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Theorem of Sternberg–Chen modulo the central manifold for Banach spaces

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Abstract. We consider C^∞-diffeomorphisms on a Banach space with a fixed point 0 and linear part L. Suppose that these diffeomorphisms have C^∞ non-contracting and non-expanding invariant manifolds, and formally conjugate along their intersection (the center). We prove that they admit local C^∞ conjugation. In particular, subject to non-resonance conditions, there exists a local C^∞ linearization of the diffeomorphisms. It also follows that a family of germs with a hyperbolic linear part admits a C^∞ linearization, which has C^∞ dependence on the parameter of the linearizing family. The results are proved under the assumption that the Banach space allows a special extension of the maps. We discuss corresponding properties of Banach spaces. The proofs of this paper are based on the technique, developed in the works of Belitskii [*Funct. Anal. Appl.* **18** (1984), 238–239; *Funct. Anal. Appl.* **8** (1974), 338–339].

1. *Introduction*

Approximations by normal forms and especially linear approximations are convenient simplifications of a physical system. Most of the physical systems are assumed to be analytic. The study of analytic conjugation was initiated by Poincaré and extended in numerous works. However, it was shown in the work of Bruno [**6**] that the class of systems which admits analytic reduction to the Poincaré–Dulac normal form is small even in $\mathbb{R}^n$. For that reason we study this problem in the class of C^∞ transformations.

This question was studied in numerous works (for example, in [**1, 7, 13**]). The first step in the proofs of these results in $\mathbb{R}^n$ is the condition of formal solvability in C^∞. The latter generates an algebraic expression, called the resonance condition. For a generic diffeomorphism, the non-resonance assumption is a necessary condition for removing nonlinear terms.

Although resonance is an obstacle for a smooth conjugation in dimensions higher than two, for diffeomorphisms on $\mathbb{R}^n$, whose hyperbolic linear part consists of only a

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contraction (or only an expansion), assumption of resonance can be removed (see [**8**] or [**5**] for a more general result). The situation is qualitatively different in Banach spaces. It was shown in the work of [**11**] that the non-resonance assumption in Banach space is essential even for contractions. The authors of the article construct an example of a contraction diffeomorphism in infinite dimensions that is not C^1 linearizable.

In this paper we study the question of smooth conjugation of diffeomorphisms with linear part Λ. The diffeomorphisms are defined on a Banach space B. We assume that they have invariant smooth non-contracting and non-expanding manifolds $W_{U,0}$ and $W_{S,0}$, such that their intersection $W_{U,0} \cap W_{S,0} = W_0$ is the central manifold. If Λ is hyperbolic, then $W_{U,0} \cap W_{S,0} = \{0\}$.

It is also interesting to consider the situation when $W_{U,0} \cap W_{S,0}$ is a set of parameters of the family of diffeomorphisms with hyperbolic linear part. In this case, $B = B_1 \times B_2$ and

$$F(x, t) = \begin{cases} \Lambda x + f(x, t), \\ t, \end{cases}$$

for $x \in B_1$ and $t \in B_2$.

If we restrict our attention to the question of smooth dependence of linearizing homeomorphisms on the parameter, we do not need to make a non-resonance assumption for some class of such diffeomorphisms. It was shown in [**10**] that for a family of diffeomorphisms with hyperbolic linear part and with nonlinearities of the second order, there exists a family of linearizing homeomorphisms which have C^1 dependence on the parameter. Unfortunately, the methods developed in [**10**] can only be used for diffeomorphisms with resonances. Also, we only proved C^1 differentiability. The proof for higher regularity would be very routine.

In this paper we show that, subject to formal conjugation along the central manifold, diffeomorphisms admit a C^∞ local conjugation. As a consequence, we deduce that subject to the non-resonance condition, there exists a C^∞ local linearization of the family.

One of the technical difficulties is the question of C^∞ extension of a local map to the whole space B. It is connected to the question of existence of 'hat' functions, i.e. $\tau(y) \in C^\infty$ with bounded support. (One can also see the assumption regarding the 'hat' function in the proof of C^1 conjugation for contracting diffeomorphisms on Banach spaces in [**12**].) If $\tau(y)$ is such a function, then the extension of f, call it $\tilde{f}$, can be defined as $\tilde{f}(y) = f(\tau(y) \cdot y)$. In this expression $\tau(y) \cdot y$ defines a map $K(y) : B \to B$ ($K(y) = \tau(y) \cdot y$) with the following properties:

(1) $K(y) = y$ for $\|y\| < 1$;

(2) $K \in C^\infty$ and $\|K^{(j)}(y)\| \leq c_j$, $j = 0, 1, \ldots$.

Therefore, we can see that for the existence of the extension $\tilde{f}$ it is enough to have map K, defined on the Banach space.

It turns out that there exist Banach spaces (for example, $C([0, 1])$) without 'hat' functions, but with K maps. Moreover, the question of the extension of a map remains open. It is not known whether there are Banach spaces without K maps.

The methods used in this paper were developed in the two works of Belitskii [**2, 3**]. The technique of the first article allows us to consider the problem in Banach spaces, where we use a special metric, induced by a countable collection of norms. The special

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metric (with 'inner' contraction) permits us to use the Banach Contraction Principle on the bounded set of smooth maps. The idea to use the Banach Contraction Principle for the study of diffeomorphisms with non-hyperbolic linear part comes from the second paper [**3**], where the families of germs are studied.

