PARAMETRIC SURFACES

II. TANGENTIAL PROPERTIES

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1. In this paper I investigate the tangential properties of parametric surfaces, leading to the main result that *the Lebesgue area is the Hausdorff two-dimensional measure of the set of points of the surface where an approximate tangential plane exists.*

Throughout this paper I will assume the notation and results of my previous paper (5). For simplicity of exposition I will assume my parametric surface to be homoeomorphic to a disk. The general case will follow from cyclic element theory.

2. Let (P, Q) be an interior point of the surface S and consider the cylinder $C(P, \theta, r)$ whose axis is in the direction θ through P, and whose right section is a circle of radius r. Let $M'(Q, \theta, r)$ be the interior points of the closure of the component containing Q of the set of points of H whose ϕ-image is interior to $C(P, \theta, r)$. Denote by $M(Q, \theta, r)$ the simply-connected domain consisting of the interior points of the simply-connected closed set obtained from $\overline{M'(Q, \theta, r)}$ by adding all its interior complementary domains. Let $m(Q, \theta, r)$ be the boundary of $M(Q, \theta, r)$. When $\phi M(Q, \theta, r)$ does not meet the boundary of S then it will be the simply-connected neighbourhood of (P, Q) on the surface whose boundary lies on $C(P, \theta, r)$.

I will now define an approximate tangential plane.

Definition 1. *I say a plane π passing through P, whose normal is in a direction θ, is approximate tangential plane to S at (P, Q) if, for sufficiently small r, $\phi m(Q, \theta, r)$ lies on the surface of $C(P, \theta, r)$ and cannot be deformed into a point thereon; while there exists a set E in H such that*

$$L^*\{EM(Q, \theta, r)\} = o(r^2) \quad as \quad r \to 0, \tag{1}$$

where (i) *no two points of $\phi(H - E)$ have the same projection on π, and*

(ii) $\delta_S\{(P_1, Q_1); (P, Q)\} \sim D_{\text{Euc}}(P_1^*, P)$ *as the right-hand side tends to zero,*

where (P_1, Q_1) is any point of $\phi(H - E)$ and P_1^* is its projection on π.

We may note in this definition that in particular all points of $\phi(H - E)$ are simple on $\phi(H - E)$.

An alternative definition is

Definition 2. *A plane π passing through P whose normal is in a direction θ is approximate tangential plane to S at (P, Q) if, for sufficiently small r, $\phi m(Q, \theta, r)$ lies on the surface of $C(P, \theta, r)$ and cannot be deformed into a point thereon, while*

$$L^*M(Q, \theta, r) \sim \pi r^2 \quad as \quad r \to 0. \tag{2}$$

Both these definitions may be modified. We obtain Definition 1′ by replacing equation (1) by

$$\Lambda^2\{\phi EM(Q, \theta, r)\} = o(r^2) \quad \text{as} \quad r \to 0, \tag{1′}$$

and we obtain Definition 2′ by replacing equation (2) by

$$\Lambda^2 \phi M(Q, \theta, r) \sim \pi r^2 \quad \text{as} \quad r \to 0. \tag{2′}$$

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Thus we have four definitions of a tangential plane. It is clear that with none of them can we have more than one tangential plane at any point.

3. Let $S(P,r)$ be the closed sphere of centre P and radius r. Denote by $K'(Q,r)$ *the interior points of the closure of the component containing* Q *of the set of points of* H *whose* ϕ*-image is interior to* $S(P,r)$. *Let* $K(Q,r)$ *be the simply connected domain consisting of the interior points of the simply connected closed set obtained from* $\overline{K'(Q,r)}$ *by adding all its interior complementary domains. Let* $l(Q,r)$ *be the boundary of* $K(Q,r)$.

DEFINITION 3. *An interior point* (P,Q) *of the surface is termed consistent if given* $\epsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\mu\{\phi, K(Q,r)\} > (1-\epsilon)\, L\phi K(Q,r) \tag{3}$$

for almost all $r < \delta$.

4. We are now in a position to state our main theorem. Let T_1, T_2, T_1', T_2' be the sets of interior points of S where there is a tangential plane in the sense of Definitions 1, 2, 1′, 2′ respectively. Let C be the set of consistent points, J that of interior points of the surface where

$$\overline{\lim_{r\to 0}} \frac{L^*K'(Q,r)}{\pi r^2} > 0, \tag{4}$$

I the set where the above limit is greater than one and I' the set where

$$\overline{\lim_{r\to 0}} \frac{\Lambda^2\phi K'(Q,r)}{\pi r^2} > 1. \tag{5}$$

Then

THEOREM 1. *If* $L(S) < \infty$, *then*

$$L^*(T_1) = \Lambda^2(T_1) = L^*(T_2) = \Lambda^2(T_2) = L^*(C) = \Lambda^2(C) = L^*(J) = \Lambda^2(J),$$

and the common value of all these is $L(S)$. *Moreover,*

$$T_1 - I \subset T_2 \subset T_1 CJ. \tag{6}$$

THEOREM 2. *If* $\Lambda^2 S < \infty$, *then*

$$L^*(T_1') = \Lambda^2(T_1') = L^*(T_2') = \Lambda^2(T_2') = L^*(C) = \Lambda^2(C) = L^*(J) = \Lambda^2(J),$$

and the common value of all these is $L(S)$. *Moreover,*

$$T_1' - I' \subset T_2' \subset T_1' CJ.$$

As $\Lambda^2\phi(X) \geqslant L^*\phi(X)$ for $X \subset H^0$ it is clear that both the assumptions and conclusions of Theorem 2 are stronger than those of Theorem 1.

T_1, T_2, C and J are all sets of 'well-behaved' points of the surface; in consequence of Theorem 1 they are nearly coincident and their common measure is the Lebesgue area of the surface. We may note that at almost all points there is either an approximate tangential plane or $\lim_{r\to 0} \frac{L^*K'(Q,r)}{\pi r^2} = 0$, when we have very irregular behaviour. From another point of view these results may be expressed by saying that *at* L^* *almost all points of* S *there is an approximate tangential plane.*

These results have enabled me to solve the problem of Geocze, that is to show that there exist polyhedral surfaces *inscribed* in S approximating arbitrarily closely to S both in Lebesgue area and in Fréchet distance.

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5. LEMMA 1.

$$\phi K'(Q,r), \quad \phi K(Q,r), \quad \phi l(Q,r), \quad \phi M(Q,r) \quad \text{and} \quad \phi m(Q,\theta,r)$$

are independent of the representation ϕ of the surface S.

For identical surfaces are K-equivalent and for K-equivalent surfaces the lemma clearly holds.

LEMMA 2. *If (P,Q) is an interior point of S, then*

$$L^*\phi K'(Q,r) \geqslant \int_0^r \Lambda\phi l(Q,t)\,dt \quad \text{for small } r.$$

This follows at once from Lemma 4 of (5).

LEMMA 3. *If C is a parametric contour and $\mu(C)$ is the lower bound of the area of surfaces subtending C then $\Lambda C = 2\pi \Rightarrow \mu(C) \leqslant \pi$. Further, given $\epsilon > 0$, there exists $\delta > 0$ such that if $\Lambda C = 2\pi$ and $\mu(C) \geqslant \pi - \delta$ there is a plane π such that C lies within ϵ of π, and $\Lambda^2 S_1 \geqslant \pi - \epsilon$, where S_1 is the projection on π of an arbitrary surface subtending C.*

Let $\{A_n\}$ $(n = 0, 1, 2, \ldots, n_0)$ be any finite set of points on C taken in the order in which they occur on C. Denote by P the polyhedral surface consisting of the triangles $A_0A_kA_{k+1}$ $(1 \leqslant k \leqslant n_0 - 1)$. By a suitable choice of the points A_n we can make the area of P approximate as closely as we please to that of a surface subtending C. On the other hand, $\Lambda^2 P \leqslant \pi$ and hence $\mu(C) \leqslant \pi$, which establishes the first part of the lemma.

Consider the class Σ of quadrilaterals $A_1A_2A_3A_4$ such that $2\pi \geqslant A_iA_j \geqslant \epsilon$ and the distance between A_4 and the plane of $A_1A_2A_3$ is at least ϵ. Denote by A_4^* the point of the plane of $A_1A_2A_3$ such that $A_1A_4^* = A_1A_4$ and $A_3A_4^* = A_3A_4$, while A_4^* and A_2 are on opposite sides of A_1A_3. Write

$$\delta(A_1A_2A_3A_4) = \mu(A_1A_2A_3A_4^*) - \mu(A_1A_2A_3A_4).$$

Clearly $\delta(A_1A_2A_3A_4) > 0$. On the other hand, the class Σ is compact and so the lower bound of $\delta(A_1A_2A_3A_4)$ is attained and hence greater than zero.

Take a small δ such that $\inf_{\Sigma} \delta(A_1A_2A_3A_4) > \delta > 0$. Suppose there exists no plane such that C lies within ϵ of it. Let $A_1A_2A_3$ divide the curve C into three arcs of length $\frac{2}{3}\pi$. Suppose that the point A_4 lies on the arc A_1A_3 but is not within ϵ of the plane of $A_1A_2A_3$. Let Γ_{ij} be the closed contour consisting of the arc A_iA_j together with the chord A_iA_j. Then by the argument used to prove the first part of this lemma

$$\mu(\Gamma_{ij}) \leqslant \tfrac{2}{9}\pi,$$

and hence either

$$\mu(C) \leqslant \tfrac{2}{3}\pi + 2\pi\epsilon < \pi - \delta,$$

or else $A_1A_2A_3A_4 \in \Sigma$. But in this latter case we obtain by the same argument again that

$$\begin{aligned}\mu(C) &\leqslant \mu(\Gamma_{12}) + \mu(\Gamma_{23}) + \mu(\Gamma_{34}) + \mu(\Gamma_{41}) + \mu(A_1A_2A_3A_4) \\ &< \mu(\Gamma_{12}) + \mu(\Gamma_{23}) + \mu(\Gamma_{34}) + \mu(\Gamma_{41}) + \mu(A_1A_2A_3A_4^*) - \delta \leqslant \pi - \delta,\end{aligned}$$

and so the second part of the lemma is established. To prove the third part we need only note that if C lies near π then by joining each point of C to its projection on π we obtain a surface having area near $\Lambda^2 S_1$ and subtending C.

