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Pointwise ergodic theorems beyond amenable groups

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Abstract. We prove pointwise and maximal ergodic theorems for probability-measure-preserving (PMP) actions of any countable group, provided it admits an essentially free, weakly mixing amenable action of stable type III_1. We show that this class contains all irreducible lattices in connected semi-simple Lie groups without compact factors. We also establish similar results when the stable type is III_λ, $0 < \lambda < 1$, under a suitable hypothesis. Our approach is based on the following two principles. First, we show that it is possible to generalize the ergodic theory of PMP actions of amenable groups to include PMP amenable equivalence relations. Secondly, we show that it is possible to reduce the proof of ergodic theorems for PMP actions of a general group to the proof of ergodic theorems in an associated PMP amenable equivalence relation, provided the group admits an amenable action with the properties stated above.

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1. *Introduction*

1.1. *Background: ergodic theorems for group actions.* Birkhoff's classical pointwise ergodic theorem [**Bi31**] states the following. If $T : (X, \mu) \to (X, \mu)$ is a probability-measure-preserving (PMP) transformation of a standard probability space (X, μ), then, for any $f \in L^1(X, \mu)$, the averages

$$\mathbb{A}_n[f] := \frac{1}{n+1} \sum_{i=0}^{n} f \circ T$$

converge pointwise almost everywhere to $\mathbb{E}[f|\mathcal{I}]$, the conditional expectation of f on the σ-algebra $\mathcal{I}$ of T-invariant Borel subsets. Convergence in the L^1-norm had been proven earlier by von Neumann [**vN32**]. This theorem has been extended in many different directions (e.g., see [**Kr85, Te92, As03**]). Our focus here is on the possibility of replacing the semi-group $\{T^i\}_{i\geq 0}$ with a general locally compact group (see the survey [**Ne05**] for further information).

Let G be a locally compact second countable (LCSC) group with a PMP action on a probability space (X, μ). Any Borel probability measure β on G determines an operator on $L^1(X, \mu)$ defined by

$$\beta(f) := \int f \circ g \, d\beta(g) \quad \text{for all } f \in L^1(X, \mu).$$

Definition 1.1. Let $\mathbb{I}$ denote either $\mathbb{R}_{>0}$ or $\mathbb{N}$. Suppose $\{\beta_r\}_{r\in\mathbb{I}}$ is a family of probability measures on G. If for every PMP action $G \curvearrowright (X, \mu)$ and every $f \in L^p(X, \mu)$ the functions $\beta_r(f)$ converge as $r \to \infty$ pointwise almost everywhere to the conditional expectation of f on the σ-algebra of G-invariant Borel sets, then $\{\beta_r\}_{r\in\mathbb{I}}$ is a *pointwise ergodic family in* L^p.

Since the time of von Neumann and Birkhoff, much of the effort in ergodic theory has been devoted to actions of amenable groups. We now describe some of the main ergodic theorems established for these, and then some of those established in the non-amenable case.

1.1.1. *Amenable groups.* An LCSC group G is *amenable* if it admits a sequence $\mathfrak{F} = \{\mathfrak{F}_n\}_{n=1}^{\infty}$ of compact subsets such that, for every compact $Q \subset G$,

$$\lim_{n\to\infty} (|Q\mathfrak{F}_n \Delta \mathfrak{F}_n|/|\mathfrak{F}_n|) = 0,$$

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where $|\cdot|$ denotes the left Haar measure. Such a sequence is called *Følner* or *asymptotically invariant*.

A Følner sequence is doubling if it is monotonic, i.e., $\mathfrak{F}_n \subset \mathfrak{F}_{n+1}$, and satisfies the volume doubling bound, namely there is a constant $C_d > 0$ such that, for every $n > 0$,

$$|\mathfrak{F}_n^{-1}\mathfrak{F}_n| \leq C_d|\mathfrak{F}_n|.$$

This condition generalizes the doubling condition introduced by Wiener [**Wi39**] and Calderon [**Ca53**], who proved that doubling Følner sequences are pointwise ergodic in L^1.

A Følner sequence is *regular* if there is a constant $C_{\mathrm{reg}} > 0$ such that, for every $n > 0$,

$$\left|\bigcup_{i\leq n} \mathfrak{F}_i^{-1}\mathfrak{F}_n\right| \leq C_{\mathrm{reg}}|\mathfrak{F}_n|.$$

The fact that regular Følner sequences are pointwise ergodic sequences in L^1 was established by Tempelman [**Te72, Te92**], and also by Bewley [**Be71**], Chatard [**Ch70**] and Emerson [**Em74**].

A Følner sequence is *tempered* if there is a constant $C > 0$ such that, for every $n > 0$,

$$\left|\bigcup_{i<n} \mathfrak{F}_i^{-1}\mathfrak{F}_n\right| \leq C|\mathfrak{F}_n|.$$

It was shown by Lindenstrauss in [**Li01**] that every Følner sequence has a tempered subsequence and every tempered Følner sequence is a pointwise ergodic sequence in L^1. This is the most general result to date for arbitrary amenable groups. An alternative proof was given by Weiss in [**We03**]. The notion of temperedness was introduced and the L^2-case was proven earlier by Shulman [**Sh88, Te92**].

Let us mention that besides the asymptotic invariance inherent in the definition of a Følner sequence, there are two other essential ingredients that appear in the proofs of each of the pointwise results stated above. One is a case-appropriate generalization of the Wiener covering argument originally proved for ball averages on Euclidean space, which leads to a weak type (1, 1)-maximal inequality for averaging on the sets $\mathfrak{F}_n$ in the group. The other is the Calderon transference principle, which reduces the maximal inequality in a general action to the maximal inequality for convolutions on the group itself. In our discussion below, we will seek to generalize these ingredients beyond the case of actions of amenable groups.

1.1.2. *Non-amenable groups.* The question of a possible generalization of ergodic theorems to arbitrary finitely generated groups was already raised half a century ago by Arnol'd and Krylov. In [**AK63**], they generalized Weyl's equidistribution theorem from dense free groups of rotations of the unit circle to dense free groups of rotations on the unit sphere. This result motivated the generalization of von Neumann's mean ergodic theorem from the free group on one generator to the free group on any finite number of generators, established by Guivarc'h [**Gu68**] using spectral theory.

Ergodic theorems for measure-preserving actions of arbitrary countable groups were obtained by Oseledets in 1965 [**Os65**]: he showed that the convolution powers of a symmetric probability measure on Γ form a pointwise ergodic family.

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Semi-simple S-algebraic groups. Techniques based on the spectral theory of unitary representations have been developed and applied to the case where G is a connected semi-simple Lie group in [**Ne94a, Ne94b, NS94, Ne97, NS97, MNS00**]. The more general case of a semi-simple S-algebraic group and, furthermore, any lattice subgroup of such a group was established in [**GN10**], to which we refer for a more detailed account. Typically, the averaging sequences studied are the uniform averages over concentric balls (and in some cases spheres) centered at the origin. As an example, we mention that a free group was handled in [**Ne94a, NS94**] by viewing it as a lattice in a group of automorphisms of a regular tree, and in [**GN10**] as a lattice in $\mathrm{PSL}_2(\mathbb{R})$.

An important feature of the spectral methods is that the ergodic theorems derived from them often exhibit a rate of convergence to the ergodic mean, a phenomenon that cannot arise in the classical amenable context. Thus, when available, spectral methods give results far sharper than those obtained by any other technique, but their scope is limited to groups whose unitary representations are well understood, and to their lattice subgroups.

Markov groups. A most elegant proof of the pointwise ergodic theorem for a free group with respect to spherical averages was given in [**Bu00**], using Markov operator techniques developed in [**Ro62**]. This approach to the ergodic theorem was inspired by earlier related ideas in [**Gr99**]. These techniques extend to a certain extent to groups with a Markov presentation which includes all Gromov hyperbolic groups. For example, in [**BKK11**], it is proven that Cesaro averages of spherical averages converge in L^1 for every Gromov hyperbolic group with respect to an arbitrary word metric. The identification of the limit function as the ergodic average has recently been obtained in the case of surface groups in [**BS10**].

1.2. *From amenable groups to amenable equivalence relations.* The purpose of this paper is to introduce a general approach for proving pointwise ergodic theorems for countable groups Γ. This approach has the remarkable feature that it treats amenable and non-amenable groups on an equal footing, and in fact constitutes a direct generalization of the classical techniques of amenable ergodic theory which also applies to non-amenable groups. The two main ideas are as follows. First, we will show that it is possible to reduce the proof of ergodic theorems in measure-preserving Γ-actions (X, μ) to the proof of ergodic theorems in certain associated *amenable PMP* equivalence relations. The amenable equivalence relations are obtained by first choosing an amenable action of Γ, typically a Poisson boundary (B, ν), considering its extension $(X \times B, \mu \times \nu)$ by the measure-preserving Γ-action, and then constructing a PMP amenable subrelation of the Maharam extension of $X \times B$. Secondly, we will show that it is possible to establish ergodic theorems along Følner sets in PMP amenable equivalence relations, directly generalizing the classical arguments. Thus, when the Følner sequence in the equivalence relation is doubling, we proceed by generalizing the arguments of Wiener and Calderon, or more generally of Tempelman, for regular sequences in amenable groups. When the Følner sequence in the equivalence relation is tempered, we proceed by generalizing Weiss' proof of Lindenstrauss' theorem [**We03**] for tempered sequences in amenable groups. This is accomplished in §2.

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1.3. *Statement of one main result.* Next we present one of our main results, with more refined results given later in the text (specifically, Theorems 4.1, 4.2 and 5.2). Undefined terminology is explained immediately following the statement of the theorem.

THEOREM 1.1. *Let $\Gamma \curvearrowright (B, \nu)$ be a measure-class-preserving action of a countable group on a standard probability space. We assume the action is essentially free, weakly mixing and of stable type III_1. Let θ be the measure on $\mathbb{R}$ given by $d\theta(t) = e^t dt$. Let $\Gamma \curvearrowright (B \times \mathbb{R}, \nu \times \theta)$ be given by*

$$g(b, t) = \left(gb, t - \log\left(\frac{d\nu \circ g}{d\nu}(b) \right) \right).$$

Let $T > 0$ be arbitrary, $I = [0, T]$, and θ_I be the probability measure on $[0, T]$ given by $d\theta_I(t) = (e^t/(e^T - 1))\, dt$. Let $\mathcal{R}_I$ be the equivalence relation on $B \times I$ given by restricting the orbit equivalence relation on $B \times \mathbb{R}$ (so $\mathcal{R}_I$ consists of all $((b, t), g(b, t))$ with $g \in \Gamma$ and $(b, t), g(b, t) \in B \times I$).

Let $\mathfrak{F} = \{\mathfrak{F}_r\}_{r \in \mathbb{I}}$ be a Borel family of subset functions for $(B \times I, \nu \times \theta_I, \mathcal{R}_I)$. (This implies, in particular, that $\mathfrak{F}_r(b, t)$ is a finite subset of the intersection of the Γ-orbit of (b, t) with $B \times I$.) Suppose $\mathfrak{F}$ is either (asymptotically invariant and regular) or (asymptotically invariant, uniform and tempered). Let $\psi \in L^q(B)$ be a probability density function (so $\psi \geq 0$ and $\int \psi\, d\nu = 1$) and define $\zeta_r^{\psi} : \Gamma \to [0, 1]$ by

$$\zeta_r^{\psi}(\gamma) := \frac{1}{T} \int_0^T \int \frac{1}{|\mathfrak{F}_r(b, t)|} 1_{\mathfrak{F}_r(b,t)}(\gamma(b, t)) \psi(b)\, d\nu(b)\, dt.$$

Then $\{\zeta_r^{\psi}\}_{r \in \mathbb{I}}$ is a pointwise ergodic family in L^p for every $p > 1$ with $1/p + 1/q \leq 1$. If $\psi \in L^{\infty}$, then $\{\zeta_r^{\psi}\}_{r \in \mathbb{I}}$ is a pointwise ergodic family in $L \log L$.

We also obtain related maximal ergodic theorems under more general hypotheses (see Theorems 3.1 and 5.1).

Let us now explain some of the terminology. *Essential freeness* of the action $\Gamma \curvearrowright (B, \nu)$ means that for almost every (a.e.) $b \in B$, the stability group $\{g \in \Gamma : gx = x\}$ is trivial. By an *amenable action*, we mean an action amenable in the sense of Zimmer [**Zi78**]. Alternatively, by [**CFW81**], this is equivalent to the existence of a Borel transformation $S : B \to B$ such that for a.e. $b \in B$, $\Gamma b = \{S^i b : i \in \mathbb{Z}\}$. *Weakly mixing* means that if $\Gamma \curvearrowright (X, \mu)$ is any ergodic PMP action, then the product action $\Gamma \curvearrowright (B \times X, \nu \times \mu)$ is ergodic.

The action $\Gamma \curvearrowright (B, \nu)$ has *type III_1* if for every $r, \epsilon > 0$ and every positive measure Borel set $A \subset B$, there exists a positive measure Borel set $A_0 \subset A$ and an element $g \in \Gamma \backslash \{e\}$ such that $gA_0 \subset A$ and

$$\left| \frac{d\nu \circ g}{d\nu}(b) - r \right| < \epsilon$$

for every $b \in A_0$. The action $\Gamma \curvearrowright (B, \nu)$ has *stable type III_1* if for every PMP ergodic action $\Gamma \curvearrowright (X, \mu)$, the product action $\Gamma \curvearrowright (B \times X, \nu \times \mu)$ has type III_1. It should be noted that we also obtain results analogous to Theorem 1.1 for certain actions of stable type III_τ for $\tau \in (0, 1)$, a notion defined in §4.1.

A Borel family $\mathfrak{F} = \{\mathfrak{F}_r\}_{r \in \mathbb{I}}$ of subset functions for $(B \times I, \nu \times \theta_I, \mathcal{R}_I)$ satisfies the following.

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(1) For each $(b, t) \in B \times I$, $\mathfrak{F}_r(b, t)$ is a finite subset of $B \times I$ contained in the Γ-orbit of (b, t).
(2) The set $\{(b, t, b', t', r) \in B \times I \times B \times I \times \mathbb{I} : (b', t') \in \mathfrak{F}_r(b, t)\}$ is Borel.

Let $\mathcal{R}_I$ denote the equivalence relation on $B \times I$ given by $((b, t), (b', t')) \in \mathcal{R}_I$ if and only if (b', t') is in the Γ-orbit of (b, t). Let $\mathrm{Inn}(\mathcal{R}_I)$ denote the full group of $\mathcal{R}_I$; this is the group of all Borel automorphisms of $B \times I$ with a graph contained in $\mathcal{R}_I$. A set $\Psi \subset \mathrm{Inn}(\mathcal{R}_I)$ *generates* $\mathcal{R}$ with respect to $\nu \times \theta_I$ if for $\nu \times \theta_I$-a.e. (b, t) and almost every (b', t') with $((b, t), (b', t')) \in \mathcal{R}_I$, there exists ψ in the group generated by Ψ such that $\psi(b, t) = (b', t')$.

The family $\mathfrak{F}$ is *asymptotically invariant* if $|\mathfrak{F}_r(b, t)| \geq 1$ for a.e. $(b, t) \in B \times I$ and $r \in \mathbb{I}$ and there exists a countable set $\Psi \subset \mathrm{Inn}(\mathcal{R})$ which generates $\mathcal{R}$ such that, for every $\psi \in \Psi$ and $\nu \times \theta_I$-a.e. $(b, t) \in B \times I$,

$$\lim_{r \to \infty} \frac{|\mathfrak{F}_r(b, t) \Delta \psi(\mathfrak{F}_r(b, t))|}{|\mathfrak{F}_r(b, t)|} = 0.$$

The family $\mathfrak{F}$ is *regular* if there is a constant $C_{\mathrm{reg}} > 0$, also called the regularity constant, such that, for $\nu \times \theta_I$-a.e. $(b, t) \in B \times I$ and every $r > 0$,

$$\left| \bigcup_{t \leq r} \mathfrak{F}_t^{-1} \mathfrak{F}_r(b, t) \right| \leq C_{\mathrm{reg}} |\mathfrak{F}_r(b, t)|,$$

where $\mathfrak{F}_t^{-1} \mathfrak{F}_r(b, t)$ is the set of all (b', t') such that $\mathfrak{F}_t(b', t') \cap \mathfrak{F}_r(b, t) \neq \emptyset$. The concepts uniform and tempered are described in §2.1.

1.4. *About the hypotheses.* Theorem 1.1 and its refinements (Theorems 4.1, 4.2 and 5.2) each require the existence of a measure-class-preserving action $\Gamma \curvearrowright (B, \nu)$ which is essentially free, weakly mixing, amenable and either stable type III_1 or type III_λ, and stable type III_τ for some $\lambda, \tau \in (0, 1)$. So, it is natural to ask, when does such an action exist and how can we find one?

First, we note that the requirement that the action be essentially free is not very restrictive in the sense that if there is an action satisfying the other conditions, then there is an essentially free action which satisfies all the conditions. To explain, let $\Gamma \curvearrowright (X, \mu)$ be any essentially free, weakly mixing PMP action (for example, Bernoulli actions satisfy this property). If $\Gamma \curvearrowright (B, \nu)$ is weakly mixing, amenable, type III_λ and stable type III_τ (for some $\lambda, \tau \in [0, 1]$), then the product action $\Gamma \curvearrowright (X \times B, \mu \times \nu)$ is also weakly mixing, amenable, type III_λ and stable type III_τ. Moreover, the product action is essentially free.

Now, given a generating symmetric probability measure μ on Γ, we may consider the Poisson boundary (B, ν) for the random walk with μ-distributed increments. There is a natural action of Γ on (B, ν) which is always amenable [**Zi78**] and weakly mixing [**AL05**]. It is not known whether this action is always type III_λ for some $\lambda \in (0, 1]$, or whether by choosing μ appropriately, one can always require the action to be type III_λ for some $\lambda \in (0, 1]$. However, under extra hypotheses on Γ, we do have some answers. We establish in §4.4 that all irreducible lattices in connected semi-simple Lie groups without compact factors have the property that their action on the Poisson boundary $B = G/P$ has stable

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type III_1. It was shown in [**INO08**] that Poisson boundaries of Gromov hyperbolic groups are never type III_0.

In future work [**BN2**], we intend to show that if Γ is Gromov hyperbolic, then the action on its boundary with respect to the Patterson–Sullivan measure is weakly mixing, amenable, type III_λ and stable type III_τ for some $\lambda, \tau \in (0, 1]$. We will use this in [**BN2**] to obtain pointwise ergodic sequences $\{\zeta_r\}_{r=1}^{\infty}$ for general Gromov hyperbolic groups Γ, such that each ζ_r is supported in a spherical shell of constant width.

We conjecture that any countable group admits an action satisfying all the requirements above. If true, then the results of this paper apply to all countable groups.

1.5. *Organization of the paper.* In §2, we prove ergodic theorems for amenable equivalence relations. In §3, we prove maximal ergodic theorems. In §4, we use results of the previous two sections to prove pointwise ergodic theorems when $\Gamma \curvearrowright (B, \nu)$ has stable type III_1. In §5, we prove pointwise ergodic theorems when $\Gamma \curvearrowright (B, \nu)$ has type III_λ and stable type III_τ for $\lambda, \tau \in (0, 1)$.

2. *Ergodic theorems for amenable equivalence relations*

2.1. *Definition of Følner sets and their properties.* A *measured equivalence relation* is a quadruple $(X, \mathcal{B}, \mu, \mathcal{R})$, where $(X, \mathcal{B}, \mu)$ is a standard σ-finite measure space and $\mathcal{R} \subset X \times X$ is a Borel equivalence relation. It is *discrete* if every equivalence class, denoted $[x]$, is at most countable. It is a *probability-measured equivalence relation* if $\mu(X) = 1$. To reduce the notation, we will usually omit the σ-algebra from the notation and say that $(X, \mu, \mathcal{R})$ is a measured equivalence relation.

Let c denote a counting measure on X (so $c(E) = \#E$ for all $E \subset X$). The measure μ on X is *$\mathcal{R}$-invariant* if $\mu \times c$ restricted to $\mathcal{R}$ equals $c \times \mu$ restricted to $\mathcal{R}$. In this case, we say $(X, \mu, \mathcal{R})$ is a *measure-preserving equivalence relation*. A Borel map $\psi : X \to X$ is an *inner automorphism* of $\mathcal{R}$ if it is invertible with a Borel inverse and its graph is contained in $\mathcal{R}$. Let $\mathrm{Inn}(\mathcal{R})$ denote the group of inner automorphisms. This group is frequently called the *full group* and denoted by $[\mathcal{R}]$. If μ is $\mathcal{R}$-invariant, then $\psi_*\mu = \mu$ for every $\psi \in \mathrm{Inn}(\mathcal{R})$. For the rest of this section, we assume $(X, \mu, \mathcal{R})$ is a discrete PMP equivalence relation.

