The sum of powers as canonical expression. By J. BRONOWSKI, B.A., Jesus College.

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1. In this paper I discuss the expression of the general form F of order n, homogeneous in $r+1$ variables $z_0, z_1, \ldots, z_l, \ldots, z_r$, as the sum of the nth powers of $h+H$ linear forms in these variables. I take $h\,(>0)$ of these linear forms to be undetermined, namely the forms

$$\left.\begin{aligned} y_1 &\equiv a_0z_0 + a_1z_1 + \ldots + a_lz_l + \ldots + a_rz_r \\ y_2 &\equiv b_0z_0 + b_1z_1 + \ldots + b_lz_l + \ldots + b_rz_r \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_g &\equiv g_0z_0 + g_1z_1 + \ldots + g_lz_l + \ldots + g_rz_r \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_h &\equiv h_0z_0 + h_1z_1 + \ldots + h_lz_l + \ldots + h_rz_r \end{aligned}\right\}, \qquad (1)$$

whose coefficients are undetermined; and I take the remaining $H\,(\geqslant 0)$ linear forms to be assigned, namely the forms

$$\left.\begin{aligned} Y_1 &\equiv A_0z_0 + A_1z_1 + \ldots + A_lz_l + \ldots + A_rz_r \\ Y_2 &\equiv B_0z_0 + B_1z_1 + \ldots + B_lz_l + \ldots + B_rz_r \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Y_G &\equiv G_0z_0 + G_1z_1 + \ldots + G_lz_l + \ldots + G_rz_r \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Y_H &\equiv H_0z_0 + H_1z_1 + \ldots + H_lz_l + \ldots + H_rz_r \end{aligned}\right\}, \qquad (1)$$

whose coefficients are assigned. Then the expression

$$y_1{}^n + y_2{}^n + \ldots + y_g{}^n + \ldots + y_h{}^n + \lambda_1 Y_1{}^n + \lambda_2 Y_2{}^n + \ldots + \lambda_G Y_G{}^n + \ldots + \lambda_H Y_H{}^n, \qquad (2)$$

under which I seek to write the general form F, is determined by the ratios of the $h\,(r+1)+H$ variables

$$a_0, a_1, \ldots, a_l, \ldots, a_r;\ b_0, b_1, \ldots, b_l, \ldots, b_r;\ \ldots;\ g_0, g_1, \ldots, g_l, \ldots, g_r;\ \ldots;\ h_0, h_1, \ldots, h_l, \ldots, h_r;\ \lambda_1;\ \lambda_2;\ \ldots;\ \lambda_G;\ \ldots;\ \lambda_H. \qquad (3)$$

The form F, whose typical term I write

$$\theta_{\alpha\beta\ldots\mu\ldots\sigma}\,\frac{n!}{\alpha!\beta!\ldots\mu!\ldots\sigma!}\,z_0{}^{\alpha}z_1{}^{\beta}\ldots z_l{}^{\mu}\ldots z_r{}^{\sigma},$$

where

$$\alpha+\beta+\ldots+\mu+\ldots+\sigma=n,$$

is determined by the ratios of its $\binom{n+r}{r}$ coefficients θ. I take these numbers to be equal, i.e.

$$h(r+1)+H=\binom{n+r}{r}, \qquad (4)$$

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and I investigate the conditions that the general form F may be written under the expression (2) in a finite number of ways. In particular, for values of n, r such that

$$k \equiv \binom{n+r}{r} \Big/ (r+1)$$

is an integer, I investigate the conditions that the general form F may be written under the expression (2) in a finite number of ways when

$$h = k, \quad H = 0.$$

2. Developing the expression (2) in terms of $z_0, z_1, \ldots, z_l, \ldots, z_r$ by means of (1), I obtain a form F' of order n in these variables, the coefficient of whose typical term is given by

$$\begin{aligned}\theta'_{\alpha\beta\ldots\mu\ldots\sigma} = {} & a_0^{\alpha} a_1^{\beta} \ldots a_l^{\mu} \ldots a_r^{\sigma} + b_0^{\alpha} b_1^{\beta} \ldots b_l^{\mu} \ldots b_r^{\sigma} + \ldots \\ & + g_0^{\alpha} g_1^{\beta} \ldots g_l^{\mu} \ldots g_r^{\sigma} + \ldots + h_0^{\alpha} h_1^{\beta} \ldots h_l^{\mu} \ldots h_r^{\sigma} \\ & + \lambda_1 A_0^{\alpha} A_1^{\beta} \ldots A_l^{\mu} \ldots A_r^{\sigma} + \lambda_2 B_0^{\alpha} B_1^{\beta} \ldots B_l^{\mu} \ldots B_r^{\sigma} + \ldots \\ & + \lambda_G G_0^{\alpha} G_1^{\beta} \ldots G_l^{\mu} \ldots G_r^{\sigma} + \ldots + \lambda_H H_0^{\alpha} H_1^{\beta} \ldots H_l^{\mu} \ldots H_r^{\sigma}. \end{aligned} \tag{5}$$

