13. In theorems II and III, given a function $f(x)$ which satisfies (2) or (2′), there is a function $\sigma(x)$ which satisfies (3) or (3′) respectively and which is connected with $f(x)$ by (4); and conversely. The enunciation of theorem II is correct when $n-1$ is odd, but in this case, given $f(x)$ satisfying (2), though we can prove that there is a function $\sigma(x)$ satisfying (3), we do not prove that $\sigma(x)$ is connected with $f(x)$ by (4). And a similar remark applies to the converse. We shall treat this case in a subsequent note.

14. My attention has been drawn to the fact that lemma 4 is a refinement of a theorem of Hamburger and R. Nevanlinna. Cf. Hamburger, *Math. Ann.* 81 (1920), Satz 9, 268–74; R. Nevanlinna, *Ann. Acad. Sci. Fennicae*, 18 (1922), no. 5, 45–8.

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THE COMPLETE SYSTEM OF THE BINARY (3, 1) FORM

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1. The only double binary forms for which the complete systems of invariants and covariants are known appear to be the (1, 1), (2, 1) and (2, 2) forms, for which the complete systems were determined sixty years ago by Peano*. In the present paper we determine the complete system of the binary (3, 1) form and establish its irreducibility. The system proves to contain twenty forms, and is thus scarcely more complicated than that of the (2, 2) form which includes eighteen concomitants, and much simpler than that of the (2, 1, 1) form derived by Gilham†. It is hoped in a subsequent note to interpret the system geometrically.

2. Let
$$f \equiv A_x^3 \equiv a_x^3 \alpha_y \tag{1}$$
be a (3, 1) form in the two sets of binary variables (x_0, x_1), (y_0, y_1) which undergo independent linear transformations. It is convenient to use small roman letters to indicate symbolic factors involving x and small greek letters to refer to symbolic factors involving y, and to employ capital letters whenever we wish to consider a form with reference to the variable x only. Equivalent symbols will be represented by accents. If f_1 and f_2 are two double binary forms the expression $(f_1, f_2)^{\lambda,\mu}$ will denote their transvectant of indices λ, μ, the former index referring to the variables (x_0, x_1) and the latter to the variables (y_0, y_1).

* Peano, *Giorn. matematico*, 20 (1882), 79–100. For further references, including simultaneous systems, see Turnbull, *The Theory of Determinants, Matrices, and Invariants* (Blackie, 1928), p. 243.

† Gilham, *Proc. London Math. Soc.* (2), 32 (1931), 259–72 and 38 (1935), 271–2.

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Any concomitant of the form f is expressible in terms of symbolic products of factors of the form a_x, α_y, (aa'), $(\alpha\alpha')$ to which the usual conventions of symbolic algebra apply. To obtain the complete system of the form f we follow the standard method of Gordan*. Considered as a binary form in (x_0, x_1) the complete system of f consists of the four forms f, h, t, Δ where

$$h = (f,f)^{2,0}, \quad t = (f,h)^{1,0}, \quad \Delta = (h,h)^{2,0}. \tag{2}$$

Any form derived by convolution from a power of f is either expressible rationally and integrally in terms of f, h, t, Δ or else is a form with a symbolic factor of the form $(\alpha\alpha')$. Thus the system $\{A_1\}$ consisting of the forms f, h, t, Δ is relatively complete for the modulus $(\alpha\alpha')$ or, as we shall write,

$$\{A_1\} \equiv \{f, h, t, \Delta\} \equiv c \ \ [\mathrm{mod}\,(\alpha\alpha')]. \tag{3}$$

3. Any symbolic product which contains the factor $(\alpha\alpha')$ is a term in some transvectant

$$((f,f)^{\lambda,1}, P)^{\lambda',0}, \tag{4}$$

where $0 \leqslant \lambda \leqslant 3$ and P is a symbolic product not involving the symbols a, a', α, α'. Hence such a product is equal to the transvectant (4) together with the sum of certain transvectants of index less than λ' of forms obtained from $(f,f)^{\lambda,1}$ and P by convolution with respect to the variables (x_0, x_1). Since the only non-zero forms arising from convolution of $(f,f)^{\lambda,1}$ are

$$F \equiv F_x^4 = (f,f)^{1,1} = (aa')(\alpha\alpha')a_x^2 a_x'^2 \tag{5}$$

and the invariant

$$I_2 = (f,f)^{3,1} = (aa')^3(\alpha\alpha'), \tag{6}$$

any such product is the sum of terms each of which contains the symbolic factor $(aa')(\alpha\alpha')$ or the invariant factor I_2. Hence

$$\{A_1\} \equiv c \quad [\mathrm{mod}\,(aa')(\alpha\alpha'), I_2]. \tag{7}$$

In particular, any product which contains the symbolic factor $(aa')^2(\alpha\alpha')$ is expressible as the sum of terms each having the invariant factor I_2.

