COMPOSITIO MATHEMATICA

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Compositio Math. **161** (2025), 2461–2492.

doi: 10.1112/S0010437X25102406

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https://doi.org/10.1112/S0010437X25102406 Published online by Cambridge University Press

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Nonlinear Hodge flows in symplectic geometry

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Abstract

Given a symplectic class $[\omega]$ on a four torus T^4 (or a $K3$ surface), a folklore problem in symplectic geometry is whether symplectic forms in $[\omega]$ are isotopic to each other. We introduce a family of nonlinear Hodge heat flows on compact symplectic four manifolds to approach this problem, which is an adaption of nonlinear Hodge theory in symplectic geometry. As a particular example, we study a conformal Hodge heat flow in detail. We prove a stability result of the flow near an almost Kähler structure (M, ω, g). We also prove that, if $|\nabla \log u|$ stays bounded along the flow, then the flow exists for all time for any initial symplectic form $\rho \in [\omega]$ and it converges to ω smoothly along the flow with uniform control, where u is the volume potential of ρ.

1. Introduction

The geometric flows have proved to be a powerful tool in geometry and topology, in particular in Riemannian geometry, complex geometry and geometry of submanifolds. Geometric flows in symplectic geometry have gained considerable attention, in particular in the last decade; see [Don99, LW01, ST14, KS19, Kro87, He21, FY18, FPPZ21, FP21], etc., and references therein. Although there has been much progress on these flows, applications of geometric flows in symplectic geometry and topology have remained relatively few.

In this paper, we will introduce some new geometric flows which may potentially be of interest in symplectic geometry and topology. Our motivation is the following question for symplectic four manifolds. Let (M, ω) be a compact symplectic four manifold such that $a = [\omega] \in H^2(M; \mathbb{R})$. Let $\mathcal{S}_a$ denote the set of the symplectic forms in the class of a,

$$\mathcal{S}_a = \{\rho \in \Omega^2(M) : \rho^2 > 0, d\rho = 0, [\rho] = a\}.$$

The following problem is a long-standing open problem in symplectic topology [MDS27, Chapter 13].

Problem 1.1. Is $\mathcal{S}_a$ path-connected?

It is generally believed that Problem 1.1 holds true for hyperKähler surfaces and $\mathbb{CP}^2$. A related problem on Kähler surfaces was proposed by Donaldson [Don06].

Problem 1.2. Suppose X is a compact Kähler surface with Kähler form ω_0. If ω is any other symplectic form on X, with the same Chern class and with $[\omega] = [\omega_0]$, is there a diffeomorphism f of X with $f^*\omega = \omega_0$?

Received 29 March 2024, accepted in final form 12 March 2025, published online 17 December 2025.
2020 Mathematics Subject Classification 53D05 (primary), 35K40 (secondary).
Keywords: uniqueness of symplectic structure; nonlinear Hodge flows; the conformal Hodge flow.
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If Problem 1.1 is confirmed, then Problem 1.2 follows as a consequence via the Moser isotropy. It is known, by the deep theory in symplectic topology, that certain symplectic structures in the same cohomology class in dimension four are unique modulo diffeomorphisms, including the projective surface $\mathbb{CP}^2$; see [MDS27, Section 13.4] for a detailed discussion of examples and references. Hence, in particular, if Problem 1.1 were confirmed for $\mathbb{CP}^2$, it would imply that every diffeomorphism of $\mathbb{CP}^2$ that acts as the identity on cohomology is isotopic to the identity. However, Problem 1.1 remains completely open for any compact symplectic four manifolds. We refer to the survey paper by Salamon [Sal13] and the book [MDS27, Chapters 13 and 14] by McDuff and Salamon for an excellent overview.

Donaldson described a geometric flow approach for Problem 1.1 in [Don99]. We roughly recall the setup and refer the readers to [KS19] for a detailed discussion. Associate (M, ω) with a compatible Riemannian metric g. For any $\rho \in \mathcal{S}_a$, define

$$\mathrm{dvol}_\rho := \frac{\rho \wedge \rho}{2}, \quad u := \frac{\rho \wedge \rho}{\omega \wedge \omega} = \frac{\mathrm{dvol}_\rho}{\mathrm{dvol}_g}, \quad g(A\cdot, \cdot) := \rho, \quad g_\rho := u^{-1} g(A\cdot, A\cdot).$$

The metric g_ρ induces the Hodge star $*_\rho$ and the Donaldson metric on the tangent space $T_\rho \mathcal{S}_a$ at ρ. The Donaldson flow, defined by

$$\partial_t \rho = d *_\rho d(\mathrm{II}(u^{-1} * \rho)), \quad \mathrm{II}(\tau) = \tau - \frac{\tau \wedge \rho}{\rho \wedge \rho}\rho \quad \text{for } \tau \in \Omega^2(M), \tag{1.1}$$

can be viewed as the negative gradient flow of the energy functional (with respect to the Donaldson metric),

$$\mathcal{E}(\rho) = \int_M \frac{|\rho^+|^2}{u}\, dv.$$

Note that $u^{-1}|\rho^+|^2 \geq 2$ pointwise and it achieves the absolute minimum when $*\rho = \rho$. Hence, ω is the unique minimum of $\mathcal{E}(\rho)$ over $a = [\omega]$. If one could prove that the Donaldson flow exists for all time and converges to ω for a *generic* $\rho \in a$, Problem 1.1 can be answered affirmatively. For the progress in Donaldson's geometric flow, see [KS19, Kro87].

In this paper, we introduce a family of parabolic flow of symplectic forms, motivated by the Hodge theory, to approach Problem 1.1. Let (M, ω, g) be an almost Kähler manifold of dimension four. Given a symplectic form $\rho \in [\omega]$, one can associate a Riemannian metric $\tilde{g}$ depending on ρ and g. In this paper, we mainly discuss a conformal metric $\tilde{g} = \sqrt{u} g$, where u is the volume potential of ρ defined above. Let $*$ denote the Hodge star of g, and $\tilde{*}$ the Hodge star of $\tilde{g}$. Given a choice of $\tilde{g} = \tilde{g}(\rho, g)$, we consider the parabolic flow of ρ by

$$\partial_t \rho = d\tilde{*} d * \rho. \tag{1.2}$$

There is flexibility to choose the metric $\tilde{g}$. For any such choice, the energy functional

$$E(\rho) = \int_M \rho \wedge *\rho$$

is decreasing along the flow and ω is the unique minimizer of $E(\rho)$, by the Hodge theory. In dimension four, (1.2) has a description as a gradient flow of $E(\rho)$ if one uses the metric $\tilde{g}$ to define the metric on the tangent space of $\rho \in \mathcal{S}_a$.

When $\tilde{g} = g_u := \sqrt{u} g$, the flow equation reads

$$\partial_t \rho = -d\left(\frac{d^* \rho}{\sqrt{u}}\right). \tag{1.3}$$

The first result regarding (1.3) is the following stability near (M, ω, g).

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THEOREM 1.3. *There exists $\epsilon > 0$ depending on (M, ω, g) such that for any $\rho_0 \in [\omega]$ satisfying $\|\rho_0 - \omega\|_{C^{1,\alpha}} < \epsilon$, (1.3) exists for all time with initial data ρ_0 and converges to ω at the exponential decay rate when $t \to \infty$.*

Remark 1.4. With reasonable effort, Theorem 1.3 should be true by assuming $\rho_0 \in [\omega]$ such that for some $p > 6$, $\|\rho_0 - \omega\|_{W^{1,p}} < \epsilon$. It remains a challenge to prove the stability if we assume $\rho_0 \in L^\infty \cap W^{1,4}$ such that $\|\rho_0 - \omega\|_{W^{1,4}} < \epsilon$.

One key point for Theorem 1.3 is to prove that the normalized energy functional

$$E_0(\rho) = E(\rho) - E(\omega)$$

is decreasing along the flow at exponential rate; see Proposition 3.6. The second result is the following long time existence and convergence criterion.

THEOREM 1.5. *Suppose $|\nabla \log u|$ stays uniformly bounded along the flow. Then (1.3) has a long time solution such that ρ converges to ω smoothly with uniform control along the flow. The convergence is at an exponential decay rate for a set of suitable positive constants A_0, A_1, C satisfying the estimate*

$$\left(\int_M |d^*\rho|^2 + A_1 |\rho_0 - \omega|^2 \right)(t) \leq A_0 e^{-Ct}. \tag{1.4}$$

With the uniform control on $|\rho|$, $|\nabla \rho|$ and exponential decay (1.4), it is standard practice to get higher-order exponential decay on $\|(d^*d)^k \rho\|_{L^2}$ for each $k \geq 1$. The key to Theorem 1.5 is to explore the nonlinear structure of the equation and consider the function f of the form for suitable positive constants A and B:

$$f = |\nabla \rho|^2 + A|\nabla u|^2 + B|\rho|^2 + 1. \tag{1.5}$$

Assuming the uniform control of $|\nabla \log u|$, one can obtain a differential inequality for f:

$$\partial_t f \leq \frac{\Delta f}{\sqrt{u}} + C_1 f. \tag{1.6}$$

A maximum principle argument then implies that $f(t) \leq C_0 e^{C_1 t}$. With this bound, it is now standard to obtain a Shi-type estimate on higher derivatives of ρ and to obtain the long time existence. The exponential decay estimate (1.4) is a refinement of the exponential decay of energy. To obtain the smooth convergence along the flow with uniform control, we need to bound $|\rho|$ and $|\nabla \rho|$ pointwise. The key point is to use the differential inequality (1.6) to bound f via a delicate parabolic Moser iteration. Hence, if $|\nabla \log u|$ were bounded along the flow, Theorem 1.5 would solve Problem 1.1 with an affirmative answer. However, this remains a very challenging problem. Instead of assuming C^1 bound on the potential u, it would be a very interesting question to relate the long time existence of the flow to the geometry of the conformal metric $g_u = \sqrt{u} g$.

It is not clear whether (1.3) will develop a (finite time) singularity or not in dimension four. Our result implies that $|\nabla \log u|$ blows up at a possible finite time singularity. If the dimension of M is $n = 6$, McDuff [MD87] constructed a symplectic class on $T^2 \times S^2 \times S^2$ such that $\mathcal{S}_a$ is not connected. One can consider a nonlinear Hodge flow

$$\partial_t \rho = -d\left(\frac{d^*\rho}{u^r} \right) \tag{1.7}$$

for any $r > 0$. The energy functional $E(\rho)$ is still decreasing. If (1.7) exists for all time with uniform control, then the flow will converge to ω, the unique minimizer of $E(\rho)$. This implies that (1.7) would have to develop singularities for McDuff's example. Hence, one might expect

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that there would be singularities along the flow (1.3) in general. If that is the case, it would be an important question to study the formation of singularity of (1.3).

Problem 1.6. Does the flow (1.3) form a (finite time) singularity or does the flow exist for all time in general?

The nonlinear structure of the conformal Hodge flow is of the form of a nonlinear harmonic heat flow type with a conformal factor $u^{-1/2}$ on the leading term:

$$\partial_t \rho = \frac{\Delta \rho}{\sqrt{u}} + \nabla(u^{-1/2}) \circ \nabla \rho + \text{l.o.t.}$$

There are two major difficulties in the study of (1.2) and (1.3). The first is whether the conformal factor $u^{-1/2}$ stays uniformly positive and bounded along the flow. The second is related to the nonlinear structure of the harmonic heat flow type. These two difficulties are intertwined, and it remains a very interesting and challenging problem to go beyond Theorem 1.5 with more flexible assumptions on u.

Compare with the linear Hodge flow in the setting

$$\partial_t \rho = -d(d^* \rho) = d * d * \rho.$$

By the standard parabolic theory and the Hodge theory, the flow exists for all time with uniform control on ρ and converges to ω, the unique minimizer of $E(\rho)$. The potential function u stays uniformly bounded and converges to 1. However, u might not stay uniformly positive, and hence ρ would not stay as a nondegenerate two form along the linear Hodge flow. Our method is a nonlinear modification of the linear Hodge flow, which ensures that ρ stays nondegenerate before it develops a singularity.

Problem 1.7. Is there a particular choice of $\tilde{g}$ such that (1.2) exists for all time and converges to ω?

In particular, one might ask whether there is a choice of $\tilde{g} = \tilde{g}(\rho, g)$ such that u stays uniformly positive along the flow (1.2). We will discuss some specific choices of $\tilde{g}$ in Section 4. Our computations suggest that this is a very delicate problem. In particular, we construct an example (Example 3.20) of a sequence of symplectic forms on T^4 with the normalized volume potential $u = 1$; along the linear Hodge flow, the volume potential can turn negative almost instantly (the time can be as small as possible for the sequence).

2. A nonlinear Hodge theory in symplectic geometry

We introduce a family of geometric flows for Problem 1.1, motivated by the Hodge theory. Such flows can also be defined in higher dimensions. Let (M, ω, g) be a compact symplectic manifold with a compatible metric g of dimension $2n$. By the Hodge theory, ω is the unique harmonic form in the class $a = [\omega]$ that minimizes the norm over a:

$$E(\alpha) = \int_M |\alpha|^2 \, dv = \int_M \alpha \wedge * \alpha.$$

The motivation is then to minimize $E(\rho)$ over $\mathcal{S}_\alpha$ while enforcing the nondegenerate condition $\rho^n > 0$. Suppose $\delta \rho = d\xi$, for $\xi \in \Omega^1(M)$. The variation of $E(\rho)$ is given by

$$\delta E(\rho) = \int_M \xi \wedge d * \rho.$$

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Let $\tilde{g}_\rho$ be a Riemannian metric induced by g and ρ, with the Hodge star $\tilde{*}$. We compute

$$\delta E(\rho) = -\int_M \xi \wedge \tilde{*}(\tilde{*} d * \rho) = -\int_M \langle \xi, \tilde{*} d * \rho \rangle \, d\tilde{v}.$$

By taking $\xi = \tilde{*} d * \rho$ and hence $\delta\rho = d(\tilde{*} d * \rho)$, we obtain

$$\delta E(\rho) = -\int_M |\tilde{*} d * \rho|^2_{\tilde{g}_\rho} \, d\tilde{v}.$$

We consider the flow

$$\partial_t \rho = d(\tilde{*} d * \rho). \tag{2.1}$$

If we fix $\tilde{g}_\rho = g$, then (2.1) is reduced to the Hodge Laplacian heat flow. The Hodge Laplacian heat flow exists for all time and converges to ω for any initial two form $\alpha \in a$, but the nondegenerate condition $\rho^n > 0$ cannot be assured. We use the choice of $\tilde{g}_\rho$ to enforce $\rho^n > 0$, which is evolved along the flow (2.1) as well. By choosing the metric $\tilde{g}_\rho$, (2.1) can be viewed as a gradient flow of $E(\rho)$ if we use $g_{\tilde{\rho}}$ to define a metric on the forms in the tangent space at $\rho \in \mathcal{S}_a$, as in the setting of the Donaldson geometric flow [KS19]. Being allowed to choose $\tilde{g}_\rho$ seems to be valuable. A critical point remains the same for $E(\rho)$ (or for $\mathcal{E}(\rho)$). In particular, the flow (2.1) decreases the functional $E(\rho)$ and the only fixed point of the flow satisfies $d * \rho = 0$; by the Hodge theory, such a fixed point ρ must be a harmonic form and equal to ω. Hence, if the gradient flow (2.1) has a long time solution with the $C^{1,\alpha}$ norm uniformly bounded, the flow will converge to ω.

