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Continuum structures. II

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Abstract

A continuum structure function is a nondecreasing mapping from the unit hypercube to the unit interval. This paper continues the author's work on the subject, extending Griffith's definitions of coherency to such functions and studying the analytic properties of a continuum structure function based on Natvig's 'second suggestion'.

1. *Introduction*

A binary coherent structure function is a mapping $\phi: \{0,1\}^n \mapsto \{0,1\}$ which is nondecreasing in each argument and for which no component is irrelevant, i.e.

$$\min_{1 \leqslant i \leqslant n} \max_{\mathbf{X} \in \{0,1\}^n} \{\phi(1_i, \mathbf{X}) - \phi(0_i, \mathbf{X})\} = 1,$$

where $(\delta_i, \mathbf{X})$ denotes $(X_1, \ldots, X_{i-1}, \delta, X_{i+1}, \ldots, X_n)$. Recently, a number of papers have studied multistate structure functions, mappings of the form $\Phi: S^n \mapsto S$, where $S = \{0, 1, \ldots, M\}$, which are nondecreasing in each argument. State M denotes perfect operation and state 0 denotes complete failure, the remaining $M-1$ states denoting various undefined intermediate stages. Unlike the binary case, where all the various definitions of coherency are essentially equivalent, there is a variety of possible definitions. Griffith [7], for example, gives the following definition.

Definition (Griffith).

G. 1. If, for any component i and state j, there exists an $\mathbf{X} \in S^n$ such that $\Phi(j_i, \mathbf{X}) = j$ whereas $\Phi(l_i, \mathbf{X}) \neq j$ for $l \neq j$, Φ is said to be *strongly coherent*.

G. 2. If, for any component i and state $j \geqslant 1$, there exists an $\mathbf{X} \in S^n$ such that $\Phi((j-1)_i, \mathbf{X}) < \Phi(j_i, \mathbf{X})$, Φ is said to be *coherent*.

G. 3. If, for any component i, there exists an $\mathbf{X} \in S^n$ such that $\Phi(0_i, \mathbf{X}) < \Phi(M_i, \mathbf{X})$, Φ is said to be *weakly coherent*.

Natvig [8] gives a slightly different definition of coherency (the 'MCS of type 1') which is stronger than G. 2, but weaker than G. 1. El-Neweihi *et al.* [6] study the general properties of strongly coherent multistate structure functions. Barlow and Wu [2] consider a special, but natural, strongly coherent structure function and Block and Savits [4] obtain a decomposition for arbitrary multistate structures.

Structure functions whose domain is a continuum have received less attention. Block and Savits [5] derive a topological decomposition for a large class of upper semicontinuous, nondecreasing functions defined on subsets of $\mathbb{R}^n$ and Baxter [3]

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studies properties of two special cases of *continuum structure functions* (CSFs), non-decreasing mappings from the unit hypercube to the unit interval, viz.

$$\zeta(\mathbf{X}) = \max_{1 \leqslant r \leqslant R} \min_{i \in P_r} X_i \tag{1·1}$$

and

$$\eta(\mathbf{X}) = \max_{1 \leqslant r \leqslant R} \prod_{i \in P_r} X_i, \tag{1·2}$$

where $P_1, P_2, \ldots, P_R$ are the R minimal paths of a binary coherent structure function.

In this paper, the earlier ideas are extended by defining various classes of coherency for CSFs and by a study of the analytic properties of a CSF based on Natvig's 'MCS of type 2' [8].

2. *Continuum structures on the unit hypercube*

A *continuum structure function on the unit hypercube* is a mapping $\gamma: [0,1]^n \mapsto [0,1]$ which is nondecreasing in each X_i where, without loss of generality, we suppose that $\gamma(\mathbf{0}) = 0$ and $\gamma(\mathbf{1}) = 1$. In order to define coherency for such functions, a rather different approach to that of Griffith [7] is required since the state space is now uncountable. It is, however, desirable to retain those intuitively satisfying characteristics of Griffith's definitions for which a parallel can be found. This suggests defining continuum coherency in terms of the measure of the set of points x for which $\gamma(x_i, \mathbf{X})$ is a strictly increasing function of x. In the following, $C = \{1, 2, 3, \ldots\}$ denotes a subset of the positive integers of cardinality n, Δ_n denotes the class of vectors $(X_1, X_2, X_3, \ldots)$ with $0 \leqslant X_i \leqslant 1$ for each $i \in C$ and μ denotes Lebesgue measure.

