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## 102.15 A Faulhaber polynomial by stacking cubes

It was first observed by Faulhaber that for every odd integer $k$,

$$1^k + 2^k + \ldots + n^k$$

is a polynomial function of

$$1 + 2 + \ldots + n = \tfrac{1}{2}n(n+1).$$

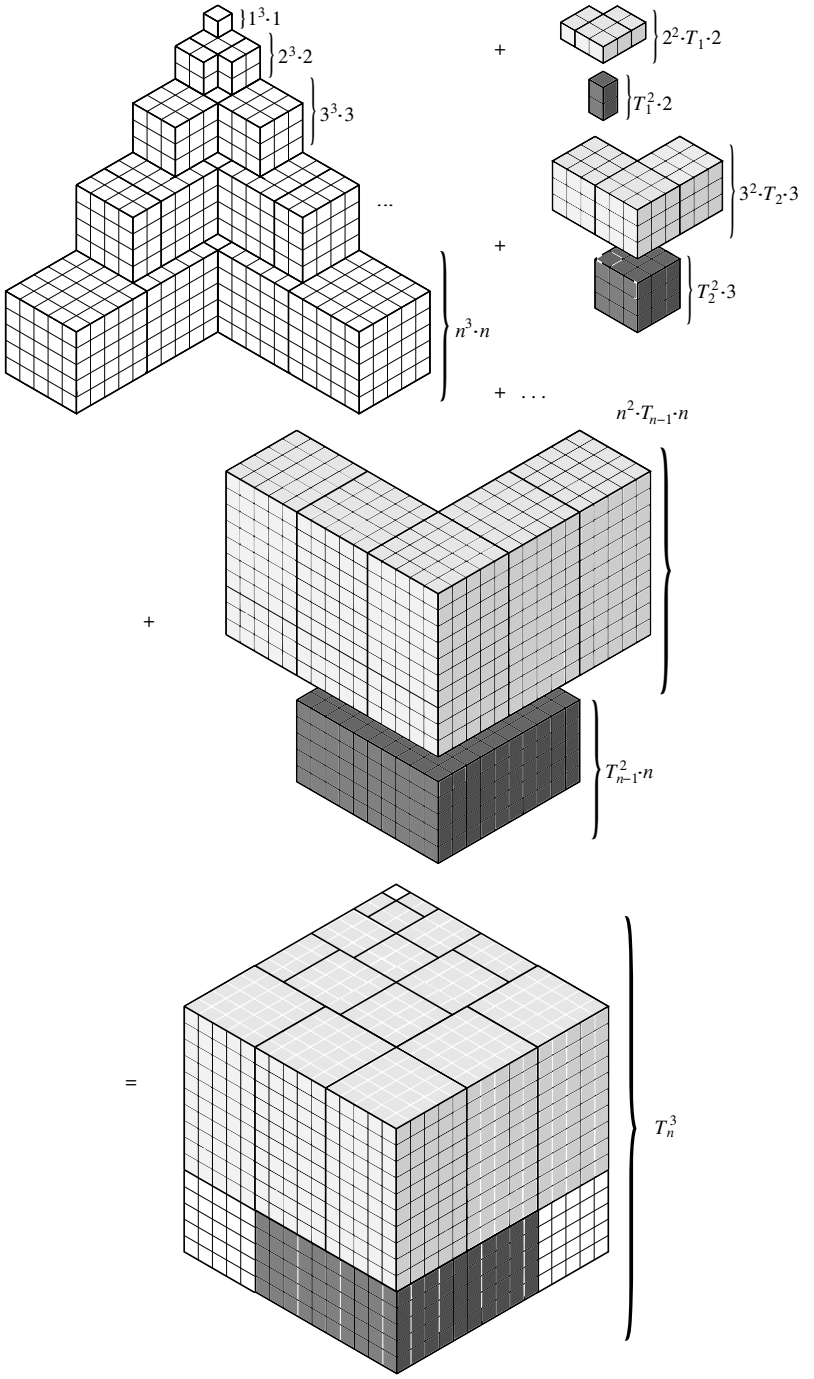
Such polynomial functions, denoted $p_k(x)$, are called *Faulhaber polynomials*. For example, $p_1(x) = x$ trivially, and $p_3(x) = x^2$, since

$$1^3 + 2^3 + \ldots + n^3 = \left(\tfrac{1}{2}n(n+1)\right)^2,$$

which is proved wordlessly in [1]. We prove that $p_5 = \frac{1}{3}\left(4x^3 - x^2\right)$ with the aid of stacking cubes.

*Proof*: Let $\sum$ stand for a summation from $i = 1$ to $n$. Define $T_i = \frac{1}{2}i(i+1)$.

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Therefore,

$$\underbrace{\sum i^4}_{\text{white cubes}} + \underbrace{\sum i^3 T_{i-1}}_{\text{grey cubes}} + \underbrace{\sum i T_{i-1}^2}_{\text{black cubes}} = T_n^3.$$

So that

$$\sum i^4 + \sum i^3 \times \tfrac{1}{2} i\,(i-1) + \sum i\left(\tfrac{1}{2} i\,(i-1)\right)^2 = \left(\sum i\right)^3.$$

Expanding the terms gives us

$$\frac{3}{4}\sum i^5 + \frac{1}{4}\sum i^3 = \left(\sum i\right)^3.$$

Since $\sum i^3 = \left(\sum i\right)^2$, we have

$$\sum i^5 = \frac{4}{3}\left(\sum i\right)^3 - \frac{1}{3}\left(\sum i\right)^2.$$

*Reference*

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## 102.16 An areal view of curvature

In this Note we use areal coordinates to deploy a little-known formula for the radius of curvature at any point $P$ on a plane curve.

With $Q$ a neighbouring point, the radius of curvature $\rho$ at $P$ is given by

$$2\rho = \lim_{T \to P} \frac{PT^2}{QT}, \tag{1}$$

where $T$ lies on the tangent to the curve at $P$ and $PT \perp QT$. This formula (attributed to Newton) is fairly easy to check in the context of Cartesian coordinates.

As an exploratory study, our treatment will be restricted to the class of conics which circumscribe the triangle of reference.

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