). \tag{102}$$

To ease the notation we let $X := X^{\mathrm{gen},\psi_1,\mathcal{X}}_{G,\overline{\rho}^{\mathrm{ss}}}$, let $\overline{X} := \overline{X}^{\mathrm{gen},\psi_1,\mathcal{X}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and let $Y := Y^{\psi_1,\mathcal{X}}_{\overline{\rho}^{\mathrm{ss}}}$.

If $x \in Y$ then Corollary 15.5 applied to $Z = \overline{X}$ gives us $\dim \mathcal{O}_{\overline{X},x} \le \dim \overline{X}$. Lemma 15.11 implies that $R^{\square,\psi_1,\mathcal{X}}_{\rho_x}/\varpi \cong \hat{\mathcal{O}}_{\overline{X},x}$ and this implies (101).

If $x \in \overline{X} \setminus Y$ then $R^{\square,\psi,\mathcal{X}}_{\rho_x}/\varpi \cong \hat{\mathcal{O}}_{\overline{X},x}[\![T]\!]$ by Lemma 15.11. Corollary 15.5 gives $\dim \mathcal{O}_{\overline{X},x}+1 \le \dim \overline{X}$ and this implies (101).

If $x \in X[1/p]$ then $R^{\square,\psi_1,\mathcal{X}}_{G,\rho_x} \cong \hat{\mathcal{O}}_{X,x}$ by Lemma 15.11 and $\dim \mathcal{O}_{X,x} \le \dim X - 1$ by Corollary 15.5 applied to $Z = X$. Since by Corollary 15.5 every irreducible component of $X$ intersects the special fibre we have $\dim X - 1 \le \dim \overline{X}$ and (101) follows. □

**Remark 15.13.** The proof of Corollary 15.12 also applies verbatim to the rings $R^{\square,\psi_1}_{G,\rho_x}$ after replacing $X^{\mathrm{gen},\psi_1,\mathcal{X}}_{G,\overline{\rho}^{\mathrm{ss}}}$ with $X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}}$ everywhere.

**Corollary 15.14.** *Let $d = \dim \overline{G}_k([F:\mathbb{Q}_p]+1)$. Then the following hold:*

(1) *$X^{\mathrm{gen},\psi_1,\mathcal{X}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is $\mathcal{O}$-flat of relative dimension $d$ and is locally complete intersection;*
(2) *$\overline{X}^{\mathrm{gen},\psi_1,\mathcal{X}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is locally complete intersection of dimension $d$.*

*Proof.* The proof is the same as the proof of Corollary 13.28 using Corollary 15.12 and Lemma 15.11. □

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**Corollary 15.15.** *Let $c$ be one of the following:*

(1) *the codimension of the $(\mathrm{Lie}\, G'_{\mathrm{sc}})^*$-special locus in $X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$;*
(2) *the codimension of the $(\mathrm{Lie}\, G'_{\mathrm{sc}})^*_k$-special locus in $\overline{X}^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$;*
(3) *the codimension of the $(\mathrm{Lie}\, G'_{\mathrm{sc}})^*_L$-special locus in $X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$.*

*Then $c \geq 1 + [F : \mathbb{Q}_p]$. Moreover, if there is no $\overline{\rho}^{\mathrm{ss}}$-compatible R-Levi subgroup $L$ of $G$ such that $\dim G - \dim L = 2$ then $c \geq 2[F : \mathbb{Q}_p]$.*

*Proof.* Let $Z^{\mathrm{spcl}}_G$ be the $(\mathrm{Lie}\, G'_{\mathrm{sc}})^*$-special locus in $X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$ and let $Z^{\mathrm{spcl}}_{\overline{G}}$ be the $(\mathrm{Lie}\, \overline{G}'_{\mathrm{sc}})^*$-special locus in $X^{\mathrm{gen}}_{\overline{G},\overline{\rho}^{\mathrm{ss}}}$. If $x' \in X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$ maps to $x \in X^{\mathrm{gen}}_{\overline{G},\overline{\rho}^{\mathrm{ss}}}$ then the representations $(\mathrm{Lie}\, G'_{\mathrm{sc}})^*_{x'}$ and $(\mathrm{Lie}\, \overline{G}'_{\mathrm{sc}})^*_x \otimes_{\kappa(x)} \kappa(x')$ are isomorphic as representations of $G$ as $G' \to \overline{G}'$ is a central isogeny of semisimple groups and the adjoint action of $G$ on both factors through the action of $\overline{G}$. Thus $h^0(\Gamma_F, (\mathrm{Lie}\, G'_{\mathrm{sc}})^*_{x'}(1)) = h^0(\Gamma_F, (\mathrm{Lie}\, \overline{G}'_{\mathrm{sc}})^*_x(1))$, and so $x$ is special if and only if $x'$ is special. Thus $Z^{\mathrm{spcl}}_G$ is equal to the preimage of $Z^{\mathrm{spcl}}_{\overline{G}}$.

Since $X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{gen}}_{\overline{G},\overline{\rho}^{\mathrm{ss}}}$ is finite by Proposition 15.1, the map $Z^{\mathrm{spcl}}_G \to Z^{\mathrm{spcl}}_{\overline{G}}$ is finite, and hence $\dim Z^{\mathrm{spcl}}_G \leq \dim Z^{\mathrm{spcl}}_{\overline{G}}$. Since $\dim X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}} = \dim X^{\mathrm{gen}}_{\overline{G},\overline{\rho}^{\mathrm{ss}}}$ by Corollary 15.14 and Corollary 13.28 the assertion follows from Theorem 14.6 applied to $\overline{G}$. The argument for the special and generic fibres is the same. □

**Lemma 15.16.** *The $(\mathrm{Lie}\, G'_{\mathrm{sc}})^*_L$-non-special locus in $X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$ is regular. Moreover, if $\pi_1(G')$ is étale then the following hold:*

(1) *$(\mathrm{Lie}\, G'_{\mathrm{sc}})^*$-non-special locus in $X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$ is regular;*
(2) *$(\mathrm{Lie}\, G'_{\mathrm{sc}})^*_k$-non-special locus in $\overline{X}^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$ is regular.*

*Proof.* Let $Z$ be the $(\mathrm{Lie}\, G'_{\mathrm{sc}})^*$-special locus in $X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$. Let $x$ be a closed point in the complement of $Z$. Then $\kappa(x)$ is either a local or a finite field and local Tate duality induces an isomorphism $H^2(\Gamma_F, \mathrm{ad}^0\, \rho_x) \cong H^0(\Gamma_F, (\mathrm{ad}^0\, \rho_x)^*(1))$.

If $\pi_1(G')$ is étale or $x \in X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$ then $(\mathrm{Lie}\, G'_{\mathrm{sc}})_x = (\mathrm{Lie}\, G')_x$. Since $x$ is not contained in the $(\mathrm{Lie}\, G'_{\mathrm{sc}})^*$-special locus $H^2(\Gamma_F, \mathrm{ad}^0\, \rho_x) = 0$ and thus $s = 0$ in the presentation (100), which implies that $R^{\square,\psi_1,\chi}_{G,\rho_x}$ is formally smooth over $\Lambda$. It follows from Lemma 15.11 that the local ring of $X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$ at $x$ is regular. If $x$ is in the special fibre then we deduce that $R^{\square,\psi_1,\chi}_{G,\rho_x}/\varpi$ is formally smooth over $\kappa(x)$ and the same argument applies. □

If there is no $\overline{\rho}^{\mathrm{ss}}$-compatible R-Levi subgroup $L$ of $G$ such that $\dim G - \dim L = 2$ then we let $c(\overline{\rho}^{\mathrm{ss}}) = 2[F : \mathbb{Q}_p] - 1$ and let $c(\overline{\rho}^{\mathrm{ss}}) = [F : \mathbb{Q}_p]$, otherwise.

**Lemma 15.17.** *$X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$ is regular in codimension $c(\overline{\rho}^{\mathrm{ss}})$. Moreover, if $\pi_1(G')$ is étale then $X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$ and $\overline{X}^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$ are also regular in codimension $c(\overline{\rho}^{\mathrm{ss}})$.*

*Proof.* The assertion follows from Corollary 15.15 and Lemma 15.16. □

**Theorem 15.18.** *If $\pi_1(G')$ is étale then $X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$ and $\overline{X}^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$ are normal. Moreover, if the orders of $(G/G^0)(\overline{k})$ and $(Z_1 \cap Z_2 G')(\overline{k})$ are coprime then they are integral.*

*Proof.* Corollary 15.14 and Lemma 15.17 imply that Serre's criterion for normality holds and hence $X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$ and $\overline{X}^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$ are normal.