A *subset function* (for $\mathcal{R}$) is a map $\mathfrak{U}$ on X such that $\mathfrak{U}(x) \subset [x]$ for all $x \in X$. The inverse of $\mathfrak{U}$ is the subset function $\mathfrak{U}^{-1}(y) := \{x \in X : y \in \mathfrak{U}(x)\}$. If $\mathfrak{U}_1$, $\mathfrak{U}_2$ are two subset functions, then their product $\mathfrak{U}_1\mathfrak{U}_2$ is the subset function defined by

$$\mathfrak{U}_1\mathfrak{U}_2(x) := \bigcup \{\mathfrak{U}_1(y) : y \in \mathfrak{U}_2(x)\}.$$

Their difference $\mathfrak{U}_1 \backslash \mathfrak{U}_2$ is defined by $\mathfrak{U}_1 \backslash \mathfrak{U}_2(x) := \mathfrak{U}_1(x) \backslash \mathfrak{U}_2(x)$. We write $\mathfrak{U}_1 \subset \mathfrak{U}_2$ if $\mathfrak{U}_1(x) \subset \mathfrak{U}_2(x)$ for a.e. x. If $\{\mathfrak{U}_i\}_{i \in I}$ is a family of subset functions, their union $\bigcup_{i \in I} \mathfrak{U}_i$ is the subset function defined by

$$\left(\bigcup_{i \in I} \mathfrak{U}_i\right)(x) := \bigcup_{i \in I} \mathfrak{U}_i(x).$$

A *Borel family of subset functions* $\mathfrak{F} = \{\mathfrak{F}_r\}_{r \in \mathbb{I}}$ (for $\mathcal{R}$) is a family of subset functions $\mathfrak{F}_r$ indexed by a set $\mathbb{I} \in \{\mathbb{N}, \mathbb{R}_{>0}\}$ such that $\{(x, y, r) \in X \times X \times \mathbb{I} : y \in \mathfrak{F}_r(x)\}$ is a Borel

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subset of $\mathcal{R} \times \mathbb{I}$. As noted already, we will always assume that $\bigcup_{s \le r} \mathfrak{F}_s(x) \subset [x]$ is finite for every $x \in X$ and $r \in \mathbb{I}$. As a result, we also have that $\bigcup_{t \le r} \mathfrak{F}_t^{-1} \mathfrak{F}_r(x)$ is finite for every $r \in \mathbb{I}$ and every $x \in X$.

Let $\mathfrak{F}$ be a Borel family of subset functions. The definitions below generalize classical concepts.

(1) A set $\Psi \subset \mathrm{Inn}(\mathcal{R})$ *generates* $\mathcal{R}$ with respect to μ if for $\mu \times c$-a.e. $(x_1, x_2) \in \mathcal{R}$ there exists $\psi \in \langle \Psi \rangle$ such that $\psi(x_1) = x_2$ (where $\langle \Psi \rangle$ denotes the subgroup of $\mathrm{Inn}(\mathcal{R})$ generated by Ψ). Equivalently, $\mathcal{R} = \bigcup_{\psi \in \langle \Psi \rangle} \mathrm{Graph}\,(\psi)$, up to a set of $\mu \times c$-measure zero.

(2) $\mathfrak{F}$ is *asymptotically invariant* if $|\mathfrak{F}_r(x)| \ge 1$ for a.e. $x \in X$ and $r \in \mathbb{I}$ and there exists a countable set $\Psi \subset \mathrm{Inn}(\mathcal{R})$ which generates $\mathcal{R}$ such that, for every $\psi \in \Psi$ and μ-a.e. $x \in X$,
$$\lim_{r \to \infty} \frac{|\mathfrak{F}_r(x) \Delta \psi(\mathfrak{F}_r(x))|}{|\mathfrak{F}_r(x)|} = 0.$$
We say that $\mathfrak{F}$ is *Følner* if it is asymptotically invariant.

(3) $\mathfrak{F}$ is *uniform* if there are constants C_u, a_r, $b_r > 0$ (for $r \in \mathbb{I}$) such that:
   (a) $b_r \le C_u a_r$ for every $r \in \mathbb{I}$;
   (b) $a_r \le |\mathfrak{F}_r(x)| \le b_r$ for a.e. $x \in X$; and
   (c) $a_r \le |\mathfrak{F}_r^{-1}(x)| \le b_r$ for a.e. $x \in X$.
   The constant C_u is called the *uniformity* constant.

(4) $\mathfrak{F}$ is *doubling* if:
   (i) $\mathfrak{F}$ is a monotonic family, namely for $s < r$, we have $\mathfrak{F}_s(x) \subset \mathfrak{F}_r(x)$ almost everywhere; and
   (ii) $\mathfrak{F}$ satisfies the volume doubling bound, namely there is a constant $C_d > 0$, called the doubling constant, such that, for μ-a.e. $x \in X$ and every $r > 0$,
   $$|\mathfrak{F}_r(x)^{-1} \mathfrak{F}_r(x)| \le C_d |\mathfrak{F}_r(x)|.$$

(5) $\mathfrak{F}$ is *regular* if there is a constant $C_{\mathrm{reg}} > 0$, also called the regularity constant, such that, for μ-a.e. $x \in X$ and every $r > 0$,
$$\left| \bigcup_{t \le r} \mathfrak{F}_t^{-1} \mathfrak{F}_r(x) \right| \le C_{\mathrm{reg}} |\mathfrak{F}_r(x)|.$$

(6) $\mathfrak{F}$ is *tempered* if the index set $\mathbb{I} = \mathbb{N}$ and there is a constant C_t such that, for μ-a.e. $x \in X$ and every $n > 0$,
$$\left| \bigcup_{m \le n-1} \mathfrak{F}_m^{-1} \mathfrak{F}_n(x) \right| \le C_t |\mathfrak{F}_n(x)|.$$
   C_t is called the *tempered* constant.

For a function f on X, consider the averages $\mathbb{A}[f|\mathfrak{F}_r]$ defined by
$$\mathbb{A}[f|\mathfrak{F}_r](x) := \frac{1}{|\mathfrak{F}_r(x)|} \sum_{x' \in \mathfrak{F}_r(x)} f(x') \quad \text{for all } x \in X.$$

A subset $E \subset X$ is $\mathcal{R}$-*invariant* (or $\mathcal{R}$-saturated) if $(E \times X) \cap \mathcal{R} = (X \times E) \cap \mathcal{R} = (E \times E) \cap \mathcal{R}$ (up to $\mu \times c$-measure zero). For a Borel function $f \in L^1(X)$, let $\mathbb{E}[f|\mathcal{I}(\mathcal{R})]$ denote the conditional expectation of f with respect to the σ-algebra $\mathcal{I}(\mathcal{R})$ of $\mathcal{R}$-invariant measurable sets.

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2.2. *Statement of ergodic theorems for equivalence relations.* Keeping the notation introduced in §2.1, in §§2.3–2.6, we prove the following results.

THEOREM 2.1. *If $\mathfrak{F}$ is either (asymptotically invariant and regular) or (asymptotically invariant, uniform and tempered), then $\mathfrak{F}$ is a (restricted) pointwise ergodic family in L^1. Namely, for every $f \in L^1(X, \mu)$, $\mathbb{A}[f|\mathfrak{F}_r]$ converges pointwise almost everywhere to $\mathbb{E}[f|\mathcal{I}(\mathcal{R})]$ as $r \to \infty$.*

Theorem 2.5 below extends Theorem 2.1 and shows that in fact $\mathfrak{F}$ is an (unrestricted) pointwise ergodic family in L^1, namely that the result passes to class-bijective ergodic extensions. This will be established in §2.6.

Our method of proof of Theorem 2.1 follows the classical pattern and is based on the next two theorems.

THEOREM 2.2. (Dense subset of good functions) *If $\mathfrak{F}$ is asymptotically invariant, then there exists a dense subset $\mathcal{G} \subset L^1(X)$ such that for all $f \in \mathcal{G}$, $\mathbb{A}[f|\mathfrak{F}_r]$ converges pointwise almost everywhere to $\mathbb{E}[f|\mathcal{I}(\mathcal{R})]$ as $r \to \infty$. Moreover, if $L^1_0(X)$ is the set of all functions $f \in L^1(X)$ with $\mathbb{E}[f|\mathcal{I}(\mathcal{R})] = 0$ almost everywhere, then there exists a dense subset $\mathcal{G}_0 \subset L^1_0(X)$ such that for all $f \in \mathcal{G}_0$, $\mathbb{A}[f|\mathfrak{F}_r]$ converges pointwise almost everywhere to 0 as $r \to \infty$.*

For $f \in L^1(X)$, let $\mathbb{M}[f|\mathfrak{F}] = \sup_{r\in\mathbb{I}} \mathbb{A}[|f||\mathfrak{F}_r]$, where $|f|$ denotes the absolute value of f. $\mathbb{M}[\cdot|\mathfrak{F}]$ is the *maximal operator* associated with the family $\{\mathbb{A}[\cdot|\mathfrak{F}_r]\}_{r\in\mathbb{I}}$. As we shall see below in the proof of the maximal inequality, the maximal function $\mathbb{M}[\cdot|\mathfrak{F}]$ is Borel measurable, even when the index set $\mathbb{I} = \mathbb{R}$. We can now state the following theorem.

THEOREM 2.3. (Weak (1, 1)-type maximal inequality) *Suppose that $\mathfrak{F}$ is either regular or (asymptotically invariant, uniform and tempered). Then there exists a constant $C > 0$ such that, for any $f \in L^1(X)$ and any $\lambda > 0$,*

$$\mu(\{x \in X : \mathbb{M}[f|\mathfrak{F}] > \lambda\}) \leq \frac{C\|f\|_1}{\lambda}.$$

In fact, C can be taken to be $8C_u^4(1 + C_tC_u)$ in the tempered case and C_{reg} in the regular case.

Theorem 2.4 below extends Theorem 2.3 to arbitrary class-bijective extensions of $(X, \mu, \mathcal{R})$.

We recall that the concept of an amenable group action has been defined in great generality in [**Zi78**], and several characterizations of this property have been established, including in [**CFW81**]. We will not elaborate on these results here, since they are not directly relevant to our discussion. Instead, we will just note that it has been shown in [**Zi78**] that the Poisson boundary associated with a non-degenerate random walk on Γ is an amenable ergodic action of Γ (but it is still an open problem whether the type can be III_0). Furthermore, in [**CFW81**], it was shown that the amenability of a general equivalence relation is equivalent to the existence of an asymptotically invariant sequence of subset functions (defined taking the Radon–Nikodym derivative into account). Focusing on the case of PMP equivalence relations, which is our main concern, let us formulate a

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general existence result for doubling Følner sequences, based on another important fact from [**CFW81**], namely that amenability of an equivalence relation is equivalent to the relation being the orbit relation of a $\mathbb{Z}$-action, as follows.

PROPOSITION 2.1. *Let $(X, \mathfrak{B}, \mu, \mathcal{R})$ be an amenable discrete PMP equivalence relation. Then there exists a sequence $\mathfrak{F} = \{\mathfrak{F}_n\}_{n=1}^{\infty}$ of subset functions which is asymptotically invariant, uniform and doubling.*

Proof. Without loss of generality, we may assume $(X, \mathfrak{B}, \mu, \mathcal{R})$ is ergodic. If X is finite, then we may choose $\mathfrak{F}_n(x) = X$ for every n, x. So, let us assume X is infinite. According to [**CFW81, Dy59, Dy63**], there exists a measure-preserving Borel transformation $T : X \to X$ so that $\mathcal{R} = \{(x, T^i x) : x \in X, i \in \mathbb{Z}\}$ (up to a $\mu \times c$-measure zero subset). Then we may let $\mathfrak{F}_n(x) := \{T^i x : |i| \le n\}$. It is easy to check that $\mathfrak{F} = \{\mathfrak{F}_n\}_{n=1}^{\infty}$ is asymptotically invariant, uniform and doubling. □

2.3. *Dense set of good functions.* In this subsection, we prove Theorem 2.2. Assume $\mathfrak{F}$ is asymptotically invariant. Let $\Psi \subset \mathrm{Inn}(\mathcal{R})$ be a countable set generating the relation $\mathcal{R}$ that witnesses the asymptotic invariance.

LEMMA 2.2. *Let ψ be in the subgroup of* $\mathrm{Inn}(\mathcal{R})$ *generated by Ψ. Then*

$$\lim_{r\to\infty} \frac{|\mathfrak{F}_r(x) \Delta \psi(\mathfrak{F}_r(x))|}{|\mathfrak{F}_r(x)|} = 0.$$

Proof. Observe that if $\psi_1, \psi_2 \in \Psi$, then $|\mathfrak{F}_r(x) \Delta \psi_i^{-1}(\mathfrak{F}_r(x))| = |\psi_i(\mathfrak{F}_r(x)) \Delta \mathfrak{F}_r(x)|$. So,

$$\lim_{r\to\infty} \frac{|\mathfrak{F}_r(x) \Delta \psi_i^{-1}(\mathfrak{F}_r(x))|}{|\mathfrak{F}_r(x)|} = 0, \quad i = 1, 2.$$

Also,

$$\begin{aligned} |\mathfrak{F}_r(x) \Delta \psi_1\psi_2(\mathfrak{F}_r(x))| &\le |\mathfrak{F}_r(x) \Delta \psi_1(\mathfrak{F}_r(x))| + |\psi_1(\mathfrak{F}_r(x)) \Delta \psi_1\psi_2(\mathfrak{F}_r(x))| \\ &= |\mathfrak{F}_r(x) \Delta \psi_1(\mathfrak{F}_r(x))| + |\mathfrak{F}_r(x) \Delta \psi_2(\mathfrak{F}_r(x))|. \end{aligned}$$

Therefore,

$$\lim_{r\to\infty} \frac{|\mathfrak{F}_r(x) \Delta \psi_1\psi_2(\mathfrak{F}_r(x))|}{|\mathfrak{F}_r(x)|} = 0.$$

Since $\psi_1, \psi_2 \in \Psi$ are arbitrary, this proves the lemma. □

LEMMA 2.3. *Let $\psi \in \langle\Psi\rangle$, $f \in L^\infty(X)$ and define $f' := f - f \circ \psi$. Then $\mathbb{A}[f'|\mathfrak{F}_r]$ converges pointwise almost everywhere to $\mathbb{E}[f'|\mathcal{I}(\mathcal{R})] = 0$ as $r \to \infty$.*

Proof. For a.e. $x \in X$, the previous lemma implies

$$\begin{aligned} \lim_{r\to\infty} |\mathbb{A}[f'|\mathfrak{F}_r](x)| &= \lim_{r\to\infty} \left| \frac{1}{|\mathfrak{F}_r(x)|} \sum_{x' \in \mathfrak{F}_r(x)} f(x') - f(\psi(x')) \right| \\ &\le 2\|f\|_\infty \lim_{r\to\infty} \frac{|\mathfrak{F}_r(x) \Delta \psi(\mathfrak{F}_r(x))|}{|\mathfrak{F}_r(x)|} = 0. \end{aligned}$$

By definition, $\mathbb{E}[f|\mathcal{I}(\mathcal{R})] = \mathbb{E}[f \circ \psi|\mathcal{I}(\mathcal{R})]$. Hence, $\mathbb{E}[f'|\mathcal{I}(\mathcal{R})] = 0$ almost everywhere. This proves the lemma. □

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LEMMA 2.4. *Let f be a measurable function on X such that for every $\psi \in \langle\Psi\rangle$, $f = f \circ \psi$ almost everywhere. Then f is $\mathcal{R}$-invariant, namely $f(x) = f(x')$ for $\mu \times c$-a.e. $(x, x') \in \mathcal{R}$.*

Proof. For each $\psi \in \langle\Psi\rangle$, let

$$X_\psi := \{x \in X : f(x) \neq f \circ \psi(x)\}.$$

Since Ψ is countable, $\langle\Psi\rangle$ is also countable and

$$\mu\left(\bigcup_{\psi\in\langle\Psi\rangle} X_\psi\right) = 0.$$

By definition, if $x \notin \bigcup_{\psi\in\langle\Psi\rangle} X_\psi$, then $f(x) = f(\psi(x))$ for all $\psi \in \langle\Psi\rangle$, but this implies $f(x) = f(x')$ for $\mu \times c$-a.e. $(x, x') \in \mathcal{R}$, since the union of the graphs of $\psi \in \Psi$ coincides with $\mathcal{R}$ up to a set of $\mu \times c$-measure zero. □

Proof of Theorem 2.2. Let $\mathcal{I} \subset L^2(X)$ be the space of $\mathcal{R}$-invariant L^2 functions, i.e., $f \in \mathcal{I}$ if and only if $f(x) = f(x')$ for a.e. $(x, x') \in \mathcal{R}$. Let $\mathcal{G} \subset L^2(X)$ be the space of all functions of the form $f - f \circ \psi$ for $f \in L^\infty(X)$ and $\psi \in \langle\Psi\rangle$. We claim that the span of $\mathcal{I}$ and $\mathcal{G}$ is dense in $L^2(X)$. To see this, let f_* be a function in the orthocomplement of $\mathcal{G}$. Denoting the L^2 inner product by $\langle\cdot, \cdot\rangle$, we have

$$\begin{aligned} 0 = \langle f_*, f - f \circ \psi\rangle &= \langle f_*, f\rangle - \langle f_*, f \circ \psi\rangle \\ &= \langle f_*, f\rangle - \langle f_* \circ \psi^{-1}, f\rangle \\ &= \langle f_* - f_* \circ \psi^{-1}, f\rangle \end{aligned}$$

for any $f \in L^\infty(X)$ and $\psi \in \langle\Psi\rangle$. Since $L^\infty(X)$ is dense in $L^2(X)$, we have $f_* = f_* \circ \psi^{-1}$ for all $\psi \in \langle\Psi\rangle$. So, the previous lemma implies f_* is $\mathcal{R}$-invariant, i.e., $f_* \in \mathcal{I}$. This implies $\mathcal{I} + \mathcal{G}$ is dense in $L^2(X)$, as claimed.

By Lemma 2.3, for every $f \in \mathcal{I} + \mathcal{G}$, $\mathbb{A}[f|\mathfrak{F}_r]$ converges pointwise almost everywhere to $\mathbb{E}[f|\mathcal{I}(\mathcal{R})]$. Since $\mathcal{I} + \mathcal{G}$ is dense in $L^2(X)$, which is dense in $L^1(X)$, the first statement follows. The second is similar. □

2.4. *Maximal inequality: the regular case.* To prove the maximal inequality for a regular Følner family, we begin with the following basic covering argument, motivated by the classical case.

LEMMA 2.5. *Suppose $\mathfrak{F}$ satisfies the regularity condition with constant $C_{\mathrm{reg}} > 0$. Let $\rho : Y \to \mathbb{I}$ be a bounded measurable function where $Y \subset X$ is Borel. Then there exists a measurable set $Z \subset Y$ such that the family of sets $\mathfrak{F}_{\rho(z)}(z)$, $z \in Z$ satisfy the following.*

(1) *For all $z_1 \neq z_2 \in Z$, $\mathfrak{F}_{\rho(z_1)}(z_1) \cap \mathfrak{F}_{\rho(z_2)}(z_2) = \emptyset$, namely the family is disjoint.*

(2) *The union of the sets in the family covers at least a fixed fraction of the measure of Y:*

$$C_{\mathrm{reg}}\mu\left(\bigcup_{z\in Z} \mathfrak{F}_{\rho(z)}(z)\right) \geq \mu(Y).$$

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Proof. Let $T : X \to \mathbb{R}$ be an injective Borel function. We will use T to break 'ties' in what follows.

If $Y' \subset Y$ is a Borel set, then we let $M(Y') \subset Y'$ be the set of all 'maximal' elements of Y'. Specifically, $y_1 \in M(Y')$ if $y_1 \in Y'$ and for all $y_2 \in Y'$ *different from* y_1 either:

(1) $\mathfrak{F}_{\rho(y_1)}(y_1) \cap \mathfrak{F}_{\rho(y_2)}(y_2) = \emptyset$;

(2) $\rho(y_1) > \rho(y_2)$; or

(3) $\mathfrak{F}_{\rho(y_1)}(y_1) \cap \mathfrak{F}_{\rho(y_2)}(y_2) \neq \emptyset$, $\rho(y_1) = \rho(y_2)$ and $T(y_1) > T(y_2)$.

Because ρ is bounded, the equivalence relation has countable classes, and $\mathfrak{F}$ is regular, it follows that for any y_1, the set of y_2 with $\mathfrak{F}_{\rho(y_1)}(y_1) \cap \mathfrak{F}_{\rho(y_2)}(y_2) \neq \emptyset$ is finite. Thus, in case (3), there exists a point y_1 with $T(y_1)$ maximal, so that if Y' is non-empty, then $M(Y')$ is non-empty.