The form F' so obtained may therefore be identified with the general form F if, and only if, the coefficients typified by (5) are independent, regarded as functions of the variables (3). Therefore a necessary and sufficient condition that the expressions (2) may be identified with the general form F is that the Jacobian determinant j of the functions θ', of which typical terms are

$$\partial\theta'_{\alpha\beta\ldots\mu\ldots\sigma}/\partial g_l, \quad \partial\theta'_{\alpha\beta\ldots\mu\ldots\sigma}/\partial\lambda_G,$$

does not vanish identically. Substituting from the equations typified by (5), the determinant j is given by

$$j \equiv \begin{vmatrix} n a_0^{n-1}, & 0, & \ldots\ldots \\ (n-1) a_0^{n-2} a_1, & a_0^{n-1}, & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \alpha a_0^{\alpha-1} a_1^{\beta} \ldots a_l^{\mu} \ldots a_r^{\sigma}, & \beta a_0^{\alpha} a_1^{\beta-1} \ldots a_l^{\mu} \ldots a_r^{\sigma}, & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0, & 0, & \ldots\ldots \end{vmatrix}$$

$$\begin{vmatrix} 0, & \ldots\ldots & H_0^n \\ 0, & \ldots\ldots & H_0^{n-1} H_1 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \mu g_0^{\alpha} g_1^{\beta} \ldots g_l^{\mu-1} \ldots g_r^{\sigma}, & \ldots\ldots & H_0^{\alpha} H_1^{\beta} \ldots H_l^{\mu} \ldots H_r^{\sigma} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0, & \ldots\ldots & H_r^n \end{vmatrix}.$$

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But, reading this determinant j by columns, its non-vanishing is the necessary and sufficient condition that the following $\binom{n+r}{r}$ equations, linear in the undetermined coefficients θ of the general form F, are linearly independent:

$$\left(\frac{\partial F}{\partial z_0}\right)_{(a)} = 0, \quad \left(\frac{\partial F}{\partial z_1}\right)_{(a)} = 0, \quad \ldots, \quad \left(\frac{\partial F}{\partial z_l}\right)_{(g)} = 0, \quad \ldots, \quad (F)_{(H)} = 0, \tag{6}$$

where I write the symbol $\left(\frac{\partial F}{\partial z_l}\right)_{(g)}$ for the expression obtained by substituting

$$z_0 = g_0, \quad z_1 = g_1, \quad \ldots, \quad z_l = g_l, \quad \ldots, \quad z_r = g_r$$

in the expression $\frac{\partial F}{\partial z_l}$. Therefore, interpreting the equations (6) for the general primal $F = 0$ of order n in space of r dimensions, I conclude that:

A necessary and sufficient condition that the general form F can be written under the expression (2) is that there exists no primal of order n in space of r dimensions, which has simple points at the H assigned points of the space,

$$(A_0, A_1, \ldots, A_l, \ldots, A_r), (B_0, B_1, \ldots, B_l, \ldots, B_r), \ldots,$$
$$(G_0, G_1, \ldots, G_l, \ldots, G_r), \ldots, (H_0, H_1, \ldots, H_l, \ldots, H_r), \tag{7}$$

and which has nodes at h further, arbitrarily assignable points of the space.

In particular, putting $h = k$, $H = 0$, there follows the corollary that:

A necessary and sufficient condition that the general form of order n, homogeneous in $r+1$ variables, can be expressed as the sum of the nth powers of $k \equiv \binom{n+r}{r} \Big/ (r+1)$ linear forms in these variables is that there exists no primal of order n in space of r dimensions, which has nodes at k arbitrarily assignable points of the space*.

3. I now consider the system S formed by primals of order n in space R of r dimensions which have simple points at the H points (7), and which further have nodes at $h-1$ arbitrarily assigned points of the space. Assuming, as I shall throughout this paper,

* This corollary, sometimes ascribed to E. K. Wakeford, *Proc. London Math. Soc.* (2), 18 (1919), 403, was given by Em. Lasker, *Math. Annalen*, 58 (1904), 434, whose paper substantially anticipates that of E. K. Wakeford. It can be proved geometrically by applying to the manifold $L_r{}^{n^r}$ of § 10 the theorem of A. Terracini, *Rend. di Palermo*, 31 (1911), 392.

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that the points (7) impose independent conditions on the primals of order n, the system S is therefore a linear system of freedom

$$\binom{n+r}{r} - 1 - H - (h-1)(r+1),$$

which number, by the equation (4) assumed, is equal to r. Hence the primals of the system S which have a further node form a system of freedom $r-1$; and the locus of such further nodes is in general a primal J of the space R. This primal J is the Jacobian primal of the system S: and if $a_0, a_1, \ldots, a_l, \ldots, a_r$ are taken for current coordinates in R, and

$$(b_0, b_1, \ldots, b_l, \ldots, b_r), \ldots, (g_0, g_1, \ldots, g_l, \ldots, g_r), \ldots, (h_0, h_1, \ldots, h_l, \ldots, h_r)$$

for the $h-1$ assigned base-nodes of S, then the equation of J is

$$j = 0.$$

Whenever J is determinate, a primal of the system S can have an hth node only upon J; and therefore cannot have an hth node at an arbitrarily assigned point of the space R. In every such case the expression (2) is therefore canonical, that is, represents the general form F of order n, homogeneous in $r+1$ variables.