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It follows from [6, Lemma 2.1] that their connected components are $Z_2G'$-invariant. If there was more than one then [57, Theorem 3 (iii)] implies that their images would disconnect $X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}} /\!\!/ Z_2G'$ (resp. $X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}} /\!\!/ Z_2G'$). If the orders of $(G/G^0)(\overline{k})$ and $(Z_1 \cap Z_2G')(\overline{k})$ are coprime then this is not possible as the GIT quotients are spectra of local rings in both cases by Corollary 15.4. □

**Theorem 15.19.** *$X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$ is normal.*

*Proof.* Corollary 15.14, Lemma 15.17 and Serre's criterion for normality. □

**Corollary 15.20.** *Assume that $\pi_1(G')$ is étale. Let $x$ be either a closed point of $\overline{X}^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}} \setminus Y^{\psi_1,\chi}_{\overline{\rho}^{\mathrm{ss}}}$ or a closed point of $Y^{\psi_1,\chi}_{\overline{\rho}^{\mathrm{ss}}}$. Then $R^{\square,\psi_1,\chi}_{G,\rho_x}$ and $R^{\square,\psi_1,\chi}_{G,\rho_x}/\varpi$ are normal domains, which are regular in codimension $c(\overline{\rho}^{\mathrm{ss}})$.*

*Proof.* Let $A$ be either $A^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$ or $A^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}/\varpi$. Since $A$ is of finite type over a complete local noetherian ring, it is excellent by [59, Tag 07QW]. Let $\mathfrak{p}$ be a prime of $A$. If $A$ is regular in codimension $i$ then $A_{\mathfrak{p}}$ is regular in codimension $i$, and since $A_{\mathfrak{p}}$ is again excellent, its completion $\hat{A}_{\mathfrak{p}}$ is also regular in codimension $i$ by [43, Theorem 23.9 (ii)]. (Since $A_{\mathfrak{p}}$ is excellent, it is a G-ring, and this implies that the assumptions made in the reference are satisfied. This is used implicitly in the proof of [43, Theorem 32.2 (i)].) By repeating the same argument with $A[T]$ we deduce that $\hat{A}_{\mathfrak{p}}[\![T]\!]$ is regular in codimension $i$. Lemma 15.17 and Lemma 5.17 imply that $R^{\square,\psi_1,\chi}_{G,\rho_x}$ and $R^{\square,\psi_1,\chi}_{G,\rho_x}$ are regular in codimension $c(\overline{\rho}^{\mathrm{ss}})$. Since $c(\overline{\rho}^{\mathrm{ss}}) \geq 1$, Corollary 15.12 imply that Serre's criterion for normality holds and we deduce that $R^{\square,\psi_1,\chi}_{G,\rho_x}$ and $R^{\square,\psi_1,\chi}_{G,\rho_x}/\varpi$ are normal domains. □

**Corollary 15.21.** *If $x$ is a closed point of $X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$ then $R^{\square,\psi_1,\chi}_{G,\rho_x}$ is a normal domain, which is regular in codimension $c(\overline{\rho}^{\mathrm{ss}})$.*

*Proof.* The same proof as the proof of Corollary 15.20. □

**Corollary 15.22.** *Assume that either $[F:\mathbb{Q}_p] \geq 3$ or $[F:\mathbb{Q}_p] = 2$ and there is no $\overline{\rho}^{\mathrm{ss}}$-compatible $R$-Levi subgroup $L$ of $G$ such that $\dim G - \dim L = 2$.*

*If $x$ is a closed point of $X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$ then $R^{\square,\psi_1,\chi}_{G,\rho_x}$ is factorial. If $x$ is a closed point of $\overline{X}^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}} \setminus Y^{\psi_1,\chi}_{\overline{\rho}^{\mathrm{ss}}}$ or a closed point of $Y^{\psi_1,\chi}_{\overline{\rho}^{\mathrm{ss}}}$ and we additionally assume that $\pi_1(G')$ is étale then $R^{\square,\psi_1,\chi}_{G,\rho_x}$ and $R^{\square,\psi_1,\chi}_{G,\rho_x}/\varpi$ are also factorial.*

*Proof.* The assumptions together with Corollary 15.20 imply that the rings are regular in codimension 3. Since they are complete intersection by Corollary 15.12, a theorem of Grothendieck implies the assertion; see [14] for a quick proof. □

### *15.4. Irreducible components*

We now focus on the case, when $M_1 = 0$ and $M_2 = M$, so that $X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}} = X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$.

**Corollary 15.23.** *$X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is reduced and $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$ is normal.*

*Proof.* The normality of $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$ follows from Theorem 15.19. This implies that $A^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$ is reduced. Since $A^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is $\mathcal{O}$-flat by Corollary 13.28, it is a subring of $A^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$ and hence is reduced. □

**Corollary 15.24.** *$X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is reduced, $\mathcal{O}$-flat and $X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$ is normal. The map $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$ induces a bijection between the sets of irreducible components.*

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*Proof.* Since $R^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$ is a subring of $A^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ it is $\mathcal{O}$-torsion free (and hence $\mathcal{O}$-flat) and reduced. Hence, the irreducible components of $R^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}$ and $R^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$ coincide. Since $A^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$ is normal by Theorem 15.19 and $R^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p] = (A^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p])^{G^0}$, [12, Proposition 6.4.1] implies that $R^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$ is normal. Since $G^0$ is connected the irreducible components of $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$ are $G^0$-invariant and since $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$ is normal its irreducible and connected components coincide. Thus irreducible components of $X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$ correspond to the GIT quotients of irreducible components of $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$. □

**Corollary 15.25.** *The reduced subscheme* $(X^{\mathrm{ps}}_G[1/p])^{\mathrm{red}}$ *is normal.*

*Proof.* Corollary 15.24 implies that $X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$ is reduced. Corollary 7.6 gives us an isomorphism of schemes between $(X^{\mathrm{ps}}_G[1/p])^{\mathrm{red}}$ and $X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$. The assertion follows from Corollary 15.24. □

**Corollary 15.26.** *If* $\pi_1(G')$ *is étale then the map* $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{gen}}_{G/G',\overline{\psi}^{\mathrm{ss}}}$ *induces a bijection between the sets of irreducible components.*

*Proof.* Since both schemes are $\mathcal{O}$-flat it is enough to verify the assertion after inverting $p$. Since $\mathcal{O}[\mu][1/p] \cong \prod_{\chi\in \mathrm{X}(\mu)} L$ we have

$$A^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p] = \prod_{\chi\in \mathrm{X}(\mu)} A^{\mathrm{gen},\chi}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p], \quad A^{\mathrm{gen}}_{G/G',\overline{\psi}^{\mathrm{ss}}}[1/p] = \prod_{\chi\in \mathrm{X}(\mu)} A^{\mathrm{gen},\chi}_{G/G',\overline{\psi}^{\mathrm{ss}}}[1/p]. \tag{103}$$

It follows from (98) that $A^{\mathrm{gen},\chi}_{G/G',\overline{\psi}^{\mathrm{ss}}}$ are $\mathcal{O}$-flat integral domains. As $Z_1$ is trivial in our case, Theorem 15.18, Lemma 15.9 and Corollary 15.14 imply that $A^{\mathrm{gen},\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$ are $\mathcal{O}$-flat integral domains. The claim follows from (103). □

**Corollary 15.27.** *If* $\pi_1(G')$ *is étale then the natural maps* $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{ps}}_G \to X^{\mathrm{ps}}_{G/G'}$ *induce bijections between the sets of irreducible components.*

*Proof.* Since $X^{\mathrm{git}}_{G,\overline{\rho}^{\mathrm{ss}}} \to X^{\mathrm{ps}}_G$ is a homeomorphism by Proposition 7.4, the assertion follows from Corollaries 15.24, 15.26. □

### *15.5. Conjecture of Böckle–Juschka*

We fix $z \in X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}}(k)$ and to ease the notation we write $\overline{\rho} := \rho_z$, $R^{\square}_{\overline{\rho}} := R^{\square,\psi_1}_{G,\rho_z}$, $\overline{\psi} := \varphi \circ \overline{\rho}$ and $R^{\square}_{\overline{\psi}} := R^{\square,\psi_1}_{G/G',\overline{\psi}}$. It follows from (96) and Proposition 5.16 that

$$R^{\square}_{\overline{\psi}} \cong \mathcal{O}[\mu][\![z_1,\dots,z_t]\!], \tag{104}$$

where $t = \dim(H_2)_k([F:\mathbb{Q}_p]+1)$. We let $X^{\square}_{\overline{\rho}} := \operatorname{Spec} R^{\square}_{\overline{\rho}}$ and $X^{\square}_{\overline{\psi}} := \operatorname{Spec} R^{\square}_{\overline{\psi}}$.