Given a symplectic form ρ and a metric g, there are many ways to produce a (family of) Riemannian metric(s) $\tilde{g}_\rho$. We define the linear transformation $A : TM \to TM$ by

$$\rho = g(A\cdot, \cdot).$$

It follows that $B = -A^2$ is symmetric and positive definite, and hence B^r is uniquely well defined for any real number r. Note that $B^r A = A B^r$. Suppose the eigenvalues of A read $\{\pm\sqrt{-1}\lambda_1, \ldots, \pm\sqrt{-1}\lambda_n\}$ with $\lambda_i > 0$ for $i = 1, \ldots, n$. The eigenvalues of B read $\{\lambda_1^2, \ldots, \lambda_n^2\}$, with each repeated. We define $\tilde{g}_\rho = g(Q\cdot, \cdot)$ with $Q = Q(\lambda_1^2, \ldots, \lambda_n^2)$ to be a positive definite symmetric matrix such that $QA = AQ$. A typical choice of Q is of the form fB^r for a positive function $f = f(\lambda_1, \ldots, \lambda_n)$ and $r \in \mathbb{R}$. Since our primary interest is on symplectic manifolds of dimension four, we focus on the case $n = 2$.

Proposition 2.1. *Let (M, ω, g) be an almost Kähler manifold of dimension four. For any initial data $\rho_0 \in a$, (2.1) has a unique smooth short time solution $\rho(t) \in a$ in $[0, t_0)$ with $\rho(0) = \rho_0$, for any choice of Q.*

If we apply the Hamilton–Nash–Moser implicit function theorem [Ham82], the short time existence of (2.1) follows immediately since the system has an obvious integrability operator $L = d$ and $d\tilde{*}d*$ is an elliptic operator on $\mathrm{Ker}(d)$. Here, we give an elementary argument for the short time existence, and for simplicity, we consider only $n = 2$. Suppose, for the moment, that ρ is only a nondegenerate two form. We consider

$$\partial_t \rho = d\tilde{*} d * \rho + * d \tilde{*} d \rho. \tag{2.2}$$

Proposition 2.2. *Let (M, g) be a four manifold with a nondegenerate two form ρ_0 such that $\rho_0^2 > 0$. Then, for any choice of $\tilde{g}_\rho$ described above, the flow (2.2) is strictly parabolic and admits a unique short time smooth solution. Along the flow we have*

$$\partial_t \int_M \rho \wedge * \rho = -\int_M (|d\rho|^2_{\tilde{g}} + |d * \rho|^2_{\tilde{g}}) \, d\tilde{v}. \tag{2.3}$$

Moreover, if $d\rho_0 = 0$, then $d\rho = 0$ along the flow. Proposition 2.1 follows as a consequence.

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Proof. Recall $\tilde{g}_\rho = g(Q\cdot,\cdot)$ with $Q = Q(\lambda_1^2, \lambda_2^2)$. For any $\theta \in \Omega^3(M)$, we compute

$$\tilde{*}\theta = (\det Q)^{-1/2}(*\theta)\circ Q.$$

We have

$$\tilde{*}\circ *|_{\Lambda^1} = *\circ \tilde{*}|_{\Lambda^3} = -(\det Q)^{-1/2}Q.$$

Denote $\tilde{Q} = (\det Q)^{-1/2}Q$. We rewrite (2.2) as

$$\partial_t \rho = -d(\tilde{Q}\circ d^*\rho) - d^*(\tilde{Q}\circ d\rho).$$

We only need to consider the leading terms, and hence, at one point, we can assume that $\tilde{Q}_{kl} = \tilde{\lambda}_k \delta_{kl}$. Then the leading term reads

$$\partial_t \rho_{ij} = \sum_k \tilde{\lambda}_k \partial_k^2 \rho_{ij} + \text{l.o.t.},$$

and hence (2.2) is a strictly parabolic equation on ρ. This proves the short time existence when $n = 2$. Hence, for any initial nondegenerate two form ρ_0, there exists a unique smooth short time solution $\rho(t)$ in $[0, T)$. For any $t_0 \in (0, T)$, $\rho(t)$ is a smooth nondegenerate two form in $[0, t_0]$. We compute

$$\begin{aligned}\partial_t \int_M |d\rho|^2 dv &= 2\int_M \langle d*d\tilde{*}d\rho, d\rho\rangle dv\\ &= 2\int_M \langle *d\tilde{*}d\rho, d^*d\rho\rangle dv\\ &= -2\int_M \langle d^*\tilde{Q}d\rho, d^*d\rho\rangle dv.\end{aligned}$$

Hence, we compute

$$\begin{aligned}\partial_t \int_M |d\rho|^2\, dv &= -2\int_M \langle d^*(\tilde{Q}d\rho), d^*d\rho\rangle dv\\ &\le -C_0\int_M \langle d^*d\rho, d^*d\rho\rangle dv + C_1\int_M |d\rho||d^*d\rho|\, dv\\ &\le C_2\int_M |d\rho|^2\, dv, \end{aligned} \tag{2.4}$$

where C_0, C_1 and C_2 are positive constants depending on t_0.

Now, suppose $\rho_0 \in a$ and $d\rho_0 = 0$. By (2.4) and Gronwall's inequality, it implies that $d\rho = 0$ over $[0, t_0]$. Hence, (2.2) reduces to (2.1), and this proves Proposition 2.1. Note that along the flow (2.2), we compute

$$\begin{aligned}\partial_t \int_M \rho\wedge *\rho &= -\int_M d\rho\wedge \tilde{*}d\rho - \int_M (d*\rho)\wedge\tilde{*}(d*\rho)\\ &= -\int_M (|d\rho|_{\tilde{g}}^2 + |d*\rho|_{\tilde{g}}^2)d\tilde{v}.\end{aligned} \tag{2.5}$$

Hence, the energy functional $E(\rho)$ is still decreasing. □

We describe several interesting choices of Q given ρ and g. First, we introduce some notation. The almost Kähler structure (ω, g) is fixed. We choose an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ and its dual coframe $\{\theta^1, \theta^2, \theta^3, \theta^4\}$. We write

$$\rho = \frac{1}{2}\rho_{ij}\theta^i\wedge\theta^j = \sum_{i<j}\rho_{ij}\theta^i\wedge\theta^j.$$

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Recall that $\rho = g(A\cdot,\cdot)$. Define B to be $*\rho = g(B\cdot,\cdot)$:

$$A = \begin{pmatrix} 0 & \rho_{12} & \rho_{13} & \rho_{14} \\ -\rho_{12} & 0 & \rho_{23} & \rho_{24} \\ -\rho_{13} & -\rho_{23} & 0 & \rho_{34} \\ -\rho_{14} & -\rho_{24} & -\rho_{34} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \rho_{34} & -\rho_{24} & \rho_{23} \\ -\rho_{34} & 0 & \rho_{14} & -\rho_{13} \\ \rho_{24} & -\rho_{14} & 0 & \rho_{12} \\ -\rho_{23} & \rho_{13} & -\rho_{12} & 0 \end{pmatrix}.$$

We have $AB = -u\,\mathrm{I}$, $B = -uA^{-1}$, with $u = \rho_{12}\rho_{34} - \rho_{13}\rho_{24} + \rho_{14}\rho_{23} = \sqrt{\det A}$. Denote

$$\rho^+ = \tfrac{1}{2}(\rho + *\rho), \quad \rho^- = \tfrac{1}{2}(\rho - *\rho).$$

We have

$$|\rho|^2 = \sum_{i<j} \rho_{ij}^2 = |\rho^+|^2 + |\rho^-|^2, \quad 2u = \sum_{i<j} \rho_{ij}(*\rho)_{ij} = \langle \rho, *\rho\rangle = |\rho^+|^2 - |\rho^-|^2.$$

PROPOSITION 2.3. *The eigenvalues of A are $\pm\sqrt{-1}\lambda_1$, $\pm\sqrt{-1}\lambda_2$, with*

$$\lambda_1 = \tfrac{1}{\sqrt{2}}(|\rho^+| + |\rho^-|), \quad \lambda_2 = \tfrac{1}{\sqrt{2}}(|\rho^+| - |\rho^-|).$$

The eigenvalues of $(a_{ij}) = -A^2$ and $(b_{ij}) = -B^2$ are $\{\lambda_1^2, \lambda_1^2, \lambda_2^2, \lambda_2^2\}$ and $u = \sqrt{\det A} = \lambda_1\lambda_2$. Moreover,

$$a_{ij} + b_{ij} = |\rho|^2 \delta_{ij}.$$

Proof. Since A is skew-symmetric and nondegenerate, the eigenvalues of A are purely imaginary, denoted by $\pm\sqrt{-1}\lambda_1, \pm\sqrt{-1}\lambda_2$. We can assume λ_1, λ_2 are both positive. It is then straightforward to compute its eigenvalues in terms of $|\rho|^2$ and u. In particular,

$$|\rho|^2 = \lambda_1^2 + \lambda_2^2, \quad u = \lambda_1\lambda_2. \qquad \square$$

Example 2.4. We describe several examples of Q.

(1) Take $Q_{ij} = u^{-1}a_{ij}$, then $\det Q = 1$ and the metric $g_\rho = u^{-1}g(a,\cdot) = u^{-1}g(A\cdot, A\cdot)$. This is the metric used in [Don99] to define Donaldson's geometric flow. One computes [KS19, Proposition 3.1]

$$*_\rho\rho = \frac{|\rho|^2}{u}\rho - *\rho \quad \text{and} \quad \rho \wedge *_\rho\rho = \rho \wedge *\rho.$$

(2) Take $Q = e^{2\varphi}\mathrm{I}$ for a conformal factor $\varphi = \varphi(\rho, g)$; then the metric $g_\varphi = e^{2\varphi}g$ is a conformal change. A particularly interesting choice is $\varphi = (\log u)/4$; hence, $\mathrm{dvol}_\varphi = \mathrm{dvol}_\rho$. For any conformal change, the Hodge star satisfies

$$*_\varphi = e^{(4-2p)\varphi} *$$

for a p-form. Hence,

$$\rho \wedge *_\varphi\rho = \rho \wedge *\rho.$$

(3) There are many other choices of $Q = Q(\lambda_1, \lambda_2)$. An important question is whether there is a choice of Q such that u stays uniformly positive along the flow.

3. A conformal Hodge flow in dimension four

Let (M, ω, g) be an almost Kähler manifold of dimension four. In this section, we consider $g_\varphi = e^{2\varphi}g$ with $\varphi = (1/4)\log u$ and, hence, $\mathrm{dvol}_\varphi = u\,\mathrm{dvol}_g = \mathrm{dvol}_\rho$. In this case, since $*\rho = \tilde{*}\rho$,

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Equation (2.1) can also be written as a Hodge Laplacian flow with respect to $\tilde{g}$,

$$\partial_t \rho = d\tilde{*}d\tilde{*}\rho = -\tilde{\Delta}^d \rho.$$

We choose a normalization condition such that

$$\mathrm{Vol}(M, g) = \int_M \mathrm{dvol}_g = \int_M \mathrm{dvol}_\rho = 1.$$

Note that the Hodge star of $e^{2\varphi}g$ is related to that of g by, for a p form θ,

$$*_\varphi \theta = e^{(4-2p)\varphi} * \theta.$$

Hence, for the conformal metric $\sqrt{u}g$, the flow reads

$$\partial_t \rho = -d\left(\frac{d^*\rho}{\sqrt{u}}\right) = -\frac{dd^*\rho}{\sqrt{u}} + \frac{du \wedge d^*\rho}{2\sqrt{u^3}} = \frac{\Delta_g \rho + \mathcal{R}(\rho)}{\sqrt{u}} + \frac{du \wedge d^*\rho}{2\sqrt{u^3}}, \tag{3.1}$$

where $\mathcal{R}(\rho)$ is the curvature tensor acting on ρ. To be more precise, for an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ and the corresponding coframe $\{\theta^1, \theta^2, \theta^3, \theta^4\}$ and a form ζ,

$$\mathcal{R}(\zeta) = \theta^i \wedge \iota_{e_j} R(e_i, e_j)\zeta.$$

We use the following sign convention for the curvature operator:

$$R(e_i, e_j)e_k = (\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i} - \nabla_{[e_i,e_j]})e_k,$$
$$Ric(X, X) = \langle R(e_i, X)X, e_i\rangle.$$

Let Δ denote the Laplace–Beltrami operator of g; for any tensor T,

$$\Delta T = \mathrm{Tr}(\nabla^2 T) = \sum (\nabla_{e_i}\nabla_{e_i} T - \nabla_{\nabla_{e_i} e_i} T).$$

We recall the Bochner–Weitzenböck identities as follows.

Proposition 3.1. *For any k-form ζ,*

$$\Delta_d \zeta = -\Delta\zeta - \mathcal{R}(\zeta),$$
$$\tfrac{1}{2}\Delta(|\zeta|^2) = \langle -\Delta_d\zeta, \zeta\rangle + |\nabla\zeta|^2 + \langle \mathcal{R}(\zeta), \zeta\rangle, \tag{3.2}$$

where $\mathcal{R}(\zeta) = \theta^i \wedge \iota_{e_j} R(e_i, e_j)\zeta$.

Note that (3.1) is a quasilinear parabolic system, and if the volume potential u is uniformly positive and Hölder continuous, the system has the structure

$$\partial_t \rho = \frac{\Delta\rho}{\sqrt{u}} + du^{-1/2} \wedge d^*\rho + \text{l.o.t.} = \frac{\Delta\rho}{\sqrt{u}} + \frac{1}{\sqrt{u^3}}\rho \circ \nabla\rho \circ \nabla\rho + \text{l.o.t.},$$

which is similar to the structure of a nonlinear harmonic map heat flow, with the leading term involved with a conformal factor of u. A standard parabolic theory implies the following proposition.

Proposition 3.2. *Let ρ_0 be a $C^{1,\alpha}$ symplectic form in $[\omega]$. There exists a unique solution of (3.1) over a short time $[0, T)$ for some $T > 0$ which is smooth for any $t \in (0, T)$.*

3.1 The stability near the energy minimizer ω

In the class of $[\omega]$, ω is the unique harmonic form with respect to g that minimizes the energy functional. When we restrict to the symplectic forms in $[\omega]$, the flow (3.1) can be viewed as a gradient flow of the energy functional $E(\rho)$. Hence, it is expected that if one starts with an initial symplectic form ρ_0 near ω, the flow exists for all time and converges to ω. We justify the

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expectation in this section, and the key point is the following monotonicity of the normalized energy. Denote

$$E_0 = E(\omega) = \int_M \omega \wedge \omega = \int_M \omega \wedge *\omega = \int_M |\omega|^2 \, \mathrm{dvol}_g.$$

For notational simplicity, we will drop the volume elements dvol_g in the integrals when there is no ambiguity. For a closed form $\alpha \in [\omega]$, we have

$$\int_M |\alpha - \omega|^2 = \int_M |\alpha|^2 - 2\int_M \alpha \wedge \omega + \int_M |\omega|^2 = E(\alpha) - E_0.$$

Denote $E_0(\alpha) = E(\alpha) - E_0$. We have the following Poincaré inequality with $c_1 = c_1(g)$:

$$E_0(\alpha) \le c_1 \int_M |d^*\alpha|^2. \tag{3.3}$$

We need the following equivalence for $W^{1,p}$ norms on the differential forms. For $p \ge 1$, one defines the geometric $W^{1,p}$ norm of a form α by

$$\|\alpha\|^p_{W^{1,p}} = \int_M |\alpha|^p + \int_M |d\alpha|^p + \int_M |d^*\alpha|^p,$$

and the analytic $W^{1,p}_A$ norm is defined as

$$\|\alpha\|^p_{W^{1,p}_A} = \int_M |\alpha|^p + \int_M |\nabla\alpha|^p.$$

When $p = 2$, it is standard practice to show that two Sobolev norms are equivalent. A key point we need is that these two norms are equivalent for any $p \ge 1$; see [Sco95, Corollary 12].

