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A dam with inverse Gaussian input

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1. *Introduction.* Gani and Prabhu (1) have presented a general theory of storage for dams subject to a steady release. At time $t \geqslant 0$, let a dam of infinite capacity have a continuous random content $Z(t) \geqslant 0$; the input $X(t) \geqslant 0$ which has entered it during time t follows an infinitely divisible distribution, while the dam is subject to a steady release at constant unit rate except if it is empty, when the release ceases. A procedure is obtained whereby various functions describing the transient behaviour of $Z(t)$ can be calculated in principle.

The most interesting case in this theory, with no direct analogue in queuing theory, is the case when almost all sample input functions have an infinite number of jumps in any finite interval. Such an input was studied by Moran (4) who took a density function of the Gamma type

$$[\rho\Gamma(t)]^{-1}(x/\rho)^{t-1}e^{-x/\rho}.$$

Unfortunately, in the case of the Gamma input, Gani and Prabhu's procedure cannot be carried out in closed form. The purpose of this note is to introduce a storage model with an inverse Gaussian type input, and to carry out explicitly Gani and Prabhu's procedure for obtaining the transient behaviour of the dam. It will be found that, with this type of input, the procedure can be carried out explicitly much further than with the Gamma input, or, for that matter, with any other type of input so far studied, as far as is known to the author. In particular, closed expressions will be obtained for the probability of emptiness, the mean content of the dam, and the Laplace–Stieltjes transform of the distribution of the content, enabling their asymptotic behaviour to be easily studied.

All formulae in the sequel will be given in the notation of Gani and Prabhu's paper (1).

2. *Some properties of the Inverse Gaussian distribution.* The Inverse Gaussian distribution has been studied by Tweedie (5) who has given various forms for it. We shall, however, use a different form from the ones he suggests, and use as parameters the mean ρ and variance σ^2 per unit time. The density function $k(x,t)$ of the input $X(t)$ will be written

$$k(x,t) = \frac{t}{\sigma\sqrt{2\pi}}\left(\frac{\rho}{x}\right)^{\frac{3}{2}}\exp\left[-\frac{\rho}{2\sigma^2 x}(x-\rho t)^2\right].$$

The Laplace Transform of $k(x,t)$ is of the form $\exp[-\xi(\theta)t]$ where

$$\xi(\theta) = \frac{\rho}{\sigma^2}(\sqrt{(2\rho\sigma^2\theta+\rho^2)}-\rho).$$

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The function $\xi(\theta)$ can be represented in the form

$$\xi(\theta) = \int_0^\infty (1-e^{-\theta u})\,\lambda(u)\,du,$$

where

$$\lambda(u) = \frac{1}{\sigma\sqrt{2\pi}}\left(\frac{\rho}{x}\right)^{\frac{3}{2}} \exp\left(-\frac{\rho x}{2\sigma^2}\right).$$

3. *The transient behaviour of the dam.* The transient behaviour of the dam is governed by the function $\eta(s)$ which satisfies the equation

$$\eta(s) = s + \xi[\eta(s)], \tag{1}$$

and is such that $\eta(s) > 0$ when s is real and positive.

In the case of the Inverse Gaussian input, equation (1) reduces to a quadratic, and only one solution satisfies the condition $\eta(s) > 0$ when s is real and positive, namely

$$\eta(s) = s - \frac{\rho^2}{\sigma^2}(1-\rho) + \sqrt{\left(\frac{\rho^4}{\sigma^4}(1-\rho)^2 + \frac{2\rho^3}{\sigma^2}s\right)}.$$

We can now calculate the density function $g(z_0, T)$ of the time of first emptiness $T(z_0)$, where z_0 is the initial content of the dam, by inverting its Laplace Transform $g^*(z_0, s) = \exp[-z_0\eta(s)]$. We find

$$g(z_0, T) = \begin{cases} \dfrac{z_0}{\sigma\sqrt{2\pi}}\left(\dfrac{\rho}{T-z_0}\right)^{\frac{3}{2}} \exp\left\{-\dfrac{\rho}{2\sigma^2(T-z_0)}[\rho T-(T-z_0)]^2\right\} & \text{for} \quad T \geqslant z_0 > 0, \\ 0 & \text{for} \quad z_0 > T, \end{cases}$$

which establishes in this case the correctness of the formula

$$g(z_0, T) = \begin{cases} (z_0/T)\,k(T-z_0, T) & \text{for} \quad T \geqslant z_0 > 0, \\ 0 & \text{for} \quad z_0 > T, \end{cases}$$