**Corollary 15.28.** *The natural map* $R^{\square}_{\overline{\psi}} \to R^{\square}_{\overline{\rho}}$ *is flat.*

*Proof.* It follows from (104) that there is an $\mathcal{O}$-algebra homomorphism $x : R^{\square}_{\overline{\psi}} \to \mathcal{O}$. Let $\psi : \Gamma_F \to (G/G')(\mathcal{O})$ be the deformation of $\overline{\psi}$ corresponding to $x$. Then

$$R^{\square,\psi}_{\overline{\rho}} \cong R^{\square}_{\overline{\rho}} \otimes_{R^{\square}_{\overline{\psi}},x} \mathcal{O}, \tag{105}$$

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where the left hand side is defined taking $M_1 = M$ and $M_2 = 0$. Reducing this identity modulo $\varpi$ we obtain an isomorphism

$$R^{\square,\psi}_{\overline{\rho}}/\varpi \cong R^{\square}_{\overline{\rho}} \otimes_{R^{\square}_{\overline{\psi}}} k. \tag{106}$$

Proposition 3.6 gives us a presentation

$$R^{\square}_{\overline{\rho}} \cong R^{\square}_{\overline{\psi}}[\![x_1, \dots, x_r]\!]/(f_1, \dots, f_s) \tag{107}$$

with $r - s = \dim G'_k([F : \mathbb{Q}_p] + 1) = \dim(R^{\square,\psi}_{\overline{\rho}}/\varpi)$, where the last equality is given by Corollary 15.12 (2) and (90) noting that $\dim(H_2)_k = 0$ in the 'fixed determinant' case. Lemma 3.9 applied with $A = R^{\square}_{\overline{\psi}}$ and $B = R^{\square}_{\overline{\rho}}$ implies the assertion. □

**Corollary 15.29.** *$R^{\square}_{\overline{\rho}}[1/p]$ is normal and $R^{\square}_{\overline{\rho}}$ is reduced.*

*Proof.* Since $R^{\square}_{\overline{\rho}}$ is $\mathcal{O}$-flat either by Lemma 15.28 or by Remark 15.13, it is a subring of $R^{\square}_{\overline{\rho}}[1/p]$ and hence it is enough to show that $R^{\square}_{\overline{\rho}}[1/p]$ is normal. Further, it is enough to show that $R^{\square,\chi}_{\overline{\rho}}[1/p]$ is normal as $R^{\square}_{\overline{\rho}}[1/p]$ is a finite product of such rings. Since normality is a local property, it is enough to show that the localisations of $R^{\square,\chi}_{\overline{\rho}}[1/p]$ at maximal ideals are normal. Since $R^{\square}_{\overline{\rho}}$ is a complete noetherian local ring, it is excellent and hence it is enough to show that the completions of $R^{\square,\chi}_{\overline{\rho}}[1/p]$ at maximal ideals are normal. These are isomorphic to $R^{\square,\psi_1,\chi}_{G,\rho_x}$ by Lemma 3.11, which are normal by Corollary 15.21. □

The following result proves an analogue of the conjecture of Böckle–Juschka posed originally for $G = \mathrm{GL}_d$ (and proved in that case in [6]).

**Corollary 15.30.** *If $\pi_1(G')$ is étale then the natural map $R^{\square}_{\overline{\psi}} \to R^{\square}_{\overline{\rho}}$ induces a bijection between the sets of irreducible components.*

*Proof.* It follows from (104) that $R^{\square}_{\overline{\psi}}$ is $\mathcal{O}$-flat and its irreducible components are given by $R^{\square,\chi}_{\overline{\psi}}$ for $\chi \in \mathrm{X}(\mu)$, which are formally smooth and hence flat over $\mathcal{O}$.

The natural map $R^{\square}_{\overline{\psi}} \to R^{\square}_{\overline{\rho}}$ induces a map of local rings $R^{\square,\chi}_{\overline{\psi}} \to R^{\square,\chi}_{\overline{\rho}}$. Since the reduced special fibres of of $X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$ and of $X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}}$ coincide by Lemma 15.6, the point $z$ lies in $X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}(k)$ for every $\chi \in \mathrm{X}(\mu)$. Hence, $R^{\square,\chi}_{\overline{\rho}}$ is an $\mathcal{O}$-flat integral domain by Corollaries 15.12, 15.20. Thus $R^{\square,\chi}_{\overline{\rho}}[1/p]$ is an integral domain for every $\chi \in \mathrm{X}(\mu)$. Since $R^{\square}_{\overline{\psi}}$ is $\mathcal{O}$-flat, Lemma 15.28 implies that $R^{\square}_{\overline{\rho}}$ is $\mathcal{O}$-flat (this also follows from Remark 15.13), hence the irreducible components of $R^{\square}_{\overline{\rho}}$ and $R^{\square}_{\overline{\rho}}[1/p]$ coincide. Since the order of $\mu$ becomes invertible once we invert $p$ we have $R^{\square}_{\overline{\rho}}[1/p] \cong \prod_{\chi} R^{\square,\chi}_{\overline{\rho}}[1/p]$, which implies the assertion. □

**Remark 15.31.** If $x$ is a closed point of $X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}}[1/p]$ then the isomorphism $\mathcal{O}[\mu][1/p] \cong \prod_{\chi \in \mathrm{X}(\mu)} L$ implies that there is a unique $\chi \in \mathrm{X}(\mu)$ such that $x \in X^{\mathrm{gen},\psi_1,\chi}_{G,\overline{\rho}^{\mathrm{ss}}}$. Then $R^{\square,\psi_1}_{G,\rho_x} = R^{\square,\psi_1,\chi}_{G,\rho_x}$ is an integral domain by Corollary 15.21. If $\chi' \neq \chi$ then $R^{\square,\psi_1,\chi'}_{G,\rho_x}$ is the zero ring.

### *15.6. Labelling irreducible components*

We have seen in the proofs of Corollaries 15.26, 15.27, 15.30 that if $\pi_1(G')$ is étale then the set of irreducible components of corresponding schemes are in bijection with the group of characters $\mathrm{X}(\mu)$. This bijection is non-canonical in general as the isomorphism $\alpha$ in (93) is non-canonical.

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We show in [48, Section 7.4] that there is a canonical action of $\mathrm{X}(\mu)$ on the set of irreducible components of $X^{\mathrm{ps}}_{H_2} = \operatorname{Spec} R^{\mathrm{ps}}_{H_2,\overline{\psi}_2^{\mathrm{ss}}}$, and this action is faithful and transitive. So to fix an $\mathrm{X}(\mu)$-equivariant bijection between $\mathrm{X}(\mu)$ and the set of irreducible components it is enough to specify to which component the identity in $\mathrm{X}(\mu)$ corresponds to. We don't know how to do this canonically in general, except in one case, which we will now discuss.

Assume that $H_2 \cong H_2^0 \rtimes \Delta$. We have already identified $\Delta = \mathrm{Gal}(E/F)$ and $\overline{\psi}_2 : \Gamma_F \to H_2(k)$ is a representation, such that composition of $\overline{\psi}_2$ with the projection to $\mathrm{Gal}(E/F)$ is the map $\gamma \mapsto \gamma|_E$. The surjection

$$H_2^0(\mathcal{O}) = \mathrm{Hom}(M_2, \mathcal{O}^\times) \twoheadrightarrow \mathrm{Hom}(M_2, k^\times) = H_2^0(k)$$

has a canonical section given by the Teichmüller lift. This section is $\Delta$-equivariant and induces a section $\sigma : H_2(k) \to H_2(\mathcal{O})$. The composition $\sigma \circ \overline{\psi}_2 : \Gamma_F \to H_2(\mathcal{O})$ is a minimally ramified lift of $\overline{\psi}_2$. It is canonical and defines an $\mathcal{O}$-valued point on a unique irreducible component of $X^{\mathrm{gen}}_{H_2,\overline{\psi}_2^{\mathrm{ss}}}$ and its $H_2$-pseudocharacter defines an $\mathcal{O}$-valued point on the unique irreducible component of $X^{\mathrm{ps}}_{H_2}$. We thus obtain a canonical bijection between the set of irreducible components of $X^{\mathrm{ps}}_{H_2}$ and $\mathrm{X}(\mu)$, where the identity in $\mathrm{X}(\mu)$ is mapped to the distinguished component constructed above.