Definition.

C. 1. If $\inf_{i \in C} \sup_{X \in \Delta_n} \mu\{x | \gamma(x_i, \mathbf{X}) < \gamma((x+\epsilon)_i, \mathbf{X}) \text{ for any } \epsilon > 0\} = 1, \gamma$ is said to be *strictly coherent.*

C. 2. If $\inf_{i \in C} \mu\{x | \gamma(x_i, \mathbf{X}) < \gamma((x+\epsilon)_i, \mathbf{X})$ for any $\epsilon > 0$ for some $\mathbf{X} \in \Delta_n\} = 1, \gamma$ is said to be *coherent.*

C. 3. If $\inf_{i \in C} \sup_{X \in \Delta_n} [\gamma(1_i, \mathbf{X}) - \gamma(0_i, \mathbf{X})] > 0$, γ is said to be *weakly coherent.*

Remarks.

1. It is obvious that C. 1 $\Rightarrow$ C. 2 $\Rightarrow$ C. 3 and that C. 3 $\not\Rightarrow$ C. 2. The construction of a CSF which satisfies C. 2 but not C. 1 is not entirely trivial and is deferred to Section 3 below.

2. Definition C. 1 is, in a sense, stronger than definition G. 1 since the equivalent requirement for multistate structures would be the existence of a vector $\mathbf{X} \in S^n$ such that $\Phi(j_i, \mathbf{X}) = j$ for $j = 0, 1, \ldots, M$ for each $i \in C$. In another sense, however, the definition is weaker as there is no requirement for γ to be surjective..

Definition C. 2. mimics G. 2: almost every $x \in [0, 1)$ is a point of increase of $\gamma(x_i, \mathbf{X})$ for some $\mathbf{X}$ for every i, and C. 3 similarly mimics G. 3.

The foregoing does not fully specify a structure function. Consider the case $n = 1$: any increasing function $\gamma: [0, 1] \mapsto [0, 1]$ is a CSF, but it is perhaps undesirable for the state of a system comprising a single component to be other than that of the component. This motivates the following definition.

Definition. A CSF γ is said to be *complete* if, for every set A of positive measure for which $\gamma(x_i, \mathbf{0})$ is strictly increasing in x for all $x \in A$, $\gamma(x_i, \mathbf{0}) = x$ for all $x \in A$. This must hold for each $i \in C$.

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Similarly, a CSF is *co-complete* if it satisfies the above definition with $\gamma(x_i, \mathbf{0})$ replaced by $\gamma(x_i, \mathbf{1})$. Intuitively, if $\mu(A) > 0$, component i acts as a 'path (cut) set' of size one on A in a complete (co-complete) CSF.

Example 2·1. The structure functions ζ and η are both strictly coherent and complete; further, ζ is also co-complete. |

It is obvious that $\gamma(\mathbf{X} \vee \mathbf{Y}) \geqslant \gamma(\mathbf{X}) \vee \gamma(\mathbf{Y})$ and that $\gamma(\mathbf{X} \wedge \mathbf{Y}) \leqslant \gamma(\mathbf{X}) \wedge \gamma(\mathbf{Y})$ for all $\mathbf{X}, \mathbf{Y} \in \Delta_n$. The conditions under which equality occurs are given by the following generalization of Griffith's Proposition 2·2.

THEOREM 2·2. (i) *Suppose that γ is a complete, coherent* CSF. *Then*

$$\gamma(\mathbf{X} \vee \mathbf{Y}) = \gamma(\mathbf{X}) \vee \gamma(\mathbf{Y})$$

if and only if $\gamma(\mathbf{X}) = \sup_{i \in C} X_i$.

(ii) *Suppose that γ is a co-complete, coherent* CSF. *Then* $\gamma(\mathbf{X} \wedge \mathbf{Y}) = \gamma(\mathbf{X}) \wedge \gamma(\mathbf{Y})$ *if and only if* $\gamma(\mathbf{X}) = \inf_{i \in C} X_i$.