2. *The main theorem*

Let us consider a linear operator Λ on a Banach space B. Denote by $L_{-,0}$, L_-, $L_{+,0}$ and L_+ the linear subspaces of B such that $|\sigma(\Lambda_{|L_{-,0}})| \leq 1$, $|\sigma(\Lambda_{|L_-})| < 1$, $|\sigma(\Lambda_{|L_{+,0}})| \geq 1$ and $|\sigma(\Lambda_{|L_-})| > 1$. (Here, $\sigma(\Lambda_{|L_{\pm,0}})$ and $\sigma(\Lambda_{|L_\pm})$ are the spectrums of the restrictions of the operator Λ on $L_{\pm,0}$ and $L_\pm$.) The linear operator Λ divides B into non-contracting and non-expanding subspaces $L_{+,0}$ and $L_{-,0}$ (see, for example, [**9**]). Assume that there exist invariant contracting, expanding and central manifolds W_U, W_S and W_0 of F which are C^∞ graphs over L_+, L_- and L_0, respectively.

Since $W_{U,0}$ and $W_{S,0}$ are C^∞ graphs over $L_{+,0}$ and $L_{-,0}$, we can smoothly straighten $W_{U,0}$ and $W_{S,0}$. That is, we assume that $W_{U,0} = L_{+,0}$ and $W_{S,0} = L_{-,0}$. Also, let us write $B = B_1 \times B_2$, where $B_1 = L_- \oplus L_+$ and $B_2 = L_0$.

Let us assume that there exist K_+, K_0 and K_-, defined on L_+, L_0 and L_-, respectively, with the properties 1 and 2, discussed above. Then, it is clear that the map $K : B \to B$ can be defined as $K = (K_+, K_0, K_-)$. However, it is unknown whether the existence of K on B implies the existence of $K_\pm$, K_0 on the subspaces. Thus, assuming that there exist maps $K_\pm$, K_0 we can prove the following theorem.

THEOREM 1. *Assume that there exist C^∞ invariant expanding, contracting and central manifolds, and there exist maps $K_\pm$ and K_0 on $L_\pm \subset B_1$ and on $L_0 \subset B_2$. Let*

$$F(y) = \Lambda y + f(y)$$

and

$$G(y) = \Lambda y + g(y)$$

be two $C^\infty(B)$ diffeomorphisms. Assume that $f(y) = o(\|y\|)$, $g(y) = o(\|y\|)$ and $f(y) - g(y) = o((\mathrm{dist}(y, L_0))^n)$ for all $n \in \mathbb{N}$, as $\mathrm{dist}(y, L_0) \to 0$.

Then, there exists a $C^\infty(B)$ diffeomorphism $H(y) = y + h(y)$, with $h(y) = o(\|y\|)$, as $\|y\| \to 0$ and $h(y) = o((\mathrm{dist}(y, L_0))^n)$ for all $n \in \mathbb{N}$, as $\mathrm{dist}(y, L_0) \to 0$ (non-uniformly in n), such that for small values of $\|y\|$

$$H \circ F = G \circ H. \tag{1}$$

Proof. We are interested in the local properties of the maps F and G. Thus, we can modify the maps F and G with the help of K, so that the new maps coincide with the original maps in a small δ neighborhood of the origin. From now on we assume that

$$F(y) = \Lambda y + f(\delta \cdot K(y/\delta)) \tag{2}$$

and

$$G(y) = \Lambda y + g(\delta \cdot K(y/\delta)). \tag{3}$$

The first derivatives of the nonlinear parts of these maps are close to zero and all other derivatives are bounded. Thus, we have obtained the desired extensions of these maps to B.

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It is enough to prove the existence of a C^∞ solution h for the equation

$$(\mathrm{Id} + h) \circ (\Lambda + f) = (\Lambda + g) \circ (\mathrm{Id} + h). \tag{4}$$

Consider closed subspaces of C^∞ maps:

$$\mathcal{E}_{L_{+,0}} = \{\phi \in C^\infty(B) \mid \phi^{(k)}_{y \in L_{+,0}} = 0,\ k \in \mathbb{N}\}$$

and

$$\mathcal{E}_{L_{-,0}} = \{\phi \in C^\infty(B) \mid \phi^{(k)}_{y \in L_{-,0}} = 0,\ k \in \mathbb{N}\}.$$

We say that ϕ is flat on $L_{-,0}$ (or $L_{+,0}$), if $\phi \in \mathcal{E}_{L_{-,0}}$ (or $\phi \in \mathcal{E}_{L_{+,0}}$).

Equation (4) can be reduced to

$$h(\Lambda y + f(y)) - \Lambda h(y) - (g(y + h(y)) - g(y)) = g(y) - f(y). \tag{5}$$

The right-hand side of this equation (call it γ) is flat on L_0. By Lemma 10, γ can be written as a sum of two maps, γ_+ and γ_-, flat on $L_{-,0}$ and $L_{+,0}$, respectively. We search for a solution h of (5) as a sum of two solutions, $h_+ \in \mathcal{E}_{L_{-,0}}$ and $h_- \in \mathcal{E}_{L_{+,0}}$, where h_+ satisfies

$$h_+(\Lambda y + f(y)) = \Lambda h_+(y) + g(y + h_+(y)) - g(y) + \gamma_+(y). \tag{6}$$

Then, for every solution h_+, the second addend, h_-, will satisfy the equation

$$h_-(\Lambda y + f(y)) = \Lambda h_-(y) + g(y + h_-(y) + h_+(y)) - g(y + h_+(y)) + \gamma_-(y).$$