We will now give a different description of the bijection in the semi-direct product case. Let $X$ be an irreducible component of $X^{\mathrm{ps}}_{H_2}$. It follows from (93) and (95) that every point $x \in X^{\mathrm{ps}}_{H_2}(\overline{\mathbb{Q}}_p)$ lies on a unique irreducible component and every irreducible component contains a $\overline{\mathbb{Q}}_p$-valued point. We choose $x \in X(\overline{\mathbb{Q}}_p)$ and let $\psi_2 : \Gamma_F \to H_2(\overline{\mathbb{Q}}_p)$ be a continuous representation such that its $H_2$-pseudocharacter corresponds to $x$. The representation $\psi_2$ is uniquely determined by $x$ up to $H_2^0(\overline{\mathbb{Q}}_p)$-conjugation. Using that $H_2$ is a semi-direct product, we may write $\psi_2(\gamma) = (\Phi(\gamma), \gamma|_E)$ for all $\gamma \in \Gamma_F$. Since $\psi_2$ is a continuous representation one may check that $\Phi : \Gamma_F \to H_2^0(\overline{\mathbb{Q}}_p)$ defines a continuous 1-cocycle and the $H_2^0(\overline{\mathbb{Q}}_p)$-conjugacy class of $\psi_2$ defines a cohomology class in $H^1(\Gamma_F, H^0(\overline{\mathbb{Q}}_p))$. We show in [48] using arguments that go back to the work of Langlands [38] on his correspondence for tori that

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathrm{Group}}((\Gamma_F^{\mathrm{ab},p} \otimes M_2)^\Delta, \overline{\mathbb{Q}}_p^\times) \cong H^1(\Gamma_F, H_2^0(\overline{\mathbb{Q}}_p)).$$

Thus to $x \in X(\overline{\mathbb{Q}}_p)$ we may canonically attach a continuous character

$$\tilde{\chi} : (\Gamma_F^{\mathrm{ab},p} \otimes M_2)^\Delta \to \overline{\mathbb{Q}}_p^\times.$$

The Artin map induces an isomorphism between $\mu := (\mu_{p^\infty}(E) \otimes M_2)^\Delta$ and the $p$-power torsion subgroup of $(\Gamma_F^{\mathrm{ab},p} \otimes M_2)^\Delta$. Let $\chi : \mu \to \overline{\mathbb{Q}}_p^\times$ be the restriction of $\tilde{\chi}$ to $\mu$. Then $\chi$ takes values in $\mathcal{O}^\times \subset \overline{\mathbb{Q}}_p^\times$ and we show in [48, Section 7.4] that the irreducible component $X$ is uniquely determined by $\chi$.

## 16. Deformations into *L*-groups and *C*-groups

In this section we spell out that the results proved in sections 13.4 and 15 are applicable, when $G$ is an $L$-group or a $C$-group. Galois representations with values in such groups appear in the Langlands correspondence.

Let $H$ be a connected reductive group defined over $F$ and let $E$ be a finite Galois extension of $F$ such that $H_E$ is split. Let $T$ be a maximal split torus in $H_E$ and let $B$ be a Borel subgroup of $H_E$ containing $T$. To the triple $(H_E, B, T)$ one may attach a *based root datum* $(X^*(T), \Phi, \Delta, X_*(T), \Phi^\vee, \Delta^\vee)$, where the symbols denote characters of $T$, the set of all roots, the set of positive roots corresponding to $B$, cocharacters of $T$, the set of coroots, and the set of positive coroots corresponding to $B$, respectively, see [19, Section 1.5]. The dual group is a triple $(\widehat{H}, \widehat{B}, \widehat{T})$, where $\widehat{H}$ is a connected split reductive group scheme defined over $\mathbb{Z}$, $\widehat{B}$ is a Borel subgroup scheme of $\widehat{H}$ containing a maximal split torus $\widehat{T}$, such

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that the based root datum of $(\widehat{H}, \widehat{B}, \widehat{T})$ is isomorphic to $(X_*(T), \Phi^\vee, \Delta^\vee, X^*(T), \Phi, \Delta)$. Out of the action of $\mathrm{Gal}(E/F)$ on $H_E$ one may construct a group of homomorphism of $\mathrm{Gal}(E/F)$ to the automorphism group of the based root datum. After fixing additional datum, called a pinning, one rigidifies the situation to obtain a group homomorphism $\mathrm{Gal}(E/F) \to \mathrm{Aut}(\widehat{H})$. We refer the reader to [8, Section I.1], or [23, Section 2] for further details. The $L$-group is defined as

$$^L H := \widehat{H} \rtimes \mathrm{Gal}(E/F),$$

which we will consider as a group scheme over $\mathcal{O}$. An important consequence of the construction is that there are canonical $\mathrm{Gal}(E/F)$-equivariant identifications

$$X_*(T) = X^*(\widehat{T}), \quad X^*(T) = X_*(\widehat{T}).$$

**Lemma 16.1.** *Let $Z(H)^0$ be the neutral component of the centre of $H$ and let $M$ be the character lattice of $\widehat{H}/\widehat{H}'$. Then there is a canonical identification*

$$Z(H)^0(F) = (E^\times \otimes M)^{\mathrm{Gal}(E/F)}.$$

*In particular, $\mu := (\mu_{p^\infty}(E) \otimes M)^{\mathrm{Gal}(E/F)}$ gets canonically identified with $p$-power torsion subgroup $Z(H)^0(F)_{p^\infty}$ of $Z(H)^0(F)$.*

*Proof.* The base change of $Z(H)^0$ to $E$ is equal to $Z(H_E)^0$. It follows from [9, Proposition 14.2] that

$$X_*(Z(H_E)^0) = \{\beta^\vee \in X_*(T) : \langle \beta^\vee, \alpha \rangle = 0, \forall \alpha \in \Phi\}.$$

Thus the canonical identification $X_*(T) = X^*(\widehat{T})$ identifies $X_*(Z(H_E)^0)$ with $M$. Since this identification is $\mathrm{Gal}(E/F)$-equivariant we obtain

$$\begin{aligned} Z(H)^0(F) = Z(H)^0(E)^{\mathrm{Gal}(E/F)} &= \mathrm{Hom}(X^*(Z(H_E)^0), E^\times)^{\mathrm{Gal}(E/F)} \\ &= (X_*(Z(H_E)^0) \otimes E^\times)^{\mathrm{Gal}(E/F)} = (M \otimes E^\times)^{\mathrm{Gal}(E/F)}. \end{aligned} \tag{108}$$

□

We fix a continuous representation $\overline{\rho} : \Gamma_F \to {}^L H(k)$ such that the composition of $\overline{\rho}$ with the projection onto $\mathrm{Gal}(E/F)$ is the map $\gamma \mapsto \gamma|_E$. Representations satisfying this condition are called *admissible*.

Let $D_{\overline{\rho}} : \mathfrak{A}_{\mathcal{O}} \to \mathrm{Set}$ be the functor which maps $(A, \mathfrak{m}_A)$ to the set of continuous representations $\rho_A : \Gamma_F \to {}^L H(A)$ such that $\rho_A(\gamma) \equiv \overline{\rho}(\gamma) \pmod{\mathfrak{m}_A}$ for all $\gamma \in \Gamma_F$. This functor is representable by a complete local noetherian $\mathcal{O}$-algebra $R^\square_{\overline{\rho}}$. The results of Section 13.4 apply to $R^\square_{\overline{\rho}}$. We have $R^\square_{\overline{\rho}} = R^\square_{G,\rho_x}$, where $G = {}^L H$ and $x \in X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}(k)$ is the point corresponding to $\overline{\rho}$. We highlight a few of the results below.