Proof. (i) The 'if' assertion follows immediately from the identity

$$\sup_{i \in C} (X_i \vee Y_i) = (\sup_{i \in C} X_i) \vee (\sup_{i \in C} Y_i).$$

The 'only if' assertion is proved in two stages: firstly, we show that

$$\gamma(\mathbf{X}) = \sup_{i \in C} \gamma((X_i)_i, \mathbf{0});$$

then we prove that $\gamma((X_i)_i, \mathbf{0}) = X_i$ for each $i \in C$.

By the monotonicity of γ, $\gamma(\mathbf{X}) \geqslant \gamma((X_i)_i, \mathbf{0})$ for each $i \in C$, i.e. $\gamma(\mathbf{X})$ is an upper bound for the sequence $\{\gamma((X_i)_i, \mathbf{0})\}$, and hence $\gamma(\mathbf{X}) \geqslant \sup_{i \in C} \gamma((X_i)_i, \mathbf{0})$. Further, for $m = 1, 2, 3, \ldots$, we can write

$$\gamma(\mathbf{X}) = [\max_{1 \leqslant i \leqslant m} \gamma((X_i)_i, \mathbf{0})] \vee \gamma(\mathbf{0}^m, \mathbf{X}^m),$$

where $\mathbf{0}^m$ denotes a vector of m zeroes and $\mathbf{X}^m = (X_{m+1}, X_{m+2}, X_{m+3}, \ldots)$. By the monotonicity of γ, $\{\gamma(\mathbf{0}^m, \mathbf{X}^m)\}$ is a nonincreasing sequence with limit $\gamma(\mathbf{0}) = 0$ and hence there exists an $N = N(\mathbf{X})$ such that $\gamma(\mathbf{0}^m, \mathbf{X}^m) \leqslant \max_{1 \leqslant i \leqslant m} \gamma((X_i)_i, \mathbf{0})$ for all $m \geqslant N$. Thus, for $m \geqslant N$,

$$\gamma(\mathbf{X}) = \max_{1 \leqslant i \leqslant m} \gamma((X_i)_i, \mathbf{0}) \leqslant \sup_{i \in C} \gamma((X_i)_i, \mathbf{0}).$$

Combining this with the previous inequality, we see that $\gamma(\mathbf{X}) = \sup_{i \in C} \gamma((X_i)_i, \mathbf{0})$ as claimed.

To prove that $\gamma((X_i)_i, \mathbf{0}) \equiv X_i$, observe that, for each $x \in [0, 1) - A$, where A is a set of measure zero, there exists, by coherency, an $\mathbf{X} \in \Delta_n$ such that $\gamma(x_i, \mathbf{X}) < \gamma(y_i, \mathbf{X})$ for any $y > x$. Now $(x_i, \mathbf{X}) \equiv (x_i, \mathbf{0}) \vee (0_i, \mathbf{X})$ and $(y_i, \mathbf{X}) \equiv (y_i, \mathbf{0}) \vee (0_i, \mathbf{X})$ so

$$\gamma(x_i, \mathbf{X}) = \gamma(x_i, \mathbf{0}) \vee \gamma(0_i, \mathbf{X}) < \gamma(y_i, \mathbf{0}) \vee \gamma(0_i, \mathbf{X}) = \gamma(y_i, \mathbf{X}) \quad \text{if} \quad y > x.$$

Thus, if $x \in [0, 1) - A$ and $y > x$, then $\gamma(x_i, \mathbf{0}) < \gamma(y_i, \mathbf{0})$ and so, by completeness, $\gamma(x_i, \mathbf{0}) = x$ for all $x \in [0, 1) - A$. The monotonicity of γ ensures that A is the empty set and hence $\gamma(\mathbf{X}) = \sup_{i \in C} X_i$ as claimed.