If we put

$$\tilde{g}(y + h_-(y)) := g(y + h_-(y) + h_+(y))$$

and

$$(\Lambda y + f(y))^{-1} = \Lambda^{-1} y + \tilde{f}(y),$$

we can see that h_- must satisfy the equation

$$\begin{aligned} h_-(y) = {} & \Lambda h_-(\Lambda^{-1} y + \tilde{f}(y)) + \tilde{g}(\Lambda^{-1} y + \tilde{f}(y) + h_-(\Lambda^{-1} y + \tilde{f}(y))) \\ & - \tilde{g}(\Lambda^{-1} y + \tilde{f}(y)) + \gamma_-((\Lambda y + \tilde{f}(y))^{-1}). \end{aligned} \tag{7}$$

It is also easy to see that if we define $\tilde{\tilde{g}}(\Lambda y + f(y)) := -\Lambda^{-1} g(y)$, Equation (6) is equivalent to

$$\begin{aligned} h_+(y) = {} & \Lambda^{-1} h_+(\Lambda y + f(y)) \\ & + \tilde{\tilde{g}}(\Lambda y + f(y) + h_+(\Lambda y + f(y))) - \tilde{\tilde{g}}(\Lambda y + f(y)) - \Lambda^{-1} \gamma_+(y). \end{aligned} \tag{8}$$

The existence of solutions to (8) and (7) follows from Lemma 6. □

It is known that one can define a 'hat' function on a Hilbert space. It can be defined in the following way: $\tau(y) := \tilde{\tau}(\|y\|^2)$, where $\tilde{\tau}(t)$ is a 'hat' function in $\mathbb{R}$. Moreover, there exist maps $K_\pm$, K_0, which can be defined as $K_\pm := (y_\pm \cdot \tau(y_\pm))$, $K_0 := (y_0 \cdot \tau(y_0))$ and there exists a map $K := (K_+, K_-, K_0)$. Then, the following corollary follows immediately from Theorem 1.

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COROLLARY 2. *Assume that B is a Hilbert space. Let*

$$F(y) = \Lambda y + f(y)$$

and

$$G(y) = \Lambda y + g(y)$$

be two $C^\infty(B)$ diffeomorphisms. Assume that there exist C^∞ expanding, contracting and central invariant manifolds. Also, assume that $f(y) = o(\|y\|)$, $g(y) = o(\|y\|)$ and $f(y) - g(y) = o((\mathrm{dist}(y, L_0))^n)$ for all $n \in \mathbb{N}$, as $\mathrm{dist}(y, L_0) \to 0$ (non-uniformly in n).

Then, there exists a $C^\infty(B)$ diffeomorphism $H(y) = y + h(y)$ with $h(y) = o(\|y\|)$, $h(y) = o((\mathrm{dist}(y, L_0))^n)$ for all $n \in \mathbb{N}$, as $\mathrm{dist}(y, L_0) \to 0$, such that for small values of $\|y\|$

$$H \circ F = G \circ H. \tag{9}$$

Remark 3. Note that there exists K on $C([0, 1])$. It can be defined as $K(x)(t) = \tau(x(t)) \cdot x(t)$. However, it is unclear whether there exist $K_\pm$, K_0 on the subspaces.

Remark 4. In fact, our proofs require the existence of only two K maps: one K map, defined on the entire space B (for an extension of a diffeomorphism), and another K map, defined on one of the subspaces L_+ or L_- (for positive and negative splitting of some map, flat on L_0). If a diffeomorphism does not have either a contracting or an expanding part, then, we do not need to split the flat map into the positive and negative parts.

Remark 5. Let $B = C([0, 1])$. Consider purely non-contracting (or non-expanding) diffeomorphisms on B. Also, suppose that $\sigma(\Lambda_{|L_-})$ (or $\sigma(\Lambda_{|L_+})$) is closed. Then, if the two diffeomorphisms formally conjugate, they smoothly conjugate in the class $C^\infty(B)$.

The same conclusion is true if at least one of the above-mentioned subspaces is finite.

3. *Solvability in flat maps*

The following lemma is presented for the 'non-positive' subspaces $L_{-,0}$, $\mathcal{E}_{L_{-,0}}$, etc. Proof for the 'non-negative' subspaces is similar.

LEMMA 6. *Consider the equation, defined on a Banach space B,*

$$\begin{aligned}\phi(y) = {} & A\phi(\Lambda y + f(y)) \\ & + g(\Lambda y + f(y) + \phi(\Lambda y + f(y))) - g(\Lambda y + f(y)) + \rho(y)\end{aligned} \tag{10}$$

with a linear operator A, invertible linear operator Λ, a map $f \in C^\infty(B)$, $f(y) = o(\|y\|)$, as $\|y\| \to 0$ and a map ρ, which is flat on the linear non-expanding invariant manifold $L_{-,0} \subset B$ of Λ. Then, there exists flat on $L_{-,0}$ solution $\phi \in C^\infty(B)$.

Proof. We search for a fixed point of the operator T, defined by the right-hand side of (10):

$$\begin{aligned}T_\phi = {} & A\phi(\Lambda y + f(y)) \\ & + g(\Lambda y + f(y) + \phi(\Lambda y + f(y))) - g(\Lambda y + f(y)) + \rho(y).\end{aligned} \tag{11}$$

Consider the space $\mathcal{E}_{L_{-,0}}$ of smooth maps, flat on the manifold $L_{-,0}$. To simplify our notation, we call it $\mathcal{E}$.