**Theorem 16.2.** *The ring $R^\square_{\overline{\rho}}$ is $\mathcal{O}$-flat, complete intersection of relative dimension* $\dim \widehat{H}_k([F : \mathbb{Q}_p]+1)$. *Its special fibre $R^\square_{\overline{\rho}}/\varpi$ is complete intersection.*

*Proof.* This follows from Corollary 13.27, observing that $G^0 = \widehat{H}$. □

**Theorem 16.3** (Corollary 15.29). *$R^\square_{\overline{\rho}}$ is reduced and $R^\square_{\overline{\rho}}[1/p]$ is normal.*

We assume that $\mathcal{O}$ contains all $p^m$-th roots of unity, where $p^m$ is the order of $Z(H)^0(F)_{p^\infty}$. We let X be the group of characters $\chi : Z(H)^0(F)_{p^\infty} \to \mathcal{O}^\times$.

**Theorem 16.4.** *If $\pi_1(\widehat{H}')$ is étale then there is a canonical bijection*

$$\chi \mapsto \mathrm{Spec}\, R^{\square,\chi}_{\overline{\rho}}$$

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*between* X *and the set of irreducible components of* $\operatorname{Spec} R^{\square}_{\overline{\rho}}$. *The rings* $R^{\square,\chi}_{\overline{\rho}}$ *and their special fibres* $R^{\square,\chi}_{\overline{\rho}}/\varpi$ *are complete intersection normal domains.*

*Proof.* We have $G/G' \cong (\widehat{H}/\widehat{H}') \rtimes \operatorname{Gal}(E/F)$. The assertion about the bijection follows from Corollary 15.30, Section 15.6 and Lemma 16.1.

Let us make the bijection more explicit. If $X$ is an irreducible component of $\operatorname{Spec} R^{\square}_{\overline{\rho}}$ then we may pick a geometric point $y \in X(\overline{\mathbb{Q}}_p)$. Then $y$ corresponds to a continuous representation $\rho : \Gamma_F \to {}^L H(\overline{\mathbb{Q}}_p)$. By quotienting out $\widehat{H}'(\overline{\mathbb{Q}}_p)$ we obtain a continuous admissible representation

$$\psi : \Gamma_F \to ({}^L H/\widehat{H}')(\overline{\mathbb{Q}}_p) = (\widehat{H}/\widehat{H}')(\overline{\mathbb{Q}}_p) \rtimes \operatorname{Gal}(E/F).$$

The group ${}^L H/\widehat{H}'$ is the $L$-group for $Z(H)^0(F)$. The $\widehat{H}(\overline{\mathbb{Q}}_p)$-conjugacy of $\psi$ corresponds to a continuous character $\tilde{\chi} : Z(H)^0(F) \to \overline{\mathbb{Q}}_p$ under the local Langlands correspondence for tori, [38], [4]. By restricting $\tilde{\chi}$ to $Z(H)^0(F)_{p^\infty}$ we obtain the required character $\chi$.

The rest of the theorem follows from the results of Section 15. Namely, $R^{\square,\chi}_{\overline{\rho}} = R^{\square,\psi_1,\chi}_{G,\rho_x}$, where $G = {}^L H$, $x \in X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}(k)$ corresponds to $\overline{\rho}$, $M_1 = 0$, so that $H_1 = \operatorname{Gal}(E/F)$, and $\psi_1 : \Gamma_F \to H_1(\mathcal{O})$ is the character $\gamma \mapsto \gamma|_E$. The assertions about the rings $R^{\square,\chi}_{\overline{\rho}}$ and $R^{\square,\chi}_{\overline{\rho}}/\varpi$ follow from Corollaries 15.12, 15.20. □

The $C$-group ${}^C H$ of $H$ is a variant of an $L$-group. It has been defined in [13, Definition 5.3.2]; a nice exposition is given in [66, Section 1.1]. For us only the following properties are important: there is an exact sequence

$$0 \to {}^L H \to {}^C H \xrightarrow{d} \mathbb{G}_m \to 0. \tag{109}$$

The derived subgroup of the neutral component of ${}^C H$ is equal to $\widehat{H}'$ and (109) induces an isomorphism:

$${}^C H/\widehat{H}' \cong {}^L H/\widehat{H}' \times \mathbb{G}_m. \tag{110}$$

These properties of the $C$-group easily follow from its construction, see for example [23, Section 2.1]. It follows from (110) that the subgroup scheme $\widehat{H}$ is normal in ${}^C H$ and (110) induces an isomorphism

$${}^C H/\widehat{H} \cong \operatorname{Gal}(E/F) \times \mathbb{G}_m. \tag{111}$$

Let $\overline{\rho} : \Gamma_F \to {}^C H(k)$ be a continuous representation such that $d \circ \overline{\rho} \equiv \chi_{\mathrm{cyc}} \pmod{\varpi}$, where $\chi_{\mathrm{cyc}}$ is the $p$-adic cyclotomic character, and the composition $\Gamma_F \xrightarrow{\overline{\rho}} {}^C H(k) \to ({}^C H/\widehat{H})(k)$ with the projection onto $\operatorname{Gal}(E/F)$ under (111) is the map $\gamma \mapsto \gamma|_E$.

Let $D^C_{\overline{\rho}} : \mathfrak{A}_{\mathcal{O}} \to \mathrm{Set}$ be the functor such that $D^C_{\overline{\rho}}(A)$ is the set of continuous representations $\rho_A : \Gamma_F \to {}^C H(A)$, such that $\rho_A \equiv \overline{\rho} \pmod{\mathfrak{m}_A}$ and $d \circ \rho_A = \chi_{\mathrm{cyc}} \otimes_{\mathcal{O}} A$. The definition of this deformation problem is motivated by the first two bullet points in [13, Conjecture 5.3.4], which describes Galois representations attached to $C$-algebraic automorphic forms. The functor $D^C_{\overline{\rho}}$ is representable by a complete local noetherian $\mathcal{O}$-algebra $R^{\square,C}_{\overline{\rho}}$ with residue field $k$.

**Theorem 16.5.** *Theorems 16.2, 16.3 and 16.4 hold with* $R^{\square,C}_{\overline{\rho}}$ *instead of* $R^{\square}_{\overline{\rho}}$.

*Proof.* It follows from (110) that we are in the setting of Section 15 with $G = {}^C H$, $M_1 = \mathbb{Z}$ with the trivial $\operatorname{Gal}(E/F)$-action, $M_2 = X^*(\widehat{H}/\widehat{H}')$ with the natural $\operatorname{Gal}(E/F)$-action. Moreover,

$$H_1 = \operatorname{Gal}(E/F) \times \mathbb{G}_m, \quad H_2 = (\widehat{H}/\widehat{H}') \rtimes \operatorname{Gal}(E/F)$$

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and $\psi_1 : \Gamma_F \to H_1(\mathcal{O})$ the character $\gamma \mapsto (\gamma|_E, \chi_{\mathrm{cyc}}(\gamma))$. The ring $R^{\square,C}_{\overline{\rho}}$ is the ring $R^{\square,\psi_1}_{G,\rho_x}$ with $x \in X^{\mathrm{gen},\psi_1}_{G,\overline{\rho}^{\mathrm{ss}}}(k)$ corresponding to $\overline{\rho}$.

The assertions of Theorem 16.2 follow from Remark 15.13, which indicates how to modify the proof of Corollary 15.12 to get the assertion.

The assertions of Theorem 16.3 follow from Corollary 15.29.

Lemma 16.1 implies that we may canonically identify $(\mu_{p^\infty}(E) \otimes M_2)^{\mathrm{Gal}(E/F)}$ with $Z(H)^0(F)_{p^\infty}$. It follows from the proof of Corollary 15.30 that the set of irreducible components of $R^{\square,C}_{\overline{\rho}}$ is in bijection of the character group of $Z(H)^0(F)_{p^\infty}$. The irreducible component corresponding to $\chi$ is the spectrum of $R^{\square,\psi_1,\chi}_{G,\rho_x}$. The assertion of Theorem 16.4 follows from Corollaries 15.14, 15.20 and observing that (90) implies $\dim \overline{G}_k = \dim \widehat{H}_k$.