The expression (2) is therefore not canonical if, and only if, the Jacobian J of the system S is indeterminate, namely, in that it vanishes identically. But a necessary and sufficient condition that this Jacobian vanishes is that the system S has the grade zero*.

I thus obtain

THEOREM 1. *A necessary and sufficient condition that the general form of order n, homogeneous in $r+1$ variables, can be written under the expression* (2) *is that the system of primals of order n in space of r dimensions which have simple points at the H assigned points* (7) *and which have nodes at $h-1$ further, arbitrarily assigned points of the space, has a grade not zero.*

4. In particular, I deduce

COROLLARY 1. *A necessary and sufficient condition that the general form of order n, homogeneous in $r+1$ variables, can be expressed as the sum of the nth powers of $k \equiv \binom{n+r}{r} \Big/ (r+1)$ linear forms in these variables is that the system of primals of order n in space of r dimensions which have nodes at $k-1$ arbitrarily assigned points of the space, has a grade not zero.*

* E. Bertini, *Rend. Lincei* (5), 10 (1901), 73 or *Geometria proiettiva degli iperspazi*, 2nd edn. (1923), 277. For a proof of the theorem without the aid of this lemma, see § 8.

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Now the grade of this system, say S', of primals of order n in space R which have nodes at $k-1$ arbitrarily assigned points, is in general

$$n^r - 2^r(k-1) = n^r + 2^r - 2^r\binom{n+r}{r}\Big/(r+1). \qquad (8)$$

Since $$\frac{\binom{n+r}{r}}{r+1} = \frac{n+r}{r}\,\frac{n+r-1}{r-1}\,\cdots\,\frac{n+2}{2}\,\frac{n+1}{r+1} < \binom{n}{2}^r$$

for $n>3$, $r>2$, the grade (8) is definitely positive, and in fact is greater than 2^r, for all such values of n, r; while, for $r=2$, it vanishes only when $n=2$ and $n=4$. Apart therefore from the latter two cases, the grade of the system S' can vanish only when, exceptionally, it is no longer given by the formula (8): among these exceptions are included the cases $n \leqslant 3$ which have not yet been discussed. Such exceptions can arise only when the primals of the system S', in virtue of their $k-1$ base-nodes, possess a base-curve or base-locus; or when the system S' reduces to a system of freedom less than r. These two possibilities I discuss under §§ 5 *a* and 5 *b* respectively.

5 *a*. The primals of the system S' certainly possess a base-curve or base-locus when $n=2$ and when $n=3$. When $n=2$ (r even), the primals of the system S' are quadric cones with a common vertex-space of $\frac{1}{2}(r-2)$ dimensions; and therefore the grade of the system S' is always zero. When $n=3$ ($r \not\equiv 2$, mod. 3), the primals of the system S' have for base-curve the $\frac{1}{2}(k-1)(k-2)$ joins of pairs of the base-nodes. This base-curve is therefore of order

$$\tfrac{1}{2}(k-1)(k-2),$$

and has a point of multiplicity $k-2$ at each of the base-nodes. Thus, subtracting this base-curve, the residual intersection of $r-1$ general primals of the system S' is a curve of order

$$3^{r-1} - \tfrac{1}{2}(k-1)(k-2),$$

which has a point of multiplicity

$$2^{r-1} - (k-2)$$

at each of the base-nodes. This curve does not meet the joins of pairs of the base-nodes again. An rth primal of the system S' therefore meets it in

$$3^r - \tfrac{3}{2}(k-1)(k-2)$$

points, of which

$$2^r(k-1) - 2(k-1)(k-2)$$

are absorbed altogether at the base-nodes. Therefore the grade of the system S' is

$$3^r - 2^r(k-1) + \tfrac{1}{2}(k-1)(k-2)$$
$$= 3^r - 2^{r-1}\tfrac{1}{3}r(r+5) + \tfrac{1}{72}r(r-1)(r+5)(r+6). \qquad (9)$$

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Since $$3^r > 2^{r-1}\tfrac{1}{3}r(r+5)$$
for all $r > 5$, this grade (9) is certainly positive for $r > 5$: and it is equal to 1, 0 respectively for the remaining cases, $r = 3$ and $r = 4$.