Let $Y_0 := Y$ and $M_0 := M(Y_0)$. Assuming that Y_n, $M_n \subset Y$ have been defined, let

$$Y_{n+1} := \{y \in Y : \mathfrak{F}_{\rho(y)}(y) \cap \mathfrak{F}_{\rho(z)}(z) = \emptyset \ \forall z \in M_n\}$$

and $M_{n+1} := M(Y_{n+1})$. Let

$$Z := \bigcup_n M_n, \quad \widetilde{Z} := \bigcup_{z \in Z} \mathfrak{F}_{\rho(z)}(z).$$

By construction, for all $z_1 \neq z_2 \in Z$, $\mathfrak{F}_{\rho(z_1)}(z_1) \cap \mathfrak{F}_{\rho(z_2)}(z_2) = \emptyset$. Also,

$$Y \subset W := \bigcup_{z \in Z} \bigcup_{r \leq \rho(z)} \mathfrak{F}_r^{-1} \mathfrak{F}_{\rho(z)}(z).$$

So, it suffices to show that $C_{\mathrm{reg}} \mu(\widetilde{Z}) \geq \mu(W)$.

Define $K : \mathcal{R} \to \mathbb{R}$ by

$$K(x, y) = \left| \bigcup_{r \leq \rho(z)} \mathfrak{F}_r^{-1} \mathfrak{F}_{\rho(z)}(z) \right|^{-1}$$

if there is a (necessarily unique) $z \in Z$ such that $y \in \mathfrak{F}_{\rho(z)}(z)$ and $x \in \bigcup_{r \leq \rho(z)} \mathfrak{F}_r^{-1} \mathfrak{F}_{\rho(z)}(z)$. Let $K(x, y) = 0$ otherwise. Because $\mu \times c|_{\mathcal{R}} = c \times \mu|_{\mathcal{R}}$,

$$\mu(\widetilde{Z}) = \int \sum_{x \in [y]} K(x, y) \, d\mu(y) = \int \sum_{y \in [x]} K(x, y) \, d\mu(x).$$

Observe that $\sum_{y \in [x]} K(x, y) = 0$, unless $x \in W$, in which case

$$\sum_{y \in [x]} K(x, y) \geq \frac{|\mathfrak{F}_{\rho(z)}(z)|}{|\bigcup_{r \leq \rho(z)} \mathfrak{F}_r^{-1} \mathfrak{F}_{\rho(z)}(z)|} \geq C_{\mathrm{reg}}^{-1},$$

where $z \in Z$ is any element such that

$$x \in \bigcup_{r \leq \rho(z)} \mathfrak{F}_r^{-1} \mathfrak{F}_{\rho(z)}(z).$$

Thus,

$$\mu(\widetilde{Z}) = \int \sum_{y \in [x]} K(x, y) \, d\mu(x) \geq C_{\mathrm{reg}}^{-1} \mu(W),$$

which implies the lemma. □

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We can now prove the weak type (1, 1)-maximal inequality (and measurability of the maximal function) for a regular family.

LEMMA 2.6. *Suppose that $\mathfrak{F}$ is regular with regularity constant $C_{\mathrm{reg}} > 0$. Then, for any $f \in L^1(X)$ and any $t > 0$,*

$$\mu(\{x \in X : \mathbb{M}[f|\mathfrak{F}] > t\}) \le \frac{C_{\mathrm{reg}}\|f\|_1}{t}.$$

Proof. For $n > 0$, let

$$\mathbb{M}_n[f|\mathfrak{F}](x) := \max_{0<r\le n} \mathbb{A}[|f||\mathfrak{F}_r](x).$$

Given x, for $s \le n$, the family $\mathfrak{F}_s(x)$ comprises of a finite number of finite subsets of $[x]$, by our standing assumption on the family $\mathfrak{F}$, and, furthermore, $(x, s) \mapsto \mathfrak{F}_s(x)$ is Borel. Hence, $\mathbb{M}_n[f|\mathfrak{F}](x)$ is measurable, and since $\mathbb{M}[f|\mathfrak{F}](x) = \lim_{n\to\infty} \mathbb{M}_n[f|\mathfrak{F}](x)$, so is $\mathbb{M}[f|\mathfrak{F}](x)$.

Now let $D_{n,t} := \{x \in X : \mathbb{M}_n[f|\mathfrak{F}](x) > t\}$. It suffices to show that

$$\mu(D_{n,t}) \le \frac{C_{\mathrm{reg}}\|f\|_1}{t} \quad \text{for each } n > 0.$$

Let $\rho : D_{n,t} \to \mathbb{I}$ be a Borel function such that $\mathbb{A}[|f||\mathfrak{F}_{\rho(x)}](x) > t$ and $\rho(x) \le n$ for all $x \in D_{n,t}$. Let $Z \subset D_{n,t}$ be the subset given by the previous lemma, where $Y = D_{n,t}$. As before, let $\widetilde{Z} = \bigcup\{\mathfrak{F}_{\rho(z)}(z) : z \in Z\}$. The previous lemma implies that $\mu(D_{n,t}) \le C_{\mathrm{reg}}\mu(\widetilde{Z})$.

The disjointness property of Z implies that for every $z \in \widetilde{Z}$, there exists a unique element $\pi(z) \in Z$ with $z \in \mathfrak{F}_{\rho(\pi(z))}(\pi(z))$. By definition of ρ,

$$\mu(D_{n,t}) \le C_{\mathrm{reg}}\mu(\widetilde{Z}) \le \frac{C_{\mathrm{reg}}}{t}\int_{\widetilde{Z}} \mathbb{A}[|f||\mathfrak{F}_{\rho(\pi(z))}](\pi(z))\, d\mu(z).$$

Let $K : \mathcal{R} \to \mathbb{R}_+$ be the function

$$K(y, z) = \frac{|f(y)|}{|\mathfrak{F}_{\rho(\pi(z))}(\pi(z))|}$$

if $z \in \widetilde{Z}$ and $y \in \mathfrak{F}_{\rho(\pi(z))}(\pi(z))$, and let $K(y, z) = 0$ otherwise. Note that for a given $y \in \widetilde{Z}$, the number of elements $z \in [y]$ such that $y \in \mathfrak{F}_{\rho(\pi(z))}(\pi(z))$ is precisely $|\mathfrak{F}_{\rho(\pi(z))}(\pi(z))|$. Since $\mu \times c|_{\mathcal{R}} = c \times \mu|_{\mathcal{R}}$, we conclude

$$\begin{aligned}\int_{y\in X}\sum_{z\in[y]} K(y, z)\, d\mu(y) &= \int_{y\in\widetilde{Z}} |f(y)|\, d\mu(y)\\ &= \int_{z\in X}\sum_{y\in[z]} K(y, z)\, d\mu(z)\\ &= \int_{\widetilde{Z}} \mathbb{A}[|f||\mathfrak{F}_{\rho(\pi(z))}](\pi(z))\, d\mu(z).\end{aligned}$$

So,

$$\begin{aligned}\mu(D_{n,t}) &\le \frac{C_{\mathrm{reg}}}{t}\int_{\widetilde{Z}} \mathbb{A}[|f||\mathfrak{F}_{\rho(\pi(z))}](\pi(z))\, d\mu(z)\\ &= \frac{C_{\mathrm{reg}}}{t}\int_{\widetilde{Z}} |f(y)|\, d\mu(y) \le \frac{C_{\mathrm{reg}}\|f\|_1}{t},\end{aligned}$$

and the proof of the maximal inequality is complete. □

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2.5. *Maximal inequality: the tempered case.* This subsection completes the proofs of Theorems 2.3 and 2.1, using [**We03**] as a model. Having considered the regular case in the previous lemma, it suffices to assume $\mathfrak{F}$ is asymptotically invariant, uniform and tempered.

LEMMA 2.7. *Suppose $\mathfrak{F}$ is uniform with uniformity constant $C_u > 0$. If $f \in L^1(X)$ with $f \geq 0$ and $r > 0$, then*

$$C_u^{-1} \int f(x)\, d\mu(x) \leq \int \mathbb{A}[f|\mathfrak{F}_r](x)\, d\mu(x) \leq C_u \int f(x)\, d\mu(x).$$

Proof. Define a function F on $\mathcal{R}$ by $F(x, y) := f(y)/|\mathfrak{F}_r(x)|$ if $y \in \mathfrak{F}_r(x)$, and $F(x, y) := 0$ otherwise. Because $\mu \times c|_{\mathcal{R}} = c \times \mu|_{\mathcal{R}}$,

$$\begin{aligned}\int \mathbb{A}[f|\mathfrak{F}_r](x)\, d\mu(x) &= \int F(x, y)\, d\mu \times c(x, y) = \int F(x, y)\, dc \times \mu(x, y)\\ &= \int f(y) \sum_{x \in \mathfrak{F}_r^{-1}(y)} |\mathfrak{F}_r(x)|^{-1}\, d\mu(y).\end{aligned}$$

Let a_r, b_r be the constants in the definition of uniformity. Then $a_r \leq |\mathfrak{F}_r^{-1}(y)| \leq b_r$ and $a_r \leq |\mathfrak{F}_r(x)| \leq b_r$ for a.e. $x, y \in X$. Therefore,

$$C_u^{-1} \leq a_r/b_r \leq \sum_{x \in \mathfrak{F}_r^{-1}(y)} |\mathfrak{F}_r(x)|^{-1} \leq b_r/a_r \leq C_u.$$

These inequalities and the equality above imply the lemma. □

We now turn to establishing the important fact that in the uniform case, asymptotic invariance under a generating set implies asymptotic invariance (in mean) under the entire group of inner automorphisms.

LEMMA 2.8. *If $\mathfrak{F}$ is uniform and asymptotically invariant, then for every $\phi \in \mathrm{Inn}(\mathcal{R})$,*

$$\lim_{r \to \infty} \int \frac{|\mathfrak{F}_r(x) \Delta \phi(\mathfrak{F}_r(x))|}{|\mathfrak{F}_r(x)|}\, d\mu(x) = 0.$$

Proof. Because

$$\begin{aligned}|\mathfrak{F}_r(x) \Delta \phi(\mathfrak{F}_r(x))| &= |\mathfrak{F}_r(x) \backslash \phi(\mathfrak{F}_r(x))| + |\phi(\mathfrak{F}_r(x)) \backslash \mathfrak{F}_r(x)|\\ &= |\mathfrak{F}_r(x) \backslash \phi(\mathfrak{F}_r(x))| + |\mathfrak{F}_r(x) \backslash \phi^{-1}(\mathfrak{F}_r(x))|\end{aligned}$$

and $\phi \in \mathrm{Inn}(\mathcal{R})$ is arbitrary, it suffices to show that

$$\lim_{r \to \infty} \int \frac{|\mathfrak{F}_r(x) \backslash \phi(\mathfrak{F}_r(x))|}{|\mathfrak{F}_r(x)|}\, d\mu(x) = 0.$$

Let $\Psi \subset \mathrm{Inn}(\mathcal{R})$ be a countable generating set witnessing the asymptotic invariance. So, Ψ generates $\mathcal{R}$ and, for a.e. $x \in X$,

$$\lim_{r \to \infty} \frac{|\mathfrak{F}_r(x) \Delta \psi(\mathfrak{F}_r(x))|}{|\mathfrak{F}_r(x)|} = 0 \quad \text{for all } \psi \in \Psi.$$

By Lemma 2.2, we may assume, without loss of generality, that Ψ is a subgroup of $\mathrm{Inn}(\mathcal{R})$. Because Ψ generates $\mathcal{R}$, this means that for $\mu \times c$-a.e. $(x, y) \in \mathcal{R}$, there is a $\psi \in \Psi$ such

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that $\psi(x) = y$. Because Ψ is countable, this implies that there is a Borel partition $\{X_i\}_{i=1}^{\infty}$ of X and elements $\psi_i \in \Psi$ such that $\phi(x) = \psi_i(x)$ for a.e. $x \in X_i$.

Let $\epsilon > 0$. Choose $N > 0$ so that $\mu(\bigcup_{i=1}^{N} X_i) \geq 1 - \epsilon$. Let $Y = \bigcup_{i>N} X_i$, so $\mu(Y) \leq \epsilon$. Let U be the subset function $U(x) = \{\psi_i(x);\ 1 \leq i \leq N\}$. Recall that we have defined in §2.1 the product of two arbitrary subset functions, and therefore the expression $U\mathfrak{F}_r(x) = \{\psi_i(y);\ y \in \mathfrak{F}_r(x),\ 1 \leq i \leq N\}$ makes sense and is also a subset function. Using the foregoing pointwise convergence results established for $\psi \in \langle\Psi\rangle$, Lebesgue's bounded convergence theorem implies

$$\lim_{r\to\infty} \int \frac{|\mathfrak{F}_r(x) \backslash U\mathfrak{F}_r(x)|}{|\mathfrak{F}_r(x)|}\, d\mu(x) = 0.$$

However,

$$|\mathfrak{F}_r(x) \backslash \phi(\mathfrak{F}_r(x))| \leq |\mathfrak{F}_r(x) \backslash U\mathfrak{F}_r(x)| + |\mathfrak{F}_r(x) \cap Y|.$$

Thus,

$$\begin{aligned}
&\lim_{r\to\infty} \int \frac{|\mathfrak{F}_r(x) \backslash \phi(\mathfrak{F}_r(x))|}{|\mathfrak{F}_r(x)|}\, d\mu(x) \\
&\quad \leq \lim_{r\to\infty} \int \frac{|\mathfrak{F}_r(x) \backslash U\mathfrak{F}_r(x)|}{|\mathfrak{F}_r(x)|}\, d\mu(x) + \lim_{r\to\infty} \int \frac{|\mathfrak{F}_r(x) \cap Y|}{|\mathfrak{F}_r(x)|}\, d\mu(x) \\
&\quad = 0 + \lim_{r\to\infty} \int \mathbb{A}[1_Y | \mathfrak{F}_r]\, d\mu(x) \leq C_u \mu(Y) \leq C_u \epsilon,
\end{aligned}$$

where C_u is the uniformity constant of $\mathfrak{F}$. The second to last inequality above follows from Lemma 2.7. Since $\epsilon > 0$ is arbitrary, this implies the lemma. □

We can formulate the following useful fact which will be used in the proof of the maximal inequality in the tempered case.

LEMMA 2.9. *If $\mathfrak{F}$ is uniform and asymptotically invariant and U is a Borel subset function with $1 \leq |U(x)|$ for a.e. x and $|U| \in L^{\infty}(X)$, then*

$$\lim_{r\to\infty} \int \frac{|U\mathfrak{F}_r(x)|}{|\mathfrak{F}_r(x)|}\, d\mu(x) = 1.$$

Proof. Let $E = \{(x, y) \in \mathcal{R} : x \in U(y) \text{ or } y \in U(x)\}$. Because U is bounded, this a bounded-degree graph. By [**KST99**], this implies that the Borel edge-chromatic number of (X, E) is finite, i.e., there exists a Borel map $\alpha : E \to A$ (where A is a finite set) such that if $(x, y), (y, z) \in E$ and $x \neq z$, then $\alpha((x, y)) \neq \alpha((y, z))$. We can also assume, without loss of generality, that $\alpha(x, y) = \alpha(y, x)$.

For each element $a \in A$, define $\phi_a : X \to X$ as follows. If $x \in X$ and there is a $y \neq x$ such that $(x, y) \in E$ and $\alpha(x, y) = a$, then define $\phi_a(x) = y$ and $\phi_a(y) = x$. Otherwise, let $\phi_a(x) = x$. Then ϕ is a Borel bijection and $\phi_a \in \mathrm{Inn}(\mathcal{R})$.

So, we have proven that there is a finite collection of automorphisms $\phi_1, \ldots, \phi_m \in \mathrm{Inn}(\mathcal{R})$ such that, for a.e. $x \in X$,

$$U(x) \subset \bigcup_{i=1}^{m} \phi_i(x).$$

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Lemma 2.8 implies that, for every i,

$$\lim_{r\to\infty}\int \frac{|\mathfrak{F}_r(x)\Delta\phi_i(\mathfrak{F}_r(x))|}{|\mathfrak{F}_r(x)|}\,d\mu(x)=0.$$

Since this is true for every i, it follows that

$$\lim_{r\to\infty}\int \frac{|\mathfrak{F}_r(x)\Delta U\mathfrak{F}_r(x))|}{|\mathfrak{F}_r(x)|}\,d\mu(x)=0,$$

which implies the lemma. □

We now state the following combinatorial result from [**We03**], together with its proof, which will serve as a model in the more complicated set-up of measured equivalence relations.

LEMMA 2.10. (Basic Lemma [**We03**]) *Let Ω be a countable set, $V_1,\dots,V_m\subset\Omega$ be non-empty finite subsets, κ be a positive measure on Ω and $C_u\geq 1$, $\lambda>0$ be constants. Suppose:*

(1) $|V_i|/|V_j|\leq C_u$ *for every* i, j;
(2) $\kappa(V_i)\geq\lambda|V_i|$ *for every* i; *and*
(3) $\sum_{i=1}^m 1_{V_i}(\omega)\leq C_u|V_1|$ *for every* $\omega\in\Omega$.

Then there is a subset $I\subset\{1,\dots,m\}$ such that:

(1) $\kappa(\bigcup_{i\in I}V_i)\geq\lambda m/4C_u^2$; *and*
(2) $\kappa(\bigcup_{i\in I}V_i)\geq\lambda|I||V_1|/4C_u^2$.

Proof. Beginning with $i(1)=1$, inductively define $i(k+1)$ to be the least integer $\leq m$, greater than $i(k)$, such that

$$\kappa\left(V_{i(k+1)}\Big\backslash\bigcup_{1\leq j\leq k}V_{i(j)}\right)\geq\frac{1}{2}\kappa(V_{i(k+1)})$$

if such an integer exists, otherwise stop and call $\{i(1),\dots,i(k)\}=:I$. We distinguish two cases.

Case 1. $|I|\geq m/2|V_1|$. In this case, clearly,

$$\kappa\left(\bigcup_{i\in I}V_i\right)\geq\frac{1}{2}\sum_{i\in I}\kappa(V_i)\geq\frac{|I|\lambda|V_1|}{2C_u}\geq\frac{\lambda m}{4C_u}.$$

Case 2. $|I|<m/2|V_1|$. Let $I^c=\{1,\dots,m\}\backslash I$. By the definition of I, if $j\in I^c$, then

$$\kappa\left(V_j\cap\bigcup_{i\in I}V_i\right)\geq\frac{1}{2}\kappa(V_j).$$

Sum over all $j\in I^c$ and use hypothesis 3 to obtain

$$\frac{1}{2}\sum_{j\in I^c}\kappa(V_j)\leq\sum_{j\in I^c}\kappa\left(V_j\cap\bigcup_{i\in I}V_i\right)\leq C_u|V_1|\kappa\left(\bigcup_{i\in I}V_i\right).$$

Now use hypothesis 2 and divide by $C_u|V_1|$ to obtain

$$\begin{aligned}\kappa\left(\bigcup_{i\in I}V_i\right)&\geq\frac{1}{2C_u|V_1|}\sum_{j\in I^c}\kappa(V_j)\geq\frac{|I^c|\lambda}{2C_u^2}\geq\frac{(m-m/2|V_1|)\lambda}{2C_u^2}\\&=\frac{1}{2C_u^2}(1-2^{-1}|V_1|^{-1})m\lambda.\end{aligned}$$

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Because $|V_1| \geq 1$, $(1 - 2^{-1}|V_1|^{-1})/2 \geq 1/4$. So, this implies

$$\kappa\left(\bigcup_{i\in I} V_i\right) \geq \frac{1}{4C_u^2} m\lambda.$$

This proves the first conclusion. The second one follows from the inequality above and the hypothesis $|I| < m/2|V_1|$. □

For the next proposition, we let Ω be a countable set and $\{V_i\}_{i=1}^N$ a sequence of subset functions on Ω. Thus, each V_i is a map $V_i : \Omega \to 2^\Omega$. We define the inverse $V_i^{-1} : \Omega \to 2^\Omega$ by $V_i^{-1}(y) = \{x \in \Omega : V_i(x) \ni y\}$. We also define the product, union, intersection and difference of subset functions as in §2.1, which considers the special case of subset functions for equivalence relations.

PROPOSITION 2.11. *Let Ω be a countable set, $I_1, \ldots, I_N \subset \Omega$ be pairwise disjoint finite subsets, $\{V_i : 1 \leq i \leq N\}$ a collection of subset functions of Ω, κ be a positive measure on Ω and C_t, C_u, $\lambda > 0$ be constants. Suppose:*

(1) $|V_i(\omega)|/|V_i(\omega')| \leq C_u$ *for every i and every $\omega, \omega' \in \Omega$;*
(2) $\kappa(V_i(\omega)) \geq \lambda|V_i(\omega)|$ *for every i and every $\omega \in I_i$;*
(3) $|V_i^{-1}|(\omega) \leq C_u V_i(\omega)$ *for every i and $\omega \in \Omega$; and*
(4) *for every j, $|\bigcup_{i<j} V_i^{-1} V_j(\omega)| \leq C_t|V_j(\omega)|$.*

Then

$$\sum_{i=1}^N |I_i| \leq \left(\frac{8C_u^2 + 8C_tC_u^3}{\lambda}\right)\kappa\left(\bigcup_{i=1}^N \bigcup_{\omega\in I_i} V_i(\omega)\right).$$

Proof. Without loss of generality, we may assume each I_i is non-empty. For each i with $1 \leq i \leq N$, choose $\omega_i \in I_i$. We construct a partition $\{L, K\}$ of $\{1, \ldots, N\}$ and sets $D_i \subset I_i$ for $i \in L$ using the following algorithm.