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Define a norm

$$N_{k,p}(\phi) := \max_{0\le j\le k} \sup\left\{ \frac{\|\phi^{(j)}(y)\|}{\|\xi\|^p}, \phi\in\mathcal{E},\ \xi\in L_-,\ y\in B \right\}.$$

PROPOSITION 7. *There exist a real number $q<1$, a closed, invariant under T subset $C\subset\mathcal{E}$ and sequences of $b_{k,p}\in\mathbb{R}_+$ and $p_k\in\mathbb{N}$, such that for all $p,k\in\mathbb{N}$, $p>p_k$ and $\phi,\psi\in C$:*

(1) $N_{0,p}(T_\phi)\le qN_{0,p}(\phi)+N_{0,p}(\rho)$,
$N_{k,p}(T_\phi)\le qN_{k,p}(\phi)+b_{k,p}N_{k-1,p}(\phi)+N_{k,p}(\rho)$ *for* $k=1,2,\ldots$ *and;*

(2) $N_{0,p}(T_\phi-T_\psi)\le qN_{0,p}(\phi-\psi)$,
$N_{k,p}(T_\phi-T_\psi)\le qN_{k,p}(\phi-\psi)+b_{k,p}N_{k-1,p}(\phi-\psi)$ *for* $k=1,2,\ldots$.

Proof.

$$N_{0,p}[T_\phi]=N_{0,p}[A\phi(\Lambda y+f(y))+\int_0^1 g'_{\sigma(s)}\,ds\cdot\phi(\Lambda y+f(y))+\rho(y)]$$
$$\le\left(\|A\|+\sup_y\|g'(y)\|\right)\cdot\ N_{0,p}[\phi(\Lambda y+f(y))]+N_{0,p}[\rho].$$

Here $\sigma:[0,1]\to B$ denotes a path, connecting the points

$$\sigma(0)=(\Lambda y+f(y)+\phi(\Lambda y+f(y)))$$

and

$$\sigma(1)=(\Lambda y+f(y)+\psi(\Lambda y+f(y))).$$

Let us recall that choosing sufficiently small δ in the K function, we can make $\sup_y\|g'(y)\|$ less than an arbitrary Δ. (See formula (3).) Then,

$$N_{0,p}[T_\phi]\le(\|A\|+\Delta)\cdot\ N_{0,p}[\phi(y)]\cdot\left(\sup_{y,\,\xi}\frac{\|(\Lambda y+f(y))_{|L_-}\|}{\|\xi\|}\right)^p+N_{0,p}[\rho].$$

Since the spectrum of $\Lambda_{|L_-}$ is less than one, and with sufficiently small δ in the K function, we can make $\|f(y)\|$ as small as we want (see formula (2)); so there exists $\alpha<1$ such that

$$\sup_{y,\xi}\frac{\|(\Lambda y+f(y))_{|L_-}\|}{\|\xi\|}<\alpha<1.$$

Then there exists p_0 such that for all $p>p_0$

$$(\|A\|+\Delta)\cdot\left(\sup_{y,\xi}\frac{\|(\Lambda y+f(y))_{|L_-}\|}{\|\xi\|}\right)^p<q\ll 1.$$

Thus,

$$N_{0,p}[T_\phi]\le qN_{0,p}[\phi]+N_{0,p}[\rho].$$

The estimates for $k=1$ are more complex.

$$N_{1,p}[T_\phi]=N_{0,p}[((A+\sup_y\|g'(y)\|)(\phi'(\Lambda y+f(y)))$$
$$+\ \sup_y\|g''(y)\|\cdot\phi(\Lambda y+f(y)))\cdot(\Lambda y+f(y))']+N_{1,p}[\rho]$$
$$\le[(\|A\|+\Delta)\cdot N_{1,p}[\phi]+\sup_y\|g''(y)\|\cdot N_{0,p}[\phi]]$$
$$\cdot\left(\sup_{y,\,\xi}\frac{\|(\Lambda y+f(y))_{|L_-}\|}{\|\xi\|}\right)^p\sup_y\|(\Lambda y+f(y))'\|+N_{1,p}[\rho].$$

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Again, there exists p_1 such that for all $p > p_1$

$$N_{1,p}[T_\phi] \le qN_{1,p}[T_\phi] + \sup_y \|g''(y)\| \cdot N_{0,p}[T_\phi] + N_{1,p}[\rho].$$

In the general case

$$N_{j,p}[T_\phi] \le [(\|A\| + \Delta) \cdot N_{j,p}[\phi] + R_{j,p}(N_{j-1,p}[\phi])] \cdot \left(\sup_{y,\,\xi} \frac{\|(\Lambda y + f(y))_{|L_-}\|}{\|\xi\|}\right)^p \sup_y \|(\Lambda y + f(y))'\| + N_{j,p}[\rho],$$

where $R_{j,p}$ is a polynomial function with positive coefficients. Then, for sufficiently large p_k and $p > p_k$,

$$N_{j,p}[T_\phi] \le qN_{j,p}[T_\phi] + R_{j,p}(N_{j-1,p}[T_\phi]).$$

Choose a number $q_0 \in (q, 1)$. Let

$$a_{0,p} := \frac{N_{0,p}[\rho]}{q_0 - q} \quad \text{and} \quad a_{j,p} := \frac{R_{j,p}(a_{j-1,p})}{q_0 - q}.$$

Define a closed set

$$C := \{\phi \in \mathcal{E} \mid N_{j,p}[\phi] \le a_{j,p}\}.$$

Then, it is easy to see that $T(K) \subset K$. Indeed, for every $\phi \in C$

$$N_{0,p}[T(\phi)] \le a_{0,p}$$

and

$$N_{j,p}[T(\phi)] \le qN_{j,p}[\phi] + a_{j,p}(q_0 - q) < a_{j,p}.$$

We have proved the first part of the lemma and we are starting the proof of the second pair of inequalities:

$$\begin{aligned} N_{0,p}[T_\phi - T_\psi] &= N_{0,p}\Big[A\phi(\Lambda y + f(y)) - A\psi(\Lambda y + f(y)) \\ &\quad + \int_0^1 g'_{\sigma(s)}\, ds \cdot (\phi(\Lambda y + f(y) - \psi(\Lambda y + f(y))\Big] \\ &\le \left(\|A\| + \sup_y \|g'(y)\|\right) \cdot N_{0,p}[\phi(\Lambda y + f(y))] \\ &\le (\|A\| + \Delta) \cdot \left(\sup_{y,\,\xi} \frac{\|(\Lambda y + f(y))_{|L_-}\|}{\|\xi\|}\right)^p \cdot N_{0,p}[\phi - \psi]. \end{aligned}$$