5 *b*. The system S' of primals of order n which have nodes at $k-1$ arbitrarily assigned points of the space R is determined, in general, by $r+1$ of its members. But if, by these $k-1$ points, there is determined a linear system, of freedom greater than 0, of primals of order $n' \leqslant \frac{1}{2}n$, then the product of any two primals of this system (suitably augmented by an arbitrary primal of order $n-2n'$) is a primal of S'; and hence S' is determined by primals, of orders n and n' respectively, in total number not greater than r. In this case, therefore, r primals of the system S' meet only in the base-points (which in particular form a base-curve or base-locus if the freedom of S' is less than $r-1$), and the grade of S' is zero. For this case to arise, the freedom of primals of order $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$ (n even or odd) through $k-1$ general points of R must be greater than zero: that is

$$\binom{\frac{1}{2}n+r}{r} - k = \binom{\frac{1}{2}n+r}{r} - \binom{n+r}{r}\Big/(r+1) > 0 \quad (n \text{ even}),$$

or

$$\binom{\frac{1}{2}n-\frac{1}{2}+r}{r} - k = \binom{\frac{1}{2}n-\frac{1}{2}+r}{r} - \binom{n+r}{r}\Big/(r+1) > 0 \quad (n \text{ odd}).$$

The former inequality is satisfied by $n = 2$, for all (even) r; and, further, only by $n = 4$, $r = 2$ and $n = 4$, $r = 4$. The latter inequality has no solutions.

6. The cases discussed under §§ 5 *a* and 5 *b* are the only cases for which the grade of the system S' is not given by the formula (8). Therefore, summarising the results of §§ 4, 5 *a* and 5 *b*, I find

COROLLARY 2. *The general form of order n, homogeneous in* $r+1$ *variables, can always be expressed as the sum of the nth powers of* $k \equiv \binom{n+r}{r}\Big/(r+1)$ *linear forms, except in the following known cases**:

$$n = 2, \quad r \text{ even};$$
$$n = 3, \quad r = 4;$$
$$n = 4, \ r = 2 \ \text{ and } \ n = 4, \ r = 4.$$

7. I add that the criterion for the adequacy or failure of a supposed canonical expression which I have here advanced, namely to determine the vanishing or the non-vanishing of the Jacobian of an appropriate system by means of the vanishing or non-

* See for example F. Palatini, *Rend. Lincei* (5), 12 (1903), 378, and *Atti Acc. Torino*, 38 (1902), 43.

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vanishing of the grade of this system, is clearly of general application. In particular, if the expression w sought contains the nth powers of one or more undetermined linear forms, say

$$w \equiv \epsilon(\lambda_1, \lambda_2, \ldots, \lambda_{\binom{n+r}{r}-1-r}) \, \mathfrak{L} \, z_0, z_1, \ldots, z_r)^n + y_1{}^n,$$

then the appropriate system may again be taken to be a linear system, of freedom r, of primals of order n in space of r dimensions; namely, the system of primals of order n apolar to the following primals of class n:

$$\frac{\partial}{\partial \lambda_1} \epsilon(\lambda_1, \lambda_2, \ldots, \lambda_{\binom{n+r}{r}-1-r}) \, \mathfrak{L} \, \zeta_0, \zeta_1, \ldots, \zeta_r)^n = 0, \ldots$$

$$\ldots \frac{\partial}{\partial \lambda_{\binom{n+r}{r}-1-r}} \epsilon(\lambda_1, \lambda_2, \ldots, \lambda_{\binom{n+r}{r}-1-r}) \, \mathfrak{L} \, \zeta_0, \zeta_1, \ldots, \zeta_r)^n = 0.$$

So if, in the expression (2), for the terms

$$\lambda_1 Y_1{}^n + \lambda_2 Y_2{}^n + \ldots + \lambda_G Y_G{}^n + \ldots + \lambda_H Y_H{}^n$$

there are substituted the terms

$$\psi^{t-1} Y_1{}^{n-t+1} + \lambda_2 Y_2{}^n + \ldots + \lambda_G Y_G{}^n + \ldots + \lambda_{H'} Y_{H'}{}^n,$$

where ψ^{t-1} is an undetermined form of order $t-1$, and

$$\binom{t-1+r}{r} + H' - 1 = H,$$

then in the system S constructed in § 3, for the simple base-points (7) there is substituted a point of multiplicity t at

$$(A_0, A_1, \ldots, A_l, \ldots, A_r),$$

together with simple base-points at the points

$$(B_0, B_1, \ldots, B_l, \ldots, B_r), \ldots, (G_0, G_1, \ldots G_l, \ldots G_r), \ldots,$$
$$(H_0', H_1', \ldots, H_l', \ldots, H_r').$$

Or again, the method may be used to show, as may readily be shown otherwise, that the general cubic form in $r+1$ variables can always be exhibited under the expression

$$\sum_0^r y_i{}^3 + \sum_0^r \lambda_{jhl} y_j y_h y_l; \qquad (j \neq h \neq l)$$

this expression containing the required number of constants, in virtue of the identity

$$\binom{r+3}{3} \equiv (r+1)^2 + \binom{r+1}{3}.$$

And hence it may be deduced in turn that the expression

$$\sum_1^4 s_i{}^3 + \sum_1^4 \lambda_{jhl} s_j s_h s_l, \qquad (j \neq h \neq l)$$

where s_1, s_2, s_3, s_4 are independent ternary quadratics, is canonical for the ternary sextic. And so on.