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COROLLARY 1. *If C_i $(i = 1, 2, \ldots)$ is a set of parametric contours and M_i $(i = 1, 2, \ldots)$ is the set of surfaces of least Λ^2 measure subtending them, then*

$$\Sigma\Lambda^2 M_i \leqslant \frac{\{\Sigma\Lambda C_i\}^2}{4\pi}.$$

COROLLARY 2. $$\mu\{\phi, K(Q,r)\} \leqslant \frac{\{\Lambda\phi l(Q,r)\}^2}{4\pi}.$$

6. Let X be any subset of $|S|$. Then $L^*(X) = L^*\phi\{\phi^{-1}X\}$ is a Caratheodory outer measure, and if $L(S) < \infty$ it is finite. Thus we obtain from (2)

LEMMA 4. *If $L(S) < \infty$ and to any P, a point of $X \subset |S|$, there corresponds a closed sphere, of radius as small as we please and centre P, belonging to a class Γ, then a disjoint subset of Γ will cover a set $X_1 \subset X$, where $L^*(X - X_1) = 0$.*

COROLLARY. *We may select a finite disjoint subset of Γ covering a set X_2 such that $L^*(X - X_2)$ is as small as we please.*

We will assume throughout the rest of this paper that

$$L(S) < \infty. \tag{7}$$

LEMMA 5. *At L^* almost all points of S*

$$\lim_{r\to 0} \frac{L^*S(P,r)}{r^4} = \infty.$$

For suppose there exists E, $L^*(E) = e > 0$, and $K < \infty$ such that for $P \in E$

$$\overline{\lim_{r\to 0}} \frac{L^*S(P,r)}{r^4} < K.$$

Take $\delta > 0$. Then by Lemma 4 we obtain a disjoint set Γ of spheres of radius at most δ such that for each sphere $S(P, r(P))$ of Γ we have

$$\frac{L^*S(P,r(P))}{r^4(P)} < 2K, \tag{8}$$

while Γ covers L^* almost all of E. Hence

$$\sum_{\Gamma} r^4(P) > \frac{e}{2K}. \tag{9}$$

That is,

$$\sum_{\Gamma} r^3(P) > \frac{e}{2K\delta}. \tag{10}$$

But all the spheres are in a bounded part of space and so we have a contradiction if δ is small.

LEMMA 6. *At L^* almost all points of S and for some r as small as we please we have*

$$L^*\phi S^{-1}(P,r) > \tfrac{1}{16} L^*\phi S^{-1}(P, 2r). \tag{11}$$

For suppose (11) fails at P for all sufficiently small r. Then for $x < r_0$,

$$L^*\phi S^{-1}(P,x) \leqslant (\tfrac{1}{16})^{(\log r_0 - \log x)/\log 2} L^*\phi S^{-1}(P, 2r_0) = \frac{x^4}{r_0^4} L^*\phi S^{-1}(P, 2r_0), \tag{12}$$

and by Lemma 5 this can happen at most at a set of points of L^* measure zero.

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7. LEMMA 7. *The set of points E_0 where* $\lim_{r\to 0}\frac{L^*\phi K'(Q,r)}{\pi r^2} = 0$ *has* $L^*(E_0) = 0$.

Suppose
$$L^*(E_0) > e > 0. \tag{13}$$

Take $\epsilon > 0$. By Lemma 6 of (5) there exists $\delta > 0$ such that if $\{r_i\}$ is any set of simply-connected Jordan domains, in H, of diameter less than δ, while the boundary of each $\phi(r_i)$ is rectifiable, then
$$\sum_i \mu\{\phi, r_i\} > \sum_i L^*\phi(r_i) - \epsilon. \tag{14}$$

As we are supposing that $S = \phi(H)$ is homoeomorphic to a disk, there will exist an $\epsilon_1 > 0$ such that if the boundary of $\phi(r_i)$ has diameter less than ϵ_1 then the diameter of r_i will be less than δ. Take $\lambda > 0$. Then there exist $\epsilon_2 > 0$ and $E_1 \subset E_0$ where $L^*(E_1) > e$ such that for $r < \epsilon_2$ and $Q \in E_1$ we have
$$\frac{L^*\phi K'(Q,r)}{\pi r^2} < \lambda. \tag{15}$$

By Lemma 6, there exists $E \subset E_1$, where $L^*(E) > e$, such that for $P \in \phi(E)$ and some r as small as we please
$$L^*\phi S^{-1}(P,r) > \tfrac{1}{16} L^*\phi S^{-1}(P, 2r). \tag{16}$$

Take $\eta < \frac{1}{2}\min(\epsilon_1, \epsilon_2)$. Consider $S(P, \frac{3}{2}r)$ where $P \in \phi(E)$ and $r < \frac{1}{2}\eta$, and let D_1 be a component of $S^{-1}(P, \frac{3}{2}r)$ containing points of E. Denote by D' the interior points of the closure of D_1. Take D' to be the $D'(r_1)$ of Theorem 3 of (5). Then in the notation of that theorem
$$\int_r^{\frac{3}{2}r} \sum_{\Delta(h)} \Lambda\phi l(i,h)\, dh < L^*\phi(D'). \tag{17}$$

Hence there exists h, $\frac{3}{2}r > h > r$, such that
$$\sum_{\Delta(h)} \Lambda\phi l(i,h) < \frac{2L^*\phi(D')}{r}. \tag{18}$$

Then by Corollary 1 of Theorem 3 we have
$$\sum_{\Delta(h)} \mu\{\phi, D(i,h)\} < \frac{\{L^*\phi(D')\}^2}{\pi r^2}. \tag{19}$$

But D' contains a point of E, Q_1 say. Then $D' \subset K'(Q_1, 4r)$, and $4r < 2\eta < \epsilon_2$, so that by (15)
$$L^*\phi(D') < 16\lambda\pi r^2. \tag{20}$$

Hence
$$\sum_{\Delta(h)} \mu\{\phi, D(i,h)\} < 16\lambda L^*\phi(D'), \tag{21}$$

where
$$h = h(P, D')$$

and
$$\Delta(h) = \Delta(h, P, D').$$

To each point P of $\phi(E)$ there corresponds an r as small as we please such that (16) is satisfied. There will by the corollary to Lemma 4 exist a finite disjoint subset of the corresponding spheres covering most of E, that is, there exists a finite set Γ of points P of $\phi(E)$, while to each P there corresponds $r = r(P) < \frac{1}{2}\eta$ such that the spheres $S(P, \frac{3}{2}r(P))$, $P \in \Gamma$ are disjoint and
$$\sum_\Gamma L^*S(P, \tfrac{3}{2}r(P)) > e.$$

Now $S^0(P, 2r(P)) \supset S(P, \frac{3}{2}r(P))$, so that
$$\sum_\Gamma L^*\phi S^{-1}(P, 2r(P)) > e. \tag{22}$$

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Denote by $\nabla(P)$ the set of those domains $D' = D'(P, r(P))$ derived from $S(P, \frac{3}{2}r(P))$ which contain points of E. Then using (21) we have

$$\sum_{P\in\Gamma}\sum_{D'\in\nabla(P)}\sum_{\Delta(h,D')}\mu\{\phi, D(i,h)\} < 16\lambda\sum_{\Gamma}L^*\phi S^{-1}(P, \tfrac{3}{2}r(P) \leqslant 16\lambda L(S) < \epsilon \tag{23}$$

if λ is small.

Consider the domains $D(i,h)$. The corresponding domains $D_1(i,h)$ are disjoint, and hence any two of the domains $D(i,h)$ are either disjoint or one contains the other. Thus we can select a disjoint subset Ω of the domains $D(i,h)$ such that

$$\sum_{P\in\Gamma}\sum_{D'\in\nabla(P)}\sum_{\Delta(h,D')}D(i,h) \subset \sum_{\Omega}D(i,h). \tag{24}$$

Then as $2h \leqslant 3r \leqslant \frac{3}{2}\eta < \epsilon_1$, we have by (14) and (23)

$$\sum_{\Omega}L^*D(i,h) < \epsilon + \sum_{\Omega}\mu\{\phi, D(i,h)\} < 2\epsilon. \tag{25}$$

Let D_1^* be a component of $S^{-1}(P, r(P))$, $P\in\Gamma$. Then if D_1^* contains points of E the domain D' containing it and derived from $S(P, \frac{3}{2}r(P))$ will belong to $\nabla(P)$. That is,

$$ES^{-1}(P, r(P)) \subset \sum_{D'\in\nabla(P)}\sum_{\Delta(h,D')}D(i,h), \tag{26}$$

and so

$$L^*\phi E\{\sum_{\Gamma}S^{-1}(P, r(P))\} \leqslant L^*\sum_{\Omega}\phi D(i,h) < 2\epsilon. \tag{27}$$

But by (16) and (22)

$$L^*\phi\sum_{\Gamma}S^{-1}(P, r(P)) \geqslant \tfrac{1}{16}L^*\phi\sum_{\Gamma}S^{-1}(P, 2r(P)) > \tfrac{1}{16}e. \tag{28}$$

Thus we may take a closed subset X_1 of $\sum_{\Gamma}S^{-1}(P, r(P))$ such that

$$L^*(X_1) > \tfrac{1}{16}e \quad \text{and} \quad L^*(EX_1) < 2\epsilon. \tag{29}$$

We may repeat this argument to construct a closed $X_2 \subset H^0 - X_1$ such that

$$L^*(X_2) > \frac{L^*(E(H^0 - X_1))}{16} \geqslant \frac{e - 2\epsilon}{16}, \tag{30}$$

while

$$L^*(EX_2) < 2\epsilon. \tag{31}$$

Thus we obtain a sequence of closed disjoint sets $X_1, X_2, X_3, \ldots, X_n, \ldots$, where $L^*(X_n) \geqslant \dfrac{e - 2n\epsilon}{16}$ and $L^*(EX_n) < 2\epsilon$.

This process will not terminate till $n > e/2\epsilon$. Take ϵ small and $n = e\epsilon^{-\frac{1}{2}}$. Then

$$L(S) \geqslant \Sigma L^*(X_n) \geqslant \frac{e - 2e\epsilon^{\frac{1}{2}}}{16}e^{\epsilon^{-\frac{1}{2}}},$$

which gives a contradiction if ϵ is small.

8. In § 6 I defined $L^*(X)$ for any subset X of $|S|$; we have

LEMMA 8. *The set of points P of $|S|$ such that $\overline{\lim}\dfrac{L^*S(P,r)}{\pi r^2} > a > 0$ is of finite Λ^2 measure.*

Let X be the set of points P of $|S|$ such that

$$\overline{\lim}\frac{L^*S(P,r)}{\pi r^2} > a.$$

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Then to each point P of X there correspond spheres as small as we please with centre P and such that
$$L^*S(P,r) > a\pi r^2. \tag{32}$$

We can select a set Γ of these spheres covering X and such that no sphere of Γ has its centre in any other. By (3) we can divide Γ into 104 subsets of disjoint spheres. Hence using (32)
$$a\pi \sum_{\Gamma} r^2 < \sum_{\Gamma} L^*S(P,r) \leqslant 104L(S), \tag{33}$$

and so, from the definition of Λ^2, we have
$$\Lambda^2(X) \leqslant \frac{104L(S)}{a} < \infty, \tag{34}$$

whence the lemma follows.