PROPOSITION 3.3 (C. Scott). *For any k-form α on a compact Riemannian manifold (M, g), two Sobolev norms $W^{1,p}$and $W^{1,p}_A$ are equivalent for $p \ge 1$.*

As a consequence of this equivalence, we have the following proposition.

PROPOSITION 3.4. *For any close form $\alpha \in [\omega]$ and $p \ge 1$, we have the Poincaré inequality,*

$$\int_M |\alpha - \omega|^p \le c_1 \int_M |d^*\alpha|^p. \tag{3.4}$$

Moreover, we have the Sobolev–Poincaré inequality on (M^4, ω, g),

$$\int_M |\alpha - \omega|^2 \le c_2 \left(\int_M |d^*\alpha|^{4/3} \right)^{3/2}. \tag{3.5}$$

Proof. Given the equivalence of two Sobolev norms on differential forms, the proof is rather standard and we sketch a brief argument. The standard Sobolev inequality implies that

$$\int_M |\alpha - \omega|^2 \le C \left(\int_M |\nabla(\alpha - \omega)|^{4/3} \right)^{3/2} + C \left(\int_M |\alpha - \omega|^{4/3} \right)^{3/2}.$$

The equivalence of the Sobolev norms implies that, with $d\alpha = 0$,

$$\int_M |\alpha - \omega|^2 \le C \left(\int_M |d^*\alpha|^{4/3} \right)^{3/2} + C \left(\int_M |\alpha - \omega|^{4/3} \right)^{3/2}.$$

Hence, (3.5) follows by the Poincaré inequality (3.4), with $p = 4/3$. The proof of (3.4) follows by a standard compactness argument and norm equivalence in Proposition 3.3. Suppose otherwise;

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then there exists a sequence of forms $\alpha_k \in [\omega]$ such that

$$\int_M |d^*\alpha_k|^p \le \frac{1}{k}\int_M |\alpha_k - \omega|^p.$$

Suppose $\int_M |\alpha_k - \omega|^p = \lambda_k^p$ for each k. We consider $\beta_k = \lambda_k^{-1}(\alpha_k - \omega)$. Then β_k is an exact two form, and

$$\int_M |d^*\beta_k|^p \le \frac{1}{k}, \quad \int_M |\beta_k|^p = 1.$$

We write $\beta_k = d\zeta_k$, where $\zeta_k \in \mathrm{Im}(d^*)$ by Hodge decomposition. By the equivalence of the Sobolev norms, $\beta_k \in W^{1,p}$ with bounded Sobolev norm; note that by the Hodge theory, ζ_k is uniquely determined by β_k and has a uniform $W^{2,p}$ bound. Hence, β_k converges by subsequence weakly in $W^{1,p}$ to an exact form $\beta_\infty = d\zeta_\infty$. Moreover, the weak convergence implies that

$$\int_M |d^*\beta_\infty|^p \le \liminf \int_M |d^*\beta_k|^p = 0.$$

It follows that $d^*\beta_\infty = d^*d\zeta_\infty = 0$, and hence $\beta_\infty = d\zeta_\infty = 0$ (we have $\zeta_\infty = 0$ as well, as a consequence of $\zeta_\infty \in \mathrm{Im}(d^*)$). However, the Rellich–Kondrachov theorem implies that β_k converges strongly in L^p. This contradicts the fact that β_k converges strongly in L^p to $\beta_\infty = 0$, since $\|\beta_k\|_{L^p} = 1$. □

With the Sobolev–Poincaré inequality, we have the following monotonicity of the normalized energy functional $E_0(\rho)$ along the flow.

Proposition 3.5. *Suppose the flow (3.1) exists in* $[0, T)$. *For any* $t < T$,

$$E_0(\rho(t)) \le E_0(\rho_0)e^{-t/c_2}. \tag{3.6}$$

Proof. We have, along the flow,

$$\partial_t \int_M |\rho - \omega|^2 = \partial_t \int_M |\rho|^2 = -\int_M \frac{|d^*\rho|^2}{\sqrt{u}}.$$

Apply the Hölder inequality:

$$\int_M fh \le \left(\int_M f^{3/2}\right)^{2/3}\left(\int_M h^3\right)^{1/3} \quad \text{with } f = \frac{|d^*\rho|^{4/3}}{u^{1/3}}, \quad h = u^{1/3}.$$

We get

$$\left(\int_M |d^*\rho|^{4/3}\right)^{3/2} \le \int_M \frac{|d^*\rho|^2}{\sqrt{u}}\left(\int_M u\right)^{1/2} = \int_M \frac{|d^*\rho|^2}{\sqrt{u}}.$$

By the Sobolev–Poincaré inequality (3.5), we have

$$\int_M |\rho - \omega|^2 \le c_2\left(\int_M |d^*\rho|^{4/3}\right)^{3/2} \le c_2\int_M \frac{|d^*\rho|^2}{\sqrt{u}}.$$

We get (3.6) as a consequence of

$$\partial_t \int_M |\rho - \omega|^2 \le -\frac{1}{c_2}\int_M |\rho - \omega|^2.$$

□

Now we are ready to state the main theorem in this section.

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THEOREM 3.6 (Stability). *There exists $\epsilon > 0$ such that if $\|\rho_0 - \omega\|_{C^{1,\alpha}} < \epsilon$, then the flow (3.1) exists over $[0, \infty)$ and converges to ω when $t \to \infty$.*

The proof of Theorem 3.6 follows the standard line of argument given the exponential decay of normalized energy in (3.6). First, we need a more precise statement of short time existence, including the smooth dependence of initial data (finite time stability). The following is a standard fact of quasi-parabolic system applied to (3.1).

PROPOSITION 3.7. *Suppose there exists $\delta_0 = \delta_0(\omega, g)$, $t_0 = t_0(\omega, g)$, $C_0 = C_0(\omega, g)$ and $C_k = C_k(\omega, g, k)$, $k \geq 2$, for any initial data ρ_0 satisfying*

$$\|\rho_0 - \omega\|_{C^{1,\alpha}} < \delta_0.$$

Then the flow exists over $[0, t_0]$, and for any $t \in [0, t_0]$,

$$\|\rho - \omega\|_{C^{1,\alpha}} \leq C_0 \|\rho_0 - \omega\|_{C^{1,\alpha}}.$$

Moreover, for any $t \in [t_0/2, t_0]$ and $k \geq 2$,

$$\|\rho - \omega\|_{C^{k,\alpha}} \leq C_k \|\rho_0 - \omega\|_{C^{1,\alpha}}.$$

We assume $\delta_0 \in (0, 1)$ is sufficiently small and denote, for $k \geq 2$,

$$V_0 = \{\rho \in [\omega] : \|\rho - \omega\|_{C^{1,\alpha}} < \delta_0, \|\rho - \omega\|_{C^{k,\alpha}} \leq C_k\}.$$

In particular, for any $\rho \in V_0$, its potential volume u is closed to 1, such that the flow (3.1) is uniformly parabolic. Now, we fix the constants δ_0, t_0, C_0, C_k and V_0. For any $t \in [t_0/2, t_0]$, $\rho(t) \in V_0$. Since $\rho(t_0) \in V_0$, we can get a solution of (3.1) over $[t_0, 2t_0]$ by applying Proposition 3.7 with initial data $\rho(t_0)$. For $t \in [t_0, 2t_0]$, we have

$$\|\rho - \omega\|_{C^{1,\alpha}} \leq C_0 \|\rho(t_0) - \omega\|_{C^{1,\alpha}} \leq C_0^2 \|\rho_0 - \omega\|_{C^{1,\alpha}}$$

and

$$\|\rho - \omega\|_{C^{k,\alpha}} \leq C_k \|\rho(t_0) - \omega\|_{C^{k,\alpha}} \leq C_k^2 \|\rho_0 - \omega\|_{C^{1,\alpha}}.$$

Now, by choosing $\epsilon << \delta_0$ sufficiently small, we can get that $\rho(t) \in V_0$ for $t \in [t_0, 2t_0]$ if $\|\rho_0 - \omega\|_{C^{1,\alpha}} < \epsilon$. Applying this repeatedly, we can get finite time stability near ω.

PROPOSITION 3.8 (Finite time stability). *For any $T \geq t_0$, there exists $\epsilon = \epsilon(T, \delta_0)$ such that if $\|\rho_0 - \omega\|_{C^{1,\alpha}} < \epsilon$, the flow (3.1) exists over $[0, T]$, and $\rho(t) \in V_0$ for $t \in [t_0, T]$.*

Now we are ready to prove Theorem 3.6.

Proof. Fix $T_0 >> 1$ such that the flow exists over $[0, T_0]$ for the initial data $\tilde{\rho}_0$ satisfying $\|\tilde{\rho}_0 - \omega\|_{C^{1,\alpha}} < \epsilon_0 = \epsilon(T_0, \delta_0)$ and $\rho(t)$ stays in V_0 over $[t_0, T_0]$. Now choose $\epsilon_1 << \epsilon_0$ such that the flow exists over $[0, T_1]$ with $T_1 >> T_0$ and $\rho \in V_0$ over $[t_0, T_1]$ if the initial data satisfies

$$\|\tilde{\rho}_0 - \omega\|_{C^{1,\alpha}} < \epsilon_1.$$

To obtain a uniform stability, we apply Proposition 3.6 to get

$$\|\rho(t) - \omega\|_{L^2}^2 \leq e^{-t/c_2} \|\rho_0 - \omega\|_{L^2}^2 \leq e^{-t/c_2}.$$

Hence, we get that $\|\rho(t) - \omega\|_{C^{k,\alpha}} \leq C_k$ (since $\rho \in V_0$) and $\|\rho(t) - \omega\|_{L^2}^2 \leq e^{-t/c_2}$, where c_2 is the Sobolev–Poincaré constant. By the interpolation inequality, we get a uniform estimate for T_1 sufficiently large and $t \geq T_1/2$:

$$\|\rho(t) - \omega\|_{C^{1,\alpha}} < \epsilon_0. \tag{3.7}$$

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In particular, $\rho(T_1)$ satisfies

$$\|\rho(T_1)-\omega\|_{C^{1,\alpha}}<\epsilon_0.$$

Now we fix T_1 and take $\epsilon=\epsilon_1$. Extend the flow with initial data $\rho(T_1)$. Then we obtain the flow solution $\rho(t)$ over $[0,T_1+T_0]$ with $\rho(t)\in V_0$ over $[t_0,T_1+T_0]$. Since the uniform estimate (3.7) on $\rho(t)$ holds for all $t\geq T_1/2$, by repeating the argument, the flow exists over $[0,\infty)$ with uniformly bounded $C^{k,\alpha}$ norm, and the energy decay implies that $\rho(t)$ converges to ω in the exponential rate. □

3.2 Shi-type estimates

It is not clear whether the flow will exist for all time or develop a finite time singularity in general. To understand the behavior of a possible finite time singularity, we develop the Shi-type estimate [Shi89] for the flow. First, we compute the evolution equations. The evolution equation of ρ reads, in local coordinates,

$$\partial_t\rho_{ij}=\frac{\Delta\rho_{ij}+(\mathcal{R}(\rho))_{ij}}{\sqrt{u}}+\frac{u_i\rho_{jl,l}-u_j\rho_{il,l}}{2\sqrt{u^3}}.$$

In this section, we assume that

$$\frac{1}{\sqrt{u}}+|\nabla\rho|^2\leq C_0. \tag{3.8}$$

Since, along the flow, $\|\rho\|_{L^2(M)}$ is uniformly bounded, the assumption (3.8) implies that $|\rho|$ is uniformly bounded. To derive the Shi-type estimates, we need only the following structures:

$$\partial_t\rho=\frac{\Delta\rho+\mathcal{R}\circ\rho}{\sqrt{u}}+\frac{\rho\circ\nabla\rho\circ\nabla\rho}{\sqrt{u^3}}, \tag{3.9}$$

where $A\circ B$ describes the possible contraction of A and B. Note that we have

$$\nabla u=\tfrac{1}{2}\langle(*\rho),\nabla\rho\rangle=\rho\circ\nabla\rho.$$

We have the following proposition.

Proposition 3.9. *Assuming (3.8), we have*

$$\partial_t|\nabla\rho|^2\leq\frac{\Delta|\nabla\rho|^2-\frac{3}{2}|\nabla^2\rho|^2}{\sqrt{u}}+C,$$

where C is a uniformly bounded constant depending on the metric g, its curvature and derivative of curvature, and the bound in (3.8). More generally, we have, for $k\geq 2$,

$$\partial_t|\nabla^k\rho|^2\leq\frac{\Delta|\nabla^k\rho|^2-\frac{3}{2}|\nabla^{k+1}\rho|^2}{\sqrt{u}}+C_k\left(\sum_{i=2}^{k}|\nabla^i\rho|^{2(k+1)/i}+1\right). \tag{3.10}$$

Proof. This is standard practice. With the symbolic notation, we have

$$\partial_t\nabla\rho=\frac{\Delta\nabla\rho}{\sqrt{u}}+\frac{\mathcal{R}\circ\nabla\rho+\nabla\mathcal{R}\circ\rho}{\sqrt{u}}+\frac{\nabla^2\rho\circ\nabla\rho\circ\rho+\mathcal{R}\circ\rho\circ\rho\circ\nabla\rho}{\sqrt{u^3}}+\frac{\rho\circ\rho\circ\nabla\rho\circ\nabla\rho\circ\nabla\rho}{\sqrt{u^5}}.$$

It follows that

$$\partial_t|\nabla\rho|^2\leq\frac{\Delta|\nabla\rho|^2-2|\nabla^2\rho|^2}{\sqrt{u}}+C(|\nabla^2\rho|+1)\leq\frac{\Delta|\nabla\rho|^2-\frac{3}{2}|\nabla^2\rho|^2}{\sqrt{u}}+C. \tag{3.11}$$

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Note that $|\nabla u|$ is assumed to be bounded. Hence, we have

$$\partial_t \nabla^2 \rho \leq \frac{\Delta \nabla^2 \rho}{\sqrt{u}} + C(|\nabla^3 \rho| + |\nabla^2 \rho|^2 + |\nabla^2 \rho| + 1).$$

And, hence,

$$\begin{aligned}\partial_t |\nabla^2 \rho|^2 &\leq \frac{\Delta |\nabla^2 \rho|^2 - 2|\nabla^3 \rho|^2}{\sqrt{u}} + C|\nabla^2 \rho|(|\nabla^3 \rho| + |\nabla^2 \rho|^2 + |\nabla^2 \rho| + 1)) \\ &\leq \frac{\Delta |\nabla^2 \rho|^2 - \frac{3}{2}|\nabla^3 \rho|^2}{\sqrt{u}} + C(|\nabla^2 \rho|^3 + 1).\end{aligned}$$

This proves (3.10) for $k = 2$. For general k, an inductive computation gives that

$$\partial_t |\nabla^k \rho|^2 \leq \frac{\Delta |\nabla^k \rho|^2 - 2|\nabla^{k+1} \rho|^2}{\sqrt{u}} + C|\nabla^k \rho| \left(\sum |\nabla^{i_1} \rho| |\nabla^{i_2} \rho| \dots |\nabla^{i_p} \rho| + 1 \right),$$

where the sum is taken such that $i_1 + i_2 + \cdots + i_p \leq k + 2$, with $1 \leq i_a \leq k + 1$. By applying Young's inequality,

$$a_1 a_2 \dots a_k \leq \sum \frac{a_i^{p_i}}{p_i}, \quad \sum p_i^{-1} = 1,$$

we can get

$$\partial_t |\nabla^k \rho|^2 \leq \frac{\Delta |\nabla^k \rho|^2 - \frac{3}{2}|\nabla^{k+1} \rho|^2}{\sqrt{u}} + C_k \left(\sum_{i=2}^{k} |\nabla^i \rho|^{2(k+1)/i} + 1 \right). \qquad \square$$

Now we state the Shi-type estimate.