Step 1. Apply the Basic Lemma to the collection $\{V_N(\omega) : \omega \in I_N\}$ to obtain a set $D_N \subset I_N$ such that:

(1) $\kappa(\bigcup_{\omega\in D_N} V_N(\omega)) \geq \lambda|I_N|/4C_u^2$; and
(2) $\kappa(\bigcup_{\omega\in D_N} V_N(\omega)) \geq \lambda|D_N||V_N(\omega_N)|/4C_u^2$.

It is convenient to rewrite these inequalities in the form:

(1) $|I_N| \leq (4C_u^2/\lambda)\kappa(\bigcup_{\omega\in D_N} V_N(\omega))$; and
(2) $|D_N||V_N(\omega_N)| \leq (4C_u^2/\lambda)\kappa(\bigcup_{\omega\in D_N} V_N(\omega))$.

Step 2. Let $L := \{N\}$, $K := \emptyset$, $i := 1$.

Step 3. If $i = N$, then stop.

Step 4. Let I'_{N-i} be the set of $\omega \in I_{N-i}$ such that $V_{N-i}(\omega)$ is disjoint from $\bigcup\{V_k(\omega') : k \in L, \omega' \in D_k\}$.

Step 5. If $|I'_{N-i}| \geq \frac{1}{2}|I_{N-i}|$, then:

(1) set $L := L \cup \{N - i\}$; and
(2) apply the Basic Lemma to obtain a set $D_{N-i} \subset I'_{N-i}$ such that:
   (a) $|I_{N-i}| \leq (8C_u^2/\lambda)\kappa(\bigcup_{\omega\in D_{N-i}} V_{N-i}(\omega))$; and
   (b) $|D_{N-i}||V_{N-i}(\omega_{N-i})| \leq (4C_u^2/\lambda)\kappa(\bigcup_{\omega\in D_{N-i}} V_{N-i}(\omega))$.

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Step 6. If $|I'_{N-i}| < \frac{1}{2}|I_{N-i}|$, then set $K := K \cup \{N-i\}$.

Step 7. Set $i := i+1$ and go to Step 3.

This algorithm produces a partition $\{L, K\}$ of $\{1, \dots, N\}$ and subsets $D_i \subset I_i$ for $i \in L$ such that:

(1) if, for $i \in L$, $H_i := \bigcup\{V_i(\omega) : \omega \in D_i\}$, then $H_i \cap H_k = \emptyset$ for all $i \neq k$;

(2) $|I_i| \leq (8C_u^2/\lambda)\kappa(\bigcup_{\omega\in D_i} V_i(\omega))$ for all $i \in L$; and

(3) $|D_i||V_i(\omega_i)| \leq (4C_u^2/\lambda)\kappa(\bigcup_{\omega\in D_i} V_i(\omega))$ for all $i \in L$.

The first two conditions above imply that

$$\sum_{i\in L}|I_i| \leq \sum_{i\in L}\frac{8C_u^2}{\lambda}\kappa\left(\bigcup_{\omega\in D_i} V_i(\omega)\right) = \frac{8C_u^2}{\lambda}\kappa\left(\bigcup_{i\in L} H_i\right).$$

Also, if $k \in K$, then there exists a set $I''_k \subset I_k$ such that $|I''_k| \geq \frac{1}{2}|I_k|$ and for every $\omega \in I''_k$, $V_k(\omega)$ is not disjoint from $\bigcup\{H_i : i > k, i \in L\}$. Therefore, $\omega \in V_k^{-1}V_j(\omega')$ for some $j > k$ with $j \in L$ and some $\omega' \in D_j$. Because $\{I_i\}_{i=1}^N$ are pairwise disjoint, hypothesis 4 implies that

$$\begin{aligned}
\sum_{k\in K}|I_k| &\leq 2\sum_{k\in K}|I''_k| \\
&\leq 2\left|\bigcup\{V_k^{-1}V_j(\omega) : j \in L,\ j > k,\ \omega \in D_j\}\right| \\
&\leq 2\sum_{j\in L,\omega\in D_j}\left|\bigcup_{i<j} V_i^{-1}V_j(\omega)\right| \\
&\leq 2C_t\sum_{j\in L,\omega\in D_j}|V_j(\omega)| \\
&\leq 2C_tC_u\sum_{j\in L}|D_j||V_j(\omega_j)| \leq 2C_tC_u\frac{4C_u^2}{\lambda}\kappa\left(\bigcup_{i\in L} H_i\right).
\end{aligned}$$

Thus,

$$\sum_{i=1}^N|I_i| = \sum_{i\in L}|I_i| + \sum_{k\in K}|I_k| \leq \frac{8C_u^2 + 8C_tC_u^3}{\lambda}\kappa\left(\bigcup_{i\in L} H_i\right),$$

which implies the result. □

Completion of the proof of Theorem 2.3. By Lemma 2.6, it suffices to assume $\mathfrak{F}$ is asymptotically invariant, uniform and tempered. Note that temperedness is defined for sequences only, so the measurability of the maximal function is obvious in this case. For $f \in L^1(X)$, define $\mathbb{M}_N[f] := \sup_{r\leq N}\mathbb{A}[|f||\mathfrak{F}_r]$. It suffices to prove the existence of a constant $C > 0$ such that, for every $\lambda > 0$, every $N > 0$ and every $f \in L^1(X)$ with $f \geq 0$,

$$\mu(\{x \in X : \mathbb{M}_N[f](x) \geq \lambda\}) \leq \frac{C\|f\|_1}{\lambda}.$$

So, fix $N > 0$, $\lambda > 0$ and $f \in L^1(X)$ with $f \geq 0$. Let

$$E_N := \{x \in X : \mathbb{M}_N[f](x) \geq \lambda\}.$$

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For $R > 0$, let $H(N, R)$ be the subset function

$$H(N, R)(x) := E_N \cap \mathfrak{F}_R(x).$$

Let 1_{E_N} be the indicator function of E_N. Observe that

$$\mathbb{A}[1_{E_N}|\mathfrak{F}_R](x) = \frac{|H(N, R)(x)|}{|\mathfrak{F}_R(x)|}.$$

By Lemma 2.7,

$$\mu(E_N) \leq C_u \int \frac{|H(N, R)(x)|}{|\mathfrak{F}_R(x)|}\, d\mu(x). \tag{2.1}$$

Let $H'_{N,R}$ be the subset function

$$H'(N, R)(x) := \{y \in X : \exists n \leq N,\ \mathbb{A}[f|\mathfrak{F}_n](y) \geq \lambda,\ \mathfrak{F}_n(y) \subset \mathfrak{F}_R(x)\}.$$

To apply Proposition 2.11, let $\Omega := [x]$, the equivalence class of x. Let κ be the measure on Ω determined by $\kappa(\{y\}) := f(y)$ (for $y \in \Omega$). For each $y \in H'(N, R)(x)$, let $k(y)$ be the smallest number such that $\mathfrak{F}_{k(y)}(y)$ satisfies $\mathbb{A}[f|\mathfrak{F}_{k(y)}](y) \geq \lambda$, $\mathfrak{F}_{k(y)}(y) \subset \mathfrak{F}_R(x)$. For each $1 \leq i \leq N$, let I_i be the set of all $y \in H'(N, R)$ such that $i = k(y)$. Let $V_i(y) := \mathfrak{F}_i(y)$ for $1 \leq i \leq N$ and $y \in \Omega$. It is easy to check that because $\mathfrak{F}$ is uniform and tempered, the hypotheses of Proposition 2.11 are satisfied. The conclusion implies that

$$|H'(N, R)(x)| \leq \frac{C}{\lambda} \sum_{y \in \mathfrak{F}_R(x)} f(y),$$

where $C = 8C_u^2 + 8C_tC_u^3$. Divide both sides by $\mathfrak{F}_R(x)$ and integrate over x to obtain

$$\int \frac{|H'(N, R)(x)|}{|\mathfrak{F}_R(x)|}\, d\mu(x) \leq \frac{C}{\lambda} \int \mathbb{A}[f|\mathfrak{F}_R](x)\, d\mu(x) \leq \frac{CC_u}{\lambda}\|f\|_1. \tag{2.2}$$

The last inequality follows from Lemma 2.7.

Let $U(N)$ and $S(N, R)$ be the subset functions

$$U_N(x) = \bigcup_{t \leq N} \mathfrak{F}_t(x), \quad S(N, R)(x) := \{y \in \mathfrak{F}_R(x) : U_N(y) \not\subseteq \mathfrak{F}_R(x)\}.$$

Observe that

$$H(N, R)\backslash H'(N, R) \subset S(N, R) \subset U_N^{-1}(U_N\mathfrak{F}_R\backslash\mathfrak{F}_R).$$

By Lemma 2.9,

$$\lim_{R\to\infty} \int \frac{|U_N\mathfrak{F}_R(x)\backslash\mathfrak{F}_R(x)|}{|\mathfrak{F}_R(x)|}\, d\mu(x) = 0.$$

Because $\mathfrak{F}$ is uniform, the function $x \mapsto |U_N^{-1}(x)|$ is essentially bounded. Therefore,

$$\lim_{R\to\infty} \int \frac{|U_N^{-1}(U_N\mathfrak{F}_R\backslash\mathfrak{F}_R)(x)|}{|\mathfrak{F}_R(x)|}\, d\mu(x) = 0.$$

Since $H(N, R)\backslash H'(N, R) \subset U_N^{-1}(U_N\mathfrak{F}_R\backslash\mathfrak{F}_R)$, it follows that

$$\lim_{R\to\infty} \int \frac{|H(N, R)(x)\backslash H'(N, R)(x)|}{|\mathfrak{F}_R(x)|}\, d\mu(x) = 0.$$

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Since $H'(N, R) \subset H(N, R)$, equations (2.1) and (2.2) now imply that

$$\mu(E_N) \leq \lim_{R\to\infty} C_u \int \frac{|H(N,R)(x)|}{|\mathfrak{F}_R(x)|}\, d\mu(x)$$
$$= \lim_{R\to\infty} C_u \int \frac{|H'(N,R)(x)|}{|\mathfrak{F}_R(x)|}\, d\mu(x) \leq \frac{CC_u^2}{\lambda}\|f\|_1.$$

Because f, N, λ are arbitrary, this implies the theorem. □

Completion of the proof of Theorem 2.1.

LEMMA 2.12. *If $\mathfrak{F}$ is any family of subset functions satisfying the conclusions of Theorems 2.2 and 2.3 (i.e., there exists a dense set of good functions and the weak* (1, 1)*-type maximal inequality is satisfied), then $\mathfrak{F}$ is a (restricted) pointwise ergodic family in L^1. Namely, for every $f \in L^1(X, \mu)$, $\mathbb{A}[f|\mathfrak{F}_r]$ converges pointwise almost everywhere to $\mathbb{E}[f|\mathcal{I}(\mathcal{R})]$ as $r \to \infty$.*

Proof. Let $f \in L^1(X)$. We will show that $\{\mathbb{A}[f|\mathfrak{F}_r]\}_{r>0}$ converges pointwise almost everywhere to $\mathbb{E}[f|\mathcal{I}(\mathcal{R})]$. After replacing f with $f - \mathbb{E}[f|\mathcal{I}(\mathcal{R})]$ if necessary, we may assume that $\mathbb{E}[f|\mathcal{I}(\mathcal{R})] = 0$ almost everywhere.

For $t > 0$, let $E_t := \{x \in X : \limsup_{r\to\infty} |\mathbb{A}[f|\mathfrak{F}_r](x)| \leq t\}$. We will show that each E_t has measure one. Let $\epsilon = t^2/4$. According to Theorem 2.2, there exists a function $f_1 \in L^1(X)$ with $\|f - f_1\|_1 < \epsilon$ such that $\{\mathbb{A}[f_1|\mathfrak{F}_r]\}_{r>0}$ converges pointwise almost everywhere to 0 as $r \to \infty$. For any $r > 0$,

$$|\mathbb{A}[f|\mathfrak{F}_r]| \leq |\mathbb{A}[f - f_1|\mathfrak{F}_r]| + |\mathbb{A}[f_1|\mathfrak{F}_r]| \leq \mathbb{M}[f - f_1|\mathfrak{F}] + |\mathbb{A}[f_1|\mathfrak{F}_r]|.$$

Let

$$D := \{x \in X : \mathbb{M}[f - f_1|\mathfrak{F}](x) \leq \sqrt{\epsilon}\}.$$

Since $\mathbb{A}[f_1|\mathfrak{F}_r]$ converges pointwise almost everywhere to zero, for a.e. $x \in D$ there is an $N > 0$ such that $r > N$ implies

$$|\mathbb{A}[f|\mathfrak{F}_r](x)| \leq \mathbb{M}[f - f_1|\mathfrak{F}](x) + |\mathbb{A}[f_1|\mathfrak{F}_r](x)| \leq 2\sqrt{\epsilon} = t.$$

Hence, $D \subset E_t$ (up to a set of measure zero). By Theorem 2.3,

$$\mu(E_t) \geq \mu(D) \geq 1 - C\epsilon^{-1/2}\|f - f_1\|_1 > 1 - \sqrt{\epsilon}C = 1 - \frac{Ct}{2}.$$

For any $s < t$, $E_s \subset E_t$. So, $\mu(E_t) \geq \mu(E_s) \geq 1 - Cs/2$ for all $s < t$, which implies $\mu(E_t) = 1$. So, the set $E := \bigcap_{n=1}^{\infty} E_{1/n}$ has full measure. This implies the result. □

Theorem 2.1 follows immediately from the lemma above and Theorems 2.2 and 2.3. □

2.6. *Extensions of Borel equivalence relations.* Together with the amenable equivalence relation on X, we must consider its class-bijective PMP extensions, whose definition we now state. For $(X, \mathfrak{B}, \mu, \mathcal{R})$, a discrete PMP equivalence relation, a *class-bijective extension* of $(X, \mu, \mathcal{R})$ is a measured equivalence relation $(\widetilde{X}, \widetilde{\mu}, \widetilde{\mathcal{R}})$ with a Borel map $\pi : \widetilde{X} \to X$ satisfying the following.

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(1) $(\widetilde{X}, \widetilde{\mu}, \tilde{\mathcal{R}})$ is a discrete PMP equivalence relation.
(2) $\pi_* \widetilde{\mu} = \mu$.
(3) $(x, x') \in \tilde{\mathcal{R}} \Rightarrow (\pi(x), \pi(x')) \in \mathcal{R}$.
(4) For a.e. $\tilde{\mathcal{R}}$-equivalence class $[x] \subset \tilde{\mathcal{R}}$, π restricted to $[x]$ is a bijection onto the $\mathcal{R}$-equivalence class $[\pi(x)]$.

Suppose $\mathfrak{F} = \{\mathfrak{F}_r\}_{r \in \mathbb{I}}$ is a family of subset functions for $(X, \mu, \mathcal{R})$. Then we may lift this family as follows. Define $\widetilde{\mathfrak{F}} = \{\widetilde{\mathfrak{F}}_r\}_{r \in \mathbb{I}}$ by

$$\widetilde{\mathfrak{F}}_r(x) := \pi^{-1}(\mathfrak{F}_r(\pi(x))) \cap [x] \quad \text{for all } x \in \widetilde{X}.$$

LEMMA 2.13. *Let P be a property in {asymptotically invariant, uniform, regular, tempered}. If $\mathfrak{F}$ has property P, then $\widetilde{\mathfrak{F}}$ also has property P.*

Proof. Case 1. Suppose $P =$ asymptotically invariant.

Let $\Psi \subset \mathrm{Inn}(\mathcal{R})$ be a countable generating set witnessing the asymptotic invariance of $\mathfrak{F}$. This means that, for a.e. $x \in X$ and $\psi \in \Psi$,

$$\lim_{r \to \infty} \frac{|\mathfrak{F}_r(x) \Delta \psi(\mathfrak{F}_r(x))|}{|\mathfrak{F}_r(x)|} = 0.$$

For any $\psi \in \Psi$, define $\tilde{\psi} : \widetilde{X} \to \widetilde{X}$ by $\tilde{\psi}(x) = x'$, where $x' \in \widetilde{X}$ is the unique element such that $(x, x') \in \tilde{\mathcal{R}}$ and $\psi(\pi(x)) = \pi(x')$. This is unique because π restricted to $[x]$ is a bijection onto its image. Let $\tilde{\Psi} = \{\tilde{\psi} : \psi \in \Psi\}$. This is a countable set of inner automorphisms of $\tilde{\mathcal{R}}$. Because π restricted to each equivalence class is a bijection, for a.e. $x \in \widetilde{X}$, $|\widetilde{\mathfrak{F}}_r(x)| = |\mathfrak{F}_r(x)|$ and $|\widetilde{\mathfrak{F}}_r(x) \Delta \tilde{\psi}(\widetilde{\mathfrak{F}}_r(x))| = |\mathfrak{F}_r(x) \Delta \psi(\mathfrak{F}_r(x))|$. So,

$$\lim_{r \to \infty} \frac{|\widetilde{\mathfrak{F}}_r(x) \Delta \tilde{\psi}(\widetilde{\mathfrak{F}}_r(x))|}{|\widetilde{\mathfrak{F}}_r(x)|} = \lim_{r \to \infty} \frac{|\mathfrak{F}_r(x) \Delta \psi(\mathfrak{F}_r(x))|}{|\mathfrak{F}_r(x)|} = 0.$$

The set $\tilde{\Psi}$ is generating because for a.e. $(x, x') \in \tilde{\mathcal{R}}$, there is an element $\psi \in \Psi$ such that $\psi(\pi(x)) = \pi(x')$, and this implies $\tilde{\psi}(x) = x'$. So, we have verified all the conditions for the asymptotic invariance of $\widetilde{\mathfrak{F}}$.

Case 2. Suppose $P =$ regular.

Let C_{reg} be a regularity constant for $\mathfrak{F}$. Because π restricted to any equivalence class is a bijection, for a.e. $x \in \widetilde{X}$ and every $r > 0$,

$$\left| \bigcup_{t \le r} \widetilde{\mathfrak{F}}_t^{-1} \widetilde{\mathfrak{F}}_r(x) \right| = \left| \bigcup_{t \le r} \mathfrak{F}_t^{-1} \mathfrak{F}_r(\pi(x)) \right| \le C_{\mathrm{reg}} |\mathfrak{F}_r(\pi(x))| = C_{\mathrm{reg}} |\widetilde{\mathfrak{F}}_r(x)|.$$

This proves $\widetilde{\mathfrak{F}}$ is regular.

The other cases, i.e., uniform and tempered, can be handled similarly. □

Recall that we have defined $\mathfrak{F}$ to be a *pointwise ergodic family* in L^p if for every class-bijective extension $(\widetilde{X}, \widetilde{\mu}, \tilde{\mathcal{R}})$ and every $f \in L^p(\widetilde{X}, \widetilde{\mu})$, $\mathbb{A}[f|\widetilde{\mathfrak{F}}_r]$ converges pointwise almost everywhere to $\mathbb{E}[f|\mathcal{I}(\tilde{R})]$. To establish this fact, first define for $f \in L^1(\widetilde{X})$, $\mathbb{M}[f|\widetilde{\mathfrak{F}}] := \sup_r \mathbb{A}[|f||\widetilde{\mathfrak{F}}_r]$, where $|f|$ denotes the absolute value of f. $\mathbb{M}[\cdot|\widetilde{\mathfrak{F}}]$ is the *maximal operator* associated with the family of operators $\mathbb{A}[\cdot|\widetilde{\mathfrak{F}}_r]$. As in the case of $\mathfrak{F}$, the maximal function $\mathbb{M}[f|\widetilde{\mathfrak{F}}]$ is Borel measurable, and using the lemma above and Theorem 2.3, we conclude the following theorem.

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THEOREM 2.4. (Weak (1, 1)-type maximal inequality) *Suppose that $\mathfrak{F}$ is either regular or (asymptotically invariant, uniform and tempered). Then there exists a constant $C > 0$ such that, for any class-bijective extension $(\widetilde{X}, \widetilde{\mu}, \widetilde{\mathcal{R}})$, any $f \in L^1(\widetilde{X}, \widetilde{\mu})$ and any $\lambda > 0$,*

$$\widetilde{\mu}(\{x \in \widetilde{X} : \mathbb{M}[f|\widetilde{\mathfrak{F}}] > \lambda\}) \leq \frac{C\|f\|_1}{\lambda}.$$

In fact, C can be taken to be $8C_u^4(1 + C_t C_u)$ in the tempered case and C_{reg} in the regular case.

Now, using the lemma above, Theorems 2.4 and 2.1, we conclude the following theorem.