LEMMA 9. *At Λ^2 almost all points of S^0*
$$\overline{\lim} \frac{L^*\phi K'(Q,r)}{\pi r^2} \leqslant 1.$$

If not, there will exist a set X of points in H^0 such that all points of $|\phi(X)|$ are simple in $\phi(X)$ and $\Lambda^2\phi(X) > 0$, while for any $Q \in X$ and a fixed $k > 0$
$$L^*\phi K'(Q,r) > \pi r^2(1+k) \tag{35}$$

for some r as small as we please. Further, by Lemma 8 we may suppose $\Lambda^2\phi(X) < \infty$. Take $\delta > 0$. Let G be any open set containing X. Then to any $Q \in X$ there will correspond arbitrarily small $r = r(Q)$ such that (35) is satisfied and $K'(Q,r) \subset G$. By Vitali's general covering principle a disjoint subset U of the spheres $S(\phi(Q), r(Q))$ will cover Λ^2 almost all of $|\phi(X)|$. We may suppose that $r(Q) < \frac{1}{2}\delta$. Then
$$\Lambda^2_\delta |\phi(X)| \leqslant \tfrac{1}{4}\pi \sum_U \{2r(Q)\}^2 \leqslant \frac{1}{1+k} \sum_U L\phi K'(Q,r) \leqslant \frac{L(G)}{1+k}. \tag{36}$$

Hence
$$\Lambda^2 |\phi(X)| \leqslant \frac{1}{1+k} L^*(X). \tag{37}$$

But $\Lambda^2 |\phi(X)| = \Lambda^2\phi(X) \geqslant L^*(X)$ by Theorem 6 of (5), a contradiction which proves the lemma.

9. LEMMA 10. *If, for all small r, $l(r) > 0$ and*
$$\int_0^r l(t)\,dt \leqslant \frac{l^2(r)}{4\pi} \quad \text{p.p.,} \quad \textit{then} \quad l(r) \geqslant 2\pi r \quad \text{p.p.}$$

Let $\int_0^r l(t)\,dt = f(r)$. Then $f(r) > 0$ and $f'(r)$ exists p.p.,
$$(4\pi)^{\frac{1}{2}} \leqslant \frac{f'(r)}{\{f(r)\}^{\frac{1}{2}}} \quad \text{p.p.,}$$

and so we obtain by integration $(4\pi)^{\frac{1}{2}} r \leqslant 2\{f(r)\}^{\frac{1}{2}}$,
whence the result.

10. LEMMA 11. *If X is a subset of the set J of points Q for which* $\overline{\lim}_{r\to 0} \frac{L^*K'(Q,r)}{\pi r^2} > 0$ *and $L^*(X) = 0$ then $\Lambda^2(X) = 0$.*

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Suppose at all points of X $\overline{\lim}\dfrac{L^*K'(Q,r)}{\pi r^2} > a > 0$, and that $\Lambda^2\phi(X) > 0$. By Lemma 8, $\Lambda^2\phi(X) < \infty$. Take $G \supset X$, an open set. To each point P of $|\phi(X)|$ there correspond r as small as we please such that $L^*K'(Q,r) > a\pi r^2$ and $K'(Q,r) \subset G$. By Vitali's general covering principle a disjoint subset U of the spheres $S(P,r)$ will cover Λ^2 almost all of $|\phi(X)|$. Then

$$\sum_U L^*K'(Q,r) > a\pi \sum_U r^2 > \tfrac{1}{2}a\Lambda^2 |\phi(X)|, \tag{38}$$

provided all the r are small. That is,

$$LG > \tfrac{1}{2}a\Lambda^2 |\phi(X)|, \tag{39}$$

and hence $\tfrac{1}{2}a\Lambda^2 |\phi(X)| \leqslant L^*(X) = 0$,

and the lemma follows.

LEMMA 12. *Given $\epsilon > 0$, then at Λ^2 almost all points (P,Q) of J there exists a simply-connected Jordan domain $R \supset Q$ such that*

$$\overline{\lim_{r\to 0}}\frac{L^*\phi RS^{-1}(P,r)}{\pi r^2} \leqslant 1+\epsilon. \tag{40}$$

For, if not, there will exist $X \subset H^0$ such that $0 < \Lambda^2\phi(X) \leqslant \Lambda^2(J) < \infty$, and for any open $G \supset X$ there will correspond to each $Q \in X$ an $r = r(Q,G)$ as small as we please, such that

$$L^*\phi GS^{-1}(P,r) > (1+\tfrac{1}{2}\epsilon)\pi r^2.$$

We may suppose that all points of $|\phi(X)|$ are simple on $\phi(X)$. A disjoint subset U of the spheres $S(P,r)$, where $(P,Q) \in \phi(X)$ and $r = r(Q,G)$, will cover Λ^2 almost all of $|\phi(X)|$. Then

$$L^*(G) \geqslant \sum_U L^*\phi GS^{-1}(P,r) \geqslant \sum_U (1+\tfrac{1}{2}\epsilon)\pi r^2 > (1+\tfrac{1}{4}\epsilon)\Lambda^2 |\phi(X)|$$

if all the $r(Q,G)$ are small. Hence

$$L^*(X) \geqslant (1+\tfrac{1}{4}\epsilon)\Lambda^2\phi(X) \geqslant (1+\tfrac{1}{4}\epsilon)L^*(X),$$

a contradiction which establishes the lemma.

11. Consider a point $P \in |S|$, and suppose $D_1(i,P,r)$ $(i = 1,2,\ldots)$ is a component of $S^{-1}(P,r)$ whose closure $\overline{D_1(i,P,r)}$ is contained in H^0. Let $D'(i,P,r)$ be the set of interior points of $\overline{D_1(i,P,r)}$. Denote by $D(i,P,r)$ the simply-connected domain consisting of the interior points of the simply-connected closed set obtained from $\overline{D_1(i,P,r)}$ by adding all its interior complementary domains. The domains $D_1(i,P,r)$ are disjoint and so any two of the domains $D(i,P,r)$ will be either disjoint or one contains the other. We may thus select a disjoint subset $\Omega(P,r)$ of the domains $D(i,P,r)$, covering $\sum_i D(i,P,r)$.

I say a point P is quasi-consistent if given $\epsilon > 0$ there exists $\delta > 0$ such that for almost all $r < \delta$

$$\sum_{\Omega(P,r)} \mu\{\phi, D(i,P,r)\} > (1-\epsilon)\sum_i L\phi D'(i,P,r). \tag{41}$$

LEMMA 13. *L^* almost all points of $|S^0|$ are quasi-consistent.*

If not, then using Theorem 3 of (5), there exist $k > 0$ and $X \subset H^0$ such that $L^*(X) > 0$, and if $Q \in X$ while $P = \phi(Q)$ then for some r as small as we please

$$\sum_{\Omega(P,r)} \mu\{\phi, D(i,P,r)\} < (1-k)\sum_i L^*\phi D'(i,P,r), \tag{42}$$

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and $$\Lambda\phi l(i,P,r)<\infty \quad \text{for} \quad D(i,P,r)\in\Omega(P,r). \tag{43}$$

Let H_1 be a disk concentric with H and such that $L^*(H_1^0X)\geqslant\frac{1}{2}L^*(X)$. We may choose r so small that any component of $S^{-1}(P,r)$ whose closure meets the boundary of H will lie entirely in $H-H_1$. By Lemma 4 we can select a disjoint set U of spheres $S(P,r)$ satisfying (42) and (43), and covering L^* almost all of $|\phi(H_1X)|$. Then by (43)

$$L^*\{\sum_U\sum_{\Omega(P,r)} D(i,P,r)\}\geqslant L^*(H_1^0X)\geqslant\tfrac{1}{2}L^*(X).$$

If any two domains $D(i,P,r)$ with different $P\in U$ meet, then one contains the other, as the ϕ-images of their boundaries lie on disjoint spheres. Thus we can select a disjoint subset Γ of the domains $D(i,P,r)$ covering all of $\sum_U\sum_{\Omega(P,r)} D(i,P,r)$. Then

$$L^*\{\sum_\Gamma D(i,P,r)\}\geqslant\tfrac{1}{2}L^*(X)>0.$$

On the other hand, as all the domains $D'(i,P,r)$ are disjoint and contained in $\sum_{\Omega(P,r)} D(i,P,r)$, we have

$$\begin{aligned}\sum_\Gamma\mu\{\phi,D(i,P,r)\}&\leqslant\sum_U\sum_{\Omega(P,r)}\mu\{\phi,D(i,P,r)\}\\&\leqslant(1-k)\sum_U\sum_i L^*\phi D'(i,P,r)\leqslant(1-k)\sum_\Gamma L^*\phi D(i,P,r).\end{aligned}\tag{44}$$

Thus $$\sum_\Gamma L^*\phi D(i,P,r)-\mu\{\phi,D(i,P,r)\}\geqslant\tfrac{1}{2}kL^*(X).$$

But if all the r are taken small the diameters of the domains $D(i,P,r)$ will be small, and so we obtain a contradiction with Lemma 6 of (5).

LEMMA 14. *At Λ^2 almost all points of J we have*

$$\lim_{r\to0}\frac{L^*K'(Q,r)}{\pi r^2}=1.$$

Suppose the contrary. Then as by Lemma 9

$$\overline{\lim_{r\to0}}\frac{L^*K'(Q,r)}{\pi r^2}\leqslant1$$

there will exist $\epsilon>0$ and a set $E_1\subset J$ such that $\Lambda^2E_1>0$ while

$$\overline{\lim_{r\to0}}\frac{L^*K'(Q,r)}{\pi r^2}<(1-3\epsilon^{\frac{1}{3}})^3, \tag{45}$$

and $$\overline{\lim_{r\to0}}\frac{L^*K'(Q,r)}{\pi r^2}>\epsilon^{\frac{1}{3}}(1-\epsilon^{\frac{1}{3}})^{-2}. \tag{46}$$

Now by Lemma 12, to almost any point (P,Q) of J there corresponds a simply-connected Jordan domain $R(Q)\supset Q$ such that

$$\overline{\lim_{r\to0}}\frac{L^*\phi R(Q)\,S^{-1}(P,r)}{\pi r^2}\leqslant1+\epsilon,$$

and hence for sufficiently small $t>0$

$$\overline{\lim_{r\to0}}\frac{L^*\phi\{K(Q,t)\,S^{-1}(P,r)\}}{\pi r^2}\leqslant1+\epsilon. \tag{47}$$

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There will exist $t_0 > 0$ and a set $E_2 \subset E_1$ such that $\Lambda^2 E_2 > 0$, and if $t \leqslant t_0$ while $(P, Q) \in E_2$ then (47) holds. Choose δ so small that if $d(X) < 2\delta$ then $d\phi(X) < t_0$. Take any point $(P_0, Q_0) \in E_2$, and denote the circle of radius δ and centre Q_0 by H_0. If $Q \subset H_0$, then by the choice of δ, $K(Q, t_0) \supset H_0$. Let $E = E_2 \phi(H_0^0)$. By (46) and Lemma 8, $0 < \Lambda^2 E < \infty$. Consider the surface $S_0 = \phi(H_0)$. On this surface we have by (47)

$$\overline{\lim_{r \to 0}} \frac{L^* \phi S^{-1}(P, r)}{\pi r^2} \leqslant 1 + \epsilon \tag{48}$$

for any $(P, Q) \in E$. By Lemmas 11 and 13 we may suppose that all points of E are quasi-consistent on S_0.

We will now consider a fixed point (P, Q) on S_0 belonging to E and will write $D_1(i, P, r)$ as $D_1(i, r)$, and similarly for $D'(i, P, r)$ and so on. This gives us the same notation as in § 8 of (5).