PROPOSITION 3.10. *Suppose (3.8) holds over $[0, T]$. Then we have the uniform bound for $\nabla^k \rho$ for $t \in (0, T]$, $k \geq 2$,*

$$|\nabla^k \rho| \leq C_k t^{-(k-1)/2}. \tag{3.12}$$

Such an estimate can be made local. Suppose (3.8) holds on $B_r(p) \times [0, T]$, where $B_r(p)$ is the geodesic ball of the background metric g. Then over $B_{r/2}(p) \times [T/2, T]$, we have

$$|\nabla^k \rho| \leq C_k(g, C_0, k, T, r). \tag{3.13}$$

Proof. Given (3.10), the proof of (3.12) follows almost line by line that of He and Li [HL21, Proposition 2.1]. The local estimate is also standard. Denote $Q = (\mu + |\nabla \rho|^2)|\nabla^2 \rho|^2$, with μ determined later. We compute

$$\begin{aligned}\partial_t Q &= \partial_t |\nabla \rho|^2 \, |\nabla^2 \rho|^2 + (\mu + |\nabla \rho|^2) \partial_t |\nabla^2 \rho|^2 \\ &\leq \left(\frac{\Delta |\nabla \rho|^2 - \frac{3}{2}|\nabla^2 \rho|^2}{\sqrt{u}} + C \right) |\nabla^2 \rho|^2 \\ &\quad + (\mu + |\nabla \rho|^2) \left(\frac{\Delta |\nabla^2 \rho|^2 - \frac{3}{2}|\nabla^3 \rho|^2}{\sqrt{u}} + C(|\nabla^2 \rho|^3 + 1) \right) \\ &\leq \frac{\Delta Q}{\sqrt{u}} - \frac{3}{2\sqrt{u}} |\nabla^2 \rho|^4 - \frac{3\mu}{2\sqrt{u}} |\nabla^3 \rho|^2 + C_1(|\nabla^3 \rho| \, |\nabla^2 \rho|^2 + |\nabla^2 \rho|^3 + 1).\end{aligned}$$

Since $u \leq C_2 = C_2(C_0, g)$ is uniformly bounded by our assumption, by choosing μ sufficiently large, depending on C_1 and C_2, we have

$$\partial_t Q - \frac{\Delta Q}{\sqrt{u}} \leq -A_0 Q^2 + A_1. \tag{3.14}$$

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With (3.14) the localization is standard as well. Choose a cut-off function ϕ supported in $B_r(p)$, and $\phi = 1$ in $B_{r/2}$. Denote, for a suitable constant $A_2 = A_2(r^{-1}, A_0, A_1, C_0)$,

$$H = \frac{A_2}{\phi^2} + \frac{1}{A_0 t} + \sqrt{A_1 A_0^{-1}}.$$

We compute

$$\partial_t H = -\frac{1}{A_0 t^2}, \quad \Delta H = \frac{A_2(-2\phi\Delta\phi + 6|\nabla\phi|^2)}{\phi^4}, \quad H^2 > \frac{A_2^2}{\phi^4} + \frac{1}{A_0^2 t^2} + \frac{A_1}{A_0}.$$

For $x \in B_r(p)$, we have $|\Delta\phi| + |\nabla\phi|^2 \le 10r^{-2}$. Hence, for A_2 sufficiently large,

$$\partial_t H - \frac{\Delta H}{\sqrt{u}} > -A_0 H^2 + A_1.$$

A maximum principle argument then implies that $Q \le H$ over $B_r(p) \times [0, T]$. This proves (3.13) for $k = 2$. For general k, take $Q_k = (\mu_k + |\nabla^{k-1}\rho|^2)|\nabla^k \rho|^2$ for sufficiently large μ_k. □

Hence, by the Shi-type estimate, if the flow exists for the maximal interval $[0, T)$ with $T < \infty$, then it develops finite singularities at $t = T$ with $1/\sqrt{u} + |\nabla\rho|^2$ blowing up. Suppose $1/\sqrt{u}$ is bounded; only the Sobolev bound $\rho \in W^{1,p}$ for $p > 4$ is needed (ρ would be Hölder continuous in particular). This is a well-known technique in the study of harmonic map theory. Even though the argument is standard for specialists, it is technically very involved. Since we do not need this result, we will not pursue such a generalization.

3.3 Doubling-time property

In this subsection, we show a doubling-time property for $|\nabla\rho|^2 + \sqrt{u}^{-1}$ as follows. Denote

$$\Lambda(x, t) = |\nabla\rho|^2(x, t) + \frac{1}{\sqrt{u(x, t)}} + 1, \quad \Lambda(t) = \max_x \Lambda(x, t), \quad \Lambda_0 = \Lambda(0).$$

Proposition 3.11. *There exists a uniformly bounded constant C_0 such that*

$$\Lambda(t) \le \frac{\Lambda_0}{\sqrt[7]{1 - 7C_0\Lambda_0^7 t}} \quad \text{for } t < (7C_0\Lambda_0^7)^{-1}. \tag{3.15}$$

Proof. Note that $\|\rho\|_{L^2}$ is uniformly bounded; $\max|\rho|(x, t) \le C\Lambda(t)$ holds for a uniformly bounded constant C depending only on the background geometry. Recall

$$\partial_t \nabla\rho = \frac{\Delta\nabla\rho}{\sqrt{u}} + \frac{\mathcal{R}\circ\nabla\rho + \nabla\mathcal{R}\circ\rho}{\sqrt{u}} + \frac{\nabla^2\rho\circ\nabla\rho\circ\rho + \mathcal{R}\circ\rho\circ\rho\circ\nabla\rho}{\sqrt{u^3}} + \frac{\rho\circ\rho\circ\nabla\rho\circ\nabla\rho\circ\nabla\rho}{\sqrt{u^5}}.$$

Hence, comparing (3.11), we compute

$$\partial_t|\nabla\rho|^2 - \frac{\Delta|\nabla\rho|^2}{\sqrt{u}} \le C(\Lambda(t))^8. \tag{3.16}$$

For the potential u, we compute

$$\partial_t u = \frac{\Delta u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\langle\nabla(*\rho), \nabla\rho\rangle + \frac{\nabla u\circ\nabla\rho\circ\rho}{\sqrt{u^3}} + \frac{\langle\mathcal{R}(\rho), *\rho\rangle}{\sqrt{u}}. \tag{3.17}$$

We can derive

$$\partial_t u^{-1/2} - \frac{\Delta u^{-1/2}}{\sqrt{u}} \le C(\Lambda(t))^8. \tag{3.18}$$

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It follows that, with a uniformly bounded constant C_0,

$$\partial_t \Lambda(x,t) - \frac{\Delta \Lambda(x,t)}{\sqrt{u}} \leq C_0 (\Lambda(t))^8. \tag{3.19}$$

Hence, by the maximum principle, we can deduce that

$$\frac{d\Lambda(t)}{dt} \leq C_0 (\Lambda(t))^8.$$

It follows that, for $t \in [0, (7C_0\Lambda_0^7)^{-1})$, we have

$$\Lambda(t) \leq \frac{\Lambda_0}{\sqrt[7]{1 - 7C_0\Lambda_0^7 t}}. \tag{3.20}$$

□

If we denote

$$\tilde{\Lambda}(x,t) = |\rho|^2(x,t) + |\nabla \rho(x,t)|^2 + \frac{1}{\sqrt{u(x,t)}} + 1,$$

we can derive a pointwise inequality of $\tilde{\Lambda}(x,t)$, compared with (3.19):

$$\partial_t \tilde{\Lambda}(x,t) \leq \frac{\Delta \tilde{\Lambda}(x,t)}{\sqrt{u}} + C_0 (\tilde{\Lambda}(x,t))^8. \tag{3.21}$$

3.4 Long time existence and convergence in $W^{1,2}$ assuming the bound of $|\nabla \log u|$

Here, we assume $|\nabla \log u| \leq C_0$. Note that we assume the normalized condition

$$\int_M u \, \mathrm{dvol} = \int_M \mathrm{dvol} = 1.$$

Hence, for any t, there exists $x_t \in M$ such that $u(x_t, t) = 1$. If $|\nabla \log u| \leq C_0$, then there exists $C_1 = C_1(C_0, g)$ such that

$$-C_1 \leq \log u \leq C_1.$$

This, in particular, implies that

$$e^{-C_1} \leq u \leq e^{C_1} \quad \text{and} \quad |\nabla u| \leq C_0 e^{C_1}.$$

We have the following proposition.

PROPOSITION 3.12. *Suppose the flow exists on $[0,T]$ with $T < \infty$ and $|\nabla \log u| \leq C_0$ over $[0,T]$. Then the flow can be extended across T.*

Proof. We need to be a bit more precise here, given the extra assumptions on u and ∇u. We write the equation as

$$\partial_t \rho = \frac{\Delta \rho}{\sqrt{u}} + \frac{\mathcal{R} \circ \rho}{\sqrt{u}} + \nabla(u^{-1/2}) \circ \nabla \rho. \tag{3.22}$$

We compute

$$\partial_t |\rho|^2 = \frac{\Delta |\rho|^2 - 2|\nabla \rho|^2}{\sqrt{u}} + \frac{\langle \mathcal{R}(\rho), \rho \rangle}{\sqrt{u}} + \frac{\nabla \log u \circ \nabla \rho \circ \rho}{\sqrt{u}}. \tag{3.23}$$

Suppose $|\nabla \log u| \leq C_0$, then we have

$$\partial_t |\rho|^2 - \frac{\Delta |\rho|^2}{\sqrt{u}} \leq \frac{-2|\nabla \rho|^2}{\sqrt{u}} + C(|\rho|^2 + |\nabla \rho||\rho|). \tag{3.24}$$

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It follows that $|\rho|^2 \leq |\rho_0|^2 e^{Ct}$ at time t. A straightforward computation gives

$$\partial_t |\nabla\rho|^2 \leq \frac{\Delta|\nabla\rho|^2 - 2|\nabla^2\rho|^2}{\sqrt{u}} + C(|\nabla^2\rho||\nabla\rho| + |\nabla\rho|^2 + |\nabla\rho||\rho| + |\nabla^2 u||\nabla\rho|). \tag{3.25}$$

For the potential u, we compute

$$\partial_t u = \frac{\Delta u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\langle\nabla(*\rho), \nabla\rho\rangle + \frac{\nabla u \circ \nabla\rho \circ \rho}{\sqrt{u^3}} + \frac{\langle\mathcal{R}(\rho), *\rho\rangle}{\sqrt{u}}. \tag{3.26}$$

A straightforward computation also gives, with uniform bounds on u and $|\nabla u|$,

$$\begin{aligned}\partial_t |\nabla u|^2 \leq{}& \frac{\Delta|\nabla u|^2 - 2|\nabla^2 u|^2}{\sqrt{u}} \\ &+ C(|\nabla^2 u| + |\nabla^2\rho||\nabla\rho| + |\nabla^2 u||\nabla\rho| + |\nabla^2\rho| + |\nabla\rho|^2 + |\nabla\rho| + |\rho|^2).\end{aligned} \tag{3.27}$$

Denote $f = |\nabla\rho|^2 + A|\nabla u|^2 + B|\rho|^2 + 1$ with $1 << A << B$, where the constants A, B depend on the bound on u and ∇u. Then we have, for a uniformly bounded constant C_1,

$$\partial_t f - \frac{\Delta f}{\sqrt{u}} \leq C_1 f. \tag{3.28}$$

A straightforward maximum principle argument implies that

$$|\nabla\rho|^2 \leq C_2 e^{C_1 t}.$$

Hence, if the flow exists over $[0, T]$ for $T < \infty$ such that $|\nabla \log u| \leq C_0$, then $|\nabla\rho|^2 \leq C_2 e^{C_1 T}$. By the Shi-type estimate, we know that the flow can be extended across T. In other words, the flow exists as long as $|\nabla \log u|$ stays bounded. □

We have the following convergence result when assuming $|\nabla \log u| \leq C_0$.

Proposition 3.13. *Suppose the flow exists for all time with $|\nabla \log u| \leq C_0$ for an initial symplectic form $\rho_0 \in [\omega_0]$. Then the flow converges in L^2 to ω_0 when $t \to \infty$.*

Proof. If one obtains a uniform estimate of $|\nabla \log u| \leq C_0$ along the flow, then the long time existence follows Proposition 3.12. The point bounds on $|\rho|^2, |\nabla\rho|^2$, etc., derived in the last subsection, assuming $|\nabla \log u| \leq C_0$ depends on a time factor e^{Ct} which is not bounded when $t \to \infty$. To derive convergence, we use the L^2 estimate. Recall that we have

$$\partial_t \int_M |\rho - \omega_0|^2 = -\int_M \frac{|d^*\rho|^2}{\sqrt{u}} \leq -C_1 \int_M |d^*\rho|^2. \tag{3.29}$$

We compute, using $|\nabla \log u| \leq C_0$,

$$\begin{aligned}\partial_t \int_M |d^*\rho|^2 &= -\int_M \left\langle d^*\rho, d^* d\left(\frac{d^*\rho}{\sqrt{u}}\right)\right\rangle = -\int_M \left\langle dd^*\rho, d\left(\frac{d^*\rho}{\sqrt{u}}\right)\right\rangle \\ &\leq -\int_M \frac{|dd^*\rho|^2}{\sqrt{u}} + C\int_M |dd^*\rho||d^*\rho|.\end{aligned} \tag{3.30}$$

Denote

$$Q_1 = \int_M |d^*\rho|^2 + A_1 \int_M |\rho - \omega_0|^2.$$

It follows that, for A_1 sufficiently large,

$$\partial_t Q_1 \leq -C_1\left(\int_M |dd^*\rho|^2 + \int_M |d^*\rho|^2\right). \tag{3.31}$$

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An application of the Poincaré inequality implies that $\partial_t Q_1 \leq -CQ_1$ for some positive constant C. It follows that, for $t \in [0, \infty)$,

$$\int_M |d^*\rho|^2 + A_1 \int_M |\rho - \omega_0|^2 \leq A_0 e^{-Ct}. \tag{3.32}$$

When $t \to \infty$, $\rho(t)$ has a uniformly bounded $W^{1,2}$ norm, and hence converges strongly in L^2 and weakly in $W^{1,2}$ to a limit form ρ_∞ with $\rho_\infty \in [\omega_0]$, $d\rho_\infty = 0$, $\rho_\infty \in W^{1,2}$. The convergence implies that

$$\int_M |d^*\rho_\infty|^2 \leq \liminf \int_M |d^*\rho|^2 = 0, \quad \int_M |\rho_\infty - \omega_0|^2 = 0.$$

It follows that $\rho_\infty = \omega_0$. □

3.5 Smooth convergence

To obtain smooth convergence, we need to obtain the pointwise bound on $|\rho|$ and $|\nabla\rho|$. This requires more careful analysis. We have the following proposition.