which was conjectured by Kendall (2) but has not so far been rigorously established (see Lloyd (3)). In the remaining part of this note, we shall assume $z_0 = 0$. The Laplace Transform $F^*(s)$ of the probability of the dam being empty, $F(0, t)$, is given by $F^*(s) = [\eta(s)]^{-1}$. Inverting, we find

$$F(0,t) = (1-\rho)\left\{1 - \tfrac{1}{2}\operatorname{erfc}\left[\left(\frac{1-\rho}{\sigma}\right)\sqrt{\frac{\rho t}{2}}\right]\right\} + \left(\frac{1+\rho}{2}\right)\operatorname{erfc}\left[\left(\frac{1+\rho}{\sigma}\right)\sqrt{\frac{\rho t}{2}}\right],$$

where $\operatorname{erfc}(x)$ is the Complementary Error Function defined by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-y^2)\,dy.$$

From this expression we can see immediately that if $\rho < 1$, $\lim_{t\to\infty} F(0,t) = 1-\rho$, but that if $\rho \geqslant 1$, $\lim_{t\to\infty} F(0,t) = 0$.

Differentiating formula (4·6) of (1) with respect to θ and putting $\theta = 0$, we obtain the following formula for the transient behaviour of the mean content of the dam

$$E[Z(t)] = z_0 - (1-\rho)\,t + \int_0^t F(0,u)\,du.$$

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Replacing $F(0,u)$ by its value, putting $z_0 = 0$ and integrating, we find

$$E[Z(t)] = \frac{\sigma^2}{2(1-\rho)} + \frac{\sigma}{\sqrt{2\pi}}\sqrt{\frac{t}{\rho}}\exp\left[-\frac{(1-\rho)^2\rho t}{2\sigma^2}\right]$$
$$-\left[\left(\frac{1-\rho}{2}\right)t + \frac{\sigma^2\rho(1+\rho^2)}{4\rho^3(1-\rho)}\right]\operatorname{erfc}\left[(1-\rho)\sqrt{\frac{\rho t}{2\sigma^2}}\right]$$
$$+\frac{\sigma^2(1+\rho)}{4\rho^2}\exp\left[\frac{2\rho^2 t}{\sigma^2}\right]\operatorname{erfc}\left[(1+\rho)\sqrt{\frac{\rho t}{2\sigma^2}}\right].$$

Here again we see immediately that if $\rho < 1$, then $\lim_{t\to\infty} E[Z(t)] = \sigma^2/2(1-\rho)$, while if $\rho \geqslant 1, \lim_{t\to\infty} E[Z(t)] = +\infty$, as expected from the general theory. Finally, we can calculate the Laplace Transform of the content distribution $\psi(\theta,t)$. We find

$$\psi(\theta,t) = \exp[-\{\xi(\theta)-\theta\}] - \theta\int_0^t F(0,\tau)\exp[-\{\xi(\theta)-\theta\}(t-\tau)]d\tau = e^{\alpha t} - \frac{\theta(1-\rho)}{\alpha}(1-e^{\alpha t})$$
$$-\frac{\theta(1-\rho)}{2}e^{\alpha t}\left[\frac{\beta}{\alpha\sqrt{(\beta^2+\alpha)}}(1-\operatorname{erfc}\sqrt{\{(\beta^2+\alpha)t\}}) + \frac{1}{\alpha}(1-e^{-\alpha t}\operatorname{erfc}\beta\sqrt{t})\right]$$
$$+\frac{\theta(1+\rho)}{2}e^{\alpha t}\left[\frac{\gamma}{\alpha\sqrt{(\gamma^2+\alpha)}}(1-\operatorname{erfc}\sqrt{\{(\gamma^2+\alpha)t\}}) + \frac{1}{\alpha}(1-e^{-\alpha t}\operatorname{erfc}\gamma\sqrt{t})\right],$$

where $$\alpha = \theta - \xi(\theta), \quad \beta = \left(\frac{1-\rho}{\sigma}\right)\sqrt{\frac{\rho}{2}}, \quad \gamma = \left(\frac{1+\rho}{\sigma}\right)\sqrt{\frac{\rho}{2}}.$$

If θ is such that $\Re(\alpha) < 0$, and if $\rho < 1$, we see that $\lim_{t\to\infty}\psi(\theta,t) = \theta(1-\rho)/[\theta-\xi(\theta)]$, which is Gani and Prabhu's formula (7·5).

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