Let $\Theta(h)$ be the class of suffixes i such that $D_1(i, h)$ is a component of $S^{-1}(P, h)$, and $\overline{D_1(i, h)} \subset H_0^0$. Consider $r > h$. Let $D_1(r)$ be a component of $S^{-1}(P, r)$. Then I defined $\Delta(h)$ in (5) as the class of i such that $D_1(i, h) \subset D_1(r)$. That is, $\Delta(h)$ depends on $D_1(r)$, which is one of the domains $D_1(j, r)$ where $j \in \Theta(r)$. Write $\Delta(h) = \Delta(h, j)$. Then

$$\sum_{j \in \Theta(r)} \Delta(h, j) = \Theta(h). \tag{49}$$

By Theorem 3 of (5),
$$\int_0^r \sum_{\Delta(h, j)} \Lambda \phi l(i, h)\, dh \leqslant L^* D'(j, r).$$

Summing over $\Theta(r)$, we obtain

$$\int_0^r \sum_{\Theta(h)} \Lambda \phi l(i, h)\, dh \leqslant \sum_{\Theta(r)} L^* D'(j, r). \tag{50}$$

On the other hand, by (41) and Lemma 3,

$$\sum_{\Theta(r)} \frac{\{\Lambda \phi l(i, r)\}^2}{4\pi} \geqslant \sum_{\Theta(r)} \mu\{\phi, D(i, r)\} \geqslant \sum_{\Omega(r)} \mu\{\phi, D(i, r)\} > (1 - \epsilon) \sum_{\Theta(r)} L^* D'(j, r)$$
$$\geqslant (1 - \epsilon) \int_0^r \sum_{\Theta(h)} \Lambda \phi l(i, h)\, dh \tag{51}$$

for almost all small r. By Lemma 10 applied to $l(t) = \frac{1}{1 - \epsilon} \sum_{\Theta(t)} l(i, t)$, we obtain

$$\sum_{\Theta(r)} \Lambda \phi l(i, r) \geqslant 2\pi r (1 - \epsilon) \quad \text{p.p. in } r. \tag{52}$$

Whence by (48) and (50) for small r

$$\pi r^2 (1 + \epsilon) \geqslant \int_0^r \sum_{\Theta(h)} \Lambda \phi l(i, h)\, dh \geqslant \pi r^2 (1 - \epsilon) \quad \text{p.p.} \tag{53}$$

Now write $k = \epsilon^{\frac{1}{4}}$, and consider the interval $[r, r(1 - k)]$. By (52) and (53) there exists ρ, $r > \rho > r(1 - k)$, such that (51) holds for ρ and

$$2\pi\rho(1 - \epsilon) \leqslant \sum_{\Theta(\rho)} \Lambda \phi l(i, \rho) \leqslant 2\pi\rho \left(1 + \frac{\epsilon}{k}\right). \tag{54}$$

But if $\Sigma a_i = b$ and $\Sigma a_i^2 = b^2$, there exists i_0 such that $a_{i_0} = b$. Elaborating this, we obtain from (51), (53) and (54) that there exists $i_0 \in \Theta(\rho)$, such that

$$\Lambda \phi l(i_0, \rho) \geqslant 2\pi\rho \frac{(1 - 2\epsilon)}{1 + k}. \tag{55}$$

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But then
$$\sum_{\Theta(\rho)-i_0} \Lambda\phi l(i,\rho) \leqslant 2\pi\rho\epsilon\left(1+\frac{2}{k}\right), \tag{56}$$

and thus by (51)
$$\sum_{\Theta(\rho)-i_0} \mu\{\phi, D(i,\rho)\} \leqslant \pi\rho^2\epsilon^2\left(1+\frac{2}{k}\right)^2 \leqslant \pi\rho^2\epsilon, \tag{57}$$

if ϵ is small.

Each interval $\{r(1-k)^n, r(1-k)^{n+1}\}$ will contain ρ_n such that (54) holds, and to each ρ_n there will correspond $i_0(n)$ such that (55)–(57) hold. Denote $\Theta(\rho_n)-i_0(n)$ by $\Theta'(\rho_n)$. To each $(P,Q)\in E$ we have thus made correspond a sequence $\rho_n\to 0$ and classes of suffixes $\Theta'(\rho_n)$.

Then, except possibly for a set of points (P,Q) of E of Λ^2 measure zero, we will have
$$\overline{\lim_{n\to\infty}} \frac{\sum_{\Theta'(\rho_n)} L^*D'(i,\rho_n)}{\rho_n^2} \leqslant k. \tag{58}$$

For if not, there exists $X\subset E$, $\Lambda^2\phi(X)>0$, such that if $Q\in X$ there will exist an infinite subsequence of the corresponding ρ_n such that
$$\sum_{\Theta'(\rho_n)} L^*D'(i,P,\rho_n) > k\rho_n^2. \tag{59}$$

(I have re-introduced the variable P as it is again relevant.) As $X\subset E$, $\Lambda^2\phi(X)<\infty$, and hence a disjoint subset U of the spheres $S(P,\rho_n)$ will cover Λ^2 almost all of $|\phi(X)|$. Then
$$\sum_U \sum_{\Theta'(P,\rho_n)} L^*D'(i,P,\rho_n) > k\sum_U \rho_n^2 \geqslant \frac{k}{2\pi}\Lambda^2|\phi(X)|, \tag{60}$$

provided we choose all the ρ_n small. If any two domains $D(i,P,\rho_n)$, $P\in U$, $i\in\Theta'(P,\rho_n)$ meet then one will contain the other. Thus a disjoint subset Ω of these domains will cover
$$\sum_U \sum_{\Theta'(P,\rho_n)} D(i,P,\rho_n), \tag{61}$$

that is,
$$\sum_\Omega L^*D(i,P,\rho_n) \geqslant \tfrac{1}{2}k\Lambda^2|\phi(X)|. \tag{62}$$

On the other hand, using (57) and (59),
$$\sum_\Omega \mu\{\phi, D(i,P,\rho_n)\} \leqslant \sum_U \sum_{\Theta'(P,\rho_n)} \mu\{\phi, D(i,P,\rho_n)\}$$
$$\leqslant \frac{\pi\epsilon}{k}\sum_U \sum_{\Theta'(P,\rho_n)} L^*D'(i,P,\rho_n) \leqslant \pi k\sum_\Omega L^*D(i,P,\rho_n), \tag{63}$$

as $k=\epsilon^{\frac{1}{2}}$. Hence
$$\sum_\Omega L^*D(i,P,\rho_n) - \mu\{\phi, D(i,P,\rho_n)\} \geqslant \tfrac{1}{2}k(1-\pi k)\Lambda^2|\phi(X)|,$$

which gives a contradiction with Lemma 6 of (5), provided all the ρ_n are small.

We may therefore assume that (58) holds at the point (P,Q) we are considering. But by (51) and (53)
$$L^*D'(i_0(n),\rho_n) > \pi\rho_n^2(1-3k) \tag{64}$$

for large n. Now consider $K'(Q,\rho_n)$. It is one of the domains $D'(i,\rho_n)$, $i\in\Theta(\rho_n)$. By (58) and (64), for large n, either
$$\frac{L^*K'(Q,\rho_n)}{\pi\rho_n^2} > 1-3k,$$

or
$$\frac{L^*K'(Q,\rho_n)}{\pi\rho_n^2} < k,$$

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according as $K'(Q,\rho_n)$ is $D'(i_0(n),\rho_n)$ or not. But if $\dfrac{L^*K'(Q,\rho_n)}{\pi\rho_n^2} < k$, then

$$\frac{L^*K'(Q,\rho_{n+1})}{\pi\rho_{n+1}^2} < \frac{k}{(1-k)^2} < 1-3k,$$

provided k is small. Hence either

$$\overline{\lim_{n\to\infty}} \frac{L^*K'(Q,\rho_n)}{\pi\rho_n^2} \leqslant k, \tag{65}$$

or

$$\underline{\lim_{n\to\infty}} \frac{L^*K'(Q,\rho_n)}{\pi\rho_n^2} \geqslant 1-3k. \tag{66}$$

Take any small $r'>0$. It will be in some interval $[r(1-k)^n, r(1-k)^{n+1}]$, and

$$\frac{1}{(1-k)^2}\frac{L^*K'(Q,\rho_n)}{\pi\rho_n^2} \geqslant \frac{L^*K'(Q,r')}{\pi r'^2} \geqslant (1-k)^2 \frac{L^*K'(Q,\rho_{n+1})}{\pi\rho_{n+1}^2}. \tag{67}$$

Hence by (65) and (66), either

$$\overline{\lim_{r'\to 0}} \frac{L^*K'(Q,r')}{\pi r'^2} \leqslant \frac{k}{(1-k)^2} = \epsilon^{\frac{1}{2}}(1-\epsilon^{\frac{1}{2}})^2, \tag{68}$$

or

$$\underline{\lim_{r'\to 0}} \frac{L^*K'(Q,r')}{\pi r'^2} \geqslant (1-3k)^3 = (1-3\epsilon^{\frac{1}{2}})^3. \tag{69}$$

But (68) and (69) contradict (45) and (46), and hence the lemma is established.

LEMMA 15. *Λ^2 almost all points of J are consistent.*

For if not, then using Lemma 14 we obtain $\lambda>0$ and $X \subset H^0$, such that $\Lambda^2\phi(X)>0$, and for any $Q \in X$ there will exist a set of arbitrarily small r, of positive measure, such that

$$\mu\{\phi, K(Q,r)\} < (1-\lambda)\, L^*\phi K(Q,r) \tag{70}$$

and

$$L^*K'(Q,r) > \tfrac{1}{2}\pi r^2. \tag{71}$$

Then as by Lemma 2 $\Lambda\phi l(Q,r) < \infty$ except for a set of r of measure zero, we may further assume that

$$\Lambda\phi l(Q,r) < \infty. \tag{72}$$

Consider the spheres $S(P,r)$, where $P=\phi(Q)$ and (70)–(72) hold. A disjoint subset U of these spheres will by Vitali's general covering principle cover Λ^2 almost all of $|\phi(X)|$. Provided we take all the r small, we will then have

$$\sum_U \pi r^2 \geqslant \tfrac{1}{2}\Lambda^2 |\phi(X)| > 0. \tag{73}$$

A disjoint subset V of the domains $K(Q,r)$ corresponding to the spheres of U will cover all of $\sum_U K'(Q,r)$, as any two domains $K(Q,r)$ will either be disjoint or one will contain the other. Then:

$$\sum_V L^*K(Q,r) \geqslant \sum_U L^*K'(Q,r) \geqslant \tfrac{1}{2}\sum_U \pi r^2 \geqslant \tfrac{1}{4}\Lambda^2 |\phi(X)|. \tag{74}$$

On the other hand,

$$\sum_V L^*K(Q,r) - \mu\{\phi, K(Q,r)\} \geqslant \lambda \sum_V L^*K(Q,r). \tag{75}$$

Equations (72), (74) and (75) give a contradiction with Lemma 6 of (5) if all the r are taken small. And so the lemma is established.

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12. DEFINITION. *Given a plane π and a finitely connected domain $D \subset H^0\, I$ define*

$$\pi\{\phi, D\}$$

as the lower bound of the Lebesgue area of the flat surface which is the projection on π of $\phi^*(D)$, where $\phi^* = \phi$ on $H - D$ and is continuous.