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8. I return to the examination of the system S of primals constructed in § 3. The primals of this system are of order n, in space R of r dimensions, and have simple points at the H points (7) of the space, and nodes at $h-1$ arbitrarily assigned points of the space. I take $a_0, a_1, \dots, a_l, \dots, a_r$ for current coordinates in the space R, and

$$(b_0, b_1, \dots, b_l, \dots, b_r), \dots, (g_0, g_1, \dots, g_l, \dots, g_r), \dots, (h_0, h_1, \dots, h_l, \dots, h_r)$$

for the coordinates of the arbitrarily assigned base-nodes of the system S. Then if $f=0$ is the equation of a general primal of the system S, the form f is of order n, homogeneous in the $r+1$ variables $a_0, a_1, \dots, a_l, \dots, a_r$, and subject to the

$$\binom{n+r}{r} - 1 - r$$

linear conditions

$$\left(\frac{\partial f}{\partial a_i}\right)_{(b)} = 0, \dots, \left(\frac{\partial f}{\partial a_i}\right)_{(g)} = 0, \dots, \left(\frac{\partial f}{\partial a_i}\right)_{(h)} = 0; \qquad (i=0, 1, \dots, r)$$

$$(f)_{(A)} = 0, \dots, \quad (f)_{(G)} = 0, \dots, \quad (f)_{(H)} = 0. \tag{10}$$

I consider again the coefficients θ', a typical one of which is $\theta'_{\alpha\beta\dots\mu\dots\sigma}$ evaluated under (5). I examine these coefficients θ' as functions of $a_0, a_1, \dots, a_l, \dots, a_r$, under the hypothesis that

$$b_0, b_1, \dots, b_l, \dots, b_r, \dots, \; g_0, g_1, \dots, g_l, \dots, g_r, \dots, \; h_0, h_1, \dots, h_l, \dots, h_r$$

are themselves independent functions

$$b_0(a_0, a_1, \dots, a_l, \dots, a_r), \dots, \; g_l(a_0, a_1, \dots a_l, \dots, a_r), \dots,$$
$$h_r(a_0, a_1, \dots, a_l, \dots, a_r) \tag{11}$$

of $a_0, a_1, \dots, a_l, \dots, a_r$. Then I consider the $r+1$ independent linear functions of the coefficients θ' which involve only

$$a_0, a_1, \dots, a_l, \dots, a_r,$$

and I write these

$$\begin{aligned}
\Theta_0 &\equiv p_{n0\dots0}\theta'_{n0\dots0} + \dots + p_{\alpha\beta\dots\mu\dots\sigma}\theta'_{\alpha\beta\dots\mu\dots\sigma} + \dots + p_{00\dots n}\theta'_{00\dots n} \\
&= p_{n0\dots0}a_0^n + \dots + p_{\alpha\beta\dots\mu\dots\sigma}a_0^\alpha a_1^\beta \dots a_l^\mu \dots a_r^\sigma + \dots + p_{00\dots n}a_r^n, \\
\Theta_1 &\equiv p'_{n0\dots0}\theta'_{n0\dots0} + \dots + p'_{\alpha\beta\dots\mu\dots\sigma}\theta'_{\alpha\beta\dots\mu\dots\sigma} + \dots + p'_{00\dots n}\theta'_{00\dots n} \\
&= p'_{n0\dots0}a_0^n + \dots + p'_{\alpha\beta\dots\mu\dots\sigma}a_0^\alpha a_1^\beta \dots a_l^\mu \dots a_r^\sigma + \dots + p'_{00\dots n}a_r^n, \\
&\dots\dots\dots

To determine the coefficients p in these expressions, I observe first that, substituting for the functions θ' from the equations typified by (5), and equating to zero the coefficients of $\lambda_1, \lambda_2, \ldots, \lambda_G, \ldots, \lambda_H$, there result for each Θ the relations

$$(\Theta)_{(A)} = 0, \ \ldots, \ (\Theta)_{(G)} = 0, \ \ldots, \ (\Theta)_{(H)} = 0.$$

Next, differentiating any of the functions Θ, substituting for the functions θ', and equating to zero the coefficients of

$$\frac{\partial b_0}{\partial a_i}, \ \ldots, \ \frac{\partial g_l}{\partial a_i}, \ \ldots, \ \frac{\partial h_r}{\partial a_i}, \qquad (i = 0, 1, \ldots, r)$$

there result for each Θ the relations

$$\left(\frac{\partial \Theta}{\partial a_i}\right)_{(b)} = 0, \ \ldots, \ \left(\frac{\partial \Theta}{\partial a_i}\right)_{(g)} = 0, \ \ldots, \ \left(\frac{\partial \Theta}{\partial a_i}\right)_{(h)} = 0. \quad (i = 0, 1, \ldots, r)$$

Therefore the forms Θ, of order n in $r+1$ variables, and constituting a linear system of freedom r, are subject precisely to the conditions (10); and are therefore identical with the forms f. Hence the system S is identical with the system of primals in R whose general member has the equation

$$\rho_0 \Theta_0 + \rho_1 \Theta_1 + \ldots + \rho_l \Theta_l + \ldots + \rho_r \Theta_r = 0.$$