PROPOSITION 3.14. *Suppose $|\nabla \log u| \leq C_0$ over $[0, \infty)$ along the flow. Then $|\rho|$ and $|\nabla\rho|$ are both uniformly bounded. It follows that the flow converges to ω smoothly when $t \to \infty$ with uniform control.*

Proof. The main point is to bound $f = |\nabla\rho|^2 + A|\nabla u|^2 + B|\rho|^2 + 1$, which satisfies (3.28) given the uniform control on $|\nabla \log u|$. We recall the inequality

$$\partial_t f \leq \frac{\Delta f}{\sqrt{u}} + C_1 f. \tag{3.33}$$

We apply the parabolic Moser iteration, which is a standard and very powerful tool for specialists. Since the argument is involved and technically delicate in its nature, we include the details for completeness. For simplicity, we work on the time interval $[0, 1]$. For $2p - 1 > 0$ (say take $p \geq (5/8)$), we compute

$$\begin{aligned}
\partial_t \int_M f^{2p} &\leq 2p \int_M f^{2p-1} \frac{\Delta f}{\sqrt{u}} + 2pC_1 \int_M f^{2p} \\
&= -2p(2p-1) \int_M f^{2p-2} \frac{|\nabla f|^2}{\sqrt{u}} - 2p \int_M f^{2p-1} \nabla f \nabla (u^{-1/2}) + 2p \int_M f^{2p} \\
&\leq -4c_0 p(2p-1) \int_M f^{2p-2} |\nabla f|^2 + pC \int_M f^{2p-1} |\nabla f| + 2p \int_M f^{2p} \\
&\leq -4c_0 \left(2 - \frac{1}{p}\right) \int_M |\nabla (f^p)|^2 + C \int_M f^{p-1} |\nabla (f^p)| + 2p \int_M f^{2p} \\
&\leq -c_0 \int_M |\nabla (f^p)|^2 + pC \int_M f^{2p},
\end{aligned} \tag{3.34}$$

where we use the Hölder inequality and Young's inequality. We use c_0 to denote a uniform positive constant depending on $\min u^{-1/2}$, and C to denote a uniformly bounded positive constant which can vary line by line. Applying a Sobolev inequality to $h = f^p$,

$$\left(\int_M h^4 \right)^{1/2} \leq c_1 \int_M (|\nabla h|^2 + h^2),$$

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we have the following inequality, for $p \in \left[\frac{5}{8}, \infty\right)$:

$$\partial_t \int_M f^{2p} + c_0 \left(\int_M f^{4p} \right)^{1/2} \leq pC \int_M f^{2p}. \tag{3.35}$$

Now, given $0 \leq a < b \leq 1$, let $\phi : [0,1] \to [0,1]$ be a Lipschitz function such that

$$\phi(s) = 0, \quad s \in [0,a]; \quad \phi(s) = 1, \quad s \in [b,1],$$

and $\phi(s)$ is linear for $s \in [a,b]$. Then

$$\phi'(s) = \frac{1}{b-a}, \quad s \in (a,b); \quad \phi' = 0, \quad s \in [0,a) \cup (b,1].$$

Taking integration with respect to t over $[0,t_0]$, for any $t_0 \in [a,1]$, we have

$$\int_0^{t_0} \phi^2(t) \partial_t \int_M f^{2p} + c_0 \int_0^{t_0} \phi^2(t) \left(\int_M f^{4p} \right)^{1/2} \leq pC \int_0^{t_0} \phi^2(t) \int_M f^{2p}. \tag{3.36}$$

Integrating by parts on the first term above, we have

$$(\phi^2 \int_M f^{2p})(t_0) + c_0 \int_0^{t_0} \phi^2 \left(\int_M f^{4p} \right)^{1/2} \leq \left(pC + \frac{2}{b-a} \right) \int_a^{t_0} \int_M f^{2p}. \tag{3.37}$$

We have, by (3.37),

$$\max_{[b,1]} \int_M f^{2p} \leq \left(pC + \frac{2}{b-a} \right) \int_a^1 \int_M f^{2p} \tag{3.38}$$

and

$$\int_b^1 \left(\int_M f^{4p} \right)^{1/2} \leq \left(pC + \frac{2}{b-a} \right) \int_a^1 \int_M f^{2p}. \tag{3.39}$$

Now, for any $t \in [b,1]$, applying the Hölder inequality, we get

$$\int_M f^{3p} \leq \left(\int_M f^{4p} \right)^{1/2} \left(\int_M f^{2p} \right)^{1/2} \leq \left(\max_{[b,1]} \int_M f^{2p} \right)^{1/2} \left(\int_M f^{4p} \right)^{1/2}. \tag{3.40}$$

It follows that over the space time $[b,1] \times M$,

$$\int_b^1 \int_M f^{3p} \leq \left(\max_{[b,1]} \int_M f^{2p} \right)^{1/2} \int_b^1 \left(\int_M f^{4p} \right)^{1/2}. \tag{3.41}$$

Applying (3.38) and (3.39) to (3.41), we have

$$\left(\int_b^1 \int_M f^{3p} \right)^2 \leq \left(pC + \frac{2}{b-a} \right)^3 \left(\int_a^1 \int_M f^{2p} \right)^3. \tag{3.42}$$

We write the above as

$$\|f\|_{L^{3p}([b,1]\times M)} \leq \left(pC + \frac{2}{b-a} \right)^{1/2p} \|f\|_{L^{2p}([a,1]\times M)}. \tag{3.43}$$

In other words, we have

$$\log \|f\|_{L^{3p}([b,1]\times M)} \leq \frac{1}{2p} \log(pC + 2(b-a)^{-1}) + \log \|f\|_{L^{2p}([a,1]\times M)}. \tag{3.44}$$

Now, take a sequence of p_i and b_i such that

$$p_i = \frac{1}{2} \left(\frac{3}{2} \right)^{i+1}, \quad b_i = \frac{1}{2} - \frac{1}{2^{i+1}}, \quad i = 0, 1, 2, \ldots$$

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and denote $q_i = 2p_i$. For each i, applying (3.44) with $p = p_i$, $a = b_i, b = b_{i+1}$, we get

$$\log \|f\|_{L^{q_{i+1}}([b_{i+1},1]\times M)} \le \left(\frac{2}{3}\right)^{i+1} \log(2^{i+2} + Cq_i) + \log \|f\|_{L^{q_i}([b_i,1]\times M)}. \tag{3.45}$$

By letting $i \to \infty$ and the fact that $\sum_i (2/3)^i i < \infty$, (3.45) gives

$$\log \|f\|_{L^\infty([\frac{1}{2},1]\times M)} \le C + \log \|f\|_{L^{3/2}([0,1]\times M)}. \tag{3.46}$$

By choosing the constants in a more careful way, we can indeed get that, for any $r > 1$, since (3.44) holds only for $2p > 1$,

$$\log \|f\|_{L^\infty([\frac{1}{2},1]\times M)} \le C_1(r) + \log \|f\|_{L^r([0,1]\times M)}, \tag{3.47}$$

where the constant C_1 depends on r, and it could blow up when $r \to 1$. To overcome this difficulty, we prove that f is indeed in $L^{3/2}([t, t+1] \times M)$ for any $t \ge 0$. Note that ρ is in $W^{1,2}(M)$ and $|\rho|$ is in $L^4(M)$. But at the moment, we have no uniform control on $\int_M |\nabla\rho|^3$ for each time slice. Note that, given bounded $|\nabla u|$,

$$\int_M f^{3/2} \le 2 \int_M |\nabla\rho|^3 + C \int_M |\rho|^3 + C.$$

Hence, to show that $f \in L^{3/2}([t, t+1] \times M)$ for each t, it suffices to show that for any t,

$$\int_t^{t+1} \int_M |\nabla\rho|^3 \le C. \tag{3.48}$$

To obtain (3.48), we need to use (3.31). Taking the integral over $[t, t+1]$, we get that

$$\int_t^{t+1} \int_M |d^*d\rho|^2 + \int_t^{t+1} \int_M |d^*\rho|^2 \le Q_1(t) - Q_1(t+1) \le C.$$

Using the Bochner–Weitzenböck formula, we obtain that

$$\int_t^{t+1} \int_M |\nabla^2\rho|^2 + \int_t^{t+1} \int_M |\nabla\rho|^2 \le C.$$

By applying the Sobolev inequality on each time slice, we obtain

$$\left(\int_M |\nabla\rho|^4\right)^{1/2} \le C \left(\int_M |\nabla^2\rho|^2 + |\nabla\rho|^2\right).$$

Hence, we have

$$\int_t^{t+1} \left(\int_M |\nabla\rho|^4\right)^{1/2} \le C.$$

Now, we have, since $|\nabla\rho| \in L^2(M)$,

$$\int_t^{t+1} \int_M |\nabla\rho|^3 \le \int_t^{t+1} \left(\int_M |\nabla\rho|^4\right)^{1/2} \left(\int_M |\nabla\rho|^2\right)^{1/2} \le C \int_t^{t+1} \left(\int_M |\nabla\rho|^4\right)^{1/2} \le C.$$

This proves (3.48). Applying (3.46) over $[t, t+1]$ and (3.48), we have the uniform control on f over $[t + (1/2), t]$ for any t. This gives the uniform bounds on $|\rho|$ and $|\nabla\rho|$, which implies the smooth convergence of $\rho \to \omega_0$ along the flow. □

The assumption $|\nabla \log u| \le C$ can be weakened considerately if we carefully refine the iteration arguments above.

PROPOSITION 3.15. *For any $p_0 > 6$, if we assume $u^{-1} + \|\nabla u\|_{L^{p_0}} \le C$ along the flow, then the flow exists for all time and converges to ω smoothly with uniform control.*

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Proof. First, we note that u is bounded above by assuming $|\nabla u| \in L^{p_0}$. The key is to bound $|\rho|$ and $|\nabla \rho|$ along the flow via the parabolic Moser iteration, and the strategy is essentially the same as for Proposition 3.14. We use the same notation and highlight only the major difference in the arguments. We have

$$\begin{aligned}\partial_t |\rho|^2 &\leq u^{-1/2}(\Delta |\rho|^2 - 2|\nabla \rho|^2) + C(|\nabla u||\nabla \rho||\rho| + |\rho|^2) \\ &\leq u^{-1/2}(\Delta |\rho|^2 - |\nabla \rho|^2) + C(|\nabla u|^2 + 1)|\rho|^2. \end{aligned} \tag{3.49}$$

Denote $h = |\rho|^2$. We compute

$$\begin{aligned}\partial_t \int_M h^{2p} &= 2p \int_M h^{2p-1} \partial_t h \\ &\leq 2p \int_M h^{2p-1} \left(u^{-1/2}(\Delta h - |\nabla \rho|^2) + C(|\nabla u|^2 + 1)h \right) \\ &\leq -c_0 \int_M |\nabla (h^p)|^2 + pC \int_M h^{2p}(1 + |\nabla u|^2), \end{aligned} \tag{3.50}$$

which is similar to (3.34); note that the only difference is that we have no pointwise bound on $|\nabla u|$, and hence we have an extra term involved with $|\nabla u|^2$ compared with (3.34). The iteration process is almost identical, but it is more involved without the pointwise bound on $|\nabla u|$. We proceed as in (3.42) to get

$$\left(\int_b^1 \int_M h^{3p} \right)^2 \leq \left(pC + \frac{2}{b-a} \right)^3 \left(\int_a^1 \int_M h^{2p}(1 + |\nabla u|^2) \right)^3. \tag{3.51}$$

Since $|\nabla u| \in L^{p_0/2}(M)$, we can obtain

$$\int_M h^{2p}(1 + |\nabla u|^2) \leq \left(\int_M h^{2pq_0} \right)^{1/q_0} \left(\int_M (1 + |\nabla u|^2)^{p_0/2} \right)^{2/p_0}, \quad \frac{1}{q_0} + \frac{2}{p_0} = 1. \tag{3.52}$$

Note that $q_0 < (3/2)$ and

$$\int_a^1 \left(\int_M h^{2pq_0} \right)^{1/q_0} dt \leq (1-a)^{p_0/2} \left(\int_a^1 \int_M h^{2pq_0} \right)^{1/q_0}. \tag{3.53}$$

Hence, we obtain

$$\left(\int_b^1 \int_M h^{3p} \right)^2 \leq \left(pC + \frac{2}{b-a} \right)^3 C_0 \left(\int_a^1 \int_M h^{2pq_0} \right)^{3/q_0}, \tag{3.54}$$

where C_0 depends on $\|\nabla u\|_{L^{p_0}}$. This implies that

$$\log \|h\|_{L^{3p}([b,1]\times M)} \leq \frac{1}{2p} \left(\log(pC + 2(b-a)^{-1}) + \log C_0 \right) + \log \|h\|_{L^{2pq_0}([a,1]\times M)}. \tag{3.55}$$

Since $q_0 < (3/2)$, we can then proceed with the iteration process to bound h over $[(1/2), 1]$, and hence to bound $|\rho|$. The arguments are almost identical from this point, and we skip the details. To bound $|\nabla \rho|$, denote again $f = |\nabla \rho|^2 + A|\nabla u|^2 + B|\rho|^2 + 1$. Without the bound on $|\nabla u|$, we can obtain

$$\partial_t f \leq \frac{\Delta f}{\sqrt{u}} + Cf(1 + |\nabla u|^2). \tag{3.56}$$

We can bound f following the same argument, given that $|\nabla u| \in L^{p_0}$ for $p_0 > 6$. □

One might ask whether it is sufficient for long time existence to assume $u^{-1} + \|\nabla u\|_{L^p} \leq C$ for any $p > 4$. The following technical question would be much more difficult and one would expect new technical ingredients to prove such a statement.

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Question 3.16. Does the flow exist for all time when assuming

$$u + u^{-1} + \|\nabla u\|_{L^4} \leq C?$$

Remark 3.17. The conformal factor $\sqrt{u}$ can be replaced by any power u^r for $r \in (0, (1/2)]$. For such r, the metric $g_r = u^r g$ has total volume

$$V_r = \int_M u^{2r} dv \leq \int_M dv.$$

The results about stability and long time existence also hold true for $r \in (0, (1/2)]$. But for $r > (1/2)$, because of the Sobolev embedding, we will not have a straightforward exponential energy decay. For stability near ω, one needs extra input.