For the given $q < 1$ there exists p_0 such that for all $p > p_0$

$$N_{0,p}[T_\phi - T_\psi] < qN_{0,p}[\phi - \psi].$$

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Now we can take the first derivative and use the fact that T maps C into itself:

$$\begin{aligned} N_{1,p}[T_\phi - T_\psi] &= N_{1,p}\Bigg[A(\phi(\Lambda y + f(y)) - \psi(\Lambda y + f(y))) \\ &\quad + \int_0^1 g'_{\sigma(s)}\, ds \cdot (\phi(\Lambda y + f(y)) - \psi(\Lambda y + f(y)))\Bigg] \\ &\le \Bigg(\left(\|A\| + \sup_y \|g'\|\right) N_{1,p}[\phi(y) - \psi(y)] \\ &\quad + (1 + N_{1,0}(\phi)) \sup_y \|g''\|\, N_{0,p}[\phi(y) - \psi(y)]\Bigg) \\ &\quad \cdot \left(\sup_{y,\,\xi} \frac{\|(\Lambda y + f(y))_{|L_-}\|}{\|\xi\|}\right)^p \cdot \|(\Lambda y + f(y))'\|. \end{aligned}$$

For $\phi \in C$ and for sufficiently large p (p is greater than some p_1), we obtain the desired estimate:

$$N_{1,p}[T_\phi - T_\psi] \le q N_{1,p}[\phi(y) - \psi(y)] + b_{1,p} N_{0,p}[\phi(y) - \psi(y)].$$

For $k > 1$

$$\begin{aligned} N_{k,p}[T_\phi - T_\psi] &= N_{k,p}\Bigg[A(\phi(\Lambda y + f(y)) - \psi(\Lambda y + f(y))) \\ &\quad + \int_0^1 g'_{\sigma(s)}\, ds \cdot (\phi(\Lambda y + f(y)) - \psi(\Lambda y + f(y)))\Bigg] \\ &\le ((\|A\| + \|g'\|) \cdot N_{k,p}[\phi(y) - \psi(y)] \\ &\quad + Q_{k,p}(N_{k,0}(\phi),\, N_{k,0}(\psi)) \cdot N_{k-1,p}[\phi(y) - \psi(y)]) \\ &\quad \cdot \left(\sup_{y,\xi} \frac{\|(\Lambda y + f(y))_{|L_-}\|}{\|\xi\|}\right)^p \cdot \|(\Lambda y + f(y))'\|^k. \end{aligned}$$

Here, $Q_{k,p}$ is a polynomial function with positive coefficients. Then, there exist positive constants $b_{k,p}$ and p_k, such that for all $p > p_k$ and for ϕ and $\psi \in C$

$$N_{k,p}(T_\phi - T_\psi) < q N_{k,p}(\phi - \psi) + b_{k,p} N_{k-1,p}(\phi - \psi).$$

Thus, we proved part 2. □

Using the above estimates, we introduce a new set of norms $\|\cdot\|_{k,p}$, defined with the help of $N_{k,p}$, such that the operator T is a contraction in $\|\cdot\|_{k,p}$.

Let $\|\cdot\|_{k,p}$ be defined on $\mathcal{E}$ according to the following rule:

$$\begin{aligned} &\|\phi\|_{0,p} := N_{0,p}(\phi) \\ &\|\phi\|_{1,p} := N_{1,p}(\phi) + A^1_{1,p} N_{0,p}(\phi) \\ &\quad\vdots \\ &\|\phi\|_{k,p} := N_{k,p}(\phi) + A^1_{k,p} N_{k-1,p}(\phi) + A^2_{k,p} N_{k-2,p}(\phi) + \cdots + A^k_{k,p} N_{0,p}(\phi). \end{aligned}$$

Choosing appropriate constants $A^j_{k,p}$, one can show that the operator T is a contraction on C in each $\|\cdot\|_{k,p}$ norm.

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PROPOSITION 8. *Let q be as in Proposition 7. Fix $\epsilon > 0$ such that $q + \epsilon < 1$. Let the constants $A^j_{k,p}$ $(j = 1, \dots, k)$ in the definition of the norms $\|\cdot\|_{k,p}$ satisfy the following inequalities:*

$$\begin{aligned} A^1_{k,p} &> \frac{b_{k,p}}{\epsilon} \\ A^2_{k,p} &> \frac{A^1_{k,p} b_{k-1,p}}{\epsilon} \\ &\vdots \\ A^k_{k,p} &> \frac{A^{k-1}_{k,p} b_{1,p}}{\epsilon}. \end{aligned} \tag{12}$$

Then, for each pair of ϕ, $\psi \in C$ and for each norm $\|\cdot\|_{k,p}$ $(p > p_k)$, the operator T admits the following estimates:

(1′) $\|T_\phi\|_{k,p} < (q + \epsilon)\|\phi\|_{k,p} + \|\rho\|_{k,p}$; *and*

(2′) $\|T_\phi - T_\psi\|_{k,p} < (q + \epsilon)\|\phi - \psi\|_{k,p}$.