(ii) This is proved similarly. |

Theorem 2·2 (ii) generalizes Theorem 2·4 (*b*) of Barlow and Proschan ([1], p. 8), which states that, if ϕ is a binary coherent structure function,

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(1) $\phi(\mathbf{x}.\mathbf{y}) \leqslant \phi(\mathbf{x})\phi(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$, and

(2) equality holds in (1) if and only if ϕ is a series structure,

where $\mathbf{x}.\mathbf{y}$ denotes the Schur (co-ordinatewise) product of $\mathbf{x}$ and $\mathbf{y}$. Since $\mathbf{x}$ and $\mathbf{y}$ are binary vectors, $\mathbf{x}.\mathbf{y} \equiv \mathbf{x} \wedge \mathbf{y}$ and $\phi(\mathbf{x})\phi(\mathbf{y}) \equiv \phi(\mathbf{x}) \wedge \phi(\mathbf{y})$, so the result for CSFs generalizes only part of the theorem for binary structures. It is not, however, the case that $\gamma(\mathbf{X}.\mathbf{Y}) \leqslant \gamma(\mathbf{X})\gamma(\mathbf{Y})$, even for quite special classes of CSFs (consider, for example, the CSF $\gamma(\mathbf{X}) = X_1 \wedge (X_2 \vee X_3)$ and let $\mathbf{X} = (0{\cdot}5,\ 0{\cdot}8,\ 0{\cdot}8)$ and $\mathbf{Y} = (0{\cdot}8,\ 0{\cdot}4,\ 0{\cdot}1)$; then $\gamma(\mathbf{X}.\mathbf{Y}) = 0{\cdot}32$ whereas $\gamma(\mathbf{X})\gamma(\mathbf{Y}) = 0{\cdot}20$). We can, however, establish the following counterpart of (2).

THEOREM 2·3. *Suppose that γ is a nonvacuously co-complete* CSF. *Then*

$$\gamma(\mathbf{X}.\mathbf{Y}) = \gamma(\mathbf{X})\gamma(\mathbf{Y})$$

if and only if $\gamma(\mathbf{X}) = \prod_{i=1}^{n} X_i$.

Proof. The 'if' assertion is immediate. The 'only if' assertion is proved by induction. Consider the case $n = 1$; the monotonicity of γ is sufficient to ensure that the only solution to the functional equation $\gamma(X_1 Y_1) = \gamma(X_1)\gamma(Y_1)$ is $\gamma(X) = X^\alpha$ where $\alpha \in \mathbb{R}$. Suppose, now, that the result is true for $n = m$, i.e.

$$\gamma(X_1, \ldots, X_m) = \prod_{i=1}^{m} X_i^\alpha \gamma(1, \ldots, 1).$$

Then
$$\gamma(X_1, \ldots, X_{m+1}) = \prod_{i=1}^{m} X_i^\alpha \gamma(1, \ldots, 1, X_{m+1})$$

and thus a similar argument verifies that

$$\gamma(\mathbf{X}) = \prod_{i=1}^{m+1} X_i^\alpha.$$

The condition of nonvacuous co-completeness is sufficient to ensure that $\alpha = 1$. ∎

3. *A coherent continuum structure*

Natvig [8] proposed a class of multistate structure functions (the 'MCS of type 2') which satisfy the condition

$$\Phi(\mathbf{X}) \geqslant j \quad \text{iff} \quad \phi_j(I_j(\mathbf{X})) = 1 \quad (j = 1, 2, \ldots, M), \tag{3·1}$$

where $\mathbf{X} \in S^n$ $(n < \infty)$, $\{\phi_j, j = 1, 2, \ldots, M\}$ is a set of binary coherent structure functions and $I_j(\mathbf{X}) = (I_j(X_1), \ldots, I_j(X_n))$ where

$$I_j(X_i) = \begin{cases} 1 & \text{if} \quad X_i \geqslant j, \\ 0 & \text{if} \quad X_i < j. \end{cases}$$

In the spirit of Natvig's suggestion, the following class of CSFs on the unit hypercube is proposed.

Definition. Let $\{\phi_\alpha, 0 < \alpha \leqslant 1\}$ be a class of binary coherent structure functions such that $\phi_\alpha(\mathbf{Y}_\alpha)$ is a left-continuous and non-increasing function of α for fixed $\mathbf{X}$ where $Y_{\alpha i}$ is the indicator of $\{X_i \geqslant \alpha\}$, $i = 1, 2, \ldots, n$. Define the structure function ξ as the function which satisfies the condition

$$\xi(\mathbf{X}) \geqslant \alpha \quad \text{iff} \quad \phi_\alpha(\mathbf{Y}_\alpha) = 1 \quad (0 < \alpha \leqslant 1) \tag{3·2}$$

where $\mathbf{X} \in \Delta_n$ $(n < \infty)$.