Therefore the grade of the system S is equal to the number of sets of values $(a_0, a_1, \ldots, a_l, \ldots, a_r)$ which satisfy an arbitrary assignment of constants

$$\Theta_0 : \Theta_1 : \ldots : \Theta_l : \ldots : \Theta_r = \nu_0' : \nu_1' : \ldots : \nu_l' : \ldots : \nu_r',$$

that is, which satisfy an arbitrary assignment of the functions (11) and an appropriate assignment of constants

$$\theta_{n0\ldots0} : \ldots : \theta_{\alpha\beta\ldots\mu\ldots\sigma} : \ldots : \theta_{00\ldots n} = \nu_{n0\ldots0} : \ldots : \nu_{\alpha\beta\ldots\mu\ldots\sigma} : \ldots : \nu_{00\ldots n}.$$

The number of such solutions $(a_0, a_1, \ldots, a_l, \ldots, a_r)$ for a suitable assignment of the coefficients θ is therefore independent of the nature of the independent functions (11), being equal to the grade of the system S which is independent of these functions. In particular, if the grade of the system S is zero, it follows that the number of sets of solutions (3) of the equations typified by (5), for an arbitrary assignment of the coefficients θ, is zero also when $b_0, b_1, \ldots, g_l, \ldots, h_r$ are independent variables. This is a proof of Theorem 1, without using the lemma of E. Bertini employed in § 3. Moreover, if the grade of the system S is unity, it follows that the number of sets of solutions (3) of the equations typified by (5) is unity also when $b_0, b_1, \ldots, g_l, \ldots, h_r$ are independent variables. This proves

THEOREM 2. *A necessary and sufficient condition that the general form of order n, homogeneous in $r+1$ variables, can be written under the expression* (2) *in only one way is that the system of*

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primals of order n in space of r dimensions which have simple points at the H assigned points (7) *and which have nodes at* $h-1$ *further, arbitrarily assigned points of the space, has a grade equal to* 1.

9. From this theorem follows

COROLLARY 3. *A necessary and sufficient condition that the general form of order n, homogeneous in* $r+1$ *variables, can be expressed in one way only as the sum of the nth powers of* $k \equiv \binom{n+r}{r} \Big/ (r+1)$ *linear forms in these variables is that the system of primals of order n in space of r dimensions which have nodes at* $k-1$ *arbitrarily assigned points of the space, has a grade equal to* 1.

Therefore, by the formulae (8) and (9) for the grades of such systems there follows

COROLLARY 4. *The general form of order n, homogeneous in* $r+1$ *variables, can be expressed in one way only as the sum of the nth powers of* $k \equiv \binom{n+r}{r} \Big/ (r+1)$ *linear forms in these variables only in the known cases*

$$n=3,\ r=3 \quad \textit{and} \quad n=5,\ r=2.$$

10. THEOREM 2 is capable of simple geometrical interpretation. Let T'' be a linear system, of freedom $h(r+1)-1$, of primals of order n in space R of r dimensions; T' being determined by the passage of its members through a set of simple base-manifolds V'. Then the manifolds V' impose upon primals of order n a number

$$\binom{n+r}{r} - h(r+1) = H$$

of independent linear conditions, namely the conditions imposed by the passage of these primals through H appropriately chosen points of V'; say, the H points (7). Let the primals of the system T' be represented by the prime-sections of a manifold M_r' of dimension r in space N of $h(r+1)-1$ dimensions. I investigate the behaviour of those sub-spaces of $h-1$ dimensions of N which are determined each by a set of h points of M_r'.

The manifold M_r' is the projection, from the H points of itself which correspond to the H points (7), of the manifold L_r^{nr} in space of $\binom{n+r}{r} - 1$ dimensions whose prime-sections represent all primals of order n in space R. Or, if reciprocally I represent the primals of order n by points of the space of $\binom{n+r}{r} - 1$ dimensions, the points of the manifold L_r^{nr} represent the primals which consist

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each of the nth power of a prime of R. In this representation, the points of the manifold M_r' therefore represent primals of order n in space R which can be thrown into the expression

$$y_1^n + \lambda_1 Y_1^n + \lambda_2 Y_2^n + \ldots + \lambda_G Y_G^n + \ldots + \lambda_H Y_H^n = 0.$$

Therefore any point of N which lies in a sub-space of $h-1$ dimensions determined by h points of M_r' represents a form of order n which can be thrown into the expression (2). Hence, by Theorems 1 and 2, a necessary and sufficient condition that through a general point of N there passes no or one sub-space of $h-1$ dimensions determined by h points of M_r' is that the linear sub-system S (constructed in § 3) of the system T' has the grade zero or unity respectively.