4. We shall next show that the form F, whose only bracket factors are $(aa')(\alpha\alpha')$, is relatively complete for the two moduli $(aF)^3$, I_2. In the first place,

$$\begin{aligned}
(f, F)^{2,0} &= (a_x^3\alpha_y, (a'a'')(\alpha'\alpha'')a_x'^2 a_x''^2)^{2,0} \\
&= \tfrac{1}{6}(a'a'')(\alpha'\alpha'')a_x\alpha_y[(aa')^2 a_x''^2 + 4(aa')(aa'')a_x'a_x'' + (aa'')^2 a_x'^2] \\
&= \tfrac{1}{6}(a'a'')(\alpha'\alpha'')a_x\alpha_y[(a'a'')^2 a_x^2 + 6(aa')(aa'')a_x'a_x''] \\
&= \tfrac{1}{6}I_2 f + (a'a'')(aa')(aa'')a_x a_x' a_x''(\alpha'\alpha'')\alpha_y \\
&= \tfrac{1}{6}I_2 f + \tfrac{1}{3}(a'a'')(aa')(aa'')a_x a_x' a_x''[(\alpha'\alpha'')\alpha_y + (\alpha''\alpha)\alpha_y' + (\alpha\alpha')\alpha_y'']
\end{aligned}$$

and so, since the second term vanishes,

$$(f, F)^{2,0} = \tfrac{1}{6}I_2 f. \tag{8}$$

* See Grace and Young, *Algebra of Invariants* (Cambridge, 1903), Chapter VI. The methods and theorems established for simple binary forms extend immediately to multiple binary forms. See, for example, Turnbull, *Proc. Edinburgh Math. Soc.* 40 (1923), 116–27.

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It follows incidentally that

$$(aa')^2(a'a'')(\alpha'\alpha'')a_x a_x''^2\alpha_y = \tfrac{1}{2}(f, F)^{2,0}. \tag{9}$$

Any symbolic product containing the factor $(aF)^2$ is a term in a transvectant

$$((aF)^2 a_x F_x^2 \alpha_y, P)^{\lambda,\mu},$$

and hence can be expressed in the form

$$((aF)^2 a_x F_x^2 \alpha_y, P)^{\lambda,\mu} + \Sigma(\overline{(aF)^2 a_x F_x^2 \alpha_y}, \overline{P})^{\lambda',\mu'},$$

where the bar, as usual, indicates convolution. Any term in this expression in which no convolution has taken place in the first member contains I_2 as a factor, and the other terms all contain the symbolic factor $(aF)^3$. Thus *any symbolic product containing the factor* $(aF)^2$ *is congruent to zero* $[\text{mod } I_2, (aF)^3]$.

Now F is a quartic involving the variables (x_0, x_1) only, and its complete system consists of the five forms

$$F, \quad H = (F, F)^{2,0}, \quad (F, F)^{4,0}, \quad (F, H)^{1,0}, \quad (F, H)^{4,0}. \tag{10}$$

And
$$H = (F, F)^{2,0} = (F_x^4, (aa')(\alpha\alpha')a_x^2 a_x'^2)^{2,0}$$
$$= (aF)^2(aa')(\alpha\alpha')a_x'^2 F_x^2 + \Sigma(F_x^4, \overline{(aa')(\alpha\alpha')a_x^2 a_x'^2})^{\lambda,0}.$$

The first term contains the factor $(aF)^2$ and the second contains the factor I_2. Thus

$$H \equiv 0 \quad [\text{mod } (aF)^3, I_2]$$

and hence, also, $\quad (F, H)^{1,0} \equiv 0, \quad (F, H)^{4,0} \equiv 0 \quad [\text{mod } (aF)^3, I_2].$

Further,
$$(F, F)^{4,0} = ((aa')(\alpha\alpha')a_x^2 a_x'^2, (a''a''')(\alpha''\alpha''')a_x''^2 a_x'''^2)^{4,0}$$
$$= (aa')(aa'')(aa''')(a'a'')(a'a''')(a''a''')(\alpha\alpha')(\alpha''\alpha''')$$
$$+ \Sigma(\overline{(aa')(\alpha\alpha')a_x^2 a_x'^2}, \overline{(a''a''')(\alpha''\alpha''')a_x''^2 a_x'''^2})^{\lambda,0}.$$

The second term on the right contains I_2 as a factor, and the first term

$$= \tfrac{1}{3}(aa')(aa'')(aa''')(a'a'')(a'a''')(a''a''')[(\alpha\alpha')(\alpha''\alpha''') + (\alpha\alpha'')(\alpha'''\alpha') + (\alpha\alpha''')(\alpha'\alpha'')]$$
$$= 0,$$

so that
$$(F, F)^{4,0} \equiv 0 \quad [\text{mod } I_2].$$

Thus every form of the complete system of F, other than F itself, is congruent to zero for the two moduli $(aF)^3$ and I_2, so that

$$\{B_1\} \equiv \{F\} \equiv c \quad [\text{mod } (aF)^3, I_2]. \tag{11}$$

5. From (7) and (11) it follows that the system $\{A_2\}$ derived by transvection from $\{A_1\}$ and $\{B_1\}$ is relatively complete for the moduli $(aF)^3$, I_2. Since F only involves the variables (x_0, x_1), only transvectants with respect to these variables come under consideration, and accordingly, in this section, we omit the second index. Since Δ does not involve the variables (x_0, x_1) we have to consider transvectants of the form

$$(f^l h^m i^n, F^s)^\lambda, \tag{12}$$

and to select such of these transvectants as are irreducible $[\text{mod } (aF)^3, I_2]$. In particular, we may neglect any such transvectant which contains a term with the symbolic factor $(aF)^2$ or $(aF)^3$.