It follows from (110) that $G/G' \cong (G^0/G') \rtimes \mathrm{Gal}(E/F)$. Hence, as we explain in section 15.6, the map associating an irreducible component to a character is canonical. Concretely, if $X$ is an irreducible component of $\mathrm{Spec}\, R^{\square,C}_{\overline{\rho}}$ then we may pick a geometric point $y \in X(\overline{\mathbb{Q}}_p)$. Then $y$ corresponds to a continuous representation $\rho : \Gamma_F \to {}^C H(\overline{\mathbb{Q}}_p)$. By quotienting out $\widehat{H}'(\overline{\mathbb{Q}}_p)$ and projecting onto $({}^L H/\widehat{H}')(\overline{\mathbb{Q}}_p)$ under (110) we obtain a continuous admissible representation

$$\psi_2 : \Gamma_F \to (\widehat{H}/\widehat{H}')(\overline{\mathbb{Q}}_p) \rtimes \mathrm{Gal}(E/F).$$

We then proceed as in the proof of Theorem 16.4. The group ${}^L H/\widehat{H}'$ is the $L$-group for $Z(H)^0(F)$. The $\widehat{H}(\overline{\mathbb{Q}}_p)$-conjugacy of $\psi_2$ corresponds to a continuous character $\tilde{\chi} : Z(H)^0(F) \to \overline{\mathbb{Q}}_p$ under the local Langlands correspondence for tori, [38], [4]. By restricting $\tilde{\chi}$ to $Z(H)^0(F)_{p^\infty}$ we obtain the required character $\chi$. □

## A. Condensed sets

In [53] Clausen and Scholze introduced the notion of a *condensed set*, which is a more algebraic notion of topological space. It turns out that condensed sets behave better in algebraic contexts and lead to clearer statements.

A condensed set can be defined as an accessible functor $X : \mathrm{ProFin}^{\mathrm{op}} \to \mathrm{Set}$ on the opposite category of profinite sets [53, Footnote 5], which satisfies the following two conditions:

(1) $X$ maps finite disjoint unions of profinite sets to finite products.
(2) For any surjection $S' \twoheadrightarrow S$ of profinite sets with projections $p_1, p_2 : S' \times_S S' \to S'$, the canonical map

$$X(S) \to \{x \in X(S') \mid p_1^*(x) = p_2^*(x) \in X(S' \times_S S')\}$$

is bijective.

A map of condensed sets is a natural transformation. The category of condensed sets can be identified with the category of functors on the full subcategory of extremally disconnected profinite sets, which only satisfy condition (1) [53, Proposition 2.7]. We will switch between the two perspectives without further comment.

It can be shown that the category of condensed sets is (up to a set-theoretic condition) a coherent topos. In particular, we can define all kinds of algebraic structures internal to condensed sets and thereby obtain a notion of condensed group, condensed ring, etc. In general we can speak about quasi-compact and quasi-separated objects of a coherent topos. We will start by giving a concrete characterisation of quasi-compact condensed sets and quasi-separated condensed sets.

**Lemma A.1.** *Profinite sets are compact objects in the category of condensed sets, that is, for every filtered colimit of condensed sets $X = \varinjlim_i X_i$ and every profinite set $S$, we have $X(S) = \varinjlim_i X_i(S)$.*

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*Proof.* We have to check that the presheaf colimit $\varinjlim_i X_i$ is already a sheaf. The sheaf condition as formulated in [53, Section 1] is in terms of finite limits, and these commute with filtered colimits. □

**Lemma A.2.** *Every surjection of condensed sets is an effective epimorphism.*

*Proof.* For a surjection $f : X \to Y$ of condensed sets, we need to see that $f$ is the coequaliser of its kernel pair $X \times_Y X \rightrightarrows X$. This is a reflexive coequaliser and therefore sifted. The condensed sets $X$ and $Y$ can be seen as finite product preserving functors on the opposite category of the category of extremally disconnected profinite sets. Since sifted coequalisers commute with finite products, we can reduce the claim to sets. Every surjection of sets is an effective epimorphism. This completes the proof. □

**Lemma A.3.** *Let $X$ be a condensed set. The following are equivalent.*

(1) *$X$ is a quasi-compact object of the topos of condensed sets. That is, for every collection of condensed sets $(T_i)_{i\in I}$ and every effective epimorphism $\coprod_{i\in I} T_i \twoheadrightarrow X$, there is a finite subset $I_0 \subseteq I$, such that $\coprod_{i\in I_0} T_i \twoheadrightarrow X$ is an effective epimorphism.*
(2) *There is a profinite set $T$ and a surjection $\underline{T} \twoheadrightarrow X$.*

*Proof.* Suppose $X$ is a quasi-compact condensed set. By definition it arises as a left Kan extension from a $\kappa$-condensed set. By the density theorem [41, Theorem III.7.1] we can write $X$ as a small colimit of $\kappa$-small profinite sets. The inclusion functor from $\kappa$-condensed sets into all condensed set preserves colimits [53, Proposition 2.9], so $X$ is a small colimit of profinite sets. In particular, there is an epimorphism $\coprod_{i\in I} \underline{S_i} \twoheadrightarrow X$, where $S_i$ are profinite sets. Since this epimorphism is effective by Lemma A.2 and $X$ is assumed quasi-compact, there is a finite subset $I_0 \subseteq I$, such that $\coprod_{i\in I_0} \underline{S_i} \twoheadrightarrow X$ is a surjection.

Suppose there is a profinite set and a surjection $\underline{T} \twoheadrightarrow X$ and let $\coprod_{i\in I} X_i \twoheadrightarrow X$ be an effective epimorphism. By pullback we obtain a surjection $\coprod_{i\in I}(X_i \times_X T) \twoheadrightarrow T$. Since by [52, Proposition 1.2 (3)] $T$ is a quasi-compact object in the category of condensed sets, there is a finite subset $I_0 \subseteq I$, such that $\coprod_{i\in I_0}(X_i \times_X T) \twoheadrightarrow T$ is an effective epimorphism. The map $\coprod_{i\in I_0}(X_i \times_X T) \twoheadrightarrow X$ factors over $\coprod_{i\in I_0} X_i \twoheadrightarrow X$, which is thus also a surjection. □

**Lemma A.4.** *Let $X$ be a condensed set. The following are equivalent.*

(1) *$X$ is a quasi-separated object of the topos of condensed sets. That is, for every pair of morphisms $Y \to X \leftarrow Z$, where $Y$ and $Z$ are quasi-compact the fibre-product $Y \times_X Z$ is quasi-compact.*
(2) *For every profinite set $S$ together with a morphism $\underline{S} \to X \times X$, the pullback $\underline{S} \times_{X\times X} X$ along the diagonal $\Delta_X : X \to X \times X$ is quasi-compact.*

*Proof.* Assume that $X$ is a quasi-separated condensed set. Let $S$ be a profinite set together with a map $\underline{S} \to X \times X$. Then $\underline{S} \times_X \underline{S}$ is quasi-compact where the maps $\underline{S} \to X$ are given by composing with the projections. In the following diagram all squares are cartesian.

$$\begin{array}{ccccc} \underline{S}_{X\times X} X & \longrightarrow & \underline{S} \times_X \underline{S} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \Delta_X} \\ \underline{S} & \longrightarrow & \underline{S} \times \underline{S} & \longrightarrow & X \times X \end{array}$$

Since $\underline{S} \times \underline{S} = \underline{S \times S}$ comes from a profinite set it is quasi-separated by [52, Proposition 1.2 (3)]. So $\underline{S}_{X\times X} X$ is quasi-compact.

Now suppose that $X$ satisfies (2). Let $Y \to X \leftarrow Z$ be a pair of morphisms, where $Y$ and $Z$ are quasi-compact. By Lemma A.3, there exist profinite sets $S$ and $T$ and surjections $\underline{S} \twoheadrightarrow Y$ and $\underline{T} \twoheadrightarrow Z$. We get

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a diagram in which are squares are cartesian:

$$\begin{array}{ccccc} \underline{S} \times_X \underline{T} & \twoheadrightarrow & Y \times_X Z & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \Delta_X} \\ \underline{S} \times \underline{T} & \twoheadrightarrow & Y \times Z & \longrightarrow & X \times X \end{array}$$

By (2) $\underline{S} \times_X \underline{T}$ is quasi-compact, hence by one further application of Lemma A.3 also $Y \times_X Z$. □

**Lemma A.5.** *A filtered colimit of quasi-compact quasi-separated condensed sets along quasi-compact injections is quasi-separated.*