In proving the foregoing result, I have postulated that the base-manifolds V' are simple, being determined by the simple base-points (7). This restriction is, however, unnecessary. For it follows from § 7 that the foregoing criterion is applicable equally to a system T with base-points

$$(A_0, A_1, \ldots, A_l, \ldots, A_r), \ldots, (G_0, G_1, \ldots, G_l, \ldots, G_r), \ldots, (H_0'', H_1'', \ldots, H_l'', \ldots, H_r'')$$

of respective multiplicities $t_1, \ldots, t_G, \ldots, t_{H''}$, where

$$\binom{t_1 - 1 + r}{r} + \ldots + \binom{t_G - 1 + r}{r} + \ldots + \binom{t_{H''} - 1 + r}{r} = H;$$

that is, to a system T with base-manifolds V of arbitrary multiplicities. I therefore have

THEOREM 3. *Let M_r' be a rational manifold in space N of $h(r+1)-1$ dimensions whose prime-sections represent primals of order n in space R of r dimensions passing through base-manifolds V of arbitrarily assigned multiplicities. Let S be a linear sub-system of the system T of such primals, such that the primals of the system S, in addition to passing through the base-manifolds V of arbitrarily assigned multiplicities, have nodes at $h-1$ arbitrarily assigned points of R. Then a necessary and sufficient condition that through a general point of N passes no or one space of $h-1$ dimensions containing h points of M_r' is that the grade of the system S is equal to 0 or 1 respectively.*

I append some examples readily verified by the theorem. An example of a surface M_2', through a general point of whose containing space N (of $3h-1$ dimensions) passes no space of $h-1$ dimensions containing h points of M_2', is the surface whose prime-sections represent plane curves of order $2h-2$ which have an assigned $(2h-4)$ple base-point and $h-2$ assigned base-nodes*: for

* F. Palatini, *Atti Acc. Torino*, 41 (1905–6), 634.

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the system S, which has $h-1$ further arbitrarily assigned base-nodes, has then the grade

$$(2h-2)^2-(2h-4)^2-4(h-2)-4(h-1)=0.$$

Surfaces M_2' in 5-dimensional space, through a general point of whose containing space passes a unique chord of M_2' (for which therefore $h=2$), are the quintic Del Pezzo surface and the normal ruled quartic surface, represented respectively by plane cubics through 4 base-points, and by plane cubics through a base-node and a base-point: for each of these systems becomes a system of unit grade on the assignment of a further base-node. Manifolds M_3' in 7-dimensional space, through a general point of whose containing space passes a unique chord of M_3', are the manifold V_3^7 represented by cubic surfaces through an elliptic twisted quartic curve, and the manifold V_3^6 represented by cubic surfaces through 3 skew lines: for each of these representative systems becomes a system of unit grade on the assignment of a further base-node. And so on*.

11. I obtained the system S in § 10 by assigning to the primals of the system T nodes at $h-1$ general points of the space R. The primals of the system S are therefore represented by the prime-sections of a manifold which is the projection of M_r' from the space joining $h-1$ general tangent spaces of M_r' (each of r dimensions). If in particular the grade of the system S is unity, so also is the order of the manifold obtained by this projection. Theorem 3 may therefore in part be re-stated as follows: a necessary and sufficient condition that through a general point of the space N passes a unique $(h-1)$-dimensional space containing h points of the rational manifold M_r' is that M_r' is birationally projected into a linear space of r dimensions from the space joining $h-1$ general tangent spaces of M_r' (each of r dimensions). I proceed to remove the restriction of the rationality of M_r' in this re-statement: and thereby show that all manifolds M_r, through a general point of whose containing space N passes a unique $(h-1)$-dimensional space containing h points of M_r, are rational, since they can be birationally projected into an r-dimensional space in the foregoing manner.

* So for the general manifolds V_r^{2r+1}, V_r^{2r} in space of $2r+1$ dimensions, a unique chord of which passes through a general point of the containing space. See W. L. Edge, *Proc. Camb. Phil. Soc.* 28 (1932), 285. I remark here that a consequence of Theorem 3 is: it is a *necessary* condition, in order that through a general point of N pass two $(h-1)$-dimensional spaces each containing h points of M_r', that the grade of the system S is equal to 2. So the surface V_2^6 in 5-dimensional space (given by W. L. Edge) has two apparent double points; it is rational, and its representative system is therefore one which the addition of a further base-node reduces to a system of grade 2—being, in fact, the system of plane quartics with a base-node and 6 simple base-points.

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Let the points of a manifold M_r in space N (of $N \equiv h(r+1)-1$ dimensions) be expressed in terms of $r+1$ homogeneous parameters $\tau_0, \tau_1, \ldots, \tau_l, \ldots, \tau_r$; any point of M_r thus has coordinates

$$[z_0(\tau_0, \tau_1, \ldots, \tau_r), \ldots, z_N(\tau_0, \tau_1, \ldots, \tau_r)] \equiv [z_0(\tau), \ldots, z_N(\tau)].$$

Then a necessary and sufficient condition that through a general point $(z_0', z_1', \ldots, z_N')$ of N passes a unique $(h-1)$-dimensional space containing h points of M_r is that the $h(r+1)$ equations

$$z_i' = \lambda_1 z_i(\tau') + \lambda_2 z_i(\tau'') + \ldots + \lambda_h z_i(\tau^{(h)}) \qquad (i = 0, 1, \ldots, N) \quad (12)$$

have a unique solution for the variables

$$\tau_0', \tau_1', \ldots, \tau_r'; \ \tau_0'', \tau_1'', \ldots, \tau_r''; \ \ldots; \ \tau_0^{(h)}, \tau_1^{(h)}, \ldots, \tau_r^{(h)}. \quad (13)$$