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Now $$t = (f, h)^{1,0} = (aa')(a'a'')^2 a_x^2 a_x'' \alpha_y \alpha_y' \alpha_y''$$

contains the symbolic factor a_x^2, and so, if g denotes any product of f, h, t, the transvectant $(tg, F^s)^\lambda$ contains a term with the symbolic factor $(aF)^2$ if $\lambda > 1$. If $\lambda = 1$ the transvectant is clearly reducible unless $g = 1$ and s is unity, and $(t, F)^1$ is reducible since t is a Jacobian. Thus no transvectants (12) in which $n > 0$ need be considered. Similarly, the only transvectant which arises when $l > 0$ is $(f, F)^1$. There remain for consideration the transvectants $(h^m, F^s)^\lambda$. The only such transvectants which are irreducible must be included in the set

$$(h, F)^1, \quad (h, F)^2, \quad (h^2, F)^3, \quad (h^2, F)^4,$$

since h is a quadratic in x and F is a quartic. In fact, all these transvectants are to be rejected. For

$$(h, F)^1 = ((aa')^2 a_x a_x' \alpha_y \alpha_y', F_x^4)^1 = (aa')^2 (aF) a_x' F_x^3 \alpha_y \alpha_y'$$
$$= (aa')(aF)[(aF) a_x' - (a'F) a_x] a_x' F_x^2 \alpha_y \alpha_y'.$$

The second term vanishes, since it changes sign when the equivalent symbols a, a' and α, α' are interchanged. And the first term contains the factor $(aF)^2$, so that

$$(h, F)^1 \equiv 0 \pmod{(aF)^3, I_2}.$$

Again,

$$(h, F)^2 = (aa')^2 (aF)(a'F) F_x^2 \alpha_y \alpha_y' = (aa')(aF)(a'F)[(aF) a_x' - (a'F) a_x] F_x \alpha_y \alpha_y'$$
$$\equiv 0 \pmod{I_2, (aF)^3},$$

since each term involves the factor $(aF)^2$ or $(a'F)^2$. And since

$$(h^2, F)^3 = (h, (h, F)^2)^1, \quad (h^2, F)^1 = (h, (h, F)^2)^2,$$

these forms are likewise congruent to zero $\pmod{(aF)^3, I_2}$.

Thus the only form which need be retained is

$$j = (f, F)^{1,0}, \tag{13}$$

and the system $\{A_2\}$ consists of f, h, t, Δ, F, j. Thus

$$\{A_2\} \equiv \{f, h, t, \Delta, F, j\} \equiv c \pmod{(aF)^3, I_2}. \tag{14}$$

6. Consider now the form

$$p \equiv p_x \pi_y = (f, F)^{3,0} = (aF)^3 F_x \alpha_y. \tag{15}$$

Then p is a bilinear form whose only bracket factor is $(aF)^3$, and its complete system consists of p and the invariant

$$I_6 = (p, p)^{1,1} = (pp')(\pi\pi'). \tag{16}$$

Hence the system $\{B_2\}$ consisting of the single form p is relatively complete for the invariant modulus I_6, i.e.

$$\{B_2\} \equiv \{p\} \equiv c \pmod{I_6}, \tag{17}$$

and so the system $\{A_3\}$ derived by transvection from $\{A_2\}$ and $\{B_2\}$ is relatively complete for the moduli I_2, I_6. Since $(p, p)^{1,1} = I_6$, any symbolic product containing (pp') or $(\pi\pi')$ as a factor $\equiv 0 \pmod{I_6}$.

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The system $\{A_3\}$ consists of the forms of $\{A_2\}$ and $\{B_2\}$ together with such transvectants of the form

$$(f^l h^m t^n \Delta^r F^s j^u, p^v)^{\lambda,\mu} \tag{18}$$

as are irreducible [mod I_2, I_6].

Now $(f,p)^{1,1} = (a_x'^3 \alpha_y', (aF)^3 F_x \alpha_y)^{1,1} = -(aF)^3 (a'F)(\alpha\alpha') a_x'^2$

is a term of $-((\alpha\alpha') a_x^3 a_x'^3, F_x^4)^{4,0},$

and so $(f,p)^{1,1} = -((\alpha\alpha') a_x^3 a_x'^3, F_x^4)^{4,0} + c_1((aa')(\alpha\alpha') a_x^2 a_x'^2, F_x^4)^{3,0}$

$$+ c_2((aa')^2 (\alpha\alpha') a_x a_x', F_x^4)^{2,0},$$

where c_1, c_2 are certain numerical constants*. Since $(f,f)^{0,1}$ and $(f,f)^{2,1}$ are zero the first and third terms vanish, and the second term is $c_1(F,F)^{3,0} = 0$. Hence

$$(f,p)^{1,1} = (ap)(\alpha\pi) a_x^2 = 0. \tag{19}$$

Since no further form can be derived from $(ap)(\alpha\pi) a_x^2$ by convolution it follows that any symbolic product containing the factor $(ap)(\alpha\pi)$ is zero, and any transvectant of the form (18) which includes a term with this factor may be ignored in constructing the system $\{A_3\}$.