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Since there are only finitely many binary coherent structure functions of order n, $\{\phi_\alpha, 0 < \alpha \leqslant 1\}$ is finite. The conditions of left-continuity and monotonicity ensure that $\{\phi_\alpha, 0 < \alpha \leqslant 1\}$ is of the form

$$\phi_\alpha = \begin{cases} \phi_1, & 0 < \alpha \leqslant \alpha_1, \\ \phi_2, & \alpha_1 < \alpha \leqslant \alpha_2, \\ \vdots & \vdots \\ \phi_k, & \alpha_{k-1} < \alpha \leqslant 1, \end{cases}$$

where $\phi_j(\mathbf{Y}) \geqslant \phi_{j+1}(\mathbf{Y})$ for all $\mathbf{Y} \in \{0,1\}^n$ $(j = 1, 2, \ldots, k-1)$ for some $k \geqslant 1$.

Note that, if $\phi_\alpha \equiv \phi$, definition (3·2) reduces to the Barlow–Wu CSF (1·1). Thus, ξ is suitable as a model for systems which are locally of the Barlow–Wu type, but for which no single underlying binary structure function is uniformly appropriate. A further advantage of this definition is that, given $\{\phi_\alpha, 0 < \alpha \leqslant 1\}$, it is easy to determine the distribution of $\xi(\mathbf{X})$ when $X_1, X_2, \ldots, X_n$ are independent random variables defined on the same probability space $(\Omega, \mathscr{F}, P)$: clearly $\{\phi_x(\mathbf{Y}_x) = 0\} \in \mathscr{F}$ for all real x and

$$P\{\xi(\mathbf{X}) < x\} = 1 - h_x(E(Y_{x1}), \ldots, E(Y_{xn})), \tag{3·3}$$

where h_x is the reliability function corresponding to ϕ_x.

LEMMA 3.1. *ξ is a complete, coherent* CSF.

Proof. The monotonicity of ξ follows from the monotonicity of the ϕ_α; that $\xi(\mathbf{0}) = 0$ and $\xi(\mathbf{1}) = 1$ is shown in Lemma 3·3 below; that ξ is complete is Corollary 4·3 below.

To see that ξ is coherent, consider $\xi(x_i, \mathbf{X})$ as a function of x for fixed $(\cdot_i, \mathbf{X})$ and, for arbitrary $x \in (0, 1]$, choose $(\cdot_i, \mathbf{X}) \in \Delta_n$ such that $\phi_x(1_i, \mathbf{Y}_x) - \phi_x(0_i, \mathbf{Y}_x) = 1$. (Some such $(\cdot_i, \mathbf{Y}_x) \in \{0,1\}^n$ must exist since, by definition, ϕ_x is a binary coherent structure function.) Then $\xi(x_i, \mathbf{X}) \geqslant x$ whereas $\xi((x-\epsilon)_i, \mathbf{X}) < x$ for all $\epsilon > 0$. This holds for all $x \in (0, 1]$ and for each $i \in C$, so ξ is coherent as claimed. ∎

The following example shows that ξ is not, however, strictly coherent.

Example 3·2.

Let $$\phi_\alpha(\mathbf{Y}_\alpha) = \begin{cases} 1 - (1 - Y_{\alpha 1})(1 - Y_{\alpha 2}) & \text{for } 0 < \alpha \leqslant \frac{1}{2}, \\ Y_{\alpha 1} Y_{\alpha 2} & \text{for } \frac{1}{2} < \alpha \leqslant 1. \end{cases}$$

The corresponding CSF ξ is *not* strictly coherent. ∎

LEMMA 3·3. *$\xi(\boldsymbol{\alpha}) = \alpha$ for all $\alpha \in [0, 1]$.*

Proof. Choose $\alpha \in [0, 1]$. It follows from definition (3·2) that $\xi(\boldsymbol{\alpha}) \geqslant \alpha$ whereas $\xi(\boldsymbol{\alpha}) < \alpha + \epsilon$ for all $\epsilon > 0$, and hence $\xi(\boldsymbol{\alpha}) = \alpha$ for all $\alpha \in [0, 1]$. ∎