Proof. We begin with the second part of the proposition. In this proof we use induction on the k-index:

$$\|T_\phi - T_\psi\|_{0,p} = N_{0,p}(T_\phi - T_\psi) < qN_{0,p}(\phi - \psi) = q\|\phi - \psi\|_{0,p}.$$

Thus, it is a contraction in $\|\cdot\|_{0,p}$.

$$\begin{aligned} \|T_\phi - T_\psi\|_{1,p} &= N_{1,p}(T_\phi - T_\psi) + A^1_{1,p} N_{0,p}(T_\phi - T_\psi) \\ &< qN_{1,p}(\phi - \psi) + b_{1,p} N_{0,p}(\phi - \psi) + A^1_{1,p} qN_{0,p}(\phi - \psi) \\ &= qN_{1,p}(\phi - \psi) + (A^1_{1,p} q + b_{1,p}) N_{0,p}(\phi - \psi) \\ &< (q + \epsilon)N_{1,p}(\phi - \psi) + A^1_{1,p}(q + \epsilon) N_{0,p}(\phi - \psi), \end{aligned}$$

if $A^1_{1,p} > b_{1,p}/\epsilon$. Finally, note that

$$(q + \epsilon)N_{1,p}(\phi - \psi) + A^1_{1,p}(q + \epsilon) N_{0,p}(\phi - \psi) = (q + \epsilon)\|\phi - \psi\|_{1,p}.$$

Thus, in this norm, the operator T is also a contraction.

Assume that the assertion of the proposition is correct for all $i = 1, \dots, k - 1$. We prove it for $i = k$, i.e. we estimate $\|T_\phi - T_\psi\|_{k,p}$:

$$\begin{aligned} \|T_\phi - T_\psi\|_{k,p} &= N_{k,p}(T_\phi - T_\psi) + A^1_{k,p} N_{k-1,p}(T_\phi - T_\psi) \\ &\quad + A^2_{k,p} N_{k-2,p}(T\phi - T\psi) + \cdots + A^k_{k,p} N_{0,p}(T\phi - T\psi) \\ &< qN_{k,p}(\phi - \psi) + b_{k,p} N_{k-1,p}(\phi - \psi) \\ &\quad + A^1_{k,p}[qN_{k-1,p}(\phi - \psi) + b_{k-1,p} N_{k-2,p}(\phi - \psi)] \\ &\quad + A^2_{k,p}[qN_{k-2,p}(\phi - \psi) + b_{k-2,p} N_{k-3,p}(\phi - \psi)] \\ &\quad \vdots \\ &\quad + A^{k-1}_{k,p}[qN_{1,p}(\phi - \psi) + b_{1,p} N_{0,p}(\phi - \psi)] \end{aligned}$$

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$$
\begin{aligned}
&+ A^k_{k,p}\, q N_{0,p}(\phi-\psi)\\
&= qN_{k,p}(\phi-\psi)\\
&\quad + (A^1_{k,p}q + b_{k,p})N_{k-1,p}(\phi-\psi)\\
&\quad + (A^2_{k,p}q + A^1_{k,p}b_{k-1,p})N_{k-2,p}(\phi-\psi)\\
&\quad \vdots\\
&\quad + (A^{k-1}_{k,p}q + A^{k-2}_{k,p}b_{2,p})N_{1,p}(\phi-\psi)\\
&\quad + (A^k_{k,p}q + A^{k-1}_{k,p}b_{1,p})N_{0,p}(\phi-\psi).
\end{aligned}
$$

From inequalities (12) it immediately follows that

$$(A^1_{k,p}q + b_{k,p}) < (q+\epsilon)A^1_{k,p}$$

and

$$A^{k-j}_{k,p}q + A^{k-j-1}_{k,p}b_{j+1,p} < (q+\epsilon)A^{k-j}_{k,p}$$

for $j = 0, \ldots, k-2$. Then,

$$
\begin{aligned}
\|T_\phi - T_\psi\|_{k,p} &< qN_{k,p}(\phi-\psi)\\
&\quad + A^1_{k,p}(q+\epsilon)N_{k-1,p}(\phi-\psi)\\
&\quad + A^2_{k,p}(q+\epsilon)N_{k-2,p}(\phi-\psi)\\
&\quad \vdots\\
&\quad + A^{k-1}_{k,p}(q+\epsilon)N_{1,p}(\phi-\psi)\\
&\quad + A^k_{k,p}(q+\epsilon)N_{0,p}(\phi-\psi),\\
&< (q+\epsilon)[N_{k,p} + A^1_{k,p}N_{k-1,p} + \cdots + A^k_{k,p}N_{0,p}]\\
&= (q+\epsilon)\|\phi-\psi\|_{k,p}.
\end{aligned}
$$

Thus, the first part of the theorem is proved. The second part follows immediately from the first part. Indeed,

$$\|T_\phi\|_{k,p} = \|T_\phi - (T_0 - \rho)\|_{k,p} \le \|T_\phi - T_0\|_{k,p} + \|\rho\|_{k,p} \le (q+\epsilon)\|\phi\|_{k,p} + \|\rho\|_{k,p}.$$

□

Now, we define a closed disc, such that it is invariant under the action of T. Choose a sequence of real positive numbers $c_{k,p}$, and define the closed disc D in the space $\mathcal{E}$ as

$$D = \{\phi \in C : \|\phi\|_{k,p} \le c_{k,p}\}.$$

Assume that for some small $\epsilon > 0$, the constants $c_{k,p}$ are such that

$$c_{k,p} \ge \frac{\|\rho\|_{k,p}}{1-q-\epsilon}.$$

COROLLARY 9. *We have*

$$T(D) \subset D.$$

Proof. The proof is an immediate consequence of the estimate (1′) in Proposition 8. □

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Thus, we have proved that in the countable collection of norms $\|\cdot\|_{k,p}$, the operator T is a contraction that maps the closed disc D into itself. Using these inequalities, we can now construct a metric on $\mathcal{E}$ with the properties, similar to those, discussed for the norms $\|\cdot\|_{k,p}$. Then, we can use Banach Contraction Principle to show the existence of a fixed point of the operator T in the disc D.