THEOREM 2.5. *If $\mathfrak{F}$ is either (asymptotically invariant and regular) or (asymptotically invariant, uniform and tempered) then $\mathfrak{F}$ is a pointwise ergodic family in L^1.*

Finally, we note the following fact, which is a standard consequence of the weak type (1, 1)-maximal inequality, and will be used below.

THEOREM 2.6. (Strong L^p-maximal inequality) *Suppose that $\mathfrak{F}$ is either regular or (asymptotically invariant, uniform and tempered). Then, for every $p > 1$, there is a constant $C_p > 0$ such that for any class-bijective extension $(\widetilde{X}, \widetilde{\mu}, \widetilde{\mathcal{R}})$ and any $f \in L^p(\widetilde{X}, \widetilde{\mu})$, $\|\mathbb{M}[f|\widetilde{\mathfrak{F}}]\|_p \leq C_p\|f\|_p$. Also, there is a constant $C_1 > 0$ such that if $f \in (L \log L)(\widetilde{X}, \widetilde{\mu})$, then $\|\mathbb{M}[f|\widetilde{\mathfrak{F}}]\|_1 \leq C_1\|f\|_{L \log L}$.*

Proof. This follows from the fact that $\mathbb{M}[\cdot|\widetilde{\mathfrak{F}}]$ satisfies a weak (1,1)-type maximal inequality (by Theorem 2.4 above) and standard interpolation arguments, i.e., since $\mathbb{M}[f|\widetilde{\mathfrak{F}}]$ is of weak type (1, 1) and is norm-bounded on L^∞, it is norm-bounded in every L^p, $1 < p < \infty$ (e.g., see [**SW71**, Ch. V, Theorem 2.4]). □

3. *Maximal inequalities for general group actions*

Let Γ be a countable group and $\Gamma \curvearrowright (B, \nu)$ an amenable action. Recall that the Maharam extension is the action $\Gamma \curvearrowright B \times \mathbb{R}$ given by

$$g(b, t) := \left(gb, t - \log\left(\frac{d\nu \circ g}{d\nu}(b)\right)\right). \tag{3.1}$$

Let θ be the measure on $\mathbb{R}$ given by $d\theta(t) = e^t dt$. The action above preserves the product measure $\nu \times \theta$. Let $T > 0$, $I = [0, T]$ and let $\mathcal{R}_I$ be the equivalence relation on $B \times I$ given by restricting the orbit equivalence relation on $B \times \mathbb{R}$ (so $\mathcal{R}_I$ consists of all $((b, t), g(b, t))$ with $g \in \Gamma$ and $(b, t), g(b, t) \in B \times I$). Let θ_I be the probability measure on $I = [0, T]$ given by $d\theta_I = (e^t/(e^T - 1))dt$. So, $\nu \times \theta_I$ is $\mathcal{R}_I$-invariant.

Notational convention. We change our notation and from now on we let (X, μ) denote an ergodic PMP action of Γ, and we consider the action of Γ on $(B \times X, \nu \times \mu)$, which is an amenable action and a class-bijective extension of the amenable action of Γ on (B, ν).

The purpose of this section is to prove the following.

THEOREM 3.1. *Let $\mathfrak{F} = \{\mathfrak{F}_r\}_{r \in \mathbb{I}}$ be a Borel family of subset functions for $(B \times I, \nu \times \theta_I, \mathcal{R}_I)$. Suppose $\mathfrak{F}$ is either regular or (asymptotically invariant, uniform and tempered).*

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Let $\Gamma \curvearrowright (X, \mu)$ be a PMP action. Let $\pi : B \times X \times I \to B \times I$ be the projection map $\pi(b, x, t) = (b, t)$ and let $\widetilde{\mathfrak{F}} = \{\widetilde{\mathfrak{F}}_r\}_{r\in\mathbb{I}}$ be the lift of $\mathfrak{F}$:

$$\widetilde{\mathfrak{F}}_r(x) := \pi^{-1}(\mathfrak{F}_r(\pi(b, x, t))) \cap [b, x, t] \quad \textit{for all } (b, x, t) \in B \times X \times I.$$

For $f \in L^1(B \times X \times I, \nu \times \mu \times \theta_I)$ and $(b, x, t) \in B \times X \times I$, define

$$\begin{aligned}
\mathbb{M}[f|\widetilde{\mathfrak{F}}](b, x, t) &:= \sup_{r\in\mathbb{I}} \mathbb{A}[|f||\widetilde{\mathfrak{F}}_r](b, x, t),\\
\overline{\mathbb{M}}[f|\widetilde{\mathfrak{F}}](b, x) &:= \sup_{r\in\mathbb{I}} \frac{1}{T}\int_0^T \mathbb{A}[|f||\widetilde{\mathfrak{F}}_r](b, x, t)\, dt,\\
\mathbb{M}[f|\widetilde{\mathfrak{F}}, \psi](x) &:= \sup_{r\in\mathbb{I}} \frac{1}{T}\int\int_0^T \mathbb{A}[|f||\widetilde{\mathfrak{F}}_r](b, x, t)\psi(b)\, dt\, d\nu(b).
\end{aligned}$$

Then:

(1) *there exist constants C_p for $p > 1$ such that, for every $f \in L^p(B \times X \times I)$,*

$$\|\mathbb{M}[f|\widetilde{\mathfrak{F}}]\|_p \le C_p\|f\|_p, \quad \|\overline{\mathbb{M}}[f|\widetilde{\mathfrak{F}}]\|_p \le C_p\left(\frac{e^T - 1}{T}\right)^{1/p}\|f\|_p;$$

also, if $1/p + 1/q = 1$ and $p > 1$, then

$$\|\mathbb{M}[f|\widetilde{\mathfrak{F}}, \psi]\|_p \le C_p\left(\frac{e^T - 1}{T}\right)^{1/p}\|\psi\|_q\|f\|_p; \textit{ and}$$

(2) *there is also a constant $C_1 > 0$ such that if $f \in L\log L(B \times X \times I)$, then*

$$\|\mathbb{M}[f|\widetilde{\mathfrak{F}}]\|_1 \le C_1\|f\|_{L\log L}, \quad \|\overline{\mathbb{M}}[f|\widetilde{\mathfrak{F}}]\|_1 \le C_1\left(\frac{e^T - 1}{T}\right)\|f\|_{L\log L};$$

if, in addition, $\psi \in L^\infty(B)$, then

$$\|\mathbb{M}[f|\widetilde{\mathfrak{F}}, \psi]\|_1 \le C_1\left(\frac{e^T - 1}{T}\right)\|\psi\|_\infty\|f\|_{L\log L}.$$

The constants C_p, for $p \ge 1$, do not depend on f or the action $\Gamma \curvearrowright (X, \mu)$.

Proof. Let us first consider the case $p > 1$ and $1/p + 1/q = 1$. By Theorem 2.6, there is a constant $C_p > 0$ (independent of f and the action $\Gamma \curvearrowright (X, \mu)$) such that $\|\mathbb{M}[f|\widetilde{\mathfrak{F}}]\|_p \le C_p\|f\|_p$. Let us therefore turn to the other two maximal operators.

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By Jensen's inequality,

$$\begin{aligned}
\|\overline{\mathbb{M}}[f|\widetilde{\mathfrak{F}}]\|_p^p &= \iint \overline{\mathbb{M}}[f|\widetilde{\mathfrak{F}}](b,x)^p \, d\nu(b)\, d\mu(x) \\
&= \iint \left(\sup_{r\in\mathbb{I}} \frac{1}{T}\int_0^T \mathbb{A}[|f||\widetilde{\mathfrak{F}}_r](b,x,t)\, dt\right)^p d\nu(b)\, d\mu(x) \\
&\le \iint \sup_{r\in\mathbb{I}} \frac{1}{T}\int_0^T \mathbb{A}[|f||\widetilde{\mathfrak{F}}_r](b,x,t)^p \, dt\, d\nu(b)\, d\mu(x) \\
&\le \iint \frac{1}{T}\int_0^T \sup_{r\in\mathbb{I}} \mathbb{A}[|f||\widetilde{\mathfrak{F}}_r](b,x,t)^p \, dt\, d\nu(b)\, d\mu(x) \\
&= \iint \frac{1}{T}\int_0^T \mathbb{M}[f|\widetilde{\mathfrak{F}}_r](b,x,t)^p \, dt\, d\nu(b) d\mu(x) \\
&\le \frac{e^T-1}{T}\int_0^T \iint \mathbb{M}[f|\widetilde{\mathfrak{F}}_r](b,x,t)^p \, d\nu(b)\, d\mu(x)\, d\theta_I(t) \\
&= \frac{e^T-1}{T}\|\mathbb{M}[f|\widetilde{\mathfrak{F}}_r]\|_p^p \le C_p^p\left(\frac{e^T-1}{T}\right)\|f\|_p^p.
\end{aligned}$$

We conclude that

$$\|\overline{\mathbb{M}}[f|\widetilde{\mathfrak{F}}]\|_p \le C_p\left(\frac{e^T-1}{T}\right)^{1/p}\|f\|_p.$$

Turning to the last maximal operator, by Hölder's inequality,

$$\begin{aligned}
\|\mathbb{M}[f|\widetilde{\mathfrak{F}},\psi]\|_p^p &= \iint \mathbb{M}[f|\widetilde{\mathfrak{F}},\psi](x)^p \, d\mu(x) \\
&= \int \left(\sup_{r\in\mathbb{I}} \int \frac{1}{T}\int_0^T \mathbb{A}[|f||\widetilde{\mathfrak{F}}_r](b,x,t)\psi(b)\, dt\, d\nu(b)\right)^p d\mu(x) \\
&\le \int \sup_{r\in\mathbb{I}}\left(\frac{1}{T}\iint_0^T \mathbb{A}[|f||\widetilde{\mathfrak{F}}_r](b,x,t)^p \, dt\, d\nu(b)\right) \\
&\quad \times \left(\iint_0^T \psi(b)^q \, dt\, d\nu(b)\right)^{p/q} d\mu(x) \\
&\le \|\psi\|_q^p \int \sup_{r\in\mathbb{I}} \frac{1}{T}\iint_0^T \mathbb{A}[|f||\widetilde{\mathfrak{F}}_r](b,x,t)^p \, dt\, d\nu(b)\, d\mu(x) \\
&\le \|\psi\|_q^p \frac{1}{T}\iiint_0^T \mathbb{M}[f|\widetilde{\mathfrak{F}}](b,x,t)^p \, dt\, d\nu(b)\, d\mu(x) \\
&\le \left(\frac{e^T-1}{T}\right)\|\psi\|_q^p\|\mathbb{M}[f|\widetilde{\mathfrak{F}}]\|_p^p \le C_p^p\left(\frac{e^T-1}{T}\right)\|\psi\|_q^p\|f\|_p^p.
\end{aligned}$$

So,

$$\|\mathbb{M}[f|\widetilde{\mathfrak{F}},\psi]\|_p \le C_p\left(\frac{e^T-1}{T}\right)^{1/p}\|\psi\|_q\|f\|_p.$$

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As to the $L \log L$ results, let us now suppose $f \in L \log L(B \times X \times I)$ and $\psi \in L^\infty(B)$. By Theorem 2.6, there is a constant $C_1 > 0$ (independent of f and the action $\Gamma \curvearrowright (X, \mu)$) such that $\|\mathbb{M}[f|\widetilde{\mathfrak{F}}]\|_1 \leq C_1 \|f\|_{L \log L}$. The proofs that

$$\|\overline{\mathbb{M}}[f|\widetilde{\mathfrak{F}}]\|_1 \leq C_1 \left(\frac{e^T - 1}{T}\right) \|f\|_{L \log L}$$

and

$$\|\mathbb{M}[f|\widetilde{\mathfrak{F}}, \psi]\|_1 \leq C_1 \left(\frac{e^T - 1}{T}\right) \|\psi\|_\infty \|f\|_{L \log L}$$

are similar to the proofs in the $p > 1$ case. □

4. *General ergodic theorems from III_1 actions*

Let $\Gamma \curvearrowright (B, \nu)$ be an action of a countable group on a standard probability space. We will assume the action is essentially free, amenable, weakly mixing and stable type III_λ for some $\lambda > 0$. From these assumptions and a choice of Følner sequence for a certain associated amenable equivalence relation, we will obtain in §4.3 and §5.2 a family of pointwise ergodic sequences for Γ.

Let us begin by explaining the terms mentioned above. *Essentially free* means that for a.e. $b \in B$, the stability group $\{g \in \Gamma : gx = x\}$ is trivial. By *amenable action* we mean amenability in the sense of Zimmer [**Zi78**]. *Weakly mixing* means that if $\Gamma \curvearrowright (X, \mu)$ is any ergodic PMP action, then the product action $\Gamma \curvearrowright (B \times X, \nu \times \mu)$ is ergodic. We now turn to defining the (new) notion of stable type.

4.1. *The stable ratio set.*

Let Γ be a countable group and (B, ν) a standard probability space on which Γ acts by non-singular transformations. The *ratio set* of the Radon–Nikodym cocycle is a set $RS(\Gamma, B, \nu) \subset [0, +\infty]$ defined as follows: a finite number $r \in RS(\Gamma, B, \nu)$ if and only if for every positive measure set $A \subset B$ and $\epsilon > 0$ there is a subset $A' \subset A$ of positive measure and an element $g \in \Gamma \backslash \{e\}$ such that:

(1) $gA' \subset A$; and

(2) $|((d\nu \circ g)/d\nu)(b) - r| < \epsilon$ for every $b \in A'$.

The extended real number $+\infty \in RS(\Gamma, B, \nu)$ if and only if for every positive measure set $A \subset B$ and $n > 0$, there is a subset $A' \subset A$ of positive measure and an element $g \in \Gamma \backslash \{e\}$ such that:

(1) $gA' \subset A$; and

(2) $((d\nu \circ g)/d\nu)(b) > n$ for every $b \in A'$.

The ratio set is also called the *asymptotic range* or *asymptotic ratio set*. By [**FM77**, Proposition 8.5], if the action $\Gamma \curvearrowright (B, \nu)$ is ergodic, then $RS(\Gamma, B, \nu)$ is a closed subset of $[0, \infty]$. Moreover, $RS(\Gamma, B, \nu) \backslash \{0, \infty\}$ is a multiplicative subgroup of $\mathbb{R}_{>0}$. In the special case in which $\Gamma \curvearrowright (B, \nu)$ is an amenable action and a.e. orbit is infinite, it is known through Krieger's work [**Kr70**] that there are four possibilities for $RS(\Gamma, B, \nu)$: either $RS(\Gamma, B, \nu) = \{1\}$, in which case the action is said to be type *II*; $RS(\Gamma, B, \nu) = \{0, 1, +\infty\}$, which is called type III_0; $RS(\Gamma, B, \nu) = \{0, \lambda^n, +\infty : n \in \mathbb{Z}\}$ for some $\lambda \in (0, 1)$, which is called type III_λ; $RS(\Gamma, B, \nu) = [0, +\infty]$, which is called type III_1.

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For a very readable review, see [**KW91**]. There is also an extension to general cocycles taking values in an arbitrary locally compact group in [**FM77**, §8].

Observe that if $\Gamma \curvearrowright (X, \mu)$ is a PMP action, then the product action $\Gamma \curvearrowright (B \times X, \nu \times \mu)$ has ratio set $RS(\Gamma, B \times X, \nu \times \mu) \subset RS(\Gamma, B, \nu)$. Therefore, it makes sense to define the *stable* ratio set of $\Gamma \curvearrowright (B, \nu)$ by $SRS(\Gamma, B, \nu) = \bigcap RS(\Gamma, B \times X, \nu \times \mu)$, where the intersection is over all PMP actions $G \curvearrowright (X, \mu)$. If $\Gamma \curvearrowright (B, \nu)$ is weakly mixing, then $SRS(\Gamma, B, \nu)$ is a closed subset of $[0, \infty]$ and $SRS(\Gamma, B, \nu)\backslash\{0, \infty\}$ is a multiplicative subgroup of $\mathbb{R}_{>0}$.

In the case we are most interested in, $\Gamma \curvearrowright (B, \nu)$ is a weakly mixing amenable action in which a.e. orbit is infinite. Therefore, $\Gamma \curvearrowright (B \times X, \nu \times \mu)$ is amenable and ergodic and there are only four possibilities for $SRS(\Gamma, B, \nu)$: either $SRS(\Gamma, B, \nu) = \{1\}$, in which case we say that $\Gamma \curvearrowright (B, \nu)$ is stable type *II*; $SRS(\Gamma, B, \nu) = \{0, 1, +\infty\}$, which is called stable type III_0; $SRS(\Gamma, B, \nu) = \{0, \lambda^n, +\infty : n \in \mathbb{Z}\}$ for some $\lambda \in (0, 1)$, which is stable type III_λ; $SRS(\Gamma, B, \nu) = [0, +\infty]$, which is called stable type III_1.

4.2. *The Maharam extension.* Suppose $\Gamma \curvearrowright (H, \eta)$ is a non-singular ergodic action on a standard probability space. The group Γ acts on $H \times \mathbb{R}$ by

$$g(h, t) := \left(gh, t - \log\left(\frac{d\eta \circ g}{d\eta}(h)\right)\right). \tag{4.1}$$

Let θ be the measure on $\mathbb{R}$ given by $d\theta(t) = e^t dt$. The action above preserves the product measure $\eta \times \theta$. This construction is called the *Maharam extension* [**Aa97, Ma64**].

The group of real numbers acts on $H \times \mathbb{R}$ by $\phi_t(h, t') := (h, t' + t)$ for $h \in H$, $t, t' \in \mathbb{R}$. This action commutes with the action of Γ and therefore descends to an $\mathbb{R}$-action on the space of ergodic components of $\eta \times \theta$. This action is called the *Mackey range* of the Radon–Nikodym cocycle [**Ma66**]. It has also been called the *Poincaré flow* [**FM77**] and the *Radon–Nikodym flow* [**Mo08**].

LEMMA 4.1. *Suppose $\Gamma \curvearrowright (H, \eta)$ is ergodic, amenable, essentially free and type III_λ for some $0 < \lambda < 1$. Then there is a probability measure η' on H which is equivalent to η such that, for a.e. $h \in H$ and every $g \in \Gamma$,*

$$\frac{d\eta' \circ g}{d\eta'}(h) \in \{\lambda^n : n \in \mathbb{Z}\}.$$

Proof. Because $\Gamma \curvearrowright (H, \eta)$ is ergodic, amenable and essentially free, this action is orbit equivalent to an action of $\mathbb{Z}$ (see [**CFW81**]). Proposition 2.2 of [**KW91**] now implies the result. □

LEMMA 4.2. *Suppose $\Gamma \curvearrowright (H, \eta)$ is ergodic, essentially free and type III_λ for some $\lambda > 0$. If $\lambda \neq 1$, then let $T = -\log(\lambda)$. If $\lambda = 1$, then let $T > 0$ be arbitrary. Then for every bounded Borel Γ-invariant function f on $H \times \mathbb{R}$, $f \circ \phi_T = f$ almost everywhere.*

Proof. This lemma follows from [**FM77**, Proposition 8.3 and Theorem 8]. To be precise, the cocycle c appearing in [**FM77**] is, for us, the logarithmic Radon–Nikodym cocycle on the Γ-orbit equivalence relation $\mathcal{R} := \{(h, gh) : h \in H, g \in \Gamma\}$ on H. So, $c : \mathcal{R} \to \mathbb{R}$,

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$c(h, h') = \log((dv \circ g)/dv)(h)$, where $g \in \Gamma$ is an element such that $gh = h'$. This element is unique for a.e. $h \in H$ because $\Gamma \curvearrowright (H, \eta)$ is essentially free. Then the asymptotic range $r_*(c)$ is, by definition, $\log(RS(\Gamma, H, \eta) \cap (0, \infty))$ and the normalized proper range $npr(c)$ is the set of all positive real numbers T such that for any Γ-invariant $f \in L^\infty(H \times \mathbb{R})$, $f(h, t) = f(h, t + T)$ for a.e. (h, t) (by [**FM77**, Proposition 8.3]). By [**FM77**, Theorem 8], $npr(c) = r_*(c)$. □

COROLLARY 4.3. *Suppose $\Gamma \curvearrowright (H, \eta)$ is ergodic, essentially free and type III_1. Then $\Gamma \curvearrowright (H \times \mathbb{R}, \eta \times \theta)$ is ergodic.*

Proof. Let $\mathcal{I}$ be the σ-algebra of Borel subsets of $H \times \mathbb{R}$ which are invariant under the Γ-action and the flow $\{\phi_t\}_{t\in\mathbb{R}}$. We claim that $\mathcal{I}$ is trivial (i.e., every set $A \in \mathcal{I}$ satisfies $\eta \times \theta(A) = 0$ or $\eta \times \theta(A^c) = 0$, where A^c denotes the complement of A). Indeed, if $A \in \mathcal{I}$ then, since A is invariant under the flow $\{\phi_t\}_{t\in\mathbb{R}}$, $A = A_0 \times \mathbb{R}$ for some Borel set $A_0 \subset H$ (up to measure zero). Since $\Gamma \curvearrowright (H, \eta)$ is ergodic, $\eta(A_0) \in \{0, 1\}$, which implies the claim.