LEMMA 3·4. *Let $m = \min_{i \in C} X_i$ and $M = \max_{i \in C} X_i$. Then $m \leqslant \xi(\mathbf{X}) \leqslant M$ for all $\mathbf{X} \in \Delta_n$.*

Proof. It follows from definition (3·2) that $\xi(\mathbf{X}) \geqslant m$ for any $\mathbf{X} \in \Delta_n$. Further, for any $\epsilon > 0$, $\xi(\mathbf{X}) < M + \epsilon$, and hence $\xi(\mathbf{X}) \leqslant M$. ∎

THEOREM 3·5. *ξ is right-continuous.*

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Proof. Let $P_1^\alpha, \ldots, P_{R(\alpha)}^\alpha$ denote the $R(\alpha)$ minimal path sets of ϕ_α and let

$$X_i^{(j)} = \begin{cases} \alpha & \text{if} \quad i \in P_j^\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, 2, \ldots, R(\alpha)$ and $i = 1, 2, \ldots, n$. Further, we define

$$A_\alpha = \bigcup_{j=1}^{R(\alpha)} \{\mathbf{X} \in \Delta_n | \mathbf{X} \geqslant \mathbf{X}^{(j)}\}.$$

We claim that $A_\alpha = U_\alpha$, where $U_\alpha = \{\mathbf{X} \in \Delta_n | \xi(\mathbf{X}) \geqslant \alpha\}$.

Suppose, firstly, that $\mathbf{X} \in U_\alpha$. Then $\xi(\mathbf{X}) \geqslant \alpha$, i.e. $\phi_\alpha(\mathbf{Y}_\alpha) = 1$, so that $Y_{\alpha i} = 1$ for all $i \in P_j^\alpha$ for some $j = 1, 2, \ldots, R(\alpha)$. Thus $\mathbf{X} \geqslant \mathbf{X}^{(j)}$ for some $j = 1, 2, \ldots, R(\alpha)$ and hence $\mathbf{X} \in A_\alpha$ so that $U_\alpha \subset A_\alpha$. Conversely, if $\mathbf{X} \in A_\alpha$, then $\mathbf{X} \geqslant \mathbf{X}^{(j)}$ for some $j = 1, 2, \ldots, R(\alpha)$ and so $\xi(\mathbf{X}) \geqslant \xi(\mathbf{X}^{(j)}) = \alpha$, i.e. $\mathbf{X} \in U_\alpha$, and hence $A_\alpha \subset U_\alpha$.

Thus $U_\alpha = A_\alpha$ and, as A_α is obviously closed (in the relative topology on Δ_n), it follows that U_α is similarly closed. Since α is arbitrary, we have shown that every U_α is closed, and thus ξ is upper semicontinuous ([9], p. 161) or, equivalently, since ξ is non-decreasing, right-continuous as claimed. |

The sets $P_\alpha = \{\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(R(\alpha))}\}$, $0 < \alpha \leqslant 1$, provide a convenient method for evaluating $\xi(\mathbf{X})$ for an arbitrary vector $\mathbf{X}$ since, if $\mathbf{X} \geqslant \mathbf{X}^{(j)} \in P_\alpha$, then $\xi(\mathbf{X}) \geqslant \alpha$. One thus compares $\mathbf{X}$ with each $\mathbf{X}^{(j)} \in P_1$ and then successively for each of the sets P_α, $0 < \alpha < 1$, until a vector which minorizes $\mathbf{X}$ is found. Then $\xi(\mathbf{X}) = \xi(\mathbf{Y})$ where $\mathbf{Y}$ is the largest such vector.

As an illustration, consider the CSF of Example (3·2). Clearly,

$$P_\alpha = \begin{cases} \{(\alpha, 0), (0, \alpha)\} & \text{for} \quad 0 < \alpha \leqslant \tfrac{1}{2}, \\ \{(\alpha, \alpha)\} & \text{for} \quad \tfrac{1}{2} < \alpha \leqslant 1, \end{cases}$$

and so $\xi(1, 0) = \xi(0, 1) = \frac{1}{2}$.