3.6 Examples and discussions

In this section, we discuss some examples. First, we consider an example on $T^4 = S^1 \times S^1 \times S^1 \times S^1$. Let $\omega_0 = dx_0 \wedge dx_1 + dx_2 \wedge dx_3$ and $g_0 = \sum dx_i \otimes dx_i$. Suppose $\rho_0 = u_0(x_1, x_2) dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ for a positive function $u_0 : T^2 = S^1 \times S^1 \to \mathbb{R}_+$ satisfying

$$\int_{T^2} u_0 dx_1 \wedge dx_2 = \int_{T^2} dx_1 \wedge dx_2. \tag{3.57}$$

Clearly, $d\rho_0 = 0$ and (3.57) imply that $\rho_0 \in [\omega_0]$. A straightforward computation implies that the flow (3.1) reduces to the parabolic flow of $u(t) : T^2 \to \mathbb{R}_+$,

$$\partial_t u = \frac{\Delta u}{\sqrt{u}} - \frac{|\nabla u|^2}{2\sqrt{u^3}}.$$

Note that one can write the equation as

$$\partial_t u = 2\Delta(u^{\frac{1}{2}}). \tag{3.58}$$

This is an example of the *fast diffusion equation*. Since u_0 is assumed to be positive, the maximum principle implies that u stays uniformly bounded and uniformly positive. By the Krylov–Safonov estimate, u has a uniformly bounded Hölder norm. Hence, (3.58) has a unique long time smooth solution with uniformly bounded Hölder norms by the standard parabolic theory. The gradient flow nature implies that u converges smoothly to $u_\infty \equiv 1$.

This extends to the example $(M^4, \omega_0 = \omega_1 \oplus \omega_2) = (\Sigma_1, \omega_1) \times (\Sigma_2, \omega_2)$ for two closed surfaces with two volume forms ω_1, ω_2. Suppose $\rho_0 \in [\omega_0]$ of the form $\rho_0 = u_0 \omega_1 \oplus \omega_2$ for $u_0 : \Sigma_1 \to \mathbb{R}_+$ such that

$$\int_{\Sigma_1} u_0 \omega_1 = \int_{\Sigma_1} \omega_1.$$

The flow (3.1) is then reduced to the fast diffusion equation in (3.58) in the classical setting. The computation is almost identical. The example above does not extend to the case when $\rho_0 = v_0(x_1, x_2) dx_1 \wedge dx_2 + w_0(x_3, x_4) dx_3 \wedge dx_4$ satisfying $v_0, w_0 > 0$, and

$$\int_{T^2} v_0 dx_1 \wedge dx_2 = \int_{T^2} w_0 dx_3 \wedge dx_4 = \int_{T^2} dx_1 \wedge dx_2 = \int_{T^2} dx_3 \wedge dx_4.$$

In this case, $\rho_0 \in [\omega_0]$, but the product structure is not preserved along the flow (3.1). Note that in this example, ρ_0 is isotropic to ω_0 in an obvious way.

Problem 3.18. Does the flow (3.1) exist for all time and converge to ω_0 on T^4 in the above example, with the initial symplectic form

$$\rho_0 = v_0(x_1, x_2) dx_1 \wedge dx_2 + w_0(x_3, x_4) dx_3 \wedge dx_4?$$

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The product example above might give some indication of how to choose the metric $\tilde{g} = g(Q, \cdot)$ and the corresponding Hodge star $\tilde{*}$. Suppose the matrices A and B are defined as $\rho = g(A\cdot, \cdot)$ and $*\rho = g(B\cdot, \cdot)$. For g_0 and ρ_0 given above on T^4, we have

$$A = \begin{pmatrix} 0 & v & 0 & 0 \\ -v & 0 & 0 & 0 \\ 0 & 0 & 0 & w \\ 0 & 0 & -w & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & w & 0 & 0 \\ -w & 0 & 0 & 0 \\ 0 & 0 & 0 & v \\ 0 & 0 & -v & 0 \end{pmatrix}.$$

Hence, $a = -A^2 = \mathrm{diag}(v^2, v^2, w^2, w^2)$ and $b = -B^2 = \mathrm{diag}(w^2, w^2, v^2, v^2)$. One can define $Q = fa^r$ or fb^r for some positive function f and some power r. For the conformal metric, we have $Q = \sqrt{u}I$. For the metric g_ρ, Q is given by $u^{-1}a$. Neither choice of $Q = \sqrt{u}I$ nor $Q = a$ (or $Q = u^{-1}a$) would preserve the product structure. In particular, the flow becomes complicated for these choices. But if we choose $Q = b$ (or b^r for $r > 0$), then the flow preserves the product structure such that

$$\partial_t v = \Delta\left(-\frac{1}{v}\right), \quad \partial_t w = \Delta\left(-\frac{1}{w}\right).$$

If we choose $Q = \sqrt{b}$, then the flow equation preserves the product structure such that

$$\partial_t v = \Delta(\log v), \quad \Delta w = \Delta(\log w).$$

For such a choice, the flow equation becomes the classical fast diffusion equation and the flow exists for all time and converges to ω_0 on T^4 by the standard theory of parabolic equations.

Problem 3.19. Let (M, ω, g) be an almost Kähler manifold of dimension four. For each symplectic form $\rho \in [\omega]$, we define the metric $\tilde{g} = g(b\cdot, \cdot)$. Consider the flow

$$\partial_t \rho = d(\tilde{*} d * \rho).$$

Does the flow exist for all time?

We will analyze the different Hodge flows in the next section. For the second example, we consider $\rho_0 \in [\omega_0]$ on (T^4, ω_0, g_0), with

$$\rho_0 = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + d\theta \quad \text{with } \theta = a(x_1, x_2)dx_3 + b(x_1, x_2)dx_4,$$

where a, b are two functions on $S^1 \times S^1$ (hence, periodic functions on $\mathbb{R} \times \mathbb{R}$). The nondegenerate condition implies that

$$u_0 = 1 - a_{x_1} b_{x_2} + a_{x_2} b_{x_1} > 0.$$

Note that for this example, ρ_0 is isotropic to ω_0 via $\rho_s = \omega_0 + s d\theta$ for $s \in [0, 1]$, with ρ_s nondegenerate for each s. The linear Hodge Laplacian flow reduces to the heat equations for a, b separately:

$$\partial_t a = \Delta a = a_{x_1 x_1} + a_{x_2 x_2}, \quad \partial_t b = \Delta b = b_{x_1 x_1} + b_{x_2 x_2}.$$

However, along the Hodge Laplacian flow, the positivity condition on $u = 1 - a_{x_1} b_{x_2} + a_{x_2} b_{x_1}$ might not hold. This fact should be well known for experts. Since we cannot find a direct reference, we construct an explicit example as follows.

Example 3.20. We consider the linear Hodge flow for $\rho = dx \wedge dy + dz \wedge dw + d(a(x, y)dz + b(x, y)dw)$ on T^4. The flow is reduced to the heat equation on T^2,

$$\partial_t a = \Delta a, \quad \partial_t b = \Delta b,$$

where a and b are 2π periodic functions for x and y, respectively. Take $a(x, y) = a(x)$, and suppose $a_x = f(x)$ while $b(x, y) = h(x)e(y)$. Along the linear flow, $f(x), h(x), e(y)$ are all periodic

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functions (functions on S^1) satisfying the heat equation

$$\partial_t f = f_{xx}, \quad \partial_t h = h_{xx} \quad \text{and } \partial_t e = e_{yy}.$$

Denote $e_y = c(y)$. For the initial data, $u_0(x,y) = 1 - f_0(x)h_0(x)c_0(y)$ and $u(x,y,t) = 1 - f(x,t)h(x,t)c(y,t)$, since $a(x,y,t) = f(x,t)$ is independent of y. There is a restriction on f and c such that

$$\int_0^{2\pi} f_0(x)\,dx = \int_0^{2\pi} c_0(y)\,dy = 0.$$

We choose f_0, h_0 such that $f_0 = 0$ for $x \in [0,\pi]$ and $f_0(x) = \sin 2x$ for $x \in [\pi, 2\pi]$, and $h_0(x) = 0$ for $x \in [\pi, 2\pi]$ and $h_0(x) = \sin(2x)$ for $x \in [0,\pi]$. Hence, $u_0(x,y) = 1$ for any $(x,y) \in S^1 \times S^1$. Let $f(x,t)$ and $h(x,t)$ solve the heat equation with initial data f_0 and h_0, respectively. Clearly, $f(x,t)h(x,t)$ is not identically zero for all $t > 0$, and $f(x,t)$ and $h(x,t)$ both converge to zero when $t \to \infty$. Let $\max f(x,1)h(x,1) = 1/A > 0$. Let $c(y,t) = A_0 e^{-t} \sin y$. Then $u(x,y,1) < 0$ for some point if $A_0 > Ae$. By choosing A_0 sufficiently large depending on t_0, $u(x,y,t_0)$ can become negative for arbitrarily small $t_0 > 0$.

The conformal flow (3.1) is reduced to the system

$$\partial_t a = \frac{\Delta a}{\sqrt{u}}, \quad \partial_t b = \frac{\Delta b}{\sqrt{u}}, \tag{3.59}$$

and the factor $1/\sqrt{u}$ forces the positivity of u before developing a singularity.

PROPOSITION 3.21. *Suppose $a(t), b(t) : T^2 \to \mathbb{R}$ satisfy (3.59). Then $\rho = \omega_0 + d\theta$ with $\theta = a dx_3 + b dx_4$ satisfies (3.1).*

Proof. This is a straightforward computation and the key point is that $d^*\theta = 0$. Suppose a, b are unique solutions of (3.59). Then

$$\partial_t \rho = d(\partial_t \theta) = d\left(\frac{\Delta a\, dx_3 + \Delta b\, dx_4}{\sqrt{u}}\right).$$

Since $d^*\theta = 0$, we have

$$-d * \rho = -d^* d\theta = \Delta\theta = \Delta a\, dx_3 + \Delta b\, dx_4.$$

Hence, we have

$$\partial_t \rho = -d\left(\frac{d^*\rho}{\sqrt{u}}\right). \qquad \square$$

The system (3.59) is a quasilinear parabolic system for (a,b) on T^2, together with a positivity restriction $u = 1 - a_{x_1} b_{x_2} + a_{x_2} b_{x_1} > 0$.

Problem 3.22. Does (3.59) have a long time solution such that ρ converges to ω_0?

It is not clear at the moment whether the flow (3.1) would develop a singularity or not when $n = 4$. The study of the two special examples above might give a hint for this problem.

3.7 Singularity and soliton

When $n = 6$, the corresponding geometric flow would have to develop singularities in general. In dimension 6, we choose $r = 1/3$ to consider the flow

$$\partial_t \rho = -d\left(\frac{d^*\rho}{u^{1/3}}\right).$$

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With almost identical arguments as for $n=4$, one can show that if $|\nabla \log u|$ stays uniformly bounded along the flow, then the flow exists for all time and converges to ω strongly in L^2. (If one assumes further that $|\nabla\rho|$ is bounded, then the convergence would be in C^∞ topology). One expects that, in general, the symplectic forms in a cohomology class $[\omega]$ are not path-connected. A particular example on $T^2\times S^2\times S^2$ is provided by McDuff. In this case, the flow has to develop singularities. The flow can be viewed as a gradient flow, and the normalized energy decreases strictly along the flow (except at ω); it would be interesting to analyze the formation of singularities along the flow. We pose a concrete question on formation of singularities.

Problem 3.23. Prove (or disprove) that, along the conformal Hodge flow, there is no type-I singularity.

One considers the solitons on $\mathbb{R}^4$. A particular type is

$$d\left(\frac{d^*\rho_0}{\sqrt{u_0}}\right)=d(i_X\rho_0), \tag{3.60}$$

where X is a Killing vector field of the Euclidean metric. Let ψ_t be the flow generated by X. Then $\rho(t)=\psi_t^*\rho_0$ satisfies the flow equation on $\mathbb{R}^4$,

$$\partial_t\rho=-d\left(\frac{d^*\rho}{\sqrt{u}}\right).$$

We can describe a family of special solutions to (3.60) as follows. Let

$$\rho_0=a(x,y)dx\wedge dy+dw\wedge dz.$$

Choose $X=(V^1,V^2,V^3,V^4)$ to be a constant vector field. Then (3.60) is reduced to

$$\left(\frac{a_x}{\sqrt{a}}\right)_x+\left(\frac{a_y}{\sqrt{a}}\right)_y+V^1a_x+V^2a_y=0. \tag{3.61}$$

It would be an interesting question to classify solutions to (3.60).

4. Other nonlinear Hodge flows in dimension four

In this section, we discuss different choices of nonlinear Hodge flows. We write the flow equation as

$$\partial_t\rho_{ij}=(h_{ik}\rho_{kl,l})_{,j}-(h_{jk}\rho_{kl,l})_{,i}, \tag{4.1}$$

where $h_{ij}=Q_{ij}\sqrt{\det Q}^{-1}$ is a positive definite symmetric matrix such that $h_{ij}\rho_{jk}=\rho_{ik}h_{jk}$. Note that h_{ij} and Q_{ij} are uniquely determined by each other. We fix the background structure (M,ω_0,g_0). For simplicity, we consider the case on the flat four torus, which will simplify computations since the curvature terms all vanish. We write $g_0=\sum_i^4 d\theta_i^2$ and $\omega_0=d\theta_1\wedge d\theta_2+d\theta_3\wedge d\theta_4$. Suppose $\rho\in[\omega_0]$ is a symplectic form with

$$\rho=\sum_{i<j}\rho_{ij}d\theta_i\wedge d\theta_j=\sum_{i\neq j}\frac{1}{2}\rho_{ij}d\theta_i\wedge d\theta_j.$$

Proposition 4.1. *Along the flow, we have*

$$\begin{aligned}\partial_t|\rho|^2=&\rho_{ij}h_{ik}\Delta\rho_{kj}+2\rho_{ij}h_{ik,\,j}\rho_{kl,\,l},\\ \partial_t u=&\tfrac{1}{2}(*\rho)_{ij}h_{ik}\Delta\rho_{kj}+(*\rho)_{ij}h_{ik,j}\rho_{kl,l}.\end{aligned} \tag{4.2}$$

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Proof. We compute

$$\begin{aligned}\partial_t|\rho|^2 &= 2\sum_{i<j}\rho_{ij}\partial_t\rho_{ij} = \sum_{i\neq j}\rho_{ij}\partial_t\rho_{ij}\\ &= 2\rho_{ij}(h_{ik}\rho_{kl,l})_j\\ &= 2\rho_{ij}h_{ik}\rho_{kl,lj} + 2\rho_{ij}h_{ik,j}\rho_{kl,l}.\end{aligned}$$

By $d\rho = 0$, we have $\rho_{kl,\,lj} = \rho_{jl,\,kl} + \Delta\rho_{kj}$. Using $\rho_{ij}h_{ik} = \rho_{ki}h_{ij}$, we deduce

$$2\rho_{ij}h_{ik}\rho_{kl,\,lj} = \rho_{ij}h_{ik}\Delta\rho_{kj}.$$

Similarly, we compute

$$\partial_t u = \tfrac{1}{2}(*\rho)_{ij}h_{ik}\Delta\rho_{kj} + (*\rho)_{ij}h_{ik,j}\rho_{kl,l}.$$

This completes the computation. □

First, we consider the linear Hodge flow.