LEMMA 16. *At Λ^2 almost all consistent points we have*

$$L\phi K(Q,r) \sim \pi r^2 \quad as \quad r \to 0 \tag{76}$$

and there exists a plane π such that for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\pi\{\phi, K(Q,r)\} > (1-\epsilon)\,\pi r^2 \tag{77}$$

for almost all $r < \delta$. And further, given $k > 0$, for sufficiently small r there will exist a set of ρ of positive measure such that $r > \rho > (1-k)\,r$ and

$$2\pi\rho(1-\epsilon) \leqslant \Lambda\phi l(Q,\rho) \leqslant 2\pi\rho(1+\epsilon). \tag{78}$$

Take $\epsilon > 0$. By Lemma 9 we may suppose

$$\overline{\lim} \frac{L^*K'(Q,r)}{\pi r^2} \leqslant 1. \tag{79}$$

Then by consistency and Lemmas 2 and 3 we have for almost all sufficiently small r,

$$\frac{\{\Lambda\phi l(Q,r)\}^2}{4\pi} \geqslant \mu\{\phi, K(Q,r)\} \geqslant (1-\epsilon)\,L^*K(Q,r)$$

$$\geqslant (1-\epsilon)\,L^*K'(Q,r) \geqslant (1-\epsilon)\int_0^r \Lambda\phi l(Q,t)\,dt. \tag{80}$$

Thus by Lemma 10 $$\Lambda\phi l(Q,r) \geqslant 2\pi r(1-\epsilon). \tag{81}$$

Hence by (79) $$L^*K'(Q,r) \sim \int_0^r \Lambda\phi l(Q,t)\,dt \sim \pi r^2 \quad \text{as} \quad r \to 0. \tag{82}$$

Take $k > 0$ and $\epsilon_0 > 0$. Suppose that $\epsilon/k\epsilon_0$ is small and that for almost all $r > \rho > r(1-k)$

$$\Lambda\phi l(Q,\rho) > 2\pi\rho(1+\epsilon_0). \tag{83}$$

Then $$\int_0^r \Lambda\phi l(Q,t)\,dt \geqslant \pi r^2(1-k)^2\,(1-\epsilon) + \pi r^2(2k-k^2)\,(1+\epsilon_0) \geqslant \pi r^2(1+k\epsilon_0) \tag{84}$$

for small ϵ. (84) gives a contradiction with (82), and hence, using (81), (78) is established.

By (80), we have, for almost any ρ satisfying (78),

$$\pi\rho^2\frac{(1+\epsilon)^2}{1-\epsilon} \geqslant L^*K(Q,\rho) \geqslant \pi\rho^2(1-\epsilon). \tag{85}$$

Now in any interval $[r, r(1-k)]$ there will exist a set of ρ, of positive measure, satisfying (78), and hence by (80)

$$\frac{\pi r^2}{(1-k)^2}\frac{(1+\epsilon)^2}{1-\epsilon} \geqslant L^*K(Q,r) \geqslant \pi r^2(1-\epsilon), \tag{86}$$

which as k is arbitrary implies (76). Take $\epsilon_1 > 0$. By (80) we have for almost all r,

$$\mu\{\phi, K(Q,r)\} \geqslant \pi r^2(1-\epsilon)^2, \tag{87}$$

and hence if r satisfies (78) we obtain by Lemma 3 that there exists a plane π_r such that

$$\pi_r\{\phi, K(Q,r)\} \geqslant \pi r^2(1-\epsilon_1). \tag{88}$$

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Thus using (76) we see that there will be r as small as we please for which

$$\frac{L^*K(Q,r)}{\pi_r\{\phi, K(Q,r)\}} \leqslant 1+\epsilon. \tag{89}$$

Suppose now $T(\alpha)$ is the set of points (P', Q') as above such that for any plane π there exists a set of arbitrarily small r' of positive measure such that

$$\frac{L^*\phi K(Q',r')}{\pi\{\phi, K(Q',r')\}} > 1+\alpha. \tag{90}$$

We may assume that simultaneously

$$\Lambda\phi l(Q',r') < \infty, \tag{91}$$

as this fails at most for a set of r' of measure zero. By (82) and Lemma 8 we have

$$\Lambda^2 T(\alpha) < \infty. \tag{92}$$

Let $(P,Q) \epsilon T(\alpha)$, and consider an r such that (89) holds. I will show that

$$\frac{\Lambda^2 \,|\, T(\alpha)\,\phi K(Q,r)\,|}{L^*K(Q,r)} < \frac{2(1+\alpha)\,\epsilon}{(1+\epsilon)\,\alpha}. \tag{93}$$

Take $\delta > 0$. Then to each (P', Q') belonging to $T(\alpha)\,\phi K(Q,r)$ there will correspond $r' < \delta$ as small as we please such that $K(Q',r') \subset K(Q,r)$ and equations (90) and (91) are satisfied, where π is the plane π_r of (89). Take $\epsilon' > 0$. By Vitali's general covering principle we can cover Λ^2 almost all of $|\, T(\alpha)\,\phi K(Q,r)\,|$ by a disjoint subset U of the spheres $S(P',r')$. Then if δ is small we may take a finite subset U_0 of U such that

$$\tfrac{1}{4}\pi \sum_{U_0} (2r')^2 \geqslant \Lambda^2 \,|\, T(\alpha)\,\phi K(Q,r)\,| - \epsilon'. \tag{94}$$

Also, by (82) we may choose r' so small that

$$L^*K'(Q',r') < \pi r'^2(1-\epsilon'), \quad S(P',r') \epsilon U_0. \tag{95}$$

Thus $$\sum_{U_0} L^*K'(Q',r') > (1-\epsilon')\{\Lambda^2 \,|\, T(\alpha)\phi K(Q,r)\,| - \epsilon'\} > \tfrac{1}{2}\Lambda^2 \,|\, T(\alpha)\phi K(Q,r)\,|, \tag{96}$$

if ϵ' is small. Now any pair of domains $K(Q',r')$ such that $S(P',r') \epsilon U_0$ are either disjoint or one is contained in the other. Thus we can select a disjoint subset U_1 of the domains $K(Q',r')$ of U_0 such that

$$\sum_{U_1} K(Q',r') \supset \sum_{U_0} K(Q',r'). \tag{97}$$

Thus by (96) $$\sum_{U_1} L^*K(Q',r') > \tfrac{1}{2}\Lambda^2 \,|\, T(\alpha)\,\phi K(Q,r)\,|. \tag{98}$$

Now by (90) $$\sum_{U_1} \pi_r\{\phi, K(Q',r')\} < \frac{1}{1+\alpha} \sum_{U_1} L^*K(Q',r'). \tag{99}$$

But by (91), and the definition of $\pi\{\phi, D\}$,

$$\pi_r\{\phi, K(Q,r)\} \leqslant \sum_{U_1} \pi_r\{\phi, K(Q',r')\} + L\phi\{K(Q,r) - \sum_{U_1} \overline{K(Q',r')}\}. \tag{100}$$

Thus by (89), (99) and (100)

$$\frac{L^*K(Q,r)}{1+\epsilon} \leqslant \frac{1}{1+\alpha} \sum_{U_1} L^*K(Q',r') + L^*K\{Q,r) - \sum_{U_1} L^*K(Q',r'). \tag{101}$$

From (98) and (101) we at once obtain equation (93).

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Take $\eta > 0$. Suppose $\Lambda^2 | T(\alpha) | > a > 0$. We may suppose that all points of $| T(\alpha) |$ are simple on $T(\alpha)$, for if not we can consider a subset of $T(\alpha)$ containing just one point (P, Q) for every P in $| T(\alpha) |$.

Now exactly as we constructed U_1 we may construct a set V_1 of disjoint domains $K(Q, r)$, satisfying equation (89), and such that

$$L^* \sum_{V_1} K(Q, r) > \tfrac{1}{2}\alpha. \tag{102}$$

Then by (93)

$$\Lambda^2 T(\alpha) \sum_{V_1} \phi K(Q, r) = \Lambda^2 | T(\alpha) \sum_{V_1} \phi K(Q, r) |$$
$$< \frac{(1+\alpha)\epsilon}{2\alpha(1+\epsilon)} \sum_{V_1} L^* K(Q, r) < \eta \sum_{V_1} L^* K(Q, r) \tag{103}$$

if ϵ is small. Then we can take a closed set $X_1 \subset \sum_{V_1} K(Q, r)$, such that

$$L^*(X_1) > \tfrac{1}{2}a \tag{104}$$

and

$$\Lambda^2\{T(\alpha)\, \phi(X_1)\} < \eta L^* X_1. \tag{105}$$

We may now apply the same argument to $\phi(H^0 - X_1)$, an open set on S, to construct $X_2 \subset H^0 - X_1$ such that

$$L^*(X_2) > \tfrac{1}{2}\Lambda^2 T(\alpha)\, \phi(H^0 - X_1) > \tfrac{1}{2}(a - \eta L^*(X_1)) \tag{106}$$

and

$$\Lambda^2\{T(\alpha)\, \phi(X_2)\} < \tfrac{1}{2}\eta L^*(X_2). \tag{107}$$

Repeat this to obtain

$$L^*(X_n) > \tfrac{1}{2}(a - \eta \sum_{1}^{n-1} 2^{-k+1} L^*(X_k)), \tag{108}$$

$$\Lambda^2\{T(\alpha)\, \phi(X_n)\} < 2^{-n}\eta L^*(X_n). \tag{109}$$

Suppose

$$\eta < \frac{a}{4L(S)}. \tag{110}$$

Then the process can never terminate, and for all X_n

$$L^*(X_n) > \tfrac{1}{4}a, \tag{111}$$

which as all the X_n are disjoint gives a contradiction. Thus $\Lambda^2 T(\alpha) = 0$ for all $\alpha > 0$. Hence at Λ^2 almost all consistent points (P, Q) there will exist a plane π such that for any $\alpha > 0$ there exists $\delta > 0$ such that for almost all $r < \delta$

$$\pi\{\phi, K(Q, r)\} > \frac{1}{1+\alpha} L^* K(Q, r). \tag{112}$$

(112) together with (76) will give (77), and so the proof of the lemma is complete.

LEMMA 17. *At Λ^2 almost all consistent points $\phi l(Q, \rho)$ will be within $o(\rho)$ of π for almost all ρ satisfying* (78), *where π is the plane satisfying* (77).

Take $\delta > 0$. Then by (76), (78), consistency and Lemma 3, we see that there exists a plane π_ρ such that

$$\phi l(Q, \rho) \text{ is within } \delta\rho \text{ of } \pi_\rho, \tag{113}$$

while

$$\pi_\rho\{\phi, K(Q, \rho)\} > (1 - \delta)\pi\rho^2. \tag{114}$$

Also, by (76) and (77), if ρ is small

$$\pi\{\phi, K(Q, \rho)\} > (1 - \delta)\pi\rho^2 \tag{115}$$

and

$$| L^* K(Q, \rho) - \pi\rho^2 | < \delta\pi\rho^2. \tag{116}$$

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I will now show that (114)–(116) imply that the angle between π and π_ρ is small, from which the lemma will follow at once.