4. *The functional form of* $\xi(x_i, \mathbf{X})$

In this section, we investigate the functional form of $\xi_i(x) = \xi(x_i, \mathbf{X})$ as a function of x for fixed $(\cdot_i, \mathbf{X})$. Where no confusion should arise, we shall omit the subscript i and write $\xi(x) = \xi_i(x)$.

Our first result is that, if $\xi(x)$ is continuous and strictly increasing on an interval of the form $[a, b)$, then $\xi(x) = x$ for all $x \in [a, b)$. Before proving this theorem, it is convenient to record the following, easily-verified lemma.

LEMMA 4·1. *If* f: $\Omega \mapsto \mathbb{R}$ *and* g: $\Omega \mapsto \mathbb{R}$, *the assertions*

(I) $f(x) \geqslant \alpha$ *if and only if* $g(x) \geqslant \alpha$ *for all* $\alpha \in \mathbb{R}$ *and* $x \in \Omega$,

(II) $f = g$ *on* Ω

are equivalent.

THEOREM 4·2. *Suppose that* $\xi(x)$ *is continuous and strictly increasing on the interval* $[a, b) \subset [0, 1]$. *Then* $\xi(x) = x$ *for all* $x \in [a, b)$.

Proof. Let A denote (a, b) and notice that A is not the empty set. Since ξ is a homeomorphism and A is an open interval, $\xi(A)$ is also an open interval. In particular, $\xi(A)$ is not empty.

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Recall that ξ is strictly increasing on A. Thus, if $x \in A$ and $\epsilon > 0$,

$$\xi(x-\epsilon) < \xi(x) < \xi(x+\epsilon)$$

so that $\phi_{\xi(x)}(1_i, \mathbf{Y}_{\xi(x)}) = 1$ and $\phi_{\xi(x)}(0_i, \mathbf{Y}_{\xi(x)}) = 0$ for each $i \in C$. Thus

$$\phi_{\xi(x)}(1_i, \mathbf{Y}_{\xi(x)}) - \phi_{\xi(x)}(0_i, \mathbf{Y}_{\xi(x)}) = 1, \tag{4·1}$$

i.e. each component is *critical* for all $x \in A$.

Choose $\eta \in \xi(A)$. Now

$$\begin{aligned} \xi(x) \geqslant \eta \quad &\text{iff} \quad \phi_\eta(\mathbf{Y}_\eta) = 1 \quad \text{by definition,} \\ &\text{iff} \quad x \geqslant \eta \quad \text{because } i \text{ is critical.} \end{aligned}$$

The events $\{\xi(x) \geqslant \eta\}$ and $\{x \geqslant \eta\}$ are thus equivalent on A. On appealing to Lemma 4·1, it follows that $\xi(x) = x$ for all $x \in A$. From Theorem 3·5, we see that $\xi(a) = a$, completing the proof. |

COROLLARY 4·3. *ξ is complete.*

Theorems 3·5 and 4·2 can be summarized by observing that, since the discontinuities of ξ_i are jumps with no isolated points, ξ_i has a unique decomposition into a step function and the identity function. In other words, there exists a unique finite partition $0 = a_0 < a_1 < a_2 < \ldots < 1$ of the unit interval such that, on each subinterval $[a_j, a_{j+1})$, $\xi_i(x)$ is constant or is equal to x. The next result identifies these constants with the endpoints of the subintervals.

THEOREM 4·4. *Suppose that $\xi(x)$ is continuously increasing on the interval $[a_{j-1}, a_j)$ and is constant on the interval $[a_j, a_{j+1})$ $(j = 1, 2, 3, \ldots)$. Then $\xi(x) = a_j$ for all $x \in [a_j, a_{j+1})$.*