4.1 The linear Hodge flow $h_{ij} = \delta_{ij}$

When $Q_{ij} = h_{ij} = \delta_{ij}$, the flow is the Hodge flow

$$\partial_t\rho = \Delta\rho.$$

The standard theory shows that the flow exists for all time with uniform control and converges to ω_0 at exponential rate. But the potential u does not remain positive in general. In the previous section, we construct an example, for any given time $t_0 > 0$, in which one can choose an initial symplectic form ρ_0 with $u_0 = 1$, but along the flow u can become negative at t_0. Nevertheless, the linear Hodge flow can indicate some key difficulties in the study of nonlinear Hodge flow, in particular for conformal Hodge flows. We have

$$\begin{aligned}&\partial_t|\rho|^2 = \Delta|\rho|^2 - 2|\nabla\rho|^2, \partial_t u = \Delta u - \langle\nabla\rho, \nabla(*\rho)\rangle,\\ &\partial_t|\rho^+|^2 = \Delta|\rho^+|^2 - 2|\nabla\rho^+|^2, \partial_t|\rho^-|^2 = \Delta|\rho^-|^2 - 2|\nabla\rho^-|^2.\end{aligned}$$

The eigenvalues of A are $\pm\sqrt{-1}\lambda_1, \pm\sqrt{-1}\lambda_2$, which are determined by

$$\lambda_1 = \frac{1}{\sqrt{2}}(|\rho^+| + |\rho^-|), \quad \lambda_2 = \frac{1}{\sqrt{2}}(|\rho^+| - |\rho^-|).$$

Note that λ_1 and λ_2 are smooth functions at points where $\rho^- \neq 0$. We compute

$$\begin{aligned}\partial_t\lambda_1 &= \Delta\lambda_1 - \frac{|\nabla\rho^+|^2 - |\nabla|\rho^+||^2}{\sqrt{2}|\rho^+|} - \frac{|\nabla\rho^-|^2 - |\nabla|\rho^-||^2}{\sqrt{2}|\rho^-|},\\ \partial_t\lambda_2 &= \Delta\lambda_2 - \frac{|\nabla\rho^+|^2 - |\nabla|\rho^+||^2}{|\rho^+|} + \frac{|\nabla\rho^-|^2 - |\nabla|\rho^-||^2}{|\rho^-|}.\end{aligned}$$

We denote, at points where $\rho^- \neq 0$,

$$J = \frac{|\nabla\rho^+|^2 - |\nabla|\rho^+||^2}{\sqrt{2}|\rho^+|}, \quad K = \frac{|\nabla\rho^-|^2 - |\nabla|\rho^-||^2}{\sqrt{2}|\rho^-|}. \tag{4.3}$$

The maximum principle argument implies the larger eigenvalue λ_1 is bounded above, but fails to preserve the positivity of the smaller eigenvalue λ_2. (Strictly speaking, our computation only works at points where $\lambda_1 \neq \lambda_2$, but with modifications the argument can work at points where $\lambda_1 = \lambda_2$ as well). Note that at the minimum of λ_2, we have

$$\nabla|\rho^+| = \nabla|\rho^-|.$$

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At the first time, $\lambda_2 = 0$, and hence $|\rho^+| = |\rho^-|$ as well; the failure to preserve the positivity (nonnegativity) of λ_2 lies in the fact that at such points (the first time $\lambda_2 = 0$), the following control does not hold:

$$|\nabla \rho^+| \leq |\nabla \rho^-|. \tag{4.4}$$

Similar consideration of the evolution equation on u leads to the same observation: that at the point where $u = 0$ for the first time, (4.4) does not hold (it has to fail in general). At the maximum point of $u^{-1}|\rho|^2$, we have

$$|\rho^-|^2 \nabla |\rho^+|^2 = |\rho^+|^2 \nabla |\rho^-|^2. \tag{4.5}$$

However, $u^{-1}|\rho|^2$ fails to be bounded above along the flow, which shows that the following control at maximum point of $u^{-1}|\rho|^2$ fails as well (when u is positive):

$$|\rho^-|^2 |\nabla \rho^+|^2 \leq |\rho^+|^2 |\nabla \rho^-|^2. \tag{4.6}$$

The failure of positivity of u along the linear Hodge flow lies in the difference between (4.5) and (4.6) from a technical perspective. Now, we compute the evolution equations of λ_1 and λ_2 with the different choices of h_{ij}.

4.2 The case when $h_{ij} = f\delta_{ij}$

The evolution equation reads

$$\partial_t \rho_{ij} = f\Delta \rho_{ij} + f_i \rho_{jl,l} - f_j \rho_{il,l}.$$

Hence, we compute

$$\begin{aligned}
\partial_t |\rho|^2 &= f(\Delta |\rho|^2 - 2|\nabla \rho|^2) + 2 f_i \rho_{ij} \rho_{jl,l}, \\
\partial_t u &= f(\Delta u - |\nabla \rho^+|^2 + |\nabla \rho^-|^2) + f_i (*\rho)_{ij} \rho_{jl,l}, \\
\partial_t |\rho^+|^2 &= f(\Delta |\rho^+|^2 - 2|\nabla \rho^+|^2) + 2 f_i \rho^+_{ij} \rho_{jl,l}, \\
\partial_t |\rho^-|^2 &= f(\Delta |\rho^-|^2 - 2|\nabla \rho^-|^2) + 2 f_i \rho^-_{ij} \rho_{jl,l}.
\end{aligned}$$

The conformal factor f can take the form of u^{-r} for some $r > 0$, or something like $u^{-1}|\rho|^2$. The structure of the evolution equation along the flow indicates the almost identical technical difficulty of preserving the uniform positivity of u (or λ_2), since the maximum principle argument fails similarly as for the linear Hodge flow. It is still possible that the flow exists for all time with $f = u^{-r}$, but it is hard to obtain the uniform positivity of u along the flow via a direct maximum principle argument.

4.3 The case when $h_{ij} = u^{-1} a_{ij}$

Here, we choose $g_\rho = u^{-1} g(A\cdot, A\cdot)$, which is the metric used in the Donaldson geometric flow. Correspondingly, we have

$$Q_{ij} = h_{ij} = \frac{a_{ij}}{u}.$$

Recall $a_{ij} = \rho_{ip}\rho_{jp}$, $b_{ij} = (*\rho)_{ip}(*\rho)_{jp}$ and $a_{ij} + b_{ij} = |\rho|^2 \delta_{ij}$. We compute

$$\begin{aligned}
\partial_t |\rho|^2 &= \left(\frac{|\rho|^2 \rho_{kj}}{u} - (*\rho)_{kj} \right) \Delta \rho_{kj} + 2\rho_{ij} h_{ik,j} \rho_{kl,l}, \\
\partial_t u &= \frac{1}{2} \rho_{kj} \Delta \rho_{kj} + (*\rho)_{ij} h_{ik,j} \rho_{kl,l}.
\end{aligned} \tag{4.7}$$

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We compute, with $x = |\rho^-|^2/|\rho^+|^2$,

$$\begin{aligned}\partial_t|\rho^+|^2 &= \frac{1+x}{1-x}\Delta|\rho^+|^2 + \frac{2}{1-x}\Delta|\rho^-|^2 - \frac{2(1+x)}{1-x}|\nabla\rho^+|^2 - \frac{4}{1-x}|\nabla\rho^-|^2\\ &\quad + \left(\frac{b_{ik}}{u} + \delta_{ik}\right)\rho_{ij,j}\rho_{kl,l} + \left(\frac{|\rho|^2}{u}\right)_j \rho_{kj}\rho_{kl,l},\\ \partial_t|\rho^-|^2 &= \frac{2x}{1-x}\Delta|\rho^+|^2 + \frac{1+x}{1-x}\Delta|\rho^-|^2 - \frac{4x}{1-x}|\nabla\rho^+|^2 - \frac{2(1+x)}{1-x}|\nabla\rho^-|^2\\ &\quad + \left(\frac{b_{ik}}{u} - \delta_{ik}\right)\rho_{ij,j}\rho_{kl,l} + \left(\frac{|\rho|^2}{u}\right)_j \rho_{kj}\rho_{kl,l}.\end{aligned} \tag{4.8}$$

We compute, at points where $|\rho^-| \neq 0$,

$$\begin{aligned}\partial_t\lambda_1 &= \frac{\lambda_1}{\lambda_2}\Delta\lambda_1 - \frac{\lambda_1}{\lambda_2}(J+K) + \frac{\lambda_1 b_{ik} - u\lambda_2\delta_{ik}}{u(\lambda_1^2-\lambda_2^2)}\rho_{ij,j}\rho_{kl,l} + \left(\frac{\lambda_1^2+\lambda_2^2}{\lambda_1\lambda_2}\right)_j \frac{\lambda_1\rho_{kj}\rho_{kl,l}}{\lambda_1^2-\lambda_2^2},\\ \partial_t\lambda_2 &= \frac{\lambda_2}{\lambda_1}\Delta\lambda_2 + \frac{\lambda_2}{\lambda_1}(K-J) + \frac{\lambda_1 u\delta_{ik} - \lambda_2 b_{ik}}{u(\lambda_1^2-\lambda_2^2)}\rho_{ij,j}\rho_{kl,l} - \left(\frac{\lambda_1^2+\lambda_2^2}{\lambda_1\lambda_2}\right)_j \frac{\lambda_2\rho_{kj}\rho_{kl,l}}{\lambda_1^2-\lambda_2^2}.\end{aligned} \tag{4.9}$$

We analyze (4.9) at points where λ_1 or λ_2 obtains its maximum or minimum, respectively. Let $\lambda_1(p,t) = \max_{x\in M}\lambda_1(x,t)$. Then at the point p, we have

$$\nabla\lambda_1 = 0, \quad \nabla^2\lambda_1 \leq 0.$$

This implies that $\nabla|\rho^+| + \nabla|\rho^-| = 0$. We compute

$$\frac{\langle\rho^+, \nabla\rho^+\rangle}{|\rho^+|} + \frac{\langle\rho^-, \nabla\rho^-\rangle}{|\rho^-|} = 0.$$

We can choose a coordinate at p such that $\rho_{12} = \lambda_1$ and $\rho_{34} = \lambda_2$ and

$$\rho_{13} = \rho_{24} = \rho_{23} = \rho_{14} = 0.$$

This implies that $\nabla\rho_{12} = \nabla\lambda_1 = 0$, $\nabla\lambda_2 = \nabla\rho_{34}$ at p. We compute

$$J + K = \frac{\lambda_1}{\lambda_1^2-\lambda_2^2}(|\nabla\rho_{13}|^2 + |\nabla\rho_{14}|^2 + |\nabla\rho_{23}|^2 + |\nabla\rho_{24}|^2) + \frac{2\lambda_2}{\lambda_1^2-\lambda_2^2}(\nabla\rho_{13}\nabla\rho_{24} - \nabla\rho_{23}\nabla\rho_{14}),$$

while for

$$\begin{aligned}\left(\frac{\lambda_1^2+\lambda_2^2}{\lambda_1\lambda_2}\right)_j \frac{\lambda_1\rho_{kj}\rho_{kl,l}}{\lambda_1^2-\lambda_2^2} &= \frac{\lambda_1}{\lambda_2^2}(\rho_{34,1}\rho_{2l,l} - \rho_{34,2}\rho_{1l,l}) + \frac{\rho_{34,3}\rho_{4l,l} - \rho_{34,4}\rho_{3l,l}}{\lambda_2},\\ \frac{\lambda_1 b_{ik} - u\lambda_2\delta_{ik}}{u(\lambda_1^2-\lambda_2^2)}\rho_{ij,j}\rho_{kl,l} &= \frac{1}{\lambda_2}[(\rho_{3j,j})^2 + (\rho_{4j,j})^2].\end{aligned}$$

It seems to be hard to bound λ_1 via a direct maximum principle argument since the term $(\lambda_1/\lambda_2^2)(\rho_{34,1}\rho_{2l,l} - \rho_{34,2}\rho_{1l,l})$ has the highest order when $\lambda_2 << 1$ and this term does not have the favorite sign. We denote

$$\partial_t\lambda_2 = \frac{\lambda_2}{\lambda_1}\Delta\lambda_2 + \mathcal{R}$$

and analyze the behavior at the minimum of λ_2, where $\lambda_2(q,t) = \min\lambda_2(\cdot,t)$. At the point q, we choose a coordinate such that $\rho_{12} = \lambda_1$ and $\rho_{34} = \lambda_2$ and

$$\rho_{13} = \rho_{24} = \rho_{23} = \rho_{14} = 0.$$

At q, $\nabla\lambda_2 = 0 = \nabla\rho_{34}, \nabla\lambda_1 = \nabla\rho_{12}$. We are not able to prove that $\mathcal{R} \geq 0$ at q.

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4.4 The case when $h_{ij} = u^{-2}a_{ij}$

We have

$$\rho_{ij}h_{ik} = \frac{|\rho|^2\delta_{kj}}{u^2} - \frac{(*\rho)_{kj}}{u}, \quad (*\rho)_{ij}h_{ik} = \frac{\rho_{kj}}{u}.$$

We compute

$$\begin{aligned} \partial_t|\rho|^2 &= \left(\frac{|\rho|^2\rho_{kj}}{u^2} - \frac{(*\rho)_{kj}}{u}\right)\Delta\rho_{kj} + 2\left(\frac{|\rho|^2\rho_{kj}}{u^2} - \frac{(*\rho)_{kj}}{u}\right)_j \rho_{kl,l} - 2h_{ik}\rho_{ij,j}\rho_{kl,l}, \\ \partial_t u &= \frac{\rho_{kj}}{2u}\Delta\rho_{kj} + \left(\frac{\rho_{kj}}{u}\right)_j \rho_{kl,l}. \end{aligned} \tag{4.10}$$

We compute

$$\partial_t(|\rho|^2 + 2u) = \left(\frac{|\rho|^2+u}{u^2}\rho_{kj} - \frac{(*\rho)_{kj}}{u}\right)\Delta\rho_{kj} + 2\left(\frac{|\rho|^2+u}{u^2}\rho_{kj} - \frac{(*\rho)_{kj}}{u}\right)_j \rho_{kl,l} - 2h_{ik}\rho_{ij,j}\rho_{kl,l},$$

$$\partial_t(|\rho|^2 - 2u) = \left(\frac{|\rho|^2-u}{u^2}\rho_{kj} - \frac{(*\rho)_{kj}}{u}\right)\Delta\rho_{kj} + 2\left(\frac{|\rho|^2-u}{u^2}\rho_{kj} - \frac{(*\rho)_{kj}}{u}\right)_j \rho_{kl,l} - 2h_{ik}\rho_{ij,j}\rho_{kl,l}.$$