Let us define the metric on D. Fix a positive $R \in \mathbb{R}$. Let $\epsilon_{k,p} < 1$ be a sequence of positive real numbers, such that $c_{k,p}\epsilon_{k,p} < R$ (here $c_{k,p}$ are the constants that define the disc D). Then,

$$\rho(\phi, \psi) := \sum_{k,p>p_k} \frac{1}{2^{kp}} \cdot \frac{\epsilon_{k,p}\|\phi - \psi\|_{k,p}}{1 + \epsilon_{k,p}\|\phi - \psi\|_{k,p}}, \quad \phi, \psi \in D.$$

Obviously, D is a complete metric space.

Consider a function $\kappa(t) = t/(1+t)$, defined for positive values of t. This is an increasing function; it is less than one for bounded values of t. Put

$$t := \epsilon_{k,p}\|\phi - \psi\|_{k,p} < R.$$

Since T is a contraction in $\|\cdot\|_{k,p}$, and $\kappa(t)$ is an increasing function,

$$\rho(T_\phi, T_\psi) < \sum_{k,p>p_k} \frac{1}{2^{kp}} \cdot \frac{(q+\epsilon)\epsilon_{k,p}\|\phi - \psi\|_{k,p}}{1 + (q+\epsilon)\epsilon_{k,p}\|\phi - \psi\|_{k,p}}.$$

Also for bounded values of t there exists $0 < r < 1$ such that

$$\kappa((q+\epsilon)t) < r\kappa(t).$$

In other words,

$$\frac{(q+\epsilon)\epsilon_{k,p}\|\phi - \psi\|_{k,p}}{1 + (q+\epsilon)\epsilon_{k,p}\|\phi - \psi\|_{k,p}} < r\frac{\epsilon_{k,p}\|\phi - \psi\|_{k,p}}{1 + \epsilon_{k,p}\|\phi - \psi\|_{k,p}}.$$

We proved that $T : D \to D$ acts as a contraction operator on D:

$$\rho(T_\phi, T_\psi) < r\rho(\phi, \psi).$$

The Banach Contraction Principle implies the existence of a fixed point in D. This fixed point is the solution of (11). □

In the proof of Theorem 1 we used the fact that any map, which is flat on the central manifold can be represented as a sum of two maps, flat on the linear subspaces. In the next lemma we discuss how to split γ into the sum of γ_+ and γ_-.

LEMMA 10. *Assume that there exist C^∞ invariant expanding, central and contracting subspaces of B: L_+, L_0 and L_-. Let $\gamma(y) \in C^\infty(B)$ be flat on L_0. Suppose that there exists a C^∞ K map on L_- or on L_+. Then, there exist maps γ_+ and γ_-, such that $\gamma = \gamma_- + \gamma_+$, and γ_+ is flat on $L_{-,0}$ and γ_- is flat on $L_{+,0}$.*

Proof. Let us write $y = (\xi, \eta, t) \in B$ with $\xi \in L_+$, $\eta \in L_-$ and $t \in L_0$. For definiteness, let us assume that there exists C^∞ K map on $L_{-,0}$. Then, we can explicitly construct γ_+ map.

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Let $\delta_j \neq 0$ be so small that the following series converges in C^∞ topology:

$$\gamma_+ := \sum_{j=0}^{\infty} \frac{\partial^j \gamma}{\partial \eta^j}(\xi, 0, t)(K(\delta_j^{-1}\eta), \ldots, K(\delta_j^{-1}\eta)) \cdot \frac{\delta_j^j}{j!}.$$

Define γ_- as

$$\gamma_- = \gamma - \gamma_+.$$

Then, it is easy to check that γ_+ is flat on $L_{-,0}$ and γ_- is flat on $L_{+,0}$. Indeed,

$$\begin{aligned}\frac{\partial^p \gamma_+}{\partial \xi^p}(0, \eta, t) &= \frac{\partial^p}{\partial \xi^p}\left(\sum_{j=0}^{\infty} \frac{\partial^j \gamma}{\partial \eta^j}(0, 0, t)(K(\delta_j^{-1}\eta), \ldots, K(\delta_j^{-1}\eta)) \cdot \frac{\delta_j^j}{j!}\right) \\ &= \frac{\partial^p}{\partial \xi^p}\left(0(K(\delta_j^{-1}\eta), \ldots, K(\delta_j^{-1}\eta)) \cdot \frac{\delta_j^j}{j!}\right) = 0.\end{aligned}$$

Also,

$$\frac{\partial^q \gamma_-}{\partial \eta^q}(\xi, 0, t) = \frac{\partial^q \gamma}{\partial \eta^q}(\xi, 0, t) - \sum_{j=0}^{\infty} \frac{\partial^q}{\partial \eta^q}\left(\frac{\partial^j \gamma}{\partial \eta^j}(\xi, 0, t)(K(\delta_j^{-1}\eta), \ldots, K(\delta_j^{-1}\eta))\right) \cdot \frac{\delta_j^j}{j!}.$$

Recall that $K^{(j)}(0) = \mathbf{0}$ for $j \neq 1$ and $K'(0) = \mathbf{1}$. Then,

$$\frac{\partial^q \gamma_-}{\partial \eta^q}(\xi, 0, t) = \frac{\partial^q \gamma}{\partial \eta^q}(\xi, 0, t) - \frac{\partial^q \gamma}{\partial \eta^q}(\xi, 0, t) \cdot \mathbf{1} \cdot \delta_1^{-1} \cdot \delta_1^1 = 0. \qquad \square$$

4. *Linearization*

In this section we prove that a linearization is a C^∞ map on a Banach space, if the non-resonance condition is satisfied.