Proof. Suppose that the theorem is false. Then there exists some $c \in (a_j, a_{j+1}]$ such that $\xi(x) = c$ for all $x \in [a_j, a_{j+1})$. In particular, $\xi(a_j) > \xi(a_j - \epsilon) + c - a_j$ for all $\epsilon > 0$. Since, however, $\xi(a_j - \epsilon) = a_j - \epsilon$ for $\epsilon < a_j - a_{j-1}$, this inequality reduces to

$$\xi(a_j) > c - \epsilon,$$

from which it follows that $\phi_{c-\epsilon}(\mathbf{Y}_{c-\epsilon}) = 1$. If $\epsilon < (c - a_j) \wedge (a_j - a_{j-1})$, then $Y_{c-\epsilon, i} = 0$ and so $1 = \phi_{c-\epsilon}(0_i, \mathbf{Y}_{c-\epsilon}) = \phi_{c-\epsilon}(1_i, \mathbf{Y}_{c-\epsilon})$. We have thus shown that i is not critical in some interval $(a_j - \epsilon, a_j)$, in contradiction to (4·1). Hence we must have $c = a_j$ as claimed. |

THEOREM 4·5. *Suppose that $\xi(x)$ is continuously increasing on the interval $[a_{j-1}, a_j)$ and is constant on each of the intervals $[a_j, a_{j+1})$ and $[a_{j+1}, a_{j+2})$. Then $\xi(x) = a_j$ for all $x \in [a_j, a_{j+2})$.*

Proof. Suppose that $\xi(x) = k > a_j$ on $[a_{j+1}, a_{j+2})$. By the previous theorem, $\xi(x) = a_j$ for all $x \in [a_j, a_{j+1})$ and so $\xi(x) < k - \epsilon$ on $[a_j, a_{j+1})$ where $0 < \epsilon < k - a_j$. It follows that $\phi_{k-\epsilon}(\mathbf{Y}_{k-\epsilon}) = 0$ for $a_j \leqslant x < a_{j+1}$ and, since i is not critical for these values of x, $\phi_{k-\epsilon}(1_i, \mathbf{Y}_{k-\epsilon}) = 0$. Since, however, $\xi(x) = k$ for all $x \in [a_{j+1}, a_{j+2})$, it follows that $\phi_k(\mathbf{Y}_k) = 1$ and hence $\phi_k(1_i, \mathbf{Y}_k) = 1 > \phi_{k-\epsilon}(1_i, \mathbf{Y}_{k-\epsilon})$. This is a contradiction and hence we must have $k = a_j$ as claimed. |

The results of this section are summarized in the following theorem.

THEOREM 4·6. *For fixed $(\cdot_i, \mathbf{X})$, there is a unique finite partition*

$$0 = a_0 < a_1 < a_2 < \ldots < 1$$

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of $[0, 1]$ *such that*

$$\xi(x) = \begin{cases} x & \text{for all } x \in [a_j, a_{j+1}), \\ a_{j+1} & \text{for all } x \in [a_{j+1}, a_{j+2}), \end{cases}$$

for either $j = 0, 2, 4, \ldots,$ or $j = 1, 3, 5, \ldots$. *In the latter case,* $\xi(x)$ *is constant on* $[0, a_1)$, *but the constant is not necessarily one of the elements of the partition.*

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REFERENCES

[1] R. E. BARLOW and F. PROSCHAN. *Statistical Theory of Reliability and Life Testing.* (Holt, Rinehart and Winston, 1975).
[2] R. E. BARLOW and A. S. WU. Coherent systems with multi-state components. *Math. Oper. Res.* **3** (1978), 275–281.
[3] L. A. BAXTER. Continuum structures I. *J. Appl. Probab.* **21** (1984), 802–815.
[4] H. W. BLOCK and T. H. SAVITS. A decomposition for multistate monotone systems. *J. Appl. Probab.* **19** (1982), 391–402.
[5] H. W. BLOCK and T. H. SAVITS. Continuous multistate structure functions. *Oper. Res.* **32** (1984), 703–714
[6] E. EL-NEWEIHI, F. PROSCHAN and J. SETHURAMAN. Multistate coherent systems. *J. Appl. Probab.* **15** (1978), 675–688.
[7] W. S. GRIFFITH. Multistate reliability models. *J. Appl. Probab.* **17** (1980), 735–744.
[8] B. NATVIG. Two suggestions of how to define a multistate coherent system. *Adv. in Appl. Probab.* **14** (1982), 434–455.
[9] H. L. ROYDEN. *Real Analysis*, second edition (MacMillan, 1968).

https://doi.org/10.1017/S0305004100064240 Published online by Cambridge University Press