We compute

$$\begin{aligned} \partial_t|\rho^+|^2 &= \frac{|\rho|^2}{2u^2}\Delta|\rho^+|^2 + \frac{|\rho|^2+2u}{2u^2}\Delta|\rho^-|^2 - \frac{|\rho|^2}{u^2}|\nabla\rho^+|^2 - \frac{|\rho|^2+2u}{u^2}|\nabla\rho^-|^2 \\ &\quad + \left(\frac{|\rho|^2+u}{u^2}\rho_{kj} - \frac{(*\rho)_{kj}}{u}\right)_j \rho_{kl,l} - h_{ik}\rho_{ij,j}\rho_{kl,l}, \\ \partial_t|\rho^-|^2 &= \frac{|\rho|^2-2u}{2u^2}\Delta|\rho^+|^2 + \frac{|\rho|^2}{2u^2}\Delta|\rho^-|^2 - \frac{|\rho|^2-2u}{u^2}|\nabla\rho^+|^2 - \frac{|\rho|^2}{u^2}|\nabla\rho^-|^2 \\ &\quad + \left(\frac{|\rho|^2-u}{u^2}\rho_{kj} - \frac{(*\rho)_{kj}}{u}\right)_j \rho_{kl,l} - h_{ik}\rho_{ij,j}\rho_{kl,l}. \end{aligned} \tag{4.11}$$

It follows that

$$\begin{aligned} \partial_t\lambda_1 &= \frac{\Delta\lambda_1}{\lambda_2^2} - \frac{1}{\lambda_2^2}(J+K) + \frac{\lambda_1 b_{ik} - u\lambda_2\delta_{ik}}{u^2(\lambda_1^2-\lambda_2^2)}\rho_{ij,j}\rho_{kl,l} + P_1, \\ \partial_t\lambda_2 &= \frac{\Delta\lambda_2}{\lambda_1^2} - \frac{1}{\lambda_1^2}(K-J) + \frac{\lambda_1 u\delta_{ik} - \lambda_2 b_{ik}}{u^2(\lambda_1^2-\lambda_2^2)}\rho_{ij,j}\rho_{kl,l} + P_2, \end{aligned} \tag{4.12}$$

where we denote

$$\begin{aligned} P_1 &= \frac{\lambda_{1,j}\rho_{kl,l}}{\lambda_1(\lambda_1^2-\lambda_2^2)}\left(\frac{(*\rho)_{kj}}{\lambda_2} - \frac{\rho_{kj}}{\lambda_1}\right) + \frac{\lambda_{2,j}\rho_{kl,l}}{\lambda_2(\lambda_1^2-\lambda_2^2)}\left(\frac{(*\rho)_{kj}}{\lambda_2} + \frac{\rho_{kj}}{\lambda_1} - \frac{2\lambda_1}{\lambda_2^2}\rho_{kj}\right), \\ P_2 &= \frac{\lambda_{1,j}\rho_{kl,l}}{\lambda_1(\lambda_1^2-\lambda_2^2)}\left(\frac{(*\rho)_{kj}}{\lambda_2} + \frac{\rho_{kj}}{\lambda_2} - \frac{2\lambda_2}{\lambda_1^2}\rho_{kj}\right) + \frac{\lambda_{2,j}\rho_{kl,l}}{\lambda_2(\lambda_1^2-\lambda_2^2)}\left(\frac{(*\rho)_{kj}}{\lambda_2} - \frac{\rho_{kj}}{\lambda_1}\right). \end{aligned}$$

4.5 The case when $h_{ij} = u^{-1}b_{ij}$

Here, we choose

$$Q_{ij} = h_{ij} = \frac{b_{ij}}{u}.$$

We have

$$\rho_{ij}h_{ik} = (*\rho)_{kj}, \quad (*\rho)_{ij}h_{ik} = \frac{|\rho|^2(*\rho)_{kj} - u\rho_{kj}}{u}.$$

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We compute

$$\begin{aligned}\partial_t|\rho|^2 &= (*\rho)_{kj}\Delta\rho_{kj} - \frac{2b_{ik}}{u}\rho_{ij,j}\rho_{kl,l},\\ \partial_t u &= \frac{|\rho|^2(*\rho)_{kj} - u\rho_{kj}}{2u}\Delta\rho_{kj} + \left(\frac{|\rho|^2(*\rho)_{kj} - u\rho_{kj}}{u}\right)_j \rho_{kl,l}.\end{aligned} \tag{4.13}$$

Hence, we have

$$\begin{aligned}\partial_t|\rho^+|^2 &= \frac{(|\rho|^2+u)(*\rho)_{kj} - u\rho_{kj}}{2u}\Delta\rho_{kj} - \frac{b_{ik}}{u}\rho_{ij,j}\rho_{kl,l} + \left(\frac{|\rho|^2(*\rho)_{kj} - u\rho_{kj}}{u}\right)_j \rho_{kl,l},\\ \partial_t|\rho^-|^2 &= \frac{(u-|\rho|^2)(*\rho)_{kj} + u\rho_{kj}}{2u}\Delta\rho_{kj} - \frac{b_{ik}}{u}\rho_{ij,j}\rho_{kl,l} - \left(\frac{|\rho|^2(*\rho)_{kj} - u\rho_{kj}}{u}\right)_j \rho_{kl,l}.\end{aligned} \tag{4.14}$$

Hence, we have

$$\begin{aligned}\partial_t\lambda_1 &= \frac{\lambda_2}{\lambda_1}\Delta\lambda_1 - \frac{\lambda_2}{\lambda_1}(J+K) - \frac{\lambda_1 b_{ik} - u\lambda_2\delta_{ik}}{u(\lambda_1^2-\lambda_2^2)}\rho_{ij,j}\rho_{kl,l} - R_1,\\ \partial_t\lambda_2 &= \frac{\lambda_1}{\lambda_2}\Delta\lambda_2 - \frac{\lambda_1}{\lambda_2}(K-J) + \frac{\lambda_2 b_{ik} - u\lambda_1\delta_{ik}}{u(\lambda_1^2-\lambda_2^2)}\rho_{ij,j}\rho_{kl,l} + R_2,\end{aligned} \tag{4.15}$$

where we denote

$$R_1 = \left(\frac{\lambda_1^2+\lambda_2^2}{\lambda_1\lambda_2}\right)_j \frac{\lambda_2(*\rho)_{kj}\rho_{kl,l}}{\lambda_1^2-\lambda_2^2}, \quad R_2 = \left(\frac{\lambda_1^2+\lambda_2^2}{\lambda_1\lambda_2}\right)_j \frac{\lambda_1(*\rho)_{kj}\rho_{kl,l}}{\lambda_1^2-\lambda_2^2}.$$

4.6 The case when $h_{ij} = u^{-2}b_{ij}$

Here, we choose

$$Q_{ij} = b_{ij}, \quad h_{ij} = \frac{b_{ij}}{u^2}. \tag{4.16}$$

We compute

$$\rho_{ij}h_{ik} = \frac{(*\rho)_{kj}}{u}, \quad (*\rho)_{ij}h_{ik} = \frac{|\rho|^2(*\rho)_{kj} - u\rho_{kj}}{u^2}.$$

Hence, we have

$$\begin{aligned}\partial_t|\rho|^2 &= \frac{(*\rho)_{kj}}{u}\Delta\rho_{kj} + 2\left(\frac{(*\rho)_{kj}}{u}\right)_{,j}\rho_{kl,l} - 2h_{ik}\rho_{ip,p}\rho_{kl,l},\\ \partial_t(2u) &= \frac{|\rho|^2(*\rho)_{kj} - u\rho_{kj}}{u^2}\Delta\rho_{kj} + 2\left(\frac{|\rho|^2(*\rho)_{kj} - u\rho_{kj}}{u^2}\right)_{,j}\rho_{kl,l},\\ \partial_t(|\rho|^2+2u) &= R_{kj}\Delta\rho_{kj} + 2R_{kj,\,j}\rho_{kl,l} - 2h_{ik}\rho_{ip,p}\rho_{kl,l},\\ \partial_t(|\rho|^2-2u) &= S_{kj}\Delta\rho_{kj} + 2S_{kj,\,j}\rho_{kl,l} - 2h_{ik}\rho_{ip,p}\rho_{kl,l},\end{aligned} \tag{4.17}$$

where we denote

$$R_{kj} = \frac{(|\rho|^2+u)(*\rho)_{kj} - u\rho_{kj}}{u^2}, \quad S_{kj} = \frac{\rho_{kj} + (*\rho)_{kj}}{u} - \frac{|\rho|^2}{u^2}(*\rho)_{kj}.$$

Recall that

$$\lambda_1 = \frac{\sqrt{|\rho|^2+2u}}{2} + \frac{\sqrt{|\rho|^2-2u}}{2}, \quad \lambda_2 = \frac{\sqrt{|\rho|^2+2u}}{2} - \frac{\sqrt{|\rho|^2-2u}}{2}.$$

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Note that λ_1 and λ_2 are both smooth functions when $|\rho^-|^2 \neq 0$. We compute

$$\begin{aligned}\partial_t \lambda_1 &= \frac{\partial_t(|\rho|^2+2u)}{4\sqrt{|\rho|^2+2u}} + \frac{\partial_t(|\rho|^2-2u)}{4\sqrt{|\rho|^2-2u}} \\ &= \left(\frac{R_{kj}\Delta\rho_{kj}}{4\sqrt{|\rho|^2+2u}} + \frac{S_{kj}\Delta\rho_{kj}}{4\sqrt{|\rho|^2-2u}}\right) + \left(\frac{R_{kj,j}\rho_{kl,l}}{2\sqrt{|\rho|^2+2u}} + \frac{S_{kj,j}\rho_{kl,l}}{2\sqrt{|\rho|^2-2u}}\right) \\ &\quad - \left(\frac{1}{2\sqrt{|\rho|^2+2u}} + \frac{1}{2\sqrt{|\rho|^2-2u}}\right) h_{ik}\rho_{ip,p}\rho_{kl,l}, \end{aligned} \tag{4.18}$$

and similarly

$$\begin{aligned}\partial_t \lambda_2 &= \frac{\partial_t(|\rho|^2+2u)}{4\sqrt{|\rho|^2+2u}} - \frac{\partial_t(|\rho|^2-2u)}{4\sqrt{|\rho|^2-2u}} \\ &= \left(\frac{R_{kj}\Delta\rho_{kj}}{4\sqrt{|\rho|^2+2u}} - \frac{S_{kj}\Delta\rho_{kj}}{4\sqrt{|\rho|^2-2u}}\right) + \left(\frac{R_{kj,j}\rho_{kl,l}}{2\sqrt{|\rho|^2+2u}} - \frac{S_{kj,j}\rho_{kl,l}}{2\sqrt{|\rho|^2-2u}}\right) \\ &\quad - \left(\frac{1}{2\sqrt{|\rho|^2+2u}} - \frac{1}{2\sqrt{|\rho|^2-2u}}\right) h_{ik}\rho_{ip,p}\rho_{kl,l}. \end{aligned} \tag{4.19}$$

By (4.18), we compute, using $\rho = \rho^+ + \rho^-$ and $*\rho = \rho^+ - \rho^-$,

$$\begin{aligned}\partial_t \lambda_1 &= \frac{\Delta\lambda_1}{\lambda_1^2} - \frac{1}{\lambda_1^2}(J+K) - \frac{\lambda_1 h_{ik}\rho_{ip,p}\rho_{kl,l}}{\lambda_1^2-\lambda_2^2} + \frac{|\rho_{kl,l}|^2}{\lambda_1(\lambda_1^2-\lambda_2^2)} \\ &\quad + \frac{\lambda_{1,j}\rho_{kl,l}}{\lambda_1(\lambda_1^2-\lambda_2^2)}\left(\left(\frac{2\lambda_2}{\lambda_1^2} - \frac{1}{\lambda_2}\right)(*\rho)_{kj} - \frac{\rho_{kj}}{\lambda_1}\right) + \frac{\lambda_{2,j}\rho_{kl,l}\left((*\rho)_{kj}/\lambda_2 - \rho_{kj}/\lambda_1\right)}{\lambda_2(\lambda_1^2-\lambda_2^2)}, \end{aligned} \tag{4.20}$$

and, similarly, we compute

$$\begin{aligned}\partial_t \lambda_2 &= \frac{\Delta\lambda_2}{\lambda_2^2} + \frac{1}{\lambda_2^2}(K-J) + \frac{\lambda_2 h_{ik}\rho_{ip,p}\rho_{kl,l}}{\lambda_1^2-\lambda_2^2} - \frac{|\rho_{kl,l}|^2}{\lambda_2(\lambda_1^2-\lambda_2^2)} \\ &\quad + \frac{\lambda_{2,j}\rho_{kl,l}}{\lambda_2(\lambda_1^2-\lambda_2^2)}\left(\frac{\rho_{kj}}{\lambda_2} + \frac{(*\rho)_{kj}}{\lambda_1} - \frac{2\lambda_1(*\rho)_{kj}}{\lambda_2^2}\right) + \frac{\lambda_{1,j}\rho_{kl,l}\left(\rho_{kj}/\lambda_2 - (*\rho)_{kj}/\lambda_1\right)}{\lambda_1(\lambda_1^2-\lambda_2^2)}. \end{aligned} \tag{4.21}$$

When $\rho = \lambda_1(x_1,x_2)dx_1\wedge dx_2 + \lambda_2(x_3,x_4)dx_3\wedge dx_4$, we have already seen that the nonlinear Hodge flow with $h_{ij} = u^{-2}b_{ij}$ preserves the structure and satisfies

$$\partial_t\lambda_1 = \Delta_{x_1,x_2}(-\lambda_1^{-1}), \partial_t\lambda_2 = \Delta_{x_3,x_4}(-\lambda_2^{-1}). \tag{4.22}$$

It is a direct computation to check that (4.22) coincides with (4.20) and (4.21). For example, $J = K = 0$ and $\rho_{1l,l} = \partial_{x_2}\lambda_1$, $\rho_{2l,l} = -\partial_{x_1}\lambda_1$, $\rho_{3l,l} = \partial_{x_4}\lambda_2$, $\rho_{4l,l} = -\partial_{x_3}\lambda_2$. Note that (4.22) is a fast diffusion equation for λ_1 and λ_2, respectively. The maximum principle indeed applies to show that λ_1, λ_2 remain uniformly bounded and uniformly positive. By the parabolic theory, the flow exists for all time with uniform control, and it will converge to $\omega = dx_1\wedge dx_2 + dx_3\wedge dx_4$. However, the maximum principle argument does not apply directly to (4.20) and (4.21) for the general case. On the other hand, (4.21) might still suggest that λ_2 stays uniformly positive along the flow. When λ_2 approaches zero, the leading term $\lambda_2^{-2}\Delta\lambda_2$ is the same as for the fast diffusion equation and it might force λ_2 to stay uniformly positive. Nevertheless, it is clear that the system is really complicated, and it remains a substantial challenge to address these difficulties.

https://doi.org/10.1112/S0010437X25102406 Published online by Cambridge University Press

Acknowledgements

It is a pleasure to acknowledge the valuable discussions with Song Sun over the years, in particular regarding the problems raised in [Don06]. We thank the referee for valuable suggestions and comments.

Conflicts of interest

None.

Journal information

Compositio Mathematica is owned by the Foundation Compositio Mathematica and published by the London Mathematical Society in partnership with Cambridge University Press. All surplus income from the publication of *Compositio Mathematica* is returned to mathematics and higher education through the charitable activities of the Foundation, the London Mathematical Society and Cambridge University Press.

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https://doi.org/10.1112/S0010437X25102406 Published online by Cambridge University Press