We now formally define the resonance for diffeomorphisms on Banach spaces. Let Λ_1 be a hyperbolic linear operator on a Banach space B_1. Consider the following linear operator L_k, defined on the space of homogeneous polynomial maps of degree k:

$$\begin{aligned}L_k &: P^{(k)}[B_1] \to P^{(k)}[B_1], \\ L_k\phi(x) &:= \Lambda_1\phi(x) - \phi(\Lambda_1 x).\end{aligned}$$

Definition 11. Suppose that for all $k \in \mathbb{N}$, $k > 1$, L_k is invertible on $P^{(k)}[B_1]$. Then, we say that the operator Λ_1 has no resonances.

In particular, when B_1 is a n-dimensional space, the non-resonance condition is equivalent to the following statement. For any $\lambda_j \in \sigma(\Lambda_1)$ and for any combination of $p_1, \ldots, p_n \in \mathbb{N}$, $p_1 + \cdots + p_n > 1$

$$\lambda_j - \lambda_1^{p_1} \ldots \lambda_n^{p_n} \neq 0.$$

Now we present C^∞ linearization theorem on Banach spaces for a family of diffeomorphisms with a hyperbolic fixed point. The proof of this theorem follows easily from Theorem 1.

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THEOREM 12. *Let*

$$F(y) = \Lambda y + f(y) = F_t(x) = \begin{cases} \Lambda_1 x + f_t(x) \\ t \end{cases}$$

be a $C^\infty(B_1 \times B_2)$ family of diffeomorphisms on the product of Banach spaces B_1 and B_2. Assume that Λ_1 is hyperbolic and $f_t(y) = o_t(\|y\|)$. Also, assume that there exist $K_\pm$ maps on the corresponding subspaces $L_\pm$, and Λ_1 has no resonances.

Then there exists a family of $C^\infty(B_1 \times B_2)$ diffeomorphisms

$$H(y) = y + h(y) = H_t(x) = \begin{cases} x + h_t(x) \\ t \end{cases}$$

with $h_t(x) = o_t(x)$, as $\|x\| \to 0$, such that for small values of $\|x\|$ and $\|t\|$

$$H_t \circ F_t = L H_t. \tag{13}$$

Proof. Let $H(y) = y + h(y)$. Then (13) can be simplified:

$$f(y) + h(\Lambda y + f(y)) = \Lambda h(y). \tag{14}$$

In order to obtain a flat residue, let us formally differentiate both sides of the hyperbolic part of this equation

$$[f_t(x) + h_t(\Lambda_1 x + f_t(x))]^{(k)}_{x=0} = \Lambda_1 h_t^{(k)}(x)_{x=0} \tag{15}$$

and solve it for $h_t^{(k)}(0)$. The solution $h_t^{(k)}(0)$ exists, if Λ_1 has no resonances.

Now, with the help of the Borel lemma, we can construct a map $\tilde{h_t}(x)$ on the whole space with the same derivatives at 0. Since there exists a K map on B, the Borel lemma is valid on the space B. The proof of the Borel lemma for Banach spaces can be found in [**4**] (these ideas were also used in [**2**]). Let us write $h_t(x)$ as the sum of $\tilde{h_t}(x)$ and a flat map $\phi_t(x)$. Then (14) can be written as

$$\Lambda_1 \phi_t(x) - \phi_t(\Lambda_1 x + f_t(x)) = f_t(x) + \tilde{h_t}(\Lambda x + f_t(x)) - \Lambda_1 \tilde{h_t}(x).$$

The right-hand side of this equation is flat at 0. Then, by Lemma 6, we can solve it for ϕ_t. □

Let us say a few words about applications of the linearization theorems. How strict is the non-resonance condition? Obviously there are many examples of non-resonant maps, if the space B has finite dimension. Here we construct a non-resonant discrete spectrum in infinite dimensions. Consider integers $\{p_j\}_{j=1}^m$ and $\{s_j\}_{j=1}^m$, which are all relatively prime. Let $r_j = p_j/s_j$. Define the spectrum of Λ as

$$\sigma(\Lambda) = \{\lambda_t = r_{j_t} e^{i\phi_t},\ j_t \in \{1, \ldots, m\},\ \phi_t \in \mathbb{T}\},$$

and

$$\Lambda = \operatorname{diag}(\lambda_t).$$

In this example, we may have infinitely many distinct eigenvalues, but their absolute values form a finite set of positive rational numbers r_j. Since the product of any combination of r_j cannot be equal to one of the r_j, the linear operator Λ has no resonances.

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We can also construct a continuous non-resonant spectrum. Suppose that the unit circle does not separate the spectrum, and $0 < a < |\sigma(\Lambda_1)| < b < \infty$. If $a > 1$, i.e. Λ is an expansion, then it is enough to assume that $a^2 > b$. Otherwise, if $b < 1$, so that Λ is a contraction, it is enough to assume that $b^2 < a$.

Similarly, one can construct an example with a contracting (expanding) discrete non-resonant spectrum.

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