In particular if g is a product of forms of the system $\{A_2\}$ which includes as a factor any one of the forms f, h, t, j, then the expression of g as a symbolic product contains a factor of the form $a_x \alpha_y$. Hence $(g, p^v)^{\lambda,\mu}$ contains a term with the symbolic factor $(ap)(\alpha\pi)$ and is to be rejected, unless either λ or μ is zero. Moreover, since p is a bilinear form, the transvectants $(g, p^v)^{0,\mu}$ and $(g, p^v)^{\lambda,0}$ contain reducible terms unless g consists of a single form (f, h, t or j as the case may be) and $\mu = v$ in the first case and $\lambda = v$ in the second. Thus the only transvectants of type (18) involving f, h, t or j which need be considered are

$$\left.\begin{array}{llllll} (f,p)^{0,1}, & (f,p)^{1,0}, & (f,p^2)^{2,0}, & (f,p^3)^{3,0}, & & \\ (h,p)^{0,1}, & (h,p^2)^{0,2}, & (h,p)^{1,0}, & (h,p^2)^{2,0}, & & \\ (t,p)^{0,1}, & (t,p^2)^{0,2}, & (t,p^3)^{0,3}, & (t,p)^{1,0}, & (t,p^2)^{2,0}, & (t,p^3)^{3,0}, \\ (j,p)^{0,1}, & (j,p)^{1,0}, & (j,p^2)^{2,0}, & (j,p^3)^{3,0}, & (j,p^4)^{4,0}, & (j,p^5)^{5,0}. \end{array}\right\} \tag{20}$$

There remain the transvectants

$$(\Delta^r F^s, p^v)^{\lambda,\mu},$$

and since Δ and F are respectively $(0,4)$ and $(4,0)$ forms and p is a $(1,1)$ form all these contain reducible terms except possibly the following:

$$\left.\begin{array}{llllll} (\Delta,p)^{0,1}, & (\Delta,p^2)^{0,2}, & (\Delta,p^3)^{0,3}, & (\Delta,p^4)^{0,4}, & & \\ (F,p)^{1,0}, & (F,p^2)^{2,0}, & (F,p^3)^{3,0}, & (F,p^4)^{4,0}, & & \\ (\Delta F,p)^{1,1}, & (\Delta F,p^2)^{1,2}, & (\Delta F,p^2)^{2,1}, & (\Delta F,p^2)^{2,2}, & & \\ (\Delta F,p^3)^{1,3}, & (\Delta F,p^3)^{2,3}, & (\Delta F,p^3)^{3,1}, & (\Delta F,p^3)^{3,2}, & (\Delta F,p^3)^{3,3}, & \\ (\Delta F,p^4)^{1,4}, & (\Delta F,p^4)^{2,4}, & (\Delta F,p^4)^{3,4}, & (\Delta F,p^4)^{4,1}, & (\Delta F,p^4)^{4,2}, & \\ (\Delta F,p^4)^{4,3}, & (\Delta F,p^4)^{4,4}. & & & & \end{array}\right\} \tag{21}$$

* We shall often use symbols c, c_1, c_2, ... in this sense. Such a symbol appearing in different contexts need not necessarily denote the *same* constant.

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We proceed to prove that nine of the forms (20) and all the twenty-four forms (21) may be neglected.

7. Since terms containing the factor $(ap)(\alpha\pi)$ vanish identically,

$$\begin{aligned}(F,p)^{1,0} &= ((aa')(\alpha\alpha')a_x^2a_x'^2, p_x\pi_y)^{1,0} = (aa')(ap)(\alpha\alpha')a_xa_x'^2\pi_y\\ &= (aa')(ap)[(\alpha\pi)\alpha_y' - (\alpha'\pi)\alpha_y]a_xa_x'^2 = -(aa')(ap)(\alpha'\pi)a_xa_x'^2\alpha_y\\ &= -(aa')[(aa')p_x + (a'p)a_x](\alpha'\pi)a_xa_x'\alpha_y = -(aa')^2(\alpha'\pi)a_xa_x'p_x\alpha_y\\ &= -(h,p)^{0,1},\end{aligned}$$

and so $(F,p)^{1,0}$ may be omitted from the system $\{A_3\}$. Again

$$(F,p^2)^{2,0} = ((aa')(\alpha\alpha')a_x^2a_x'^2, p_xp_x'\pi_y\pi_y')^{2,0}$$

contains the term $T = (aa')(\alpha\alpha')(ap)(a'p')a_xa_x'\pi_y\pi_y'$,

and so

$$\begin{aligned}(F,p^2)^{2,0} &= T + \Sigma((\overline{aa')(\alpha\alpha')a_x^2a_x'^2}, \overline{p_xp_x'\pi_y\pi_y'})^{\lambda,0} \quad \text{(the convolutions referring to } x)\\ &= T + cI_2p^2 \equiv T \pmod{I_2}.\end{aligned}$$

And

$$\begin{aligned}T &= (aa')(ap)(a'p')a_xa_x'.(\alpha\alpha')\pi_y\pi_y'\\ &= (aa')(ap)(a'p')a_xa_x'[(\alpha\pi)\alpha_y'\pi_y' + (\pi\pi')\alpha_y\alpha_y' - (\alpha'\pi')\alpha_y\pi_y]\\ &\equiv 0 \pmod{I_6},\end{aligned}$$

since the first and third terms vanish and the second contains the factor $(\pi\pi')$.