Lemma 4.2 implies that any bounded Borel Γ-invariant function f on $H \times \mathbb{R}$ is invariant under the flow. By the claim above, this implies f is constant almost everywhere. Therefore, $\Gamma \curvearrowright (H \times \mathbb{R}, \eta \times \theta)$ is ergodic. □

4.3. *Random and non-random pointwise ergodic theorems.* Let (B, ν) be a standard probability space and $\{\zeta_r\}_{r\in\mathbb{I}}$ a family of maps $\zeta_r : B \times \Gamma \to [0, 1]$ satisfying

$$\sum_{\gamma\in\Gamma} \zeta_r(b, \gamma) = 1 \quad \text{for a.e. } b \in B.$$

We say $\{\zeta_r\}_{r\in\mathbb{I}}$ is a *random pointwise ergodic family in L^p* if for every PMP action $\Gamma \curvearrowright (X, \mu)$, every $f \in L^p(X, \mu)$ and a.e. $(b, x) \in B \times X$,

$$\lim_{r\to\infty} \sum_{\gamma\in\Gamma} \zeta_r(b, \gamma) f(\gamma x) = \mathbb{E}[f|\mathcal{I}](x),$$

where $\mathcal{I}$ is the σ-algebra of Γ-invariant Borel sets in X.

THEOREM 4.1. *Let $\Gamma \curvearrowright (B, \nu)$ be an action of a countable group on a standard probability space. We assume the action is essentially free, weakly mixing and stable type III_1. Let $\Gamma \curvearrowright (B \times \mathbb{R}, \nu \times \theta)$ be the Maharam extension. Let $T > 0$ be arbitrary, $I = [0, T]$, and θ_I be the probability measure on $[0, T]$ given by $d\theta_I(t) = (e^t/(e^T - 1))dt$. Let $\mathcal{R}_I$ be the equivalence relation on $B \times I$ given by restricting the orbit equivalence relation on $B \times \mathbb{R}$ (so $\mathcal{R}_I$ consists of all $((b, t), g(b, t))$ with $g \in \Gamma$ and $(b, t), g(b, t) \in B \times I$).*

Let $\mathfrak{F} = \{\mathfrak{F}_r\}_{r\in\mathbb{I}}$ be a Borel family of subset functions for $(B \times I, \nu \times \theta_I, \mathcal{R}_I)$. Suppose $\mathfrak{F}$ is either (asymptotically invariant and regular) or (asymptotically invariant, uniform and tempered). Define $\zeta_r : B \times I \times \Gamma \to [0, 1]$ by

$$\zeta_r(b, t, \gamma) := \frac{1}{|\mathfrak{F}_r(b, t)|} 1_{\mathfrak{F}_r(b,t)}(\gamma(b, t)).$$

Then $\{\zeta_r\}_{r\in\mathbb{I}}$ is a random pointwise ergodic family for Γ in L^1.

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Proof. Let $\Gamma \curvearrowright (X, \mu)$ be a PMP action and $f \in L^1(X) \subset L^1(B \times X \times I)$. Then, for any (b, x, t),

$$\sum_{\gamma \in \Gamma} \zeta_r(b, t, \gamma) f(\gamma x) = \mathbb{A}[f|\widetilde{\mathfrak{F}}_r](b, x, t),$$

where $\widetilde{\mathfrak{F}}$ is the lift of the subset function $\mathfrak{F}$ from $B \times I$ to $B \times X \times I$.

Without loss of generality, we may assume $\Gamma \curvearrowright (X, \mu)$ is ergodic. Because $\Gamma \curvearrowright (B, \nu)$ is stable type III_1, Corollary 4.3 implies that the equivalence relation $(B \times X \times I, \nu \times \mu \times \theta_I, \mathcal{R}_I)$ is ergodic. By Theorem 2.5, when $f \in L^1(X)$,

$$\lim_{r \to \infty} \mathbb{A}[f|\widetilde{\mathfrak{F}}](b, x, t) = \int_B \int_X \int_I f \, d\nu \, d\mu \, d\theta_I = \int_X f \, d\mu.$$

This proves the result. □

We now turn to proving a (non-random) pointwise ergodic theorem for arbitrary ergodic PMP actions of Γ, with respect to a fixed sequence of probability measures supported on Γ. Here, we establish convergence for functions in L^p, $p > 1$, as well as functions in $L \log L$, but not for all functions in L^1.

THEOREM 4.2. *Let $\Gamma \curvearrowright (B, \nu)$ be an action of a countable group on a standard probability space. We assume the action is essentially free, weakly mixing and stable type III_1. Let $\Gamma \curvearrowright (B \times \mathbb{R}, \nu \times \theta)$ be the Maharam extension. Let $T > 0$ be arbitrary, $I = [0, T]$, and θ_I be the probability measure on $[0, T]$ given by $d\theta_I(t) = (e^t/(e^T - 1)) \, dt$. Let $\mathcal{R}_I$ be the equivalence relation on $B \times I$ given by restricting the orbit equivalence relation on $B \times \mathbb{R}$ (so $\mathcal{R}_I$ consists of all $((b, t), g(b, t))$ with $g \in \Gamma$ and $(b, t), g(b, t) \in B \times I$).*

Let $\mathfrak{F} = \{\mathfrak{F}_r\}_{r \in \mathbb{I}}$ be a Borel family of subset functions for $(B \times I, \nu \times \theta_I, \mathcal{R}_I)$. Suppose $\mathfrak{F}$ is either (asymptotically invariant and regular) or (asymptotically invariant, uniform and tempered). Define $\zeta_r : B \times \Gamma \to [0, 1]$ by

$$\zeta_r(b, \gamma) := \frac{1}{T} \int_0^T \frac{1}{|\mathfrak{F}_r(b, t)|} 1_{\mathfrak{F}_r(b,t)}(\gamma(b, t)) \, dt.$$

Then $\{\zeta_r\}_{r \in \mathbb{I}}$ is a random pointwise ergodic family for Γ in L^p for every $p > 1$ and in $L \log L$.

If $\psi \in L^q(B)$ is a probability density function (so $\psi \geq 0$ and $\int \psi \, d\nu = 1$) and $\zeta_r^\psi : \Gamma \to [0, 1]$ is defined by $\zeta_r^\psi(\gamma) = \int \zeta_r(b, \gamma)\psi(b) \, d\nu(b)$, then $\{\zeta_r^\psi\}_{r \in \mathbb{I}}$ is a pointwise ergodic family in L^p for every $p > 1$ with $1/p + 1/q \leq 1$. If $\psi \in L^\infty$, then $\{\zeta_r^\psi\}_{r \in \mathbb{I}}$ is a pointwise ergodic family in $L \log L$.

Proof of Theorem 4.2. Without loss of generality, we may assume $\Gamma \curvearrowright (X, \mu)$ is ergodic. The Maharam extension of the product action $\Gamma \curvearrowright (B \times X, \nu \times \mu)$ is

$$\Gamma \curvearrowright (B \times X \times \mathbb{R}, \nu \times \mu \times \theta) \simeq \Gamma \curvearrowright (B \times \mathbb{R}, \nu \times \theta) \times (X, \mu).$$

Suppose now that f depends only on its x-argument (so $f(b, x, t) = f(x)$). Then, for

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any (b, x),

$$\sum_{\gamma\in\Gamma}\zeta_r(b,\gamma)f(\gamma x)=\sum_{\gamma\in\Gamma}\frac{1}{T}\int_0^T\frac{1}{|\mathfrak{F}_r(b,t)|}1_{\mathfrak{F}_r(b,t)}(\gamma(b,t))f(\gamma x)\,dt$$
$$=\frac{1}{T}\int_0^T\mathbb{A}[f|\widetilde{\mathfrak{F}}_r](b,x,t)\,dt.$$

Similarly,

$$\sum_{\gamma\in\Gamma}\zeta_r^{\psi}(\gamma)f(\gamma x)=\frac{1}{T}\iint_0^T\mathbb{A}[f|\widetilde{\mathfrak{F}}_r](b,x,t)\psi(b)\,dt\,d\nu(b).$$

If $f\in L^\infty(X)$, then the bounded convergence theorem implies that for a.e. $(b, x)\in B\times X$,

$$\lim_{r\to\infty}\sum_{\gamma\in\Gamma}\zeta_r(b,\gamma)f(\gamma x)=\lim_{r\to\infty}\frac{1}{T}\int_0^T\mathbb{A}[f|\widetilde{\mathfrak{F}}_r](b,x,t)\,dt$$
$$=\frac{1}{T}\int_0^T\lim_{r\to\infty}\mathbb{A}[f|\widetilde{\mathfrak{F}}_r](b,x,t)\,dt$$
$$=\frac{1}{T}\int_0^T\mathbb{E}[f|\mathcal{I}(\widetilde{\mathcal{R}}_I)](b,x,t)\,dt=\int f\,d\mu(x).$$

Above, $\widetilde{\mathcal{R}}_I$ denotes the orbit-equivalence relation of the action $\Gamma\curvearrowright B\times X\times\mathbb{R}$ restricted to $B\times X\times I$ and $\mathcal{I}(\widetilde{\mathcal{R}}_I)$ is the σ-algebra of $\widetilde{\mathcal{R}}_I$-invariant measurable sets. This proves $\{\zeta_r\}_{r\in\mathbb{I}}$ is a random pointwise ergodic sequence in L^∞. Also, for a.e. $x\in X$,

$$\lim_{r\to\infty}\sum_{\gamma\in\Gamma}\zeta_r^{\psi}(\gamma)f(\gamma x)=\lim_{r\to\infty}\frac{1}{T}\iint_0^T\mathbb{A}[f|\widetilde{\mathfrak{F}}_r](b,x,t)\psi(b)\,dt\,d\nu(b)$$
$$=\frac{1}{T}\iint_0^T\lim_{r\to\infty}\mathbb{A}[f|\widetilde{\mathfrak{F}}_r](b,x,t)\psi(b)\,dt\,d\nu(b)$$
$$=\frac{1}{T}\iint_0^T\mathbb{E}[f|\mathcal{I}(\widetilde{\mathcal{R}}_I)](b,x,t)\psi(b)\,dt\,d\nu(b)$$
$$=\int f\,d\mu(x).$$

This proves $\{\zeta_r^{\psi}\}_{r\in\mathbb{I}}$ is a pointwise ergodic sequence in L^∞.

Suppose now that $f\in L^p(X)\subset L^p(B\times X\times I)$ for some $p>1$. We will show that for a.e. (b, x), $\lim_{r\to\infty}\sum_{\gamma\in\Gamma}\zeta_r(b,\gamma)f(\gamma x)=\int f\,d\mu$. By replacing f with $f-\int f\,d\mu$ if necessary, we may assume $\int f\,d\mu=0$.

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Let $\epsilon > 0$. Because $L^\infty(X)$ is dense in $L^p(X)$, there exists an element $f' \in L^\infty(X)$ such that $\|f - f'\|_p \leq \epsilon$ and $\int f'\, d\mu = 0$. So, for a.e. $(b, x) \in B \times X$,

$$\begin{aligned}\limsup_{r\to\infty}\left|\sum_{\gamma\in\Gamma}\zeta_r(b,\gamma)f(\gamma x)\right| &\leq \limsup_{r\to\infty}\left|\sum_{\gamma\in\Gamma}\zeta_r(b,\gamma)[f(\gamma x)-f'(\gamma x)]\right| \\ &\quad + \left|\lim_{r\to\infty}\sum_{\gamma\in\Gamma}\zeta_r(b,\gamma)f'(\gamma x)\right| \\ &= \limsup_{r\to\infty}\left|\sum_{\gamma\in\Gamma}\zeta_r(b,\gamma)[f(\gamma x)-f'(\gamma x)]\right| \\ &= \limsup_{r\to\infty}\left|\frac{1}{T}\int_0^T \mathbb{A}[f-f'|\mathfrak{F}_r](b,x,t)\,dt\right| \\ &\leq \overline{\mathbb{M}}[f-f'|\mathfrak{F}](b,x).\end{aligned}$$

Thus, if

$$F(b,x) := \limsup_{r\to\infty}\left|\sum_{\gamma\in\Gamma}\zeta_r(b,\gamma)f(\gamma x)\right|,$$

then

$$\|F\|_p \leq \|\overline{\mathbb{M}}[f-f'|\mathfrak{F}]\|_p \leq C'_p\|f-f'\|_p \leq C'_p\epsilon$$

for some constant $C'_p > 0$ (which is independent of f and f') by Theorem 3.1. Since ϵ is arbitrary, $\|F\|_p = 0$, which implies

$$\lim_{r\to\infty}\sum_{\gamma\in\Gamma}\zeta_r(b,\gamma)f(\gamma x) = 0$$

for a.e. (b, x), as required. This proves $\{\zeta_r\}_{r\in\mathbb{I}}$ is a random pointwise ergodic sequence in L^p for every $p > 1$.

Now suppose $p > 1$ and $1/p + 1/q = 1$. Let f, f' be as above. Then, for a.e. $x \in X$,

$$\begin{aligned}\limsup_{r\to\infty}\left|\sum_{\gamma\in\Gamma}\zeta_r^\psi(\gamma)f(\gamma x)\right| &\leq \limsup_{r\to\infty}\left|\sum_{\gamma\in\Gamma}\zeta_r^\psi(\gamma)[f(\gamma x)-f'(\gamma x)]\right| \\ &\quad + \left|\lim_{r\to\infty}\sum_{\gamma\in\Gamma}\zeta_r^\psi(\gamma)f'(\gamma x)\right| \\ &= \limsup_{r\to\infty}\left|\sum_{\gamma\in\Gamma}\zeta_r^\psi(\gamma)[f(\gamma x)-f'(\gamma x)]\right| \\ &= \limsup_{r\to\infty}\left|\frac{1}{T}\iint_0^T \mathbb{A}[f-f'|\mathfrak{F}_r](b,x,t)\psi(b)\,dt\,d\nu(b)\right| \\ &\leq \mathbb{M}[f-f'|\mathfrak{F},\psi](x).\end{aligned}$$

Thus, if $F(x) := \limsup_{r\to\infty}|\sum_{\gamma\in\Gamma}\zeta_r^\psi(\gamma)f(\gamma x)|$, then

$$\|F\|_p \leq \|\mathbb{M}[f-f'|\mathfrak{F},\psi]\|_p \leq C''_p\|f-f'\|_p \leq C''_p\epsilon$$

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for some constant $C_p'' > 0$ (which is independent of f and f' but may depend on ψ) by Theorem 3.1. Since ϵ is arbitrary, $\|F\|_p = 0$, which implies

$$\lim_{r\to\infty} \sum_{\gamma\in\Gamma} \zeta_r^{\psi}(\gamma) f(\gamma x) = 0$$

for a.e. x, as required. This proves $\{\zeta_r^{\psi}\}_{r\in\mathbb{I}}$ is a pointwise ergodic sequence in L^p for every $p > 1$.

The last two cases to be handled occur when $f \in L\log L(X)$ and $\psi \in L^\infty(B)$. The proofs of these cases are similar to the proofs above. □

Remark 4.4. (The type II_1 case) Theorem 4.2 applies, in particular, to any *amenable* group which admits a free weakly mixing action of stable type III_λ for $\lambda > 0$, for example, when a non-trivial Poisson boundary with these properties exists. However, when G is an amenable group, we can also use actions of type II_1 to produce pointwise ergodic sequences on G. Indeed, consider a weakly mixing measure-preserving action on a probability space (B, ν). This action is of course amenable, and *any* (uniform tempered, or regular) Følner sequence for the orbit equivalence relation of B induces a random pointwise ergodic sequence in L^1 for the G-action on X. By averaging a probability distribution ψ on B, we also obtain a pointwise ergodic sequence on G for its action on X. The proof is straightforward using the arguments in the proof of Theorem 4.2.

Remark 4.5. (Convergence and identification of the limit) Let B be any free amenable action of a countable group Γ, not necessarily of stable type III_λ or weakly mixing. Thus, for any (uniform tempered, or regular) Følner sequence on the Γ-orbit equivalence relation restricted to $B \times [0, T]$ (for any $T > 0$), we obtain a random sequence ζ_r of averages which converge pointwise almost surely and in L^1-norm, the limit being the conditional expectation on the σ-algebra of relation-invariant sets by Theorem 2.1. Averaging them further with respect to a probability density ψ on $B \times [0, T]$, we obtain averaging sequences ζ_r^{ψ} on Γ which converge pointwise almost surely. Thus, amenability of B suffices to obtain convergence almost surely, but may not be sufficient to identify the limit of $\zeta_r^{\psi}(f)$ as the ergodic mean. Our arguments establishing this fact in Theorem 4.2 depend crucially on weak-mixing and stable type III_1. It is interesting to note that in [**BKK11**] the authors prove pointwise convergence of uniform averages of spherical measures on Markov groups, but they do not identify the limit function.

4.4. *Lattices and actions of stable type III_1.* Summarizing our results thus far, Theorem 4.2 provides the following recipe for proving pointwise ergodic theorems for an arbitrary group Γ. First, find an essentially free, weakly mixing, amenable action $\Gamma \curvearrowright (B, \nu)$ of stable type III_1. Let $T > 0$, and choose a Følner family on $(B \times I, \nu \times \theta_I, \mathcal{R}(B \times I))$ which is uniform and tempered (or just regular). Such a family always exists by amenability, as noted in Proposition 2.1. Finally, choose a probability density ψ on B. From these objects, a pointwise ergodic sequence is constructed. The maximal inequalities for the associated averages hold more generally (they do not depend on the stable type or the weak mixing hypothesis, as shown in §3.3).

There are several choices in this construction: the action $\Gamma \curvearrowright (B, \nu)$, the Følner family $\mathfrak{F}$, and the probability density ψ. It is an interesting problem to determine

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whether a given family of probability measures $\{\mu_r\}_{r>0}$ on Γ arises from one of these constructions. For example, suppose Γ acts cocompactly by isometries on a negatively curved manifold (M, d) with a basepoint x_0 and β_r is the uniform probability measure on $\{g \in \Gamma : d(gx_0, x_0) < r\}$. Then is β_r a pointwise ergodic family? Does it arise from one of these constructions? In [**BN1**], the authors used an explicit particular instance of the present construction to prove that spherical averages form a pointwise ergodic sequence for non-abelian free groups (up to a certain well-known periodicity phenomenon).

The importance of the action $\Gamma \curvearrowright (B, \nu)$ leads to the following question.

Question 4.6. Does every discrete group have an essentially free, weakly mixing, amenable action of stable type III_λ for some $\lambda \in (0, 1]$?

The requirement that the action be essentially free can be removed by the following device. Let u be the uniform measure on $\{0, 1\}$. Γ acts on the product space $(\{0, 1\}^\Gamma, u^\Gamma)$ by $g \cdot x(f) = x(g^{-1}f)$ for all $x \in \{0, 1\}^\Gamma$, $g, f \in \Gamma$. This is a *Bernoulli shift* action. If $\Gamma \curvearrowright (B, \nu)$ is any action, then the product action $\Gamma \curvearrowright (B \times \{0, 1\}^\Gamma, \nu \times u^\Gamma)$ is essentially free. Moreover, if (B, ν) has any one of the properties {weakly mixing, amenable, stable type III_λ}, then this product action has the same property.

The action of a group on any of its Poisson boundaries is amenable [**Zi78**] and weakly mixing [**AL05**] (indeed, these actions are doubly ergodic with coefficients in Hilbert spaces by [**Ka03**]). If Γ is non-amenable, then these actions are necessarily of type III_λ for some $\lambda \in [0, 1]$. It may well be the case that the type of the action on a Poisson boundary is *never* III_0, but this problem is still open.

We are unaware of any previous study of the *stable* type of an amenable action. However, there are results on the types of boundary actions. For example, in [**INO08**], it is proven that the Poisson boundary of a random walk on a Gromov hyperbolic group induced by a non-degenerate measure on Γ of finite support is never of type III_0. In [**Su78, Su82**], Sullivan proved that the recurrent part of an action of a discrete conformal group on the sphere $\mathbb{S}^d$ relative to the Lebesgue measure is type III_1. Spatzier [**Sp87**] showed that if Γ is the fundamental group of a compact connected negatively curved manifold, then the action of Γ on the sphere at infinity of the universal cover is also of III_1 type. The types of harmonic measures on free groups were computed by Ramagge and Robertson [**RR97**] and Okayasu [**Ok03**].

An important class of discrete groups for which the type of the boundary action is known is that of irreducible lattices in connected semi-simple Lie groups with finite center and no compact factors. Let G be such a group and $\Gamma \subset G$ an irreducible lattice subgroup. The maximal boundary $B = G/P$, where P is a minimal parabolic subgroup, carries a unique G-quasi-invariant measure class, denoted ν. As to the stable type, we have the following proposition.

PROPOSITION 4.7. *The action of* Γ *on* $(G/P, \nu)$ *is amenable, weakly mixing and essentially free, and of stable type* III_1.