By Lemma 3 of (5), as $\Lambda\phi l(Q,\rho)<\infty$, $K(Q,\rho)$ is a Jordan domain. Let Ω be a polygon approximating to $\phi K(Q,\rho)$ both in area and Fréchet distance. Then the projection of Ω on any plane will be near the projection of $\phi K(Q,\rho)$ on that plane. Let Ω_1, Ω_2 be the projections of Ω on π and π_ρ respectively. Then

$$\Lambda^2\Omega \leqslant \pi\rho^2(1+2\delta), \tag{116'}$$

$$\Lambda^2\Omega_1 \geqslant \pi\rho^2(1-2\delta), \tag{115'}$$

$$\Lambda^2\Omega_2 \geqslant \pi\rho^2(1-2\delta), \tag{114'}$$

by the definition of Lebesgue area and of $\pi\{\phi, D\}$. From (114')–(116') it is at once evident that the angle between π and π_ρ is small.

13. Lemma 18. *If*

$$\Lambda^2 S<\infty,$$

Lemma 16 *may be strengthened by replacing equation* (76) *by*

$$\Lambda^2\phi K(Q,r)\sim\pi r^2. \tag{117}$$

At almost all points of S^0

$$\overline{\lim}\frac{\Lambda^2\phi K'(Q,r)}{\pi r^2}\leqslant 1. \tag{118}$$

Hence when (82) holds we may suppose

$$\Lambda^2\phi K'(Q,r)\sim\pi r^2. \tag{119}$$

Take $k>0$. Let $U(Q,r)$ be the class of interior domains $D(i,Q,r)$ of $K'(Q,r)$. Let $l(i,Q,r)$ be the boundary of $D(i,Q,r)$. By Lemma 1 of (5),

$$\int_0^r \sum_{U(Q,t)} \Lambda\phi l(i,Q,t)+\Lambda\phi l(Q,t)\,dt\leqslant\Lambda^2\phi K'(Q,r). \tag{120}$$

Hence by (81) we have

$$\sum_{U(Q,t)} \Lambda\phi l(i,Q,t)+\Lambda\phi l(Q,t)\leqslant 2\pi t(1+\epsilon) \tag{121}$$

for some t in any interval of the form

$$r>t>r(1-k), \tag{122}$$

so that

$$\Lambda\phi l(Q,t)\leqslant 2\pi t(1+\epsilon) \tag{123}$$

and

$$\sum_{U(Q,t)} \Lambda\phi l(i,Q,t)\leqslant 4\epsilon t. \tag{124}$$

Take $\delta>0$. Then, by (123) and Lemma 3, (88) will hold for t. Thus, using (119), (124) and Lemma 3, we obtain for t small,

$$\frac{\Lambda^2K'(Q,t)}{\pi_t\{\phi,K(Q,t)\}-\sum_{U(Q,t)}\mu\{\phi,D(i,Q,t)\}}\leqslant 1+\delta. \tag{125}$$

Consider now the points $T'(\alpha)$ as above, where for any plane π there will exist a set of arbitrarily small r' of positive measure such that

$$\frac{\Lambda^2\phi K(Q',r')}{\pi\{\phi,K(Q',r')\}}>1+\alpha. \tag{126}$$

We may assume that simultaneously

$$\Lambda\phi l(Q',r')<\infty, \tag{127}$$

as this holds p.p.

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I now use (125)–(127) as I used (89)–(91) in Lemma 16 to construct a finite disjoint set U_1 of domains $K(Q', r')$ satisfying (126) and (127) and such that

$$\sum_{U_1} K(Q', r') \subset K'(Q, t) \tag{128}$$

and

$$\sum_{U_1} \Lambda^2 K(Q', r') > \tfrac{1}{2}\Lambda^2 \left| T'(\alpha)\, \phi K'(Q, t) \right|. \tag{129}$$

(Compare (98).) Then, by (124), (127) and the definition of $\pi\{\phi, D\}$, we have

$$\pi_t\{\phi, K(Q,t)\} \leqslant \sum_{U(Q,t)} \mu\{\phi, D(i, Q, t)\} + \sum_{U_1} \pi_t\{\phi, K(Q', r')\} + \Lambda^2 \phi\{K'(Q,t) - \overline{\sum_{U_1} K(Q', r')}\}. \tag{130}$$

(Compare (100).) Hence, using (125), (126) and (130), we obtain

$$\frac{\Lambda^2 \phi K'(Q,t)}{1+\delta} \leqslant \sum_{U_1} \frac{\Lambda^2 \phi K(Q', r')}{1+\alpha} + \Lambda^2 \phi K'(Q,t) - \sum_{U_1} \Lambda^2 \phi K(Q', r'). \tag{131}$$

(Compare (101).) Thus, using (129),

$$\frac{\Lambda^2 \left| T'(\alpha)\, \phi K'(Q,t) \right|}{\Lambda^2 \phi K'(Q,t)} < \frac{2(1+\alpha)\,\delta}{\alpha(1+\delta)}. \tag{132}$$

We now again proceed as in Lemma 16 to obtain

$$\Lambda^2 T'(\alpha) = 0. \tag{133}$$

Hence from the definition of $T'(\alpha)$ we see that at Λ^2 almost all consistent points, given $\alpha > 0$, there exists $\epsilon > 0$ such that

$$\frac{\Lambda^2 \phi K(Q, r)}{L^* K(Q, r)} < 1 + \alpha \tag{134}$$

for almost all $r < \epsilon$. This, together with (76), implies that $1 - \alpha < \dfrac{\Lambda^2 \phi K(Q, r)}{\pi r^2} < 1 + \alpha$ for almost all small r, whence the lemma follows at once.

The reader's attention is drawn to the difference in the proofs of Lemmas 16 and 18, due to the difference between equations (89) and (125). In Lemma 18 (89) is not available, as it depends on a direct appeal to consistency establishing (76) first. On the other hand, (125) depends on (119), which was not available in Lemma 16.

14. LEMMA 19. *Given a cylinder C, and a parametric contour Γ of finite length lying on the surface of C and deformable into a point thereon, then there exists a parametric surface subtending Γ and lying on the surface of C.*

Consider a point P_0 on the contour. Let $\theta(P)$ be the angular separation of P from P_0 taken along the contour in a given sense. $\theta(P)$ is uniquely defined, for the angular separation of P from itself is $2\pi k$ for some integer k, and is continuous with respect to continuous deformations of Γ, and hence zero. Let r be the radius of C. Then

$$|\theta(P)| \leqslant \frac{\Lambda \Gamma}{2\pi r}. \tag{135}$$

Let P_1, P_2 be two points of Γ where $\theta(P)$ has its maximum and minimum respectively. Then $\Gamma = \Gamma_1 + \Gamma_2$, where each Γ_i $(i = 1, 2)$ is an arc joining P_1, P_2. Suppose both are taken in the sense $P_1 \to P_2$. Let Q_{11}, Q_{12} be the first points of Γ_1 and Γ_2 respectively on the generator of C opposite to that through P_1. If either does not exist let both be P_2.

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Join Q_{11} and Q_{12} along the generator by a curve Γ'. Denote the arc of Γ_i $(i = 1, 2)$ joining P_1 to Q_{1i} by Γ_i'.

The contour $\Gamma' + \Gamma_1' + \Gamma_2'$ lies on one-half of the cylinder between two opposite generators. This is topologically a plane. Hence there is a surface S' on the cylinder subtending $\Gamma' + \Gamma_1' + \Gamma_2'$. On the other hand, the variation of $\theta(P)$ on the part of the contour consisting of $\Gamma_1 - \Gamma_1' + \Gamma' + \Gamma_2 - \Gamma_2'$ is at least π less than the variation of $\theta(P)$ on Γ, and so if we repeat the above procedure we will exhaust Γ in a finite number of steps. The sum of the surfaces, such as S' constructed above, will be a surface subtending Γ.

15. *I will now consider a fixed point* (P_0, Q_0) *satisfying the conclusions of Lemmas* 16 *and* 17, *and in the case* $\Lambda^2 S < \infty$ *those of Lemma* 18 *also.*

According to Lemma 16 there is a fixed plane associated with (P_0, Q_0). Let it be π_0. Then I will write

$$\left.\begin{aligned} M(Q_0, \theta_0, r) &= M(r), \\ C(P_0, \theta_0, r) &= C(r), \\ m(Q_0, \theta_0, r) &= m(r), \end{aligned}\right\} \tag{136}$$

where θ_0 is the direction perpendicular to π_0.

LEMMA 20. π_0 *is an approximate tangential plane in the sense of Definition* 2 *at* (P_0, Q_0).

Take $k > 0$. By Lemmas 16 and 17 there will exist R, corresponding to any sufficiently small r, such that

$$\frac{r}{(1-k)^2} > R > \frac{r}{1-k} \tag{137}$$

and $\phi l(Q_0, R)$ is very near π_0. Then $\phi l(Q_0, R)$ will lie entirely outside $C(r)$, and so $l(Q_0, R)$ will not meet $M'(Q_0, \theta_0, r)$. If $l(Q_0, R)$ were to lie in one of the interior domains of $\overline{M'(Q_0, \theta_0, r)}$, so would $K(Q_0, R)$, and hence Q_0, which is false. Hence $l(Q_0, R)$ is contained in $H - \overline{M(r)}$, and so

$$M(r) \subset K(Q_0, R). \tag{138}$$

Thus for small r, $M(r)$ will not meet the boundary of H, and so $\phi m(r)$ lies on the surface of $C(r)$. Moreover,

$$L^*M(r) \leqslant L^*K(Q_0, R), \tag{139}$$

so that for small r we have by equation (137)

$$L^*M(r) \leqslant \frac{\pi r^2}{(1-k)^5}. \tag{140}$$

By the precise method of Lemma 4 of (5) it can be shown that

$$\int_0^r \Lambda\phi m(t)\, dt \leqslant L^*M(r). \tag{141}$$

Hence for almost all r, $m(r)$ is a simple closed Jordan curve and $\Lambda\phi m(r) < \infty$.

$\phi m(r)$ *cannot be deformed into a point on the surface of* $C(r)$. For if it can, by Lemma 19,

$$\pi_0\{\phi, M(r)\} = 0. \tag{142}$$

But then

$$\begin{aligned} \pi_0\{\phi, K(Q_0, R)\} &\leqslant \pi_0\{\phi, K(Q_0, R) - M(r)\} + \pi_0\{\phi, M(r)\} \leqslant \pi_0\{\phi, K(Q_0, R) - M(r)\} \\ &\leqslant L^*\{K(Q_0, R) - M(r)\} \leqslant L^*\{K(Q_0, R) - K(Q_0, r)\} \leqslant \pi r^2((1-k)^{-2} - 1) + o(r^2), \end{aligned}$$

which contradicts equation (77).

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As $\phi m(r)$ cannot be deformed into a point on the surface of $C(r)$,

$$L^*M(r) \geqslant \pi r^2. \tag{143}$$

As k is arbitrary, (140) and (143) imply that

$$L^*M(r) \sim \pi r^2, \tag{144}$$

and so the lemma is established.