Thus $$(F,p^2)^{2,0} \equiv 0 \pmod{I_2, I_6}.$$

Any symbolic product involving $(Fp)(Fp')$ as a factor either contains the factor $(\pi\pi')$, in which case it vanishes since p, p' and π, π' are equivalent symbols, or else is a term in a transvectant

$$((Fp)(Fp')F_x^2\pi_y\pi_y', P)^{\lambda,\mu},$$

and since no non-zero form can be derived by convolution from $(Fp)(Fp')F_x^2\pi_y\pi_y'$ it follows that any such term $\equiv 0 \pmod{I_2, I_6}$. It follows at once that all the transvectants (21) in which the first index exceeds unity may be rejected, since these transvectants all contain terms with the factor $(Fp)(Fp')$.

8. Again, if $h \equiv h_x^2\theta_y^2$,

$$\begin{aligned}(\Delta,p)^{0,1} &= ((hh')^2\theta_y^2\theta_y'^2, p_x\pi_y)^{0,1} = (hh')^2(\theta\pi)p_x\theta_y\theta_y'^2\\ &= (hh')[(hp)h_x' - (h'p)h_x](\theta\pi)\theta_y\theta_y'^2\\ &= (hh')(hp)(\theta\pi)h_x'\theta_y\theta_y'^2 - (hh')(h'p)[(\theta\theta')\pi_y + (\theta'\pi)\theta_y]h_x\theta_y\theta_y'\\ &= 2(hh')(hp)(\theta\pi)h_x'\theta_y\theta_y'^2 - (hh')(h'p)(\theta\theta')h_x\theta_y\theta_y'\\ &= 2T_1 - T_2 \quad \text{(say)},\end{aligned}$$

on interchanging the equivalent symbols h, h' and θ, θ' in the third term. Now

$$T_1 = ((h,p)^{1,1}, h)^{1,0},$$

and

$$\begin{aligned}(h,p)^{1,1} &= ((aa')^2a_xa_x'\alpha_y\alpha_y', p_x\pi_y)^{1,1}\\ &= (aa')^2(ap)(\alpha\pi)a_x'\alpha_y' + cI_2p\\ &\equiv 0 \pmod{I_2},\end{aligned} \tag{22}$$

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since the first term vanishes identically owing to the factor $(ap)(\alpha\pi)$. Thus

$$T_1 \equiv 0 \pmod{I_2}.$$

And $$T_2 = ((h,h)^{1,1}, p)^{1,0}.$$

Now
$$\begin{aligned}(h,h)^{1,1} &= ((aa')^2 a_x a'_x \alpha_y \alpha'_y, (a''a''')^2 a''_x a'''_x \alpha''_y \alpha'''_y)^{1,1} \\ &= (aa')^2 (aa'')(a''a''')^2 (\alpha\alpha'') a'_x a'''_x \alpha'_y \alpha'''_y + cI_2 h \\ &= T_3 + cI_2 h \quad \text{(say)} \\ &\equiv T_3 \pmod{I_2}.\end{aligned}$$

Since, by (9), $(aa'')(a''a''')^2(\alpha\alpha'') a_x^2 a'''_x \alpha'''_y = \frac{1}{2}(f,F)^{2,0} \equiv 0 \pmod{I_2}$,

we have $((aa'')(a''a''')^2(\alpha\alpha'') a_x^2 a'''_x \alpha'''_y, a_x'^3 \alpha'_y)^{2,0} \equiv 0 \pmod{I_2}$.

But the transvectant on the left is equal to

$$\begin{aligned}&\tfrac{1}{3}(aa'')(a''a''')^2(\alpha\alpha'')[(aa')^2 a'''_x + 2(aa')(a'''a')a_x] a'_x \alpha'_y \alpha'''_y \\ &= (aa'')(a''a''')^2(\alpha\alpha'')[(aa')^2 a'''_x - \tfrac{2}{3}(aa')(aa''')a'_x] a'_x \alpha'_y \alpha'''_y \\ &= T_3 - \tfrac{2}{3}(f,p)^{1,0}\end{aligned}$$

on expressing p in terms of symbolic factors referring to f. So

$$T_3 \equiv \tfrac{2}{3}(f,p)^{1,0} \pmod{I_2}$$

and hence
$$(h,h)^{1,1} \equiv \tfrac{2}{3}(f,p)^{1,0} \pmod{I_2}. \tag{23}$$

Thus
$$\begin{aligned}T_2 &= ((h,h)^{1,1}, p)^{1,0} \equiv \tfrac{2}{3}((f,p)^{1,0}, p)^{1,0} \pmod{I_2} \\ &= \tfrac{2}{3}(f,p^2)^{2,0},\end{aligned}$$

and so $(\Delta, p)^{0,1}$ may be replaced by $(f,p^2)^{2,0}$.