*Proof.* Let $X = \varinjlim_i X_i$ be a filtered colimit of quasi-compact quasi-separated condensed sets where all transition maps in the diagram are injections. Let $T$ be a profinite set together with a map $T \to X \times X$. The map $X_i \times X_i \to X \times X$ is still a quasi-compact injection, so the fibre product $T_i := T \times_{X \times X} (X_i \times X_i)$ is quasi-compact. We know that the diagonals $\Delta_{X_i} : X_i \to X_i \times X_i$ are quasi-compact. Since in the diagram

$$\begin{array}{ccccc} T \times_{X_i \times X_i} X_i & \longrightarrow & X_i & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \Delta_{X_i}} & & \downarrow{\scriptstyle \Delta_X} \\ T & \longrightarrow & X_i \times X_i & \longrightarrow & X \times X \end{array}$$

both squares are cartesian, we have $T \times_{X \times X} X = T \times_{X_i \times X_i} X_i$, which is quasi-compact. The claim follows from Lemma A.4. □

**Lemma A.6.** *Let $X$ be a quasi-separated condensed set and let $Y \hookrightarrow X$ be an injection. Then $Y$ is quasi-separated.*

*Proof.* Let $T$ be a profinite set. In the diagram

$$\begin{array}{ccccc} T \times_{Y \times Y} Y & \longrightarrow & Y & \hookrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \Delta_Y} & & \downarrow{\scriptstyle \Delta_X} \\ T & \longrightarrow & Y \times Y & \hookrightarrow & X \times X \end{array}$$

the right square is cartesian, since $Y \to X$ is injective. So the outer square is cartesian. Hence $T \times_{Y \times Y} Y$ is quasi-compact, so $Y$ is quasi-separated by Lemma A.4. □

**Lemma A.7.** *Let $S$ be a profinite set, let $X$ be a quasi-separated condensed set and let $f : S \to X(*)$ be a map. Then there is at most one morphism of condensed sets $\tilde{f} : \underline{S} \to X$, whose map on underlying sets is $f$.*

*Proof.* By [52, Proposition 1.2 (4)] $X$ is a filtered colimit of quasi-compact quasi-separated condensed sets $X_i$ along injections. Recall that the $X_i$ can be seen as compact Hausdorff spaces [52, Proposition 1.2 (3)]. So for any profinite set $S$

$$X(S) = \bigcup_i \mathscr{C}(S, X_i) \subseteq \mathrm{Map}(S, X(*))$$

using Lemma A.1. □

### *A.1. Condensed structure on points*

Let $X : \mathrm{CRing} \to \mathrm{Set}$ be an accessible presheaf and let $A : \mathrm{ProFin}^{\mathrm{op}} \to \mathrm{CRing}$ be a condensed commutative ring. We denote by $X(A)$ the condensed set obtained by composition $X \circ A$ with the functor of points of $X$. By construction $X \mapsto X(A)$ preserves limits.

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**Lemma A.8.** *Let $X \hookrightarrow \mathbb{A}^n_R$ be a closed immersion of schemes over a commutative ring $R$ and let $A$ be a quasi-separated condensed $R$-algebra. Then $X(A) \hookrightarrow A^n$ is a quasi-compact injection.*

*Proof.* We see $X$ as the zero set of a family of polynomials $g_1, \dots, g_r \in R[x_1, \dots, x_n]$. We can see them as a map of condensed sets $\underline{g} : A^n \to A$. Since $A$ is quasi-separated $\Delta_A : A \to A \times A$ is quasi-compact and the map $0 : * \to A$, is quasi-compact by pullback along $(0, \mathrm{id}) : A \to A \times A$. It follows from the cartesian diagram

$$\begin{array}{ccc} X(A) & \longrightarrow & \{0\} \\ \downarrow & & \downarrow \\ A^n & \xrightarrow{\underline{g}} & A \end{array}$$

that $X(A) \to A^n$ is a quasi-compact injection. □

## Glossary

$\mathrm{ad}\,\overline{\rho}$ adjoint representation associated to $\overline{\rho}$
$\mathrm{ad}^{0,\varphi}$ Proposition 3.6
$A_G^{\mathrm{gen}(,\tau)}$ coordinate ring representing $X_G^{\mathrm{gen},\tau}$
$A^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$ abbreviation of $A^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$ as an $R^{\mathrm{ps}}_G$-algebra used after Corollary 8.4
$A^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$ coordinate ring of $X^{\mathrm{gen},\tau}_{G,\overline{\rho}^{\mathrm{ss}}}$
$\mathfrak{A}_{\mathcal{O}}$ category of local Artinian $\mathcal{O}$-algebras with residue field $k$
$\mathfrak{A}_{\Lambda}$ category of local Artinian $\Lambda$-algebras with residue field $\kappa$
$A^{\chi}$ Section 15.2
${}^{C}H$ $C$-group attached to a connected reductive group $H$
$\mathrm{CH}(D)$ Cayley–Hamilton ideal of a determinant law $D$
$\mathrm{cPC}^{\Gamma}_G(A)$ set of continuous $G$-pseudocharacters of $\Gamma$ with values in $A$
$c_{P,L}$ semisimplification map $P(\kappa) \to G(\kappa)$, Section 2.2
$\underline{(-)}_{\mathrm{disc}}$ discrete condensed set associated to a set
$\overline{D}$ determinant law associated to $\overline{\rho}$
$D_\rho$ determinant law associated to $\rho$
$D^{\square}_{(\rho,)G}$ framed deformation functor of $\rho$ with values in $G$, Section 3
$D^u$ universal determinant law extended to $R^{\mathrm{ps}}_{\mathrm{GL}_d}[\![\Gamma]\!]$
$D^u_{|A}$ specialisation of $D^u$ along $R^{\mathrm{ps}}_{\mathrm{GL}_d} \to A$
$D^u_{E^u}$ determinant law induced by $D^u$ on $E^u$
$D_{\overline{\Theta}}$ deformation functor of $\overline{\Theta}$ Definition 7.2
$D^{\square,\psi_1}_{G,\rho}$ Section 15
$E$ a finite extension of $F$ such that $\Gamma_E = \ker(\Gamma_F \xrightarrow{\overline{\rho}} G(k) \to (G/G^0)(k))$
$E^u$ universal Cayley–Hamilton quotient $R^{\mathrm{ps}}_{\mathrm{GL}_d}[\![\Gamma]\!]/\mathrm{CH}(D^u)$
$\mathcal{F}_N$ free abstract group on $N$ generators
$G$ a generalised reductive group scheme over $\mathcal{O}$
$G'$ derived group scheme of $G^0$
$G'_{\mathrm{sc}}$ simply connected central cover of $G'$
$G^0$ neutral component of $G$
$L_\lambda$ R-Levi attached to a cocharacter $\lambda$, Section 2.1
${}^{L}H$ $L$-group of $H$, Section 16
$\mathfrak{m}_A$ maximal ideal of a local ring $A$, Section 3
$N$ fixed number of topological generators of $Q$, Section 5.1