LEMMA 21. *If $\Lambda^2 S < \infty$, π_0 is an approximate tangential plane in the sense of definition 2′ at (P_0, Q_0).*

The proof is the same as that of Lemma 20, using (117) instead of (76).

16. Let (r, θ) be polar coordinates in π_0 with origin at P_0. Denote by Θ the plane containing the normal to π_0 at P_0 and the radius vector θ_0. Define Θ^{-1} by

$$\Theta^{-1} = \phi^{-1}(\Theta S). \tag{145}$$

Let $c(r)$ be the open circle of centre P_0 and radius r, while $b(r)$ is its circumference. Then

LEMMA 22. *If E is a set of points in π,*

$$\int_0^r \Lambda E b(t)\, dt = \Lambda^2 E c(r). \tag{146}$$

If E consists of simple points this is clear, and if there are multiple points the result follows by addition.

LEMMA 23. *If E is any set and $E(\theta)$ is its intersection with Θ, then, if all points of E are outside $C(r_0)$, we have*

$$r_0 \int_0^r \Lambda E(\theta)\, d\theta \leqslant \Lambda^2 E. \tag{147}$$

The proof is similar to that of Lemma 1 of (5).

LEMMA 24. *Suppose $\epsilon(r) \to 0+$. Then if*

$$L^*M(r) \leqslant \pi r^2 + \epsilon^4(r)\, r^2 \tag{148}$$

for small r, we have for any set $E \subset H^0$

$$\left| L^*\phi EM(r) - \int_0^r \Lambda \phi E m(t)\, dt \right| \leqslant 7\epsilon^2(r)\, r^2 \tag{149}$$

for small r.

Take three coordinate planes, the first being π_0, and denote projections on to them by P_1, P_2, P_3. Then

$$LP_1\{\phi M(r)\} \geqslant \pi r^2. \tag{150}$$

Now by (4), for any surface S,

$$\sum_i \{LP_i(S)\}^2 \leqslant \{L(S)\}^2 \leqslant \{\sum_i LP_i(S)\}^2, \tag{151}$$

whence for any E

$$L^*\phi EM(r) \leqslant \sum_i L^*P_i\{\phi EM(r)\}. \tag{152}$$

But using (151) we have by (148)

$$LP_2\phi M(r) + LP_3\phi M(r) < 6\epsilon^2(r)\, r^2. \tag{153}$$

Hence, using Lemma 22,

$$\begin{aligned} L^*\phi EM(r) < L^*P_1\{\phi EM(r)\} + 6\epsilon^2(r)\, r^2 &\leqslant \Lambda^2 P_1\{\phi EM(r)\} \\ &+ 6r^2\epsilon^2(r) \leqslant \int_0^r \Lambda P_1 \phi(E)\, b(t)\, dt + 6r^2\epsilon^2(r). \end{aligned} \tag{154}$$

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But by Lemma 22

$$\int_0^r \Lambda P_1\{\phi M(r)\}\, b(t) - \Lambda P_1\{\phi m(t)\}\, dt \leqslant L^*M(r) - \pi r^2 \leqslant r^2\epsilon^4(r). \tag{155}$$

Hence using (154)

$$L^*\phi E M(r) \leqslant \int_0^r \Lambda P_1\, \phi E m(t)\, dt + 7r^2\epsilon^2(r) \leqslant \int_0^r \Lambda \phi E m(t)\, dt + 7r^2\epsilon^2(r). \tag{156}$$

The same equation will hold for $E' = H - E$. But

$$\int_0^r \Lambda \phi m(t)\, dt \leqslant L^*M(r). \tag{157}$$

(Compare (141).) The lemma is therefore established.

LEMMA 25. *If π_0 is a tangential plane in the sense of Definition* 2 *at (P_0, Q_0), it is a tangential plane in the sense of Definition* 1 *also.*

Take r such that $\Lambda\phi m(r)$ and $\Lambda\phi m(\frac{1}{2}r)$ are both finite, so that $m(r)$ and $m(\frac{1}{2}r)$ are simple closed Jordan curves. Suppose, further, that the plane Θ contains exactly two points of $\phi m(r)$, (P_1, Q_1) and (P_2, Q_2) say. Denote the annulus between $m(r)$ and $m(\frac{1}{2}r)$ by R, and let $S_n = \phi_n(R)$ be a topological annulus subtending $\phi m(r)$ and $\phi m(\frac{1}{2}r)$. Then either there is a continuum in $\bar{R}\phi_n^{-1}\Theta$ joining Q_1 and Q_2 or each of them is joined by such a continuum to $m(\frac{1}{2}r)$.

For if not, one of the points Q_1 and Q_2 will be the only point of the component of $\bar{R}\phi_n^{-1}(\Theta)$ containing it, which is on the boundary of R. Suppose this point is Q_1. Then there will be a splitting of $\bar{R}\phi_n^{-1}(\Theta) + m(\frac{1}{2}r)$ into two disjoint sets, open in $\bar{R}$, one of which contains Q_1, and the other $Q_2 + m(\frac{1}{2}r)$. Then points of $m(r)$ near Q_1 but on opposite sides of it can be joined by an arc in $\bar{R}$, not meeting $\phi_n^{-1}(\Theta)$. But this means that $\phi m(r)$ does not cross Θ at (P_1, Q_1), and so $\phi m(r)$ must meet Θ at least three times, which gives a contradiction.

Let $\Gamma_n(\theta)$ be the continuum in $R\phi_n^{-1}(\theta)$ joining Q_1 and Q_2, or else the sum of two continua joining Q_1 and Q_2 to $m(\frac{1}{2}r)$. Suppose now that

$$\Lambda^2 S_n \leqslant \tfrac{3}{4}\pi r^2 + r^2\epsilon^4(r). \tag{158}$$

Let E_1 be the set of θ, $0 < \theta \leqslant \pi$ such that $\Gamma_n(\theta)$ is defined and $\phi_n\Gamma_n(\theta)$ meets either $C(\frac{1}{4}r)$ or $C(2r)$. Denote by $|E_1|$ the Lebesgue measure of E_1. Then

$$\Lambda\phi_n\Gamma_n(\theta)\{[C(2r) - C(r)] + [C(\tfrac{1}{2}r) - C(\tfrac{1}{4}r)]\} \geqslant \tfrac{1}{4}r. \tag{159}$$

Hence, by (158), (159) and Lemma 23,

$$\tfrac{1}{4}r\int_{E_1} \tfrac{1}{4}r\, d\theta \leqslant \Lambda^2 S_n\{[C(2r) - C(r)] + [C(\tfrac{1}{2}r) - C(\tfrac{1}{4}r)]\} \leqslant r^2\epsilon^4(r). \tag{160}$$

Hence
$$|E_1| \leqslant 16\epsilon^4(r). \tag{161}$$

Let E_2 be the set of θ, $0 < \theta \leqslant \pi$ such that $\Gamma_n(\theta)$ is defined and contained in $C(2r) - C(\frac{1}{4}r)$. Divide $(2r, \frac{1}{4}r)$ into intervals

$$r_0 = \tfrac{1}{4}r < r_1 < r_2 < \ldots < r_n = 2r, \tag{162}$$

where
$$r_k = \tfrac{1}{2}r \quad \text{and} \quad r_{k'} = r, \tag{163}$$

and
$$|r_i - r_{i-1}| < r\epsilon^2(r). \tag{164}$$

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Let $\delta_i = 1$ for $k < i \leqslant k'$ and 0 otherwise. By Lemma 23

$$\begin{aligned}\Lambda^2\{S_n[C(r_i)-C(r_{i-1})]\}-\{\pi r_i^2-\pi r_{i-1}^2\}\,\delta_i \\ \geqslant r_{i-1}\int_{E_2}\Lambda\phi_n\,\Gamma_n(\theta)\,\{C(r_i)-C(r_{i-1})\}\,d\theta-2\delta_i r_i\int_0^{\pi}(r_i-r_{i-1})\,d\theta \\ \geqslant r_{i-1}\int_{E_2}\Lambda\phi_n\,\Gamma_n(\theta)\,\{C(r_i)-C(r_{i-1})\}-2\delta_i(r_i-r_{i-1})\,d\theta \\ -2\delta_i r_{i-1}(r_i-r_{i-1})\,(\pi-|E_2|)-2\pi\delta_i(r_i-r_{i-1})^2. \qquad (165)\end{aligned}$$

Summing this for $i = 1$ to n we get

$$\Lambda^2 S_n-\{\pi r^2-\tfrac{1}{4}\pi r^2\}\geqslant \tfrac{1}{4}r\int_{E_2}\{\Lambda\phi_n\,\Gamma_n(\theta)-r\}\,d\theta-4\pi r^2\epsilon^2(r)-8r^2(\pi-|E_2|). \qquad (166)$$

Suppose now that $$\Lambda\phi m(r) < 2\pi r + r\epsilon^2(r). \qquad (167)$$

$\Gamma_n(\theta)$ is defined whenever $\phi m(r)$ meets Θ at most twice, hence

$$|E_2|+|E_1|\geqslant \pi-\epsilon^2(r). \qquad (168)$$

Thus by (158), (161), (166) and (168),

$$r^2\epsilon^4(r)+4\pi r^2\epsilon^2(r)+8r^2(\epsilon^2(r)+16\epsilon^4(r))\geqslant \tfrac{1}{4}r\int_{E_2}\{\Lambda\phi_n\,\Gamma_n(\theta)-r\}\,d\theta. \qquad (169)$$

Let E_3 be the set of points where $\Gamma_n(\theta)$ is defined and lies in $C(2r)-C(\frac{1}{4}r)$, while

$$\Lambda\phi\Gamma_n(\theta)\geqslant r+r\epsilon(r). \qquad (170)$$

We always have for $\theta\in E_2$, $\Lambda\Gamma_n(\theta)\geqslant r$ and so by (169) if $\epsilon(r)$ is small

$$|E_3| < 90\epsilon(r). \qquad (171)$$

Thus if (167) holds then, except at a set of θ of measure at most $100\epsilon(r)$, $\Gamma_n(\theta)$ will exist and satisfy

$$\Lambda\phi\Gamma_n(\theta) < r+r\epsilon(r). \qquad (172)$$

The only assumption we have made about S_n is that equation (158) holds; as we have an approximate tangential plane in the sense of Definition 2 at (P_0, Q_0) we may therefore assume that $\phi_n\to\phi$.

Denote by $E(n)$ the set of θ where $\Gamma_n(\theta)$ is defined and (172) holds. Let

$$E=\overline{\lim_{n\to\infty}}\,E(n). \qquad (173)$$

Then $$|E| > \pi-100\epsilon(r). \qquad (174)$$

Suppose $\theta\in E$. Then by the same argument, applied to the curves $\Gamma_n(\theta)$, that we applied to the curves $l(h,n)$ in Lemma 4 of (5), we may show that the $\Gamma_n(\theta)$ converge in a subsequence to a curve $\Gamma(\theta)$ such that

$$\Lambda\phi\Gamma(\theta)\leqslant r+r\epsilon(r). \qquad (175)$$

$\Gamma(\theta)$ will consist of two arcs joining Q_1 and Q_2 to $m(\frac{1}{2}r)$, as if it were an arc joining Q_1 and Q_2 equation (175) could not hold.