9. Next,

$$\begin{aligned}(\Delta, p^2)^{0,2} &= ((hh')^2 \theta_y^2 \theta_y'^2, p_x p'_x \pi_y \pi'_y)^{0,2} \\ &= (hh')^2 (\theta\pi)(\theta'\pi') p_x p'_x \theta_y \theta'_y + c_1 (h,h)^{2,2} p^2 \\ &= T_1 + c_1 (h,h)^{2,2} p^2 \quad \text{(say)}.\end{aligned}$$

Since, by (22), $(h,p)^{1,1} \equiv 0 \pmod{I_2}$ and $(h,p)^{1,1}$ is a bilinear form, any symbolic product containing the factor $(hp)(\theta\pi) \equiv 0 \pmod{I_2}$. Hence

$$\begin{aligned}T_1 &= (hh')^2 (\theta\pi)(\theta'\pi') p_x p'_x \theta_y \theta'_y \\ &= (hh')[(hp) h'_x p'_x + (pp') h_x h'_x - (h'p') h_x p_x](\theta\pi)(\theta'\pi') \theta_y \theta'_y \\ &\equiv 0 \pmod{I_2, I_6},\end{aligned}$$

since the three terms contain, respectively, the factors $(hp)(\theta\pi)$, (pp'), $(h'p')(\theta'\pi')$. And

$$\begin{aligned}(h,h)^{2,2} &= ((aa')^2 a_x a'_x \alpha_y \alpha'_y, (a''a''')^2 a''_x a'''_x \alpha''_y \alpha'''_y)^{2,2} \\ &= (aa')^2 (aa'')(a'a''')(a''a''')^2 (\alpha\alpha'')(\alpha'\alpha''') + c_2 I_2^2 \\ &= ((aa'')(\alpha\alpha'') a_x^2 a''^2_x, (a'a''')(\alpha'\alpha''') a_x'^2 a'''^2_x)^{4,0} + c_3 I_2^2 \\ &= (F,F)^4 + c_3 I_2^2 \\ &\equiv 0 \pmod{I_2}.\end{aligned}$$

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Hence $$(\Delta, p^2)^{0,2} \equiv 0 \pmod{I_2, I_6}.$$

Any symbolic product involving $(\Delta\pi)(\Delta\pi')$ as a factor either contains the factor (pp'), in which case it vanishes since p, p' and π, π' are equivalent symbols, or else is a term in a transvectant

$$((\Delta\pi)(\Delta\pi')\, p_x p'_x \Delta_y^2, P)^{\lambda,\mu},$$

and since no non-zero form can be derived by convolution from $(\Delta\pi)(\Delta\pi')\, p_x p'_x \Delta_y^2$ it follows that any such term $\equiv 0 \pmod{I_2, I_6}$. It follows at once that all the transvectants (21) in which the second index exceeds unity may be rejected, since these transvectants all contain terms with the symbolic factor $(\Delta\pi)(\Delta\pi')$.

10. We have shown that any transvectant (21) in which either index exceeds unity is to be rejected, and that $(F, p)^{1,0}$ and $(\Delta, p)^{0,1}$ may be replaced by forms in the set (20). The only one of the transvectants (21) not so disposed of is $(\Delta F, p)^{1,1}$. Now

$$\begin{aligned}(\Delta F, p)^{1,1} &= (\Delta\pi)(Fp) F_x^3 \Delta_y^3 = -((\Delta\pi) p_x \Delta_y^3, F_x^4)^{1,0} \\ &= -((\Delta, p)^{0,1}, F)^{1,0} \\ &\equiv +\tfrac{2}{3}((f, p^2)^{2,0}, F)^{1,0} \pmod{I_2}\end{aligned}$$

by § 8. But

$$\begin{aligned}((f, p^2)^{2,0}, F)^{1,0} &= ((ap)(ap')\, a_x \alpha_y \pi_y \pi'_y, F_x^4)^{1,0} \\ &= (aF)(ap)(ap') F_x^3 \alpha_y \pi_y \pi'_y\end{aligned}$$

is a term of

$$((aF) a_x^2 F_x^3 \alpha_y, p_x p'_x \pi_y \pi'_y)^{2,0} = (j, p^2)^{2,0}$$

and so

$$\begin{aligned}(aF)(ap)(ap') F_x^3 \alpha_y \pi_y \pi'_y &= (j, p^2)^{2,0} + c_1((f, F)^{2,0}, p^2)^{1,0} + c_2 p^3 \\ &\equiv (j, p^2)^{2,0} + c_2 p^3 \pmod{I_2}\end{aligned}$$

and so $(\Delta F, p)^{1,1}$ may be replaced by $(j, p^2)^{2,0}$.

11. We shall now show that nine of the forms (20) are reducible. For this purpose we observe that any symbolic product T containing as factor either $(aa')(\alpha\pi)(\alpha'\pi') p_x p'_x$ or $(ap)(a'p')(\alpha\alpha')\pi_y \pi'_y$ is congruent to zero $\pmod{I_6}$. For, if M denotes the product of the remaining elementary symbolic factors of T we have, in the two cases,

$$T = (aa')(\alpha\pi)(\alpha'\pi') p_x p'_x M = (\alpha\pi)(\alpha'\pi') M[(ap) a'_x p'_x + (pp') a_x a'_x - (a'p') a_x p_x],$$

$$T = (ap)(a'p')(\alpha\alpha')\pi_y \pi'_y M = (ap)(a'p') M[(\alpha\pi)\alpha'_y \pi'_y + (\pi\pi')\alpha_y \alpha'_y - (\alpha'\pi')\alpha_y \pi_y],$$

and in each case the first and third terms on the right vanish on account of the factors $(ap)(\alpha\pi)$, $(a'p')(\alpha'\pi')$, and the second terms contain (pp') or $(\pi\pi')$ and so $\equiv 0 \pmod{I_6}$. Any transvectant of the form (20) containing a term with such a factor may therefore be ignored. In particular, if g is one of f, h, t, j and if g contains the symbolic factor $(aa')\alpha_y \alpha'_y$, then $(g, p^v)^{0,v}$ contains a term with the factor $(aa')(\alpha\pi)(\alpha'\pi') p_x p'_x$ if $v > 1$, and if g contains the symbolic factor $(\alpha\alpha') a_x a'_x$ then $(g, p^v)^{v,0}$ contains a term with the factor $(\alpha\alpha')(ap)(a'p)\pi_y \pi'_y$ if $v > 1$. Hence the seven forms

$$(h, p^2)^{0,2}, \quad (t, p^2)^{0,2}, \quad (t, p^3)^{0,3}, \quad (j, p^2)^{2,0}, \quad (j, p^3)^{3,0}, \quad (j, p^4)^{4,0}, \quad (j, p^5)^{5,0}$$

are all to be rejected.