I now investigate the equations (12) under the hypothesis that

$$\tau_0'', \tau_1'', \ldots, \tau_r'', \ldots, \tau_0^{(h)}, \tau_1^{(h)}, \ldots, \tau_r^{(h)}$$

are independent functions

$$\tau_0''(\tau_0', \tau_1', \ldots, \tau_r'), \ldots, \tau_r^{(h)}(\tau_0', \tau_1', \ldots, \tau_r') \quad (14)$$

of $\tau_0', \tau_1', \ldots, \tau_r'$, the ratios of the constants $z_0', z_1', \ldots, z_N'$ being appropriately chosen. Then the equations (12) can be reduced to $r+1$ linearly independent equations

$$\left.\begin{aligned} Z_0 &\equiv p_0 z_0' + p_1 z_1' + \ldots + p_N z_N' \\ &= p_0 z_0(\tau') + p_1 z_1(\tau') + \ldots + p_N z_N(\tau') \\ Z_1 &\equiv p_0' z_0' + p_1' z_1' + \ldots + p_N' z_N' \\ &= p_0' z_0(\tau') + p_1' z_1(\tau') + \ldots + p_N' z_N(\tau') \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Z_r &\equiv p_0^{(r)} z_0' + p_1^{(r)} z_1' + \ldots + p_N^{(r)} z_N' \\ &= p_0^{(r)} z_0(\tau') + p_1^{(r)} z_1(\tau') + \ldots + p_N^{(r)} z_N(\tau') \end{aligned}\right\}; \quad (15)$$

with the conditions, obtained as in § 8, for each Z,

$$\left(\frac{\partial Z}{\partial \tau_j'}\right)_{(\tau'')} = 0, \ \ldots, \ \left(\frac{\partial Z}{\partial \tau_j'}\right)_{(\tau^{(h)})} = 0. \qquad (j = 0, 1, \ldots, r) \quad (16)$$

Then if the equations (15) under the conditions (16) have a unique solution, whatever the nature of the functions (14), so also do the equations (12) when all the variables (13) are independent; and conversely.

Now consider the tangent r-dimensional space to M_r at the general point $[z_0(\tau''), z_1(\tau''), \ldots, z_N(\tau'')]$ of M_r. The coordinates of an arbitrary point of this space are $(z_0'', z_1'', \ldots, z_N'')$, where

$$z_i'' \equiv \beta_0 \left(\frac{\partial z_i(\tau)}{\partial \tau_0}\right)_{(\tau'')} + \beta_1 \left(\frac{\partial z_i(\tau)}{\partial \tau_1}\right)_{(\tau'')} + \ldots + \beta_r \left(\frac{\partial z_i(\tau)}{\partial \tau_r}\right)_{(\tau'')}.$$

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Therefore the condition that M_r can be birationally projected from the tangent r-dimensional spaces at the $h-1$ general points of M_r whose parameters are (τ''), ..., $(\tau^{(h)})$ is that the $h(r+1)$ equations

$$z_i' = \lambda_1 . z_i(\tau') + \beta_0 \left(\frac{\partial z_i(\tau)}{\partial \tau_0}\right)_{(\tau'')} + \ldots + \beta_r \left(\frac{\partial z_i(\tau)}{\partial \tau_r}\right)_{(\tau'')} + \ldots$$
$$+ \kappa_0 \left(\frac{\partial z_i(\tau)}{\partial \tau_0}\right)_{(\tau^{(h)})} + \ldots + \kappa_r \left(\frac{\partial z_i(\tau)}{\partial \tau_r}\right)_{(\tau^{(h)})} \qquad (i = 0, 1, \ldots, N)$$

have a unique solution for τ_0', τ_1', ..., τ_r', given the arbitrary point $(z_0', z_1', \ldots, z_N')$ of N. Since, however, the projection is upon an r-dimensional space, these $h(r+1)$ equations can be reduced to $r+1$ linearly independent equations: namely, precisely the equations (15), under precisely the conditions (16) obtained by the elimination of the parameters $\beta_0, \beta_1, \ldots, \beta_r, \ldots, \kappa_0, \kappa_1, \ldots, \kappa_r$. This proves

THEOREM 4. *A necessary and sufficient condition that through a general point of a space N, of $h(r+1)-1$ dimensions, there passes a unique $(h-1)$-dimensional space containing h points of a manifold M_r lying in N is that the manifold M_r is birationally projected into a linear space of r dimensions from the space joining $h-1$ general r-dimensional tangent spaces to M_r;*

with

COROLLARY 5. *All manifolds M_r, through a general point of whose containing space N (of $h(r+1)-1$ dimensions) passes a unique $(h-1)$-dimensional space containing h points of M_r, are rational.*

I illustrate Theorem 4 by the known theorem that the only surfaces M_2 in 5-dimensional space, a unique chord of which passes through a general point of the space, are: the quintic Del Pezzo surface, and the normal ruled quartic surface*. Each of these surfaces is rational, and projects into the plane from any one of its general tangent planes.

* F. Severi, *Rend. di Palermo*, 15 (1901), 33.

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