Let N be the set of points Q which lie on a curve $m(r)$ such that (167) holds and $\Lambda\phi m(\frac{1}{2}r) < \infty$, while the point $P = \phi(Q)$ is on a plane Θ such that θ belongs to the set E associated with r and defined above. From the definition of E we know that there

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are at most two points of $\phi m(r)$ on Θ and so no two points of N will have the same projection on π_0. For almost all r for which (167) holds we have

$$\Lambda\phi N m(r) \geqslant 2\pi r - 200 r\epsilon(r). \tag{176}$$

As we have an approximate tangential plane in the sense of Definition 2 we may assume that (148) holds, and that $\epsilon(r)$ tends to zero so slowly that $\epsilon(\frac{1}{2}r) > \frac{1}{2}\epsilon(r)$. Then

$$\int_0^r \Lambda\phi m(t)\,dt - \int_0^r 2\pi t\,dt \leqslant r^2\epsilon^4(r). \tag{177}$$

Hence the set of t in $(r, \frac{1}{2}r)$ where (167) fails has measure at most

$$\frac{2r\epsilon^4(r)}{\epsilon^2(\frac{1}{2}r)} \leqslant 8r\epsilon^2(r).$$

Hence
$$\int_{\frac{1}{2}r}^r \Lambda\phi N m(t)\,dt \geqslant \tfrac{3}{4}\pi r^2 - 100 r^2\epsilon(r) - 16\pi r^2\epsilon^2(r). \tag{178}$$

Thus by Lemma 24
$$L^*\{NM(r)\} \geqslant \pi r^2 - o(r^2). \tag{179}$$

Let $Q \in N$ and suppose Q is on $m(r)$. Then by (175) there is an arc Γ joining Q to a point of $m(\frac{1}{2}r)$ and such that
$$\tfrac{1}{2}r < \Lambda\phi\Gamma < \tfrac{1}{2}r + r\epsilon(r). \tag{180}$$

Take $k > 0$. Suppose that for all r' such that $\frac{3}{4}r > r' > \frac{1}{2}r$ there is no point of ϕN within kr of Γ on $\phi m(r')$, the distance being measured along $\phi m(r')$. Then

$$\Lambda\phi\{(H-N)\,m(r')\} > 2kr, \tag{181}$$

and so by Lemma 24
$$L^*(H-N)\,M(r) \geqslant \tfrac{1}{2}kr^2 - 7\epsilon^2(r)\,r^2. \tag{182}$$

This contradicts (179). Thus there exists r', $\frac{3}{4}r > r' > \frac{1}{2}r$ and a point Q_2 of N on $m(r')$, such that Q, Q_2 can be joined by an arc Γ_1, where

$$\Lambda\phi\Gamma_1 < r - r' + kr + r\epsilon(r). \tag{183}$$

We can repeat this argument with Q_2 to obtain r'', $\frac{3}{4}r' > r'' > \frac{1}{2}r'$, and Q_3 such that Q_2 and Q_3 are joined by an arc Γ_2, where

$$\Lambda\phi\Gamma_2 < r' - r'' + kr' + r'\epsilon(r'). \tag{184}$$

The sum of the arcs $\Gamma_1, \Gamma_2, \ldots$, so constructed is a curve Γ_0 joining Q to Q_0 and such that

$$\Lambda\phi\Gamma_0 < r + 4kr + 4r\epsilon(r). \tag{185}$$

But k is arbitrary, and so
$$\delta\{(P,Q),(P_0,Q_0)\} \leqslant r + 4r\epsilon(r). \tag{186}$$

This, together with (179), completes the proof of the lemma, if we take $E = H - N$ in Definition 1.

LEMMA 26. *If π is a tangential plane in the sense of Definition 2′ then it is also one in the sense of Definition 1′.*

It is clearly a tangential plane in the sense of Definition 2 and so by Lemma 25 in the sense of Definition 1 also. Thus

$$\Lambda^2\phi M(r) \sim L^*M(r) \sim L^*\{M(r)\,(H-E)\} \sim \pi r^2, \tag{187}$$

and hence
$$\Lambda^2\phi\{M(r)\,(H-E)\} \sim \pi r^2, \tag{188}$$

which implies
$$\Lambda^2\phi\{EM(r)\} = o(r^2), \tag{189}$$

and so the lemma is established.

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17. LEMMA 27. *If C is the set of consistent points,*

$$\Lambda^2 C = L(S). \tag{190}$$

Consider $X \subset C$. Take $\epsilon > 0$. By (82) of Lemma 16 we have at Λ^2 almost all points of X

$$L^*\phi K'(Q,r) \sim \pi r^2. \tag{191}$$

Choose δ so small that (1) for any set U of convex bodies of diameter $d < \delta$

$$\tfrac{1}{4}\pi \sum_U d^2 \geqslant (1-\epsilon)\,\Lambda^2 U \,|\, \phi(X) \,|. \tag{192}$$

Take an open $G \supset X$. To each $Q \in X$ there corresponds an $r < \frac{1}{2}\delta$ as small as we please such that

$$K'(Q,r) \subset G \tag{193}$$

and

$$L^*\phi K'(Q,r) \geqslant \pi r^2(1-\epsilon). \tag{194}$$

In virtue of Lemma 8, C is of finite measure, and so by Vitali's general covering principle a disjoint subset V of the corresponding spheres $S(\phi(Q), r)$ will cover Λ^2 almost all of $|\, \phi(X) \,|$. Then

$$LG \geqslant \sum_V L\phi K'(Q,r) \geqslant (1-\epsilon)\,\tfrac{1}{4}\pi \sum_V (2r)^2 \geqslant (1-\epsilon)^2\,\Lambda^2 \,|\, \phi(X) \,|. \tag{195}$$

Thus

$$L^*(X) \geqslant \Lambda^2 \,|\, \phi(X) \,|. \tag{196}$$

Let E_∞ be the points of $|\, C \,|$ of infinite multiplicity on C. Consider $E(n,k)$, the set of points P of E_∞ such that there are n ϕ-elements $Q_1, \ldots, Q_n$ in C for which $\phi(Q_i) = P$ and no two of which are within $1/k$ of each other.

$$\lim_{k\to\infty} E(n,k) = E_\infty. \tag{197}$$

Divide H into disjoint Borel sets B_i of diameter less than $1/k$. Then by (196)

$$\sum_i \Lambda^2 \,|\, \phi\{B_i \phi^{-1} E(k,n)\} \,| \leqslant \sum_i L^*(B_i) \leqslant L(S). \tag{198}$$

On the other hand, every point of $E(k,n)$ is in at least n of the sets $|\, \phi\{B_i \phi^{-1} E(k,n)\} \,|$. Hence

$$n\Lambda^2 E(k,n) \leqslant L(S). \tag{199}$$

Let $k \to \infty$ and then $n \to \infty$. We obtain by (197) that

$$\Lambda^2 E_\infty = 0. \tag{200}$$

Let E_k be the points P of $|\, C \,|$ such that no two ϕ-elements of $C\phi^{-1}(P)$ are within $1/k$ of each other. Divide H into disjoint Borel sets B_i of diameter at most $1/k$. Then on each B_i, $E_k \,|\, \phi(B_i) \,|$ will be simple, and so by (196)

$$L^* C\phi\{B_i \phi^{-1} E_k\} \geqslant \Lambda^2 C\phi\{B_i \phi^{-1} E_k\}. \tag{201}$$

Adding, we obtain

$$L(S) \geqslant \Lambda^2 C\phi\{\phi^{-1} E_k\}. \tag{202}$$

But

$$\lim_{k\to\infty} E_k \supset |\, C \,| - E_\infty, \tag{203}$$

and so by (200)

$$L(S) \geqslant \Lambda^2 C. \tag{204}$$

But by Lemmas 7 and 15 and Theorem 6 of (5)

$$\Lambda^2 C \geqslant L^*(C) = L(S), \tag{205}$$

and so the lemma is established.

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Proof of Theorem 1. Suppose π is a tangential plane in the sense of Definition 1. Take $\delta > 0$. Consider the points (P', Q') of $\phi\{M(r) - K'(Q, r+\delta r)\}$. Any curve joining (P', Q') and (P, Q) on the surface must meet $S(P, r+\delta r)$, and so

$$\delta\{(P', Q'), (P, Q)\} > r + \delta r.$$

Hence by Definition 1 $$L^*\phi\{M(r) - K'(Q, r+\delta r)\} = o(r^2). \tag{206}$$

That is, $$L^*\phi K'(Q, r+\delta r) \geqslant \pi r^2 - o(r^2). \tag{207}$$

But δ is arbitrary, and so $$T_1 \subset J. \tag{208}$$

By Lemma 25 $$T_2 \subset T_1. \tag{209}$$

Hence if π is a tangential plane in the sense of Definition 2, then, using (207),

$$L^*K(Q, r) \sim L^*M(r) \sim \pi r^2. \tag{210}$$

Also, for almost all r, $\Lambda\phi l(Q, r) < \infty$, and so,

$$\mu\{\phi, K(Q, r)\} + L^*\{M(r) - K(Q, r)\} \geqslant \pi r^2, \tag{211}$$

so that $$\mu\{\phi, K(Q, r)\} \sim \pi r^2. \tag{212}$$

Hence by (210) $$T_2 \subset C. \tag{213}$$

Further, if $(P, Q) \in T_1 - I$ then $L^*K'(Q, r) \sim \pi r^2$ and so by (206)

$$L^*M(r) \sim \pi r^2, \tag{214}$$

so that $$T_1 - I \subset T_2. \tag{215}$$

(208), (209), (213) and (215) give (6). Also, by Lemma 9, $\Lambda^2 I = 0$, while by Lemmas 15 and 20 $$\Lambda^2(C - J) = 0 \tag{216}$$

and $$\Lambda^2(T_2 - C) = 0. \tag{217}$$

These last results, together with Lemma 27, establish the theorem.

Proof of Theorem 2. The proof is similar to the above. We use Lemmas 21 and 26 instead of 20 and 25.

REFERENCES

(1) Besicovitch, A. S. On the fundamental geometrical properties of linearly measurable plane sets of points. II. *Math. Ann.* 115 (1938), 296–329.

(2) Besicovitch, A. S. A general form of the covering principle and the relative differentiation of additive functions. I. *Proc. Camb. phil. Soc.* 41 (1945), 103–10.

(3) Besicovitch, A. S. On the definition and value of the area of a surface. *Quart. J. Math.* 16 (1945), 86–102.

(4) Cesari, L. Una ugualianza fondamentale per l'area delle superficie. *Mem. R. Accad. Ital.* 14 (1944), 891–951. (See also p. 1379 of: Sui fondamenti geometrici dell' integrale classico per l'area delle superficie in forma parametrica. *Mem. R. Accad. Ital.* 13 (1943), 1323–481.)

(5) Reifenberg, E. R. Parametric surfaces. I. Area. *Proc. Camb. phil. Soc.* 47 (1951), 687–98.

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