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Next, $(j,p)^{1,0}$ contains the term

$$T = (aa')(a'a'')(a'p)(\alpha'\alpha'')a_x^2 a_x''^2 \alpha_y \pi_y$$
$$= (aa')(a'a'')(a'p)[(\alpha'\pi)\alpha_y'' - (\alpha''\pi)\alpha_y'] a_x^2 a_x''^2 \alpha_y = -(aa')(a'a'')(a'p)(\alpha''\pi) a_x^2 a_x''^2 \alpha_y \alpha_y'$$
$$= -(aa')(a'a'')[(a'a'')p_x + (a''p)a_x'](\alpha''\pi) a_x^2 a_x'' \alpha_y \alpha_y' = -(aa')(a'a'')^2 (\alpha''\pi) a_x^2 a_x'' p_x \alpha_y \alpha_y'$$

which is a term of

$$-((aa')(a'a'')^2 a_x^2 a_x'' \alpha_y \alpha_y' \alpha_y'', p_x \pi_y)^{0,1} = -(t,p)^{0,1},$$

so that $(j,p)^{1,0}$ may be replaced by $(t,p)^{0,1}$.

Finally, $(f,p^3)^{3,0}$ is reducible. For since, by (23),

$$(f,p)^{1,0} = \tfrac{3}{2}(h,h)^{1,1} \quad [\text{mod } I_2],$$

and $(h,h)^{1,1}$ contains the term

$$T = (aa')^2 (aa'')(a''a''')^2 (\alpha'\alpha''') a_x' a_x''' \alpha_y \alpha_y'',$$

so that

$$(h,h)^{1,1} \equiv T \quad [\text{mod } I_2],$$

we have

$$(f,p^3)^{3,0} = ((f,p)^{1,0}, p^2)^{2,0}$$
$$\equiv \tfrac{3}{2}(T,p^2)^{2,0} \quad [\text{mod } I_2].$$

But $(T,p^2)^{2,0}$ contains the symbolic factor

$$(\alpha'\alpha''')(a'p)(a'''p')\pi_y \pi_y'$$

and hence $\equiv 0$ [mod I_6]. Thus

$$(f,p^3)^{3,0} \equiv 0 \quad [\text{mod } I_2, I_6].$$

12. The system $\{A_3\}$ thus contains the forms f, h, t, Δ, F, j, p together with the eleven forms

$$(f,p)^{0,1}, \quad (f,p)^{1,0}, \quad (f,p^2)^{2,0}, \quad (h,p)^{0,1}, \quad (h,p)^{1,0}, \quad (h,p^2)^{2,0},$$
$$(t,p)^{0,1}, \quad (t,p)^{1,0}, \quad (t,p^2)^{2,0}, \quad (t,p^3)^{3,0}, \quad (j,p)^{0,1}.$$

Since $\{A_3\}$ is relatively complete for the invariant moduli I_2, I_6 the complete system of f consists of the forms of $\{A_3\}$ together with I_2 and I_6. The twenty forms of the system may be exhibited as follows, the symbol (m,n) indicating a form of order m in (x_0,x_1) and n in (y_0,y_1).

Degree	Forms			
1	f (3, 1)			
2	$F \equiv (f,f)^{1,1}$ (4, 0),	$h \equiv (f,f)^{2,0}$ (2, 2),	$I_2 \equiv (f,f)^{3,1}$ (0, 0)	
3	$j \equiv (f,F)^{1,0}$ (5, 1),	$t \equiv (f,h)^{1,0}$ (3, 3),	$p \equiv (f,F)^{3,0}$ (1, 1)	
4	$(f,p)^{0,1}$ (4, 0),	$(f,p)^{1,0}$ (2, 2),	$\Delta \equiv (h,h)^{2,0}$ (0, 4)	
5	$(h,p)^{0,1}$ (3, 1),	$(h,p)^{1,0}$ (1, 3)		
6	$(j,p)^{0,1}$ (6, 0),	$(t,p)^{0,1}$ (4, 2),	$(t,p)^{1,0}$ (2, 4),	$I_6 \equiv (p,p)^{1,1}$ (0, 0)
7	$(f,p^2)^{2,0}$ (1, 3)			
8	$(h,p^2)^{2,0}$ (0, 4)			
9	$(t,p^2)^{2,0}$ (1, 5)			
12	$(t,p^3)^{3,0}$ (0, 6)			

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13. The irreducibility of the system may be established by exhibiting the explicit forms of the concomitants in a special case, and observing that no relations hold between these forms which would imply reducibility of any of them. Such a system is given below. The form suggested for f is in fact canonical, but this is not relevant to our purpose.