Proof. Recall the duality principle for ergodicity on homogeneous spaces [**Mo66**]: if G is an LCSC group, and H_1, H_2 are two closed subgroups, then H_1 is ergodic on G/H_2 if and only if H_2 is ergodic on G/H_1, if and only if G is ergodic on $G/H_1 \times G/H_2$. The measure

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classes taken on G/H_1 and G/H_2 are the unique G-invariant ones, and on $G/H_1 \times G/H_2$ we take their product. A further aspect of the duality principle for homogeneous spaces is that G/H_2 is an amenable H_1-space if H_2 is an amenable subgroup [**Zi84**, Corollary 4.3.7].

The fact that the action of Γ on G/P is amenable and ergodic therefore follows from the fact that the minimal parabolic subgroup P is amenable and ergodic on G/Γ. Here, we take the G-quasi-invariant measure class ν on G/P. Let $P = MAN$ be the Levi decomposition of P. Then up to ν-measure zero $G/P \times G/P \cong G/A$, and since A is ergodic on G/Γ by the Howe–Moore ergodicity theorem, Γ is ergodic on $G/P \times G/P$, i.e., Γ is doubly ergodic. Similarly, Γ is doubly ergodic on the product with coefficients in Hilbert spaces and, in particular, the action of Γ on G/P is weakly mixing. It is well known that the Γ-action is also essentially free.

We now show that the type of the action is III_1, and then that the stable type is also III_1. First, note that the Maharam extension of the G-action on G/P, namely the action on $G/P \times \mathbb{R}$ given by $g(hP, t) = (ghP, t - \log r_\nu(g, hP))$, is a transitive G-action. Indeed, the stability group of $(P, 1)$ is the kernel of the modular homomorphism $\delta : P \to \mathbb{R}_+^*$, which we denote by L. Now, $r_\nu(p, P) = \delta(p)$ and the modular homomorphism is clearly surjective, so the well-defined map $G/L \to G/P \times \mathbb{R}$ given by $gL \mapsto (gP, \log r_\nu(g, P))$ is a G-equivariant isomorphism. In particular, G is ergodic on the Maharam extension $G/P \times \mathbb{R}$, but then so is the restriction of the G-action to Γ by [**Zi77**, Theorem 5.4]. Hence, the Mackey range of the Radon–Nikodym derivative cocycle of the Γ-action on G/P is the action of $\mathbb{R}$ on a point and the type of the Γ-action on the boundary is III_1.

Consider now the action of Γ on $(G/P \times X, \nu \times \mu)$, where (X, μ) is an ergodic Γ-action. In general, for any cocycle $\beta : \Gamma \times Y \to H$ defined on a Γ-space Y, the Mackey range of the cocycle β coincides with the Mackey range of the cocycle $\tilde{\beta}$, defined for the G-action on the induced space $\mathrm{Ind}_\Gamma^G(Y) = G/\Gamma \times_\alpha Y$ by $\tilde{\beta}(g, u\Gamma, y) = \beta(\alpha(g, u\Gamma), y)$. Here, $\alpha : G \times G/\Gamma \to \Gamma$ is a cocycle associated with a section $\tau : G/\Gamma \to G$ with $\tau(\Gamma) = e$, and the notation $\times_\alpha$ denotes that the action on the second component is via the cocycle α, namely $g(u\Gamma, y) = (gu\Gamma, \alpha(g, u\Gamma)y)$.

For a Γ-space X, consider the G-action $\mathrm{Ind}_\Gamma^G(G/P \times X)$ induced by the Γ-action on $G/P \times X$. Note that the induced action is equivariantly isomorphic to the product G-action on G/P and $\mathrm{Ind}_\Gamma^G(X)$:

$$G/\Gamma \times_\alpha (G/P \times X) = \mathrm{Ind}_\Gamma^G(G/P \times X) \cong G/P \times (\mathrm{Ind}_\Gamma^G X) = G/P \times (G/\Gamma \times_\alpha X).$$

This follows from the well-known fact that the action $G/\Gamma \times_\alpha G/P$ of G induced by the Γ-action on G/P is isomorphic to the product G-action on $G/\Gamma \times G/P$.

If (X, μ) is a *measure-preserving* probability Γ-space, the Mackey range of the Radon–Nikodym cocycle $r_{\nu\times\mu}$ on $G/P \times X$ coincides with the Mackey range of the Radon–Nikodym cocycle of the G-action on the induced space $\mathrm{Ind}_\Gamma^G(G/P \times X)$. Indeed, the latter coincides with $\tilde{r}_{\nu\times\mu}$, since the extension $G/P \times (G/\Gamma \times_\alpha X) \to G/P$ is a measure-preserving extension.

To find the Mackey range of the Radon–Nikodym cocycle in question, consider the Maharam extension $G/P \times (\mathrm{Ind}_\Gamma^G X) \times \mathbb{R}$ of the product action. The Maharam extension is clearly G-isomorphic to the product G-action on $G/L \times \mathrm{Ind}_\Gamma^G X$, since the Maharam extension $G/P \times \mathbb{R}$ of G/P is the G-action on G/L, as noted above. Now, $\mathrm{Ind}_\Gamma^G X$ is an

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ergodic PMP G-action, and its restriction to L is still ergodic. Indeed, while the G-action on $G/\Gamma \times_\alpha X$ may be a reducible action, the unipotent radical N of P acts ergodically in any ergodic G-space. This follows from the Mautner phenomenon: if G_1 is simple and non-compact, then any L^2-function invariant under the unipotent radical N_1 of a minimal parabolic subgroup P_1 of G_1 is in fact G_1-invariant. Hence, if $G = \prod_{i=1}^N G_i$ is a product of simple non-compact groups, $N = \prod_{i=1}^N N_i$ is ergodic in any ergodic G space, and hence so is the larger subgroup L.

It follows that the action of G on $G/L \times \mathrm{Ind}_\Gamma^G X$ is also ergodic. Thus, G is ergodic on the Maharam extension, and the Mackey range of the G-action is the $\mathbb{R}$-action on a point. By the foregoing arguments, this is also the Mackey range of the Radon–Nikodym cocycle of the action of Γ on $G/P \times X$, and thus the stable type is III_1. □

5. *General ergodic theorems from III_λ actions*

The purpose of this section is to obtain general ergodic theorems as in the previous section, but under a different set of hypotheses. The main difference is that we assume throughout that the Radon–Nikodym derivatives for the action $\Gamma \curvearrowright (B, \nu)$ take values in a discrete group. More precisely, we assume there is some $\lambda \in (0, 1)$ such that if

$$R_\lambda(g, b) := \log_\lambda\left(\frac{d\nu \circ g}{d\nu}(b)\right), \tag{5.1}$$

then $R_\lambda(g, b) \in \mathbb{Z}$ for every $g \in \Gamma$ and a.e. $b \in B$. Let $\Gamma \curvearrowright B \times \mathbb{Z}$ by

$$g(b, n) = (gb, n + R_\lambda(g, b)).$$

This action, called the *discrete Maharam extension*, preserves the product measure $\nu \times \theta_\lambda$, where $\theta_\lambda(\{n\}) = \lambda^{-n}$. Given an integer $N \geq 0$, let $I = \{0, \ldots, N-1\}$ and $\mathcal{R}_I$ be the equivalence relation on $B \times I$ given by restricting the orbit-equivalence relation of $\Gamma \curvearrowright B \times \mathbb{Z}$. Let $\theta_{\lambda,I}$ be the probability measure on I given by $\theta_{\lambda,I}(\{n\}) = \lambda^{-n}/(1 + \cdots + \lambda^{-N+1})$.

Suppose also that $\Gamma \curvearrowright (X, \mu)$ is a PMP action. Let $\Gamma \curvearrowright B \times X \times \mathbb{Z}$ by

$$g(b, x, n) = (gb, gx, n + R_\lambda(g, b)).$$

This action preserves the product measure $\nu \times \mu \times \theta_\lambda$. Let $\widetilde{\mathcal{R}}_I$ be the equivalence relation on $B \times X \times I$ obtained by restricting the orbit-equivalence relation of the action $\Gamma \curvearrowright B \times X \times \mathbb{Z}$.

Our first step is to prove some maximal inequalities.

THEOREM 5.1. *Let $\mathfrak{F} = \{\mathfrak{F}_r\}_{r\in\mathbb{I}}$ be a Borel family of subset functions for $(B \times I, \nu \times \theta_{\lambda,I}, \mathcal{R}_I)$. Suppose $\mathfrak{F}$ is either regular or (asymptotically invariant, uniform and tempered). We assume $\Gamma \curvearrowright (B, \nu)$ is essentially free. Let $\pi : B \times X \times I \to B \times I$ be the projection map $\pi(b, x, t) = (b, t)$ and let $\widetilde{\mathfrak{F}} = \{\widetilde{\mathfrak{F}}_r\}_{r\in\mathbb{I}}$ be the lift of $\mathfrak{F}$:*

$$\widetilde{\mathfrak{F}}_r(x) := \pi^{-1}(\mathfrak{F}_r(\pi(b, x, t))) \cap [b, x, t] \quad \textit{for all } (b, x, t) \in B \times X \times I,$$

where $[b, x, t]$ denotes the $\widetilde{\mathcal{R}}_I$-equivalence class of (b, x, t). Let $\psi \in L^1(B, \nu)$ be a probability density (i.e., $\psi \geq 0$ and $\int \psi \, d\nu = 1$). For $f \in L^1(B \times X \times I, \nu \times \mu \times \theta_{\lambda,I})$

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and $(b, x, t) \in B \times X \times I$, *define*

$$\mathbb{M}[f|\widetilde{\mathfrak{F}}, \psi](x) := \sup_{r \in \mathbb{I}} \int \frac{1}{N+1} \sum_{t=0}^{N} \mathbb{A}[|f||\widetilde{\mathfrak{F}}_r](b, x, t)\psi(b)\, d\nu(b).$$

Then:

(1) *there exist constants* C_p *for* $p > 1$ *such that for every* $f \in L^p(B \times X \times I)$, *if* $1/p + 1/q = 1$, *then* $\|\mathbb{M}[f|\widetilde{\mathfrak{F}}, \psi]\|_p \le C_p \|\psi\|_q \|f\|_p$; *and*

(2) *there is also a constant* $C_1 > 0$ *such that if* $f \in L \log L(B \times X \times I)$ *and* $\psi \in L^\infty(B)$, *then* $\|\mathbb{M}[f|\widetilde{\mathfrak{F}}, \psi]\|_1 \le C_1 \|\psi\|_\infty \|f\|_{L \log L}$.

The constants C_p, *for* $p \ge 1$, *do not depend on* f *or the action* $\Gamma \curvearrowright (X, \mu)$, *but they may depend on* p *and* N.

Proof. The proof is analogous to the proof of Theorem 3.1. We leave the details to the reader. □

5.1. *Ergodic decomposition.* We let $\Gamma \curvearrowright (B, \nu)$, $R_\lambda(g, b)$, $\lambda > 0$, etc. be as in the previous subsection. The main result of this section is Corollary 5.4, which provides a formula for a certain average of conditional expectation operators. We also obtain an explicit description of the ergodic decomposition for the Maharam-extension action $\Gamma \curvearrowright (B \times X \times \mathbb{Z}, \nu \times \mu \times \theta_\lambda)$.

LEMMA 5.1. *Let* (W, ω) *be a standard probability space,* $\Gamma \curvearrowright (W, \omega)$ *an ergodic action preserving the measure-class. Let* $\alpha : \Gamma \times W \to \mathbb{Z}$ *be a cocycle for the action. Assume that for a.e.* $x \in W$ *and* $n \in \mathbb{Z}$ *there is a* $g \in \Gamma$ *with* $\alpha(g, x) = n$. *Let* $N > 0$ *be an integer and* $\mathcal{R}' = \{(x, y) \in W : \exists g \in \Gamma,\ gx = y,\ \alpha(g, x) \equiv 0 \mod N\}$. *Then there exists a positive integer* k *such that* $k|N$ *and a partition* $\{H_i\}_{i=0}^{k-1}$ *of* W *such that:*

(1) *each* H_i *has positive measure and is* $\mathcal{R}'$*-saturated;*

(2) $\omega|_{H_i}$ *is* $\mathcal{R}'$*-ergodic; and*

(3) *for a.e.* $x \in H_i$, *for all* $g \in \Gamma$, $gx \in H_j \Leftrightarrow \alpha(g, x) \equiv j - i \mod k$.

Proof. Let $\mathcal{R}$ be the orbit-equivalence relation on W, i.e., $\mathcal{R}$ is the set of all (x, gx) for $x \in W$, $g \in \Gamma$. The relation $\mathcal{R}'$ is a subequivalence relation of $\mathcal{R}$ of index N (i.e., for a.e. $x \in W$, the $\mathcal{R}$-equivalence class of x contains N distinct $\mathcal{R}'$-equivalence classes). This is because for a.e. $x \in W$ and $n \in \mathbb{Z}$, there is a $g \in \Gamma$ with $\alpha(g, x) = n$.

Because $\Gamma \curvearrowright (W, \omega)$ is ergodic, any $\mathcal{R}'$-invariant measurable function f must take on at most N different values (after ignoring a measure-zero set). So, the ergodic decomposition of ω with respect to $\mathcal{R}'$ contains k components for some $k \le N$. By the ergodic decomposition theorem, there exists a measurable partition $\{H_i\}_{i=0}^{k-1}$ of W such that each H_i has positive measure, each H_i is $\mathcal{R}'$-saturated and $\omega|_{H_i}$ is $\mathcal{R}'$-ergodic.

For $j \in \{0, \dots, k-1\}$, let F_j be the function on W defined as follows: $F_j(x)$ is the set of all $n + N\mathbb{Z} \in \mathbb{Z}/N\mathbb{Z}$ such that there exists $g \in \Gamma$ with $gx \in H_j$ and $\alpha(g, x) \equiv n \mod N$. We claim that F_j is $\mathcal{R}'$-invariant almost everywhere. To see this, let $x \in W$, $n + N\mathbb{Z} \in F_j(x)$ and $g \in \Gamma$ with $gx \in H_j$ and $\alpha(g, x) \equiv n \mod N$. Let y be $\mathcal{R}'$-equivalent to x. So, there exists $g_0 \in \Gamma$ with $y = g_0 x$ and $\alpha(g_0, x) \equiv 0 \mod N$.

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Thus,

$$\alpha(gg_0^{-1}, y) = \alpha(gg_0^{-1}, g_0x) = \alpha(g, x) + \alpha(g_0^{-1}, g_0x)$$
$$= \alpha(g, x) - \alpha(g_0, x) \equiv \alpha(g, x) \mod N.$$

Because $gg_0^{-1}y = gx \in H_j$, this proves that $n + N\mathbb{Z} \in F_j(y)$. Since x, y, n are arbitrary, this implies F_j is $\mathcal{R}'$-invariant. By ergodicity, F_j is constant on each H_i.

Let G be the function on W defined by $G(x) = F_i(x)$ whenever $x \in H_i$. We claim that G is Γ-invariant almost everywhere. So, suppose $x \in H_i$, $y \in H_j$ and $y = g_0x$ for some $g_0 \in \Gamma$. Let $n + N\mathbb{Z} \in G(x)$. By definition, there exists $g_1 \in \Gamma$ with $g_1x \in H_i$ and $\alpha(g_1, x) \equiv n \mod N$. Because F_j is constant on H_i, there exists $g_2 \in \Gamma$ with $g_2(g_1x) \in H_j$ and $\alpha(g_2, g_1x) \equiv \alpha(g_0, x)$. Thus, $g_2g_1g_0^{-1}y \in H_j$ and

$$\alpha(g_2g_1g_0^{-1}, y) = \alpha(g_2g_1g_0^{-1}, g_0x) = \alpha(g_2, g_1x) + \alpha(g_1g_0^{-1}, g_0x)$$
$$= \alpha(g_2, g_1x) + \alpha(g_1, x) - \alpha(g_0, x) \equiv \alpha(g_1, x) \equiv n \mod N.$$

Since x, y, n are arbitrary, this shows that G is Γ-invariant. By ergodicity, G is constant almost everywhere. By the cocycle equation, there is a subgroup $G_0 < \mathbb{Z}/N\mathbb{Z}$ such that $G(x) = G_0$ for a.e. x.

Let $G(i, j) \subset \mathbb{Z}/N\mathbb{Z}$ be the subset satisfying the following: for a.e. $x \in H_i$, $F_j(x) = G(i, j)$. We claim that there is an integer $t(i, j)$ such that $t(i, j) + G_0 = G(i, j)$. Indeed, if $n + N\mathbb{Z}, m + N\mathbb{Z} \in G(i, j)$, then for a.e. $x \in H_i$ there exist $g_n, g_m \in \Gamma$ with $g_nx, g_mx \in H_j$ and $\alpha(g_n, x) \equiv n \mod N, \alpha(g_m, x) \equiv m \mod N$. Therefore,

$$\alpha(g_mg_n^{-1}, g_nx) = \alpha(g_m, x) - \alpha(g_n, x) \equiv m - n \mod N$$

and $g_mg_n^{-1}(g_nx) \in H_j$. This implies $m - n \in G_0$, which establishes the claim.

We claim that if $j_1 \neq j_2$, then $G(i, j_1) \cap G(i, j_2) = \emptyset$. Indeed, if $n + N\mathbb{Z} \in G(i, j_1) \cap G(i, j_2)$, then for a.e. $x \in H_i$ there exists $g_1, g_2 \in \Gamma$ such that $g_1x \in H_{j_1}$, $g_2x \in H_{j_2}$, $\alpha(g_1, x) \equiv \alpha(g_2, x) \equiv n \mod N$. Therefore, $g_2x \in H_{j_2}$, $g_1g_2^{-1}(g_2x) \in H_{j_1}$ and $\alpha(g_1g_2^{-1}, g_2x) \equiv 0 \mod N$. This contradicts the fact that H_{j_2} is $\mathcal{R}'$-saturated. So, the claim is proven.

For each i, $\bigcup_{j=0}^{k-1} G(i, j)$ partitions the group $\mathbb{Z}/N\mathbb{Z}$ into cosets of G_0 (and thus $k|N$ and G_0 is generated by $k + N\mathbb{Z}$). So, after re-indexing the H_is if necessary, we may assume that $G(0, j) = j + G_0$ for each j.

We claim that $G(i, j) = G(0, j - i) = j - i + G_0$ (indices mod k). Let $n + N\mathbb{Z} \in G(i, j)$. So, for a.e. $x \in H_i$, there is a $g_0 \in \Gamma$ with $g_0x \in H_j$ and $\alpha(g_0, x) \equiv n \mod N$. By ergodicity, for a.e. such x, there exists $g_1 \in \Gamma$ such that $g_1x \in H_0$. Then $g_0g_1^{-1}(g_1x) \in H_j$, so $\alpha(g_0g_1^{-1}, g_1x) + N\mathbb{Z} \in j + G_0$ and $\alpha(g_1^{-1}, g_1x) + N\mathbb{Z} \in i + G_0$, but

$$\alpha(g_0g_1^{-1}, g_1x) = \alpha(g_0, x) + \alpha(g_1^{-1}, g_1x).$$

So, $n \equiv \alpha(g_0, x) \equiv j - i \mod k$. This proves the claim. Thus, for a.e. $x \in H_i$, for all $g \in \Gamma$, $gx \in H_j \Leftrightarrow \alpha(g, x) \equiv j - i \mod k$. □

LEMMA 5.2. *Suppose $\Gamma \curvearrowright (B, \nu)$ is an essentially free, weakly mixing, amenable action of type III_λ and stable type III_τ for some $\lambda, \tau \in (0, 1)$. Suppose as well that $\lambda^N = \tau$ for*

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some integer $N \geq 1$ and $R_\lambda(g, b) \in \mathbb{Z}$, where $R_\lambda(\cdot, \cdot)$ is defined as in (5.1). Let $\Gamma \curvearrowright (X, \mu)$ be an ergodic PMP action. Let $\Gamma \curvearrowright B \times X \times \mathbb{Z}$ by

$$g(b, x, n) = (gb, gx, n + R_\lambda(g, b)).$$

Then, for every bounded Borel Γ-invariant function f on $B \times X \times \mathbb{R}$, $f(b, x, n) = f(b, x, n + N)$ for a.e. (b, x, n).

Proof. This lemma follows from [**FM77**, Proposition 8.3 and Theorem 8]. To be precise, the cocycle c appearing in [**FM77**] is, for us, R_λ. So, $c : \mathcal{R} \to \mathbb{Z}$, $c((b, x), (b', x')) = R_\lambda(g, b)$, where $g \in \Gamma$ is an element such that $gb = b'$. This element is unique for a.e. $b \in B$ because $\Gamma \curvearrowright (B, \nu)$ is essentially free. Then, the asymptotic range $r_*(c)$ is, by definition, $\log_\lambda(RS(\Gamma, B \times X, \nu \times \mu) \cap (0, \infty))$. Because $\Gamma \curvearrowright (B, \nu)$ has stable type III_τ with $\tau = \lambda^N$, $RS(\Gamma, B \times X, \nu \times \mu) \supset \{\lambda^{Ni} : i \in \mathbb{Z}\}$. So, $N\mathbb{Z} \subset r_*(c)$.