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$\mathcal{O}_\kappa$ Section 3
$\mathrm{PC}_G^\Gamma$ scheme of $G$-pseudocharacters of $\Gamma$, Section 7
$P_\lambda$ R-parabolic attached to a cocharacter $\lambda$, Section 2.1
$Q$ fixed quotient of $\Gamma$ as in Lemma 5.5
$\mathrm{Rep}_G^{\Gamma(,\square)}$ scheme of group homomorphisms $\Gamma \to G(A)$, Sections 1.4, 5.1
$R_G^{\mathrm{git},\tau}$ invariant ring $(A_G^{\mathrm{gen},\tau})^{G^0}$, coordinate ring of $X_G^{\mathrm{git},\tau}$, Section 6.1
$R_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{git}}$ abbreviation of $R_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{git},\tau}$ as an $R_G^{\mathrm{ps}}$-algebra used after Corollary 8.4
$R_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{git},\tau}$ connected component of $R_G^{\mathrm{git},\tau}$ corresponding to $\overline{\rho}^{\mathrm{ss}}$, Section 6.1
$R_{(\rho,)G}^{\square}$ universal deformation ring representing $D_{\rho,G}^{\square}$, Section 3
$R_{\overline{\psi}}^{\square}$ the universal framed deformation ring of $\overline{\psi}$
$R_{\overline{\rho}}^{\square,\psi_1}$ the universal deformation ring pro-representing $D_{\overline{\rho}}^{\square,\psi_1}$
$R_G^{\mathrm{ps}}$ the complete local noetherian $\mathcal{O}$-algebra pro-representing $D_{\overline{\Theta}}$, Section 7.2
$R_{\mathrm{GL}_d}^{\mathrm{ps}}$ universal deformation ring of $\overline{D}$, Section 5.1
$R_{G,\rho}^{\square,\psi_1}$, Section 15
$\underline{S}$ condensed set associated to a topological space $S$
$U_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{git},\tau}$ punctured space $X_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{git},\tau} \setminus \{\text{closed point}\}$
$U_{LG}$, $\overline{U}_{LG}$ Section 9.2
$U_{LG,\overline{\rho}^{\mathrm{ss}}}$, $\overline{U}_{LG,\overline{\rho}^{\mathrm{ss}}}$ component of $U_{LG}$, $\overline{U}_{LG}$ associated to $\overline{\rho}^{\mathrm{ss}}$
$U_G^{\mathrm{ps}}$ punctured spectrum $(\mathrm{Spec}R_G^{\mathrm{ps}}) \setminus \{\text{closed point}\}$
$U_\lambda$ unipotent radical of $P_\lambda$, Section 2.1
$V_{LG}$, $\overline{V}_{LG}$, Section 9.2
$V_{GG,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{n-spcl}}$, $\overline{V}_{GG,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{n-spcl}}$ Definition 13.19
$\overline{V}_{GG,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{spcl}}$ Proposition 13.18
$X /\!\!/ H$ affine GIT quotient of affine scheme $X$ by $H$, i.e. $\mathrm{Spec}(\mathcal{O}(X)^H)$
$X_{\mathrm{GL}_d}^{\mathrm{gen}}$ Wang-Erickson's space of generic matrices, Definition 5.3
$X_G^{\mathrm{gen}(,\tau)}$ closed subscheme of $X_{\mathrm{GL}_d}^{\mathrm{gen}}$ of representations factoring through $\tau$, Definition 5.7
$\overline{X}_G^{\mathrm{gen},\tau}$ special fibre of $X_G^{\mathrm{gen},\tau}$
$X_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{gen}}$ abbreviation of $X_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{gen},\tau}$ as an $R_G^{\mathrm{ps}}$-scheme used after Corollary 8.4
$X_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{gen},\tau}$ component of $X_G^{\mathrm{gen},\tau}$ corresponding to $\overline{\rho}^{\mathrm{ss}}$, Section 6.2
$X_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{gen},\psi_1}$, $\overline{X}_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{gen},\psi_1}$, Section 15
$X_{P,\rho}^{\mathrm{gen}}$ Definition 12.3
$X_G^{\mathrm{git}(,\tau)}$ GIT quotient $X_G^{\mathrm{gen},\tau} /\!\!/ G^0$, Section 6
$\overline{X}_G^{\mathrm{git}(,\tau)}$ GIT quotient $\overline{X}_G^{\mathrm{gen},\tau} /\!\!/ G^0$, Section 6
$X_{HG}^{\mathrm{git}(,\tau)}$, $\overline{X}_{HG}^{\mathrm{git}(,\tau)}$ image of $X_H^{\mathrm{git},\tau} \to X_G^{\mathrm{git},\tau}$ (resp. $\overline{X}_H^{\mathrm{git},\tau} \to \overline{X}_G^{\mathrm{git},\tau}$)
$X_{HG,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{git}(,\tau)}$ component of $X_{HG}^{\mathrm{git}(,\tau)}$ corresponding to $\overline{\rho}^{\mathrm{ss}}$
$\overline{X}_{HG}^{\mathrm{git}(,\tau)}$ special fiber of $X_{HG}^{\mathrm{git}(,\tau)}$, as a topological space, Remark 9.3
$X_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{git}}$ abbreviation of $X_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{git},\tau}$ as an $R_G^{\mathrm{ps}}$-scheme used after Corollary 8.4
$X_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{git},\tau}$ connected component of $X_G^{\mathrm{git},\tau}$ corresponding to $\overline{\rho}^{\mathrm{ss}}$, Section 6.1
$X_{\mathrm{GL}_d}^{\mathrm{ps}}$ spectrum of $R_{\mathrm{GL}_d}^{\mathrm{ps}}$, the universal deformation ring of $\overline{D}$, Section 5.1
$X_{G(,\overline{\rho}^{\mathrm{ss}})}^{\mathrm{ps}}$, $\overline{X}_{G(,\overline{\rho}^{\mathrm{ss}})}^{\mathrm{ps}}$ spectrum of $R_{G(,\overline{\rho}^{\mathrm{ss}})}^{\mathrm{ps}}$ $R_{G(,\overline{\rho}^{\mathrm{ss}})}^{\mathrm{ps}}/\varpi$
$X_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{spcl}}$, $\overline{X}_{G,\overline{\rho}^{\mathrm{ss}}}^{\mathrm{spcl}}$ Proposition 13.18

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$X(\mu)$ group of characters $\mu \to \mathcal{O}^\times$
$X^\chi$ Section 15.2
$X^\square_{G,\overline{\rho}}$, $\overline{X}^\square_{G,\overline{\rho}}$ spectrum of $R^\square_{G,\overline{\rho}}$, $R^\square_{G,\overline{\rho}}/\varpi$
$Y$ preimage of closed points of $X_G^{\mathrm{git},\tau}$ in $X_G^{\mathrm{gen},\tau}$, Section 9.2
$Y_{\overline{\rho}^{\mathrm{ss}}}$ preimage of closed point of $X_G^{\mathrm{ps}}$ in $X^{\mathrm{gen}}_{G,\overline{\rho}^{\mathrm{ss}}}$
$Y^{\psi_1}_{\overline{\rho}^{\mathrm{ss}}}$ Section 15.2
$Z_G(H)$ centraliser of a subgroup scheme $H \leq G$
$Z(H)$ centre of a group scheme $H$, Lemma 2.1
$\underline{Z}(H)$ centre functor of a group scheme $H$, Lemma 2.1
$Z^i(\Gamma, V)$ set of continuous $i$-cocycles
$\overline{Z}^{\mathrm{spcl}}_{GG,\overline{\rho}^{\mathrm{ss}}}$ Proposition 13.18
$\Delta$ image of $\pi_G \circ \overline{\rho}^{\mathrm{ss}} : \Gamma \to (G/G^0)(\overline{k})$
$\underline{\Delta}$ constant group scheme associated to $\Delta$
$\Gamma_F^{\mathrm{ab},p}$ maximal abelian pro-$p$ quotient of $\Gamma_F$
$\underline{\gamma_i}$ for $1 \leq i \leq N$ fixed topological generators of $Q$, Section 5.1
$\overline{\Theta}$ the $G$-pseudocharacter attached to $\overline{\rho}$
$\Theta_n$ homomorphism $\Theta_n : \mathcal{O}[G^n]^{G^0} \to \mathrm{Map}(\Gamma^n, A)$ part of $G$-pseudocharacter $\Theta$
$\Theta^\mu_{|A}$ specialisation of $\Theta^\mu$ along $R^{\mathrm{ps}}_G \to A$
$\Theta^\mu_{(G)}$ universal $G$-pseudocharacter over $R^{\mathrm{ps}}_G$
$\Theta_\rho$ $G$-pseudocharacter associated with $\rho$
$\kappa$ finite or local field in Section 3, otherwise a (topological) field
$\kappa(\mathfrak{p})$ residue field at prime ideal $\mathfrak{p}$
$\kappa[\varepsilon]$ ring of dual numbers $\kappa[\varepsilon] := \kappa[x]/(x^2)$ of $\kappa$
$\Lambda$ coefficient ring for $\kappa$, Section 3
$\Lambda^0$ subring of coefficient ring $\Lambda$ for $\kappa$, Section 3
$\mu_{p^\infty}(E)$ the group of all $p$-power roots of unity contained in $E$
$\pi_1(G')$ kernel of $G'_{sc} \to G'$
$\pi_G$ projection $G \to G/G^0$, Section 9
$\overline{\rho}$ a continuous representation $\Gamma_F \to G(k)$
$\rho_A$ a continuous representation $\Gamma_F \to G(A)$ lifting a given residual representation
$\rho^{\mathrm{ss}}$ $G$-semisimplification of $\rho$ Definition 2.21
$\tau$ fixed faithful representation $\tau : G \to \mathrm{GL}_d$, Section 5.1
$\chi_{\mathrm{cyc}}$ the $p$-adic cyclotomic character
$\widehat{\otimes}$ completed tensor product, [59, Tag 0AMU]
$M^{\otimes_R d}$ abbreviation for the $d$-fold tensor product $M \otimes_R \cdots \otimes_R M$
$(-)^\sharp$ for a map $f : X \to Y$ of schemes $f^\sharp : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$

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