The system of forms may be taken to be the following:

$$f \equiv ax_0^3y_0 + 3bx_0^2x_1y_1 + 3bx_0x_1^2y_0 + ax_1^3y_1,$$

$$F \equiv 2[abx_0^4 + (a^2-3b^2)\,x_0^2x_1^2 + abx_1^4],$$

$$h \equiv 2[bx_0^2(ay_0^2 - by_1^2) + (a^2-b^2)\,x_0x_1y_0y_1 + bx_1^2(-by_0^2 + ay_1^2)],$$

$$I_2 \equiv 2(a^2+3b^2),$$

$$j \equiv -2ab^2x_0^5y_1 + a(a^2-7b^2)\,x_0^4x_1y_0 - b(a^2+3b^2)\,x_0^3x_1^2y_1 + b(a^2+3b^2)\,x_0^2x_1^3y_0 - a(a^2-7b^2)\,x_0x_1^4y_1 + 2ab^2x_1^5y_0,$$

$$t \equiv x_0^3y_1[a(a^2-3b^2)\,y_0^2 + 2b^3y_1^2] + 3bx_0^2x_1y_0[-2aby_0^2 + (a^2+b^2)\,y_1^2] + 3bx_0x_1^2y_1[-(a^2+b^2)\,y_0^2 + 2aby_1^2] + x_1^3y_0[-2b^3y_0^2 - a(a^2-3b^2)\,y_1^2],$$

$$p \equiv 3b(a^2-b^2)\,(-x_0y_1 + x_1y_0),$$

$$(f,p)^{0,1} \equiv -3b(a^2-b^2)\,[ax_0^4 + 6bx_0^2x_1^2 + ax_1^4],$$

$$(f,p)^{1,0} \equiv 3b(a^2-b^2)\,[x_0^2(ay_0^2 + by_1^2) + 4bx_0x_1y_0y_1 + x_1^2(by_0^2 + ay_1^2)],$$

$$\Delta \equiv 2[-4ab^3y_0^4 + (-a^4 + 6a^2b^2 + 3b^4)\,y_0^2y_1^2 - 4ab^3y_1^4],$$

$$(h,p)^{0,1} \equiv 3b(a^2-b^2)\,[-2abx_0^3y_0 - (a^2-3b^2)\,x_0^2x_1y_1 - (a^2-3b^2)\,x_0x_1^2y_0 - 2abx_1^3y_1],$$

$$(h,p)^{1,0} \equiv 3b(a^2-b^2)\,[x_0\{2aby_0^3 + (a^2-3b^2)\,y_0y_1^2\} + x_1\{(a^2-3b^2)\,y_0^2y_1 + 2aby_1^3\}],$$

$$(j,p)^{0,1} \equiv -3ab(a^2-b^2)\,(a^2-9b^2)\,x_0x_1(x_0^4 - x_1^4),$$

$$(t,p)^{0,1} \equiv -b(a^2-b^2)\,[2a(a^2-3b^2)\,x_0^4y_0y_1 + x_0^3x_1\{a(a^2-21b^2)\,y_0^2 + 3b(a^2+3b^2)\,y_1^2\} - x_0x_1^3\{3b(a^2+3b^2)\,y_0^2 + a(a^2-21b^2)\,y_1^2\} - 2a(a^2-3b^2)\,x_1^4y_0y_1],$$

$$(t,p)^{1,0} \equiv 3b(a^2-b^2)\,[x_0^2y_0y_1\{a(a^2-5b^2)\,y_0^2 + b(a^2+3b^2)\,y_1^2\} - 4ab^2x_0x_1(y_0^4 - y_1^4) - x_1^2y_0y_1\{b(a^2+3b^2)\,y_0^2 + a(a^2-5b^2)\,y_1^2\}],$$

$$I_6 \equiv 18b^2(a^2-b^2)^2,$$

$$(f,p^2)^{2,0} \equiv 9b^2(a^2-b^2)^2\,[x_0y_0(ay_0^2 + 3by_1^2) + x_1y_1(3by_0^2 + ay_1^2)],$$

$$(h,p^2)^{2,0} \equiv 18b^2(a^2-b^2)^2\,[aby_0^4 + (a^2-3b^2)\,y_0^2y_1^2 + aby_1^4],$$

$$(t,p^2)^{2,0} \equiv 9b^2(a^2-b^2)^2\,[x_0y_1\{a(a^2-7b^2)\,y_0^4 + b(a^2+3b^2)\,y_0^2y_1^2 + 2ab^2y_1^4\} - x_1y_0\{2ab^2y_0^4 + b(a^2+3b^2)\,y_0^2y_1^2 + a(a^2-7b^2)\,y_1^4\}],$$

$$(t,p^3)^{3,0} \equiv 27ab^3(a^2-b^2)^3\,(a^2-9b^2)\,y_0y_1(y_0^4 - y_1^4).$$

It thus appears that none of the twenty forms vanishes identically. Consideration of the orders in x and y of the twenty forms and of the degrees of these forms in the coefficients of f show, as may easily be verified, that any form expressible rationally and integrally in terms of forms of lower degree must contain the factor I_2. Since in our list none of the forms other than I_2 itself possesses the factor a^2+3b^2 we conclude that the system obtained is irreducible.

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