The normalized proper range $npr(c)$ is the set of all positive integers T such that for any Γ-invariant $f \in L^\infty(B \times X \times \mathbb{Z})$, $f(b, x, n) = f(b, x, n + T)$ for a.e. (b, x, n) (by [**FM77**, Proposition 8.3]). By [**FM77**, Theorem 8], $npr(c) = r_*(c)$. □

LEMMA 5.3. *Let the hypotheses be as in the previous lemma. Then there is a partition $\{H_i\}_{i=0}^{k-1}$ of $B \times X$ such that $k|N$ and:*

(1) *if $K_i = \bigcup_{g \in \Gamma} g(H_i \times \{0\})$, then $\{K_i\}_{i=0}^{k-1}$ partitions $B \times X \times \mathbb{Z}$ up to a set of measure zero;*
(2) *$K_i = \bigcup_{j \in \mathbb{Z}} H_{i+j} \times \{j\}$, where the indices on H are taken mod k;*
(3) *for each i, $\Gamma \curvearrowright (K_i, \nu \times \mu \times \theta_\lambda|_{K_i})$ is ergodic; and*
(4) *$\nu \times \mu(H_i) = 1/k$ for all i.*

Proof. Let $\Gamma \times \mathbb{Z}$ act on $B \times X \times \mathbb{Z}$ by

$$(g, m)(b, x, n) = (gb, gx, m + n + R_\lambda(g, b)).$$

We claim that this action is ergodic. Indeed, any $\Gamma \times \mathbb{Z}$-invariant Borel set A is necessarily of the form $A = A_0 \times \mathbb{Z}$ for some Γ-invariant $A_0 \subset B \times X$. By ergodicity of the action $\Gamma \curvearrowright (B \times X, \mu \times \nu)$, $\mu \times \nu(A_0) \in \{0, 1\}$, which implies A or its complement has $\mu \times \nu \times \theta_\lambda$-measure zero, establishing the claim.

Let f be a bounded Γ-invariant Borel function on $B \times X \times \mathbb{Z}$. By Lemma 5.2, for a.e. $(b, x, n) \in B \times X \times \mathbb{Z}$, $f(b, x, n) = f(b, x, n + N)$, i.e., f is invariant under the action of the subgroup $\Gamma \times N\mathbb{Z}$.

Let $\mathcal{R} = \{((b, x), g(b, x)) \in B \times X \times B \times X : (b, x) \in B \times X, g \in \Gamma\}$ be the orbit-equivalence relation for the action $\Gamma \curvearrowright (B \times X, \nu \times \mu)$. Let $\mathcal{R}'$ be the set of all $((b, x), g(b, x)) \in \mathcal{R}$ such that $R_\lambda(g, b) \equiv 0 \mod N$. Because $\Gamma \curvearrowright (B, \nu)$ is type III_λ and $R_\lambda(g, b)$ takes values in the integers, it follows that for a.e. $b \in B$, $R_\lambda(\cdot, b)$ maps onto $\mathbb{Z}$. By Lemma 5.1, there exists a measurable partition $\{H_i\}_{i=0}^{k-1}$ of $B \times X$ such that $k|N$, each H_i has positive measure, each H_i is $\mathcal{R}'$-saturated, $\nu \times \mu|_{H_i}$ is $\mathcal{R}'$-ergodic for each i and for a.e. $(b, x) \in H_i$, for all $g \in \Gamma$, $g(b, x) \in H_j \Leftrightarrow R_\lambda(g, b) \equiv j - i \mod k$.

Let K_i be the Γ-orbit of $H_i \times \{0\} \subset B \times X \times \mathbb{Z}$. Because Γ-invariance automatically implies $\Gamma \times N\mathbb{Z}$-invariance, each K_i is $\Gamma \times N\mathbb{Z}$-invariant. Because $\{H_i\}_{i=0}^{k-1}$ partitions $B \times X$, it follows that $\{K_i\}_{i=0}^{k-1}$ partitions $B \times X \times \mathbb{Z}$ (up to measure-zero sets). Also, $K_i = \bigcup_{j \in \mathbb{Z}} H_{i+j} \times \{j\}$, where the indices on H are taken mod k (because of the last statement in the previous lemma).

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The restriction of $\nu \times \mu \times \theta_\lambda$ to K_i is ergodic for the Γ-action. To see this, let f be a bounded Γ-invariant Borel function with support in K_i. As mentioned above, f is $\Gamma \times N\mathbb{Z}$-invariant. Therefore, if $((b, x), g(b, x)) \in \mathcal{R}'$ (i.e., $R_\lambda(g, b) \in N\mathbb{Z}$) then $f(b, x, 0) = f(gb, gx, R_\lambda(g, b)) = f(gb, gx, 0)$. So, the map $(b, x) \mapsto f(b, x, 0)$ is $\mathcal{R}'$-invariant. Because $\nu \times \mu|_{H_i}$ is $\mathcal{R}'$-ergodic, f restricted to $H_i \times \{0\}$ is constant. Because K_i is the Γ-orbit of $H_i \times \{0\}$, this implies that f is constant on K_i. Because f is arbitrary, this proves the claim: $\Gamma \curvearrowright (K_i, \nu \times \mu \times \theta_\lambda|_{K_i})$ is ergodic.

Let $F : B \times X \to [0, 1]$ be the function defined almost everywhere by $F(b, x) = \mu(A_{(b,x)})$, where $A_{(b,x)}$ is defined by

$$\{b\} \times A_{(b,x)} = (\{b\} \times X) \cap H_i,$$

where i is such that $(b, x) \in H_i$. We claim that F is Γ-invariant. Indeed, if $(b, x) \in H_i$ and $g \in \Gamma$, then $g(b, x) \in H_{i+R_\lambda(g,b)}$ (index mod k). Thus, $A_{g(b,x)} = gA_{(b,x)}$, which, because μ is Γ-invariant, implies the claim. By ergodicity of $\Gamma \curvearrowright B \times X$, there is a constant $c > 0$ such that $F = c$ almost everywhere.

Let $\pi_B : B \times X \to B$ be the projection map. Fix i with $0 \le i \le k - 1$. For $b \in \pi_B(H_i)$, choose an element $x_b \in X$ with $(b, x_b) \in H_i$. Then

$$\nu \times \mu(H_i) = \int_{\pi_B(H_i)} F(b, x_b)\, d\nu(b) = c \cdot \nu(\pi_B(H_i)). \tag{5.2}$$

We claim that $\pi_B(H_i) = B$ (up to sets of measure zero) for every i. To see this, let $\mathcal{R}'_B$ be the equivalence relation on B given by $(b, gb) \in \mathcal{R}'_B$ if and only if $R_\lambda(g, b) \equiv 0 \mod N$. We claim that ν is $\mathcal{R}'_B$-ergodic. By Lemma 5.1, there is a measurable partition $\{C_i\}_{i=0}^{m-1}$ of B into $\mathcal{R}'_B$-saturated positive measure sets such that ν restricted to each C_i is $\mathcal{R}'_B$-ergodic (for some integer $m|N$). Moreover, for $b \in C_i$, $gb \in C_{i+R_\lambda(g,b)}$ with indices mod m.

Because $\Gamma \curvearrowright (B, \nu)$ is type III_λ, there exists a positive measure subset $C'_0 \subset C_0$ and an element $g \in \Gamma \backslash \{e\}$ such that $gC'_0 \subset C_0$ and for every $b \in C'_0$, $R_\lambda(g, b) = 1$. Thus, $0 \equiv 1 \mod m$. So, $m = 1$ and ν is $\mathcal{R}'_B$-ergodic as claimed.

Because H_i is $\mathcal{R}'$-saturated, it follows that each $\pi_B(H_i)$ is $\mathcal{R}'_B$-saturated. By ergodicity and since $\nu \times \mu(H_i) > 0$, this implies $\pi_B(H_i) = B$ up to a set of measure zero. By (5.2), $\nu \times \mu(H_i) = c$ for every i. Since $\sum_{i=0}^{k-1} \nu \times \mu(H_i) = 1$, this implies $c = 1/k$, which finishes the lemma. □

COROLLARY 5.4. *Let the hypotheses be as in the previous lemma. Let $I = \{0, \dots, N - 1\}$ and let $\widetilde{\mathcal{R}}_I$ be the restricted orbit-equivalence relation on $B \times X \times I$. Let $\overline{K}_i = K_i \cap B \times X \times I$. Then $\nu \times \mu \times \theta_\lambda(\overline{K}_i) = \nu \times \mu \times \theta_\lambda(\overline{K}_j)$ for every i, j. Also, let $\widetilde{\eta}_i$ be the restriction of $\nu \times \mu \times \theta_\lambda$ to $\overline{K}_i$ and normalized to have total mass 1. Then each $\widetilde{\eta}_i$ is $\widetilde{\mathcal{R}}_I$-invariant, ergodic and*

$$\nu \times \mu \times \theta_{\lambda,I} = \frac{1}{k} \sum_{i=0}^{k-1} \widetilde{\eta}_i.$$

Thus, for any $f \in L^1(B \times X \times I)$ and a.e. $(b, x) \in B \times X$,

$$\frac{1}{N} \sum_{i=0}^{N-1} \mathbb{E}[f|\mathcal{I}(\widetilde{\mathcal{R}}_I)](b, x, i) = \int f\, d(\nu \times \mu \times \theta_{\lambda,I})$$

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where $\mathbb{E}[f|\mathcal{I}(\widetilde{\mathcal{R}}_I)]$ *denotes the conditional expectation of* f *on the* σ*-algebra* $\mathcal{I}(\widetilde{\mathcal{R}}_I)$ *of* $\widetilde{\mathcal{R}}_I$*-saturated Borel sets.*

Proof. By the previous lemma,

$$\nu \times \mu \times \theta_\lambda(\overline{K}_i) = \sum_{j=0}^{N-1} \nu \times \mu(H_{i+j})\lambda^{-j} = (1 + \lambda^{-1} + \cdots + \lambda^{-N+1})/k,$$

with indices on H taken mod k. Because $\nu \times \mu \times \theta_\lambda|_{K_i}$ is ergodic for the action of Γ, it follows immediately that each $\widetilde{\eta}_i$ is $\widetilde{\mathcal{R}}_I$-ergodic. Because

$$\nu \times \mu \times \theta_\lambda = \sum_{i=0}^{k-1} \nu \times \mu \times \theta_\lambda|_{K_i},$$

it follows that

$$\nu \times \mu \times \theta_{\lambda,I} = \frac{\nu \times \mu \times \theta_\lambda|_{B\times X\times I}}{1 + \lambda^{-1} + \cdots + \lambda^{-N+1}} = \frac{\sum_{i=0}^{k-1} \nu \times \mu \times \theta_\lambda|_{\overline{K}_i}}{1 + \lambda^{-1} + \cdots + \lambda^{-N+1}} = \frac{1}{k}\sum_{i=0}^{k-1} \widetilde{\eta}_i.$$

For any $f \in L^1(B \times X \times I)$ and a.e. (b, x, n), $\mathbb{E}[f|\mathcal{I}(\widetilde{\mathcal{R}}_I)](b, x, n) = \int f\, d\widetilde{\eta}_i$ almost everywhere, where i is such that $(b, x, n) \in \widetilde{K}_i$. This is well defined because $\{\overline{K}_i\}_{i=0}^{k-1}$ partitions $B \times X \times I$ (up to measure zero). By the previous lemma, if $(b, x, n) \in \overline{K}_i$, then $(b, x, n + 1) \in \overline{K}_{i+1}$ (indices mod N and k, respectively). Thus, for a.e. $(b, x) \in B \times X$,

$$\frac{1}{N}\sum_{i=0}^{N-1} \mathbb{E}[f|\mathcal{I}(\widetilde{\mathcal{R}}_I)](b, x, i) = \frac{1}{k}\sum_{i=0}^{k-1} \int f\, d\widetilde{\eta}_i = \int f\, d(\nu \times \mu \times \theta_{\lambda,I}). \qquad \square$$

5.2. *Pointwise ergodic theorems from* III_λ *actions.*

THEOREM 5.2. *Let* $\Gamma \curvearrowright (B, \nu)$ *be an action of a countable group on a standard probability space. We assume the action is essentially free, weakly mixing, type* III_λ *and stable type* III_τ *for some* $\lambda, \tau \in (0, 1)$*, with* $\tau = \lambda^N$ *for some integer* $N \geq 1$*, and* $R_\lambda(g, b) \in \mathbb{Z}$*, where* $R_\lambda(\cdot, \cdot)$ *is defined as in (5.1). Let* $I = \{0, \ldots, N - 1\}$*. Let* $\mathfrak{F} = \{\mathfrak{F}_r\}_{r\in\mathbb{I}}$ *be a Borel family of subset functions for* $(B \times I, \nu \times \theta_{\lambda,I}, \mathcal{R}_I)$*. Suppose* $\mathfrak{F}$ *is either (asymptotically invariant and regular) or (asymptotically invariant, uniform and tempered). Define* $\zeta_r : B \times \Gamma \to [0, 1]$ *by*

$$\zeta_r(b, \gamma) := \frac{1}{N}\sum_{t=0}^{N-1} \frac{1}{|\mathfrak{F}_r(b, t)|} 1_{\mathfrak{F}_r(b,t)}(\gamma(b, t)).$$

Then $\{\zeta_r\}_{r\in\mathbb{I}}$ *is a random pointwise ergodic family for* Γ *in* L^1*.*

If $\psi \in L^q(B)$ *is a probability density function (so* $\psi \geq 0$ *and* $\int \psi\, d\nu = 1$*) and* $\zeta_r^\psi : \Gamma \to [0, 1]$ *is defined by* $\zeta_r^\psi(\gamma) = \int \zeta_r(b, \gamma)\psi(b)\, d\nu(b)$*, then* $\{\zeta_r^\psi\}_{r\in\mathbb{I}}$ *is a pointwise ergodic family in* L^p *for every* $p > 1$ *with* $1/p + 1/q \leq 1$*. If* $\psi \in L^\infty(B)$*, then* $\{\zeta_r^\psi\}_{r\in\mathbb{I}}$ *is a pointwise ergodic family in* $L \log L$*.*

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Proof of Theorem 5.2. Without loss of generality, we may assume $\Gamma \curvearrowright (X, \mu)$ is ergodic. Suppose now that $f \in L^1(B \times X \times I)$ depends only on its x-argument (so $f(b, x, t) = f(x)$). Recall that the averaging operator $\mathbb{A}[f|\widetilde{\mathfrak{F}}_r]$ is defined by

$$\mathbb{A}[f|\widetilde{\mathfrak{F}}_r](b, x, t) := |\widetilde{\mathfrak{F}}_r(b, x, t)|^{-1} \sum_{(b', x', t') \in \widetilde{\mathfrak{F}}(b,x,t)} f(b', x', t').$$

So, for any (b, x),

$$\sum_{\gamma \in \Gamma} \zeta_r(b, \gamma) f(\gamma x) = \sum_{\gamma \in \Gamma} \frac{1}{N} \sum_{t=0}^{N-1} \frac{1}{|\mathfrak{F}_r(b, t)|} 1_{\mathfrak{F}_r(b,t)}(\gamma(b, t)) f(\gamma x)$$

$$= \frac{1}{N} \sum_{t=0}^{N-1} \mathbb{A}[f|\widetilde{\mathfrak{F}}_r](b, x, t).$$

Similarly,

$$\sum_{\gamma \in \Gamma} \zeta_r^{\psi}(\gamma) f(\gamma x) = \frac{1}{N} \int \sum_{t=0}^{N-1} \mathbb{A}[f|\widetilde{\mathfrak{F}}_r](b, x, t) \psi(b) \, d\nu(b).$$

By Theorem 2.5 and Corollary 5.4, for a.e. (b, x),

$$\lim_{r \to \infty} \sum_{\gamma \in \Gamma} \zeta_r(b, \gamma) f(\gamma x) = \lim_{r \to \infty} \frac{1}{N} \sum_{t=0}^{N-1} \mathbb{A}[f|\widetilde{\mathfrak{F}}_r](b, x, t)$$

$$= \frac{1}{N} \sum_{t=0}^{N-1} \lim_{r \to \infty} \mathbb{A}[f|\widetilde{\mathfrak{F}}_r](b, x, t)$$

$$= \frac{1}{N} \sum_{t=0}^{N-1} \mathbb{E}[f|\mathcal{I}(\widetilde{\mathcal{R}}_I)](b, x, t)$$

$$= \int f \, d\mu(x).$$

Above, $\widetilde{\mathcal{R}}_I$ denotes the orbit-equivalence relation of the action $\Gamma \curvearrowright B \times X \times \mathbb{Z}$ restricted to $B \times X \times I$ and $\mathcal{I}(\widetilde{\mathcal{R}}_I)$ is the σ-algebra of $\widetilde{\mathcal{R}}_I$-invariant measurable sets. This proves $\{\zeta_r\}_{r \in \mathbb{I}}$ is a random pointwise ergodic sequence in L^1. If $f \in L^\infty$, then by the bounded convergence theorem, for a.e. $x \in X$,

$$\lim_{r \to \infty} \sum_{\gamma \in \Gamma} \zeta_r^{\psi}(\gamma) f(\gamma x) = \lim_{r \to \infty} \int \frac{1}{N} \int \sum_{t=0}^{N-1} \mathbb{A}[f|\widetilde{\mathfrak{F}}_r](b, x, t) \psi(b) \, d\nu(b)$$

$$= \frac{1}{N} \int \sum_{t=0}^{N-1} \lim_{r \to \infty} \mathbb{A}[f|\widetilde{\mathfrak{F}}_r](b, x, t) \psi(b) \, d\nu(b)$$

$$= \frac{1}{N} \int \sum_{t=0}^{N-1} \mathbb{E}[f|\mathcal{I}(\widetilde{\mathcal{R}}_I)](b, x, t) \psi(b) \, d\nu(b)$$

$$= \int f \, d\mu(x).$$

This proves $\{\zeta_r^{\psi}\}_{r \in \mathbb{I}}$ is a pointwise ergodic sequence in L^∞.

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Suppose now that $f \in L^p(B \times X \times I)$ for some $p > 1$ and $1/p + 1/q = 1$. Without loss of generality, let us assume $\int f\, d\mu = 0$. Let $\epsilon > 0$. Because $L^\infty(X)$ is dense in $L^p(X)$, there exists an element $f' \in L^\infty(X)$ such that $\|f - f'\|_p \le \epsilon$ and $\int f'\, d\mu = 0$. Then, for a.e. $x \in X$,

$$\begin{aligned}\limsup_{r\to\infty}\left|\sum_{\gamma\in\Gamma}\zeta_r^\psi(\gamma)f(\gamma x)\right| &\le \limsup_{r\to\infty}\left|\sum_{\gamma\in\Gamma}\zeta_r^\psi(\gamma)[f(\gamma x)-f'(\gamma x)]\right| \\ &\quad + \left|\lim_{r\to\infty}\sum_{\gamma\in\Gamma}\zeta_r^\psi(\gamma)f'(\gamma x)\right| \\ &= \limsup_{r\to\infty}\left|\sum_{\gamma\in\Gamma}\zeta_r^\psi(\gamma)[f(\gamma x)-f'(\gamma x)]\right| \\ &= \limsup_{r\to\infty}\left|\frac{1}{N}\int\sum_{t=0}^{N-1}\mathbb{A}[f-f'|\widetilde{\mathfrak{F}}_r](b,x,t)\psi(b)\,d\nu(b)\right| \\ &\le \mathbb{M}[f-f'|\widetilde{\mathfrak{F}},\psi](x),\end{aligned}$$

where $\mathbb{M}[f - f'|\widetilde{\mathfrak{F}}, \psi](x)$ is as defined in Theorem 5.1. Thus, if

$$F(x) := \limsup_{r\to\infty}\left|\sum_{\gamma\in\Gamma}\zeta_r^\psi(\gamma)f(\gamma x)\right|,$$

then

$$\|F\|_p \le \|\mathbb{M}[f-f'|\widetilde{\mathfrak{F}},\psi]\|_p \le C_p''\|f-f'\|_p \le C_p''\epsilon$$

for some constant $C_p'' > 0$ (which is independent of f and f' but may depend on ψ) by Theorem 5.1. Since ϵ is arbitrary, $\|F\|_p = 0$, which implies

$$\lim_{r\to\infty}\sum_{\gamma\in\Gamma}\zeta_r^\psi(\gamma)f(\gamma x) = 0$$

for a.e. x, as required. This proves $\{\zeta_r^\psi\}_{r\in\mathbb{I}}$ is a pointwise ergodic sequence in L^p for every $p > 1$.

The last two cases to be handled occur when $f \in L\log L(X)$ and $\psi \in L^\infty(B)$. The proofs of these cases are similar to the